FRENQENCY SPECTRUM OF A FM/FM SIGNAL

By J. C. HOLT

Summary. -Several authors have attempted to derive a mathematical model that will describe the frequency spectrum of a FM/FM signal.[1,2] However, to this author's knowledge, none of the mathematical models that have been published is valid when the carrier is modulated by more than one subcarrier. In this paper an expression for a FM/FM signal is derived that is valid when IRIG specifications are applied. Then this expression is manipulated into a form that will yield the frequency spectrum when the carrier is modulated by any number of subcarriers. Then an illustration of a two subcarrier frequency spectrum is presented.

Introduction. -FM/FM modulation systems with large numbers of subcarriers are used extensively today in the field of telemetry. In the design of these systems it is sometimes necessary to determine the amplitude and frequency of significant sidebands in order to predict bandwidth, calculate distortion, or otherwise measure and assess the importance of the products of modulation. The calculation of these sidebands does not follow the simple rules of linear superposition.

In this paper an expression for a FM/FM signal, subjected to certain restrictions, is derived and manipulated into a form that will yield the amplitude and phase of the individual frequency components of the signal. Then an illustration of a two subcarrier frequency spectrum is presented and discussed. This mathematical model is useful since it provides a basis for a realistic and revealing description of frequency modulation.

Mathematical Analysis. -The general expression for a frequency modulated wave is:

$$v(t) = E_C \sin\left[\omega_c t + k\int f(t)\,dt\right] \qquad (1)$$

where:

E_c = carrier voltage

ω_c = carrier frequency

k = proportionality constant

$\int f(t)\,dt$ = modulating signal

Since the purpose of this paper is to discuss a FM/FM signal, we shall let $f(t)$ be an ensemble of sinusoidal subcarriers, i.e.,

$$f(t) = \sum_{j=1}^{k} \Delta\omega_c \cos\left[\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin\omega_{m_j} t\right]$$

Mr. Holt is with the Douglas Aircraft Company, Missiles and Space Systems Division, Santa Monica, Calif.

where:

k = number of subcarriers

$\Delta\omega_c$ = carrier deviation

ω_{s_j} = frequency of the j th subcarrier

$\Delta\omega_{s_j}$ = deviation of the j th subcarrier

ω_{m_j} = frequency of the modulating signal of the j th subcarrier

A diagram of such a system is shown in Figure 1.

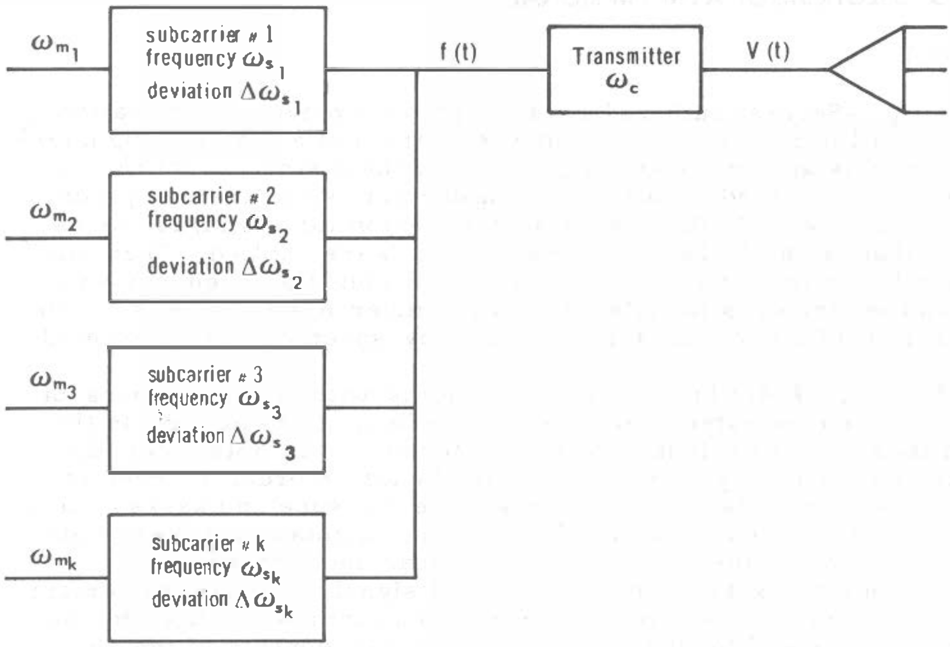

FIGURE 1

In the following paragraphs we shall proceed to integrate f(t) in order to obtain an expression for the modulating signal. We shall let:

$$I = \int f(t)\,dt = \sum_{j=1}^{k} \Delta\omega_c \int \cos\left[\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right] dt$$

By elementary trigometric separation this yields:

$$I = \Delta\omega_c \sum_{j=1}^{k} \int \left[\cos \omega_{s_j} t \cos\left(\frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right) - \sin \omega_{s_j} t \sin\left(\frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right)\right] dt$$

Then by integrating by parts we have:

$$I = \Delta\omega_c \sum_{j=1}^{k} \frac{1}{\omega_{s_j}} \cos\left(\frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right) \sin \omega_{s_j} t$$

$$- \frac{1}{\omega_{s_j}} \int \sin \omega_{s_j} t \sin\left(\frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{s_j} t\right) \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \omega_{m_j} \cos \omega_{m_j} t \, dt$$

$$+ \frac{1}{\omega_{s_j}} \sin\left(\frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right) \cos \omega_{m_j} t$$

$$- \frac{1}{\omega_{s_j}} \int \cos \omega_{s_j} t \cos\left(\frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{s_j} t\right) \cos \omega_{m_j} t \, \omega_{m_j} \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \, dt$$

By recombination we have:

$$I = \Delta\omega_c \sum \frac{1}{\omega_{s_j}} \left[\sin\left(\omega_{s_j} t - \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right) - \frac{\Delta\omega_{s_j}}{\omega_{s_j}} \int \cos \omega_{m_j} t \cos\left(\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{s_j} t\right) dt\right]$$

Now if the condition $\omega_{s_j} >> \omega_{m_j}$ is imposed the term $\cos \omega_{m_j} t$ can be considered a quasiconstant with respect to the integral and removed from the integral sign so that:

$$I = \Delta\omega_c \sum_{j=1}^{k} \frac{1}{\omega_{s_j}} \left[\sin\left(\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right) - \frac{\Delta\omega_{s_j}}{\omega_{s_j}} \cos \omega_{m_j} t \int \cos\left(\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{s_j} t\right) dt\right]$$

However, the expression in the integral is identically I so that we may write:

$$\sum_{j=1}^{k} \Delta\omega_c \left(1 + \frac{\Delta\omega_{s_j}}{\omega_{s_j}} \cos \omega_{m_j} t\right) \int \cos\left(\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{s_j} t\right) dt$$

$$= \sum_{j=1}^{k} \frac{\Delta\omega_c}{\omega_{s_j}} \sin\left(\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right)$$

Now if the condition $\Delta\omega_{s_j}/\omega_{s_j} << 1$ is also applied, then:

$$\sum_{j=1}^{k} \Delta\omega_c \int \cos\left(\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{s_j} t\right) dt = \sum_{j=1}^{k} \frac{\Delta\omega_c}{\omega_{s_j}} \sin\left(\omega_{s_j} t + \frac{\Delta\omega_{s_j}}{\omega_{m_j}} \sin \omega_{m_j} t\right)$$

and thus we have our expression for $\int f(t)\,dt$ or the modulating signal which can be submitted into expression (1) to obtain the expression for the FM/FM wave, i.e.,

$$V(t) = E_c \sin[\omega_c t + \sum_{j=1}^{k} \beta_{2_j} \sin(\omega_{s_j} t + \beta_{1_j} \sin \omega_{m_j} t)]$$

where:

$$\beta_{1_j} = \Delta\omega_{s_j}/\omega_{m_j}$$

$$\beta_{2_j} = \Delta\omega_c/\omega_{s_j}$$

This simply means that we have a sine wave modulated by k sine waves which are in turn each modulated by a sine wave. In otherwords, we have a double FM signal. Now in order to obtain the frequency spectrum we must manipulate expression (2) in such a manner that the result will be of the form:

$$V(t) = A_1 \sin \alpha_1 + A_2 \sin \alpha_2 + \cdots + A_i \sin \alpha_i + \cdots$$

where:

A_i = some amplitude factor

α_i = an argument expressed only in a linear combination of angular frequencies

With our expression in this form it is possible to say that at frequency α_i there is a sideband with energy A_i.

Since, $e^{i\chi} = \cos\chi + i\sin\chi$ then $\mathrm{Im}[e^{i\chi}] = \sin\chi$ where $\mathrm{Im}[\chi]$ denotes the imaginary part of some number χ. Then if we let:

$$\chi = \omega_c t + \sum_{j=1}^{k} \beta_{2_j} \sin(\omega_{s_j} t + \beta_{1_j} \sin \omega_{m_j} t)$$

expression (2) becomes:

$$v(t) = E_c \,\mathrm{Im}\{\exp i[\omega_c t + \sum_{j=1}^{k} \beta_{2_j} \sin(\omega_{s_j} t + \beta_{1_j} \sin \omega_{m_j} t)\}$$

or

$$v(t) = E_c \,\mathrm{Im}\{\exp[i\omega_c t]\exp[i\sum_{j=1}^{k} \beta_{2_j} \sin(\omega_{s_j} t + \beta_{1_j} \sin \omega_{m_j} t)]]$$

However $e^{\sum_i \chi_i} = \pi e^{\chi_i}$ hence,

$$v(t) = E_c \,\mathrm{Im}\{\exp(i\omega_c t) \prod_{j=1}^{k} \exp[i\beta_{2_j} \sin \omega_{s_j} t + \beta_{1_j} \sin \omega_{m_j} t)]\} \quad (2)$$

Now by the Jacobi-Anger relationship:

$$e^{i\chi \sin\theta} = \sum_{n=-\infty}^{\infty} J_n(\chi)\, e^{in\theta}$$

Where $J_n(X)$ is known as the Bessel function[3] of the first kind of argument X and order n and where:

$$J_n(X) = \sum_{k=0}^{\infty} \frac{(-1)^K X^{n+2K}}{2^{n+2K} K!(n+K)!}$$

By applying this relationship twice we have:

$$v(t) = E_c \, Im \{ \exp[i\omega_c t] \prod_{j=1}^{k} \sum_{n_j=-\infty}^{\infty} \exp[in_j \omega_{s_j} t] \sum_{v_j=-\infty}^{\infty} \exp[iv_j \omega_{m_j} t] J_{v_j}(n_j \beta_{1_j}) J_{n_j}(\beta_{2_j})$$

which can be written as:

$$V(t) = E_c \, Im \left\{ \exp(i\omega_c t) \prod_{j=1}^{k} \sum_{n_j=-\infty}^{\infty} \sum_{v_j=-\infty}^{\infty} J_{n_j}(\beta_{2_j}) J_{v_j}(n_j \beta_{1_j}) \exp i [n_j \omega_{s_j} t + v_j \omega_{m_j} t] \right|$$

Recalling the $e^{iX} = \cos X + i \sin X$ and that $Im[e^{iX}] = \sin X$ and by writing in a more compact notation our expression becomes:

$$V(t) = E_c \sum_{n_j=-\infty}^{\infty} \sum_{v_j=-\infty}^{\infty} \prod_{j=1}^{k} [J_{n_j}(\beta_{2_j}) J_{v_j}(n_j \beta_{1_j})] \sin\left(\omega_c t + \sum_{j=1}^{k} n_j \omega_{s_j} t + \Sigma v_j \omega_{m_j} t\right)$$

Now since the Bessel functions may be viewed as amplitude factors and since the argument of the sine function is expressed as a linear combination of angular frequencies, our expression is in the form necessary to yield the frequency spectrum.

An Illustration. -Now that the desired expression has been derived, an illustration of the manner in which this expression may be used shall be presented. As an example let us consider a FM/FM signal with two subcarriers. In this case expression (3) could be written as:

$$V(t) = E_c [\sum_n \sum_m J_n(\beta_1) J_m(n\beta_2) \sum_p \sum_q J_p(\beta_3) J_q(p\beta_4)]$$

$$[\sin t(\omega_c + n\omega_{s_1} + p\omega_{s_2} + m\omega_{m_1} + q\omega_{m_2})]$$

It is obvious that each sideband is proportional to the product of $2K$ Bessel functions where k is the number of subcarriers and the proportionality factor is E_c . For instance if we consider the part of the expression where m, n, p, q are all zero and $E_c = 1$ we have a sideband of amplitude $J_0(\beta_1) J_0(0\beta_2) J_0(\beta_3) J_0(0\beta_4)$ at frequency $\omega_c + 0{\cdot}\omega_{s_1} + 0{\cdot}\omega_{s_2} + 0{\cdot}\omega_{n_1} + 0{\cdot}\omega_{m_2} = \omega_c$. However, if we consider the part of the expression where n, m, p, q are all equal to one, we have a sideband of amplitude $J_1(\beta_1) J_1(\beta_2) J_1(\beta_3) J_1(\beta_4)$ at frequency $\omega_c + \omega_{s_1} + \omega_{s_2} + \omega_{m_1} + \omega_{m_2}$

The following chart illustrates how these and other sidebands could be formed.

Amplitude of Sideband	Frequency
$J_0(\beta_1)J_0(0\beta_2)J_0(\beta_3)J_0(\beta_4)$	$\omega_c + 0\cdot\omega_{s_1} + 0\cdot\omega_{s_2} + 0\cdot\omega_{m_1} + 0\cdot\omega_{m_2}$
$J_0(\beta_1)J_0(0\beta_2)J_0(\beta_3)J_1(0\beta_4)$	$\omega_c + 0\cdot\omega_{s_2} + 0\cdot\omega_{s_2} + 0\cdot\omega_{m_1} + 1\cdot\omega_{m_2}$
$J_0(\beta_1)J_0(0\beta_2)J_1(\beta_3)J_0(\beta_4)$	$\omega_c + 0\cdot\omega_{s_1} + 1\cdot\omega_{s_2} + 0\cdot\omega_{m_1} + 0\cdot\omega_{m_2}$
$J_0(\beta_1)J_1(0\beta_2)J_0(\beta_3)J_0(0\beta_4)$	$\omega_c + 0\cdot\omega_{s_1} + 0\cdot\omega_{s_2} + 1\cdot\omega_{m_1} + 0\cdot\omega_{m_2}$
$J_1(\beta_1)J_0(\beta_2)J_0(\beta_3)J_0(0\beta_4)$	$\omega_c + 1\cdot\omega_{s_1} + 0\cdot\omega_{s_2} + 0\cdot\omega_{m_1} + 0\cdot\omega_{m_2}$

Now all that need be done in order to form a sideband is have some method of approximating the Bessel functions. These functions could be approximated by summing the series:

$$J_n(X) \approx J_n'(X) = \sum_{k=0}^{L} \frac{(-1)^K X^{n+2K}}{2^{n+2K}(n+k)!\,k!}$$

a sufficient number of terms, L, where L could be chosen by requiring that $|J_n(X) - J_n'(X)| < \epsilon$. However, if the argument is fairly large $(X > 5)$ it is more convenient to approximate the Bessel functions with the quickly converging asymptotic expansion:

$$J_n(X) = \left(\frac{2}{\pi X}\right)^{1/2}\left[\cos\left(X - \frac{n\pi}{2} - \frac{\pi}{4}\right)c_n(X) - \sin\left(X - \frac{n\pi}{2} - \frac{\pi}{4}\right)s_n(X)\right]$$

where

$$c_n(X) = 1 - \frac{(4n^2-1)(4n^2-3^2)}{2!(8X)^2} + \frac{(4n^2-1)(4n^2-3^2)(4n^2-5^2)(4n^2-7^2)}{4!(8X)^4} - \cdots$$

and

$$s_n(X) = \frac{4n^2-1}{8X} - \frac{(4n^2-1)(4n^2-3^2)(4n^2-5^2)}{3!(8X)^3} + \cdots$$

For a given argument, these series need be evaluated for only two orders since the recursive relationship,

$$J_n(X) = \frac{2(n-1)}{X}J_{n-1}(X) - J_{n-2}(X)$$

can be used to obtain the Bessel functions for higher orders.

References

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3. McLachlin, N. W., Bessel Functions for Engineers