MATCHED-FILTER SYSTEMS IN RISING NOISE SPECTRUMS

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Summary.-It is common knowledge that the matched filters for signals in white noise have impulse responses whose time duration is exactly as long as that of the input signal. Nothing can be gained by extending the response of the matched filters to longer than one bit unless (1) the signal source is coded, or (2) the noise spectrum is rising or at least is other than white-Gaussian-ergodic.

This paper discusses means of improving bit error rates without coding the source.

There are essentially two ways of extending the integration time of the matched filters, each of which offers an improvement in signal-to-noise ratio. The first way is to extend the response directly to more than one bit but constrain the filter to give zero or some small preassigned intersymbol crosstalk. The second way is to build matched filters for multiple bits. Both techniques can be used simultaneously; i.e., matched filters can be constructed for bit patterns, and the responses can be extended to longer than the baud to which the filters are matched. Once again this extension is done under the constraint of zero or little crosstalk.

In this paper, the matched filters for several examples are expanded in a rapidly converging series, each term of which is identifiable with a known network. For the cases where the shape of the noise is not known analytically, an experimental technique is given for determining sufficient statistics of the noise so that the optimum matched filters can be designed.

Matched Filters in f^2 Noise.-The impulse response of the filter can be expanded in a compactly carried Fourier series.

$$h(t) = \sum_{1}^{n} A_k \sin \frac{k\pi t}{\tau_1} \quad \text{when } 0 \leq t \leq \tau_1$$

$$\text{and } = 0 \text{ when } \tau_1 < t < \infty \text{ or } t < 0. \tag{1}$$

If h(t) can be expanded in the above series, then the "weighting function," w(t), or time-reversed impulse response, can be expanded in a similar series.

$$w(t) = h(\tau_1 - t) = \sum_{0}^{n} A_k (-1)^{k+1} \sin \frac{k\pi t}{\tau_1} \quad \text{when } 0 < t < \tau$$

$$\text{and } = 0 \text{ when } \tau_1 < t < \infty \text{ or } t < 0. \tag{2}$$

The synthesis of networks for such finite-duration sine pulses is fairly well known; however, since it is such an integral part of the

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matched-filter design, we shall describe it in the appendix.

The input signal can be expanded in a similar series.

As our reference case we will use the matched filter in f^2 noise for PCM signals where the impulse response of the matched filter is of the same time duration, τ, as the PCM pulse.

The input signal may be $\pm S(t)$ where $S(t) = \sum A_k \sin \frac{k \pi t}{\tau}$. If S(t) is unit square, $S(t) = \sum_{k \text{ odd}} \frac{4}{k\pi} \sin \frac{k \pi t}{\tau}$.

The weighting function is $w(t) = \sum B_k \sin \frac{k \pi t}{\tau}$.

The peak signal is $\int_0^\tau w(t) \cdot S(t)\, dt$ (3)

$$= \frac{\tau}{2} \sum B_k A_k .$$

The noise energy in f^2 noise

$$n_0^{\,2} = \int_0^\infty c^2 w^2 H(j\omega)\, H(-j\omega)\, df = c^2 \int_0^\infty \left(\frac{dk}{dt}\right)^2 dt . \tag{4}$$

Here we have used Planceral's theorem $\int |H(j\omega)|^2 df = \int h(t)^2 dt$ and the fact that multiplication by $j\omega$ corresponds to differentiation in time. This noise energy, because of the orthogonality of the Fourier series, is:

$$n_0^{\,2} = \frac{c^2}{2\tau} \sum B_k^{\,2} k^2 \pi^2 . \tag{5}$$

Maximizing peak signal under the constraint of holding noise constant corresponds to maximizing

$$J = \frac{\tau}{2} \sum B_k A_k - \frac{\lambda c^2}{2\tau} \sum B_k^{\,2} k^2 \pi^2 \tag{6}$$

$$\frac{\delta J}{\delta B_k} = 0 \Rightarrow B_k = K \frac{A_k}{k^2 \pi^2} , \tag{7}$$

where K is any positive constant of proportionality.

Examining the peak signal to rms noise for the matched case:

$$S/N = \frac{(\tau)^{3/2}}{\sqrt{2c}} \sqrt{\sum \frac{A_k^{\,2}}{k^2 \pi^2}} , \tag{8}$$

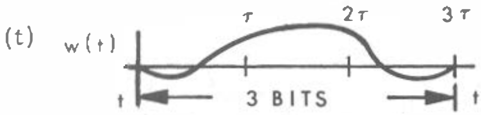
which converges much faster than the Fourier series from the signal. If the Fourier approximation to the matched filter is truncated, the S/N ratio is truncated at the same term.

For a unit-amplitude square pulse,

$$S/N \cong \frac{\tau^{3/2}}{\pi c} \sqrt{8.033} . \tag{9}$$

From the form of equation (8), one can see that any matched filter system which allows one to use a lower harmonic or lower frequencies would help the S/N ratio considerably.

Consider the matched filter with a response which lasts for three bits: The decision will be delayed one bit, and the impulse response of the filter will be constrained to give zero crosstalk from both of the extraneous bits. A sketch of such a response is shown in Figure 1.



The baud which creates an output is from τ to 2τ. 0 to τ or 2τ to 3τ gives no output.

The matched filter can be expressed;

$$w(t) = \sum B_k \sin \frac{k \pi t}{3\tau} . \tag{10}$$

$$\text{The peak signal} = \int_{\tau}^{2\tau} \sum S(t)\, B_k \sin \frac{k \pi t}{3\tau}\, dt. \tag{11}$$

$$\text{The noise} = \frac{c^2}{18\tau} \sum B_k^2\, k^2 \pi^2 . \tag{12}$$

Crosstalk from the first and third bits is:

$$X_1 = \int_0^{\tau} S_1(t) \sum B_k \sin \frac{k \pi t}{3\tau}\, dt \text{ and} \tag{13}$$

$$X_3 = \int_{2\tau}^{3\tau} S_3(t) \sum B_k \sin \frac{k \pi t}{3\tau}\, dt , \tag{14}$$

where $S(t)$, $S_1(t)$, and $S_3(t)$ are the input signals occurring during the middle bit, the first, and the third bit, respectively.

For the case where $S(t)$ is symmetrical, we have $X_1 = X_3$.

$$\text{Call} \int_0^{\tau} S_1(t) \sin \frac{k \pi t}{3\tau}\, dt = C_k = \int_2^{3} S_3(t) \sin \frac{k \pi t}{3\tau}\, dt \tag{15}$$

$$\int_{\tau}^{2\tau} S(t) \sin \frac{k \pi t}{3\tau}\, dt = A_k . \tag{16}$$

To maximize the S/N holding X Constant, one maximizes

$$J = \sum B_k A_k - \lambda_1 \sum B_k^2 k^2 \pi^2 + \lambda_2 \sum B_k C_k , \tag{17}$$

where λ_1 and λ_2 are Langrangian multipliers.

$$\frac{\delta J}{\delta B_q} = 0 = A_q - 2\lambda_1 k^2 \pi^2 B_q + \lambda_2 C_2 , \tag{18}$$

which gives $B_q = \frac{K}{q^2 \pi^2} \left(A_q + \lambda_2 C_q \right)$. (19)

where K is any positive constant.

To evaluate λ_2, multiply equation (19) by c_q and sum over q, obtaining:

$$X_1 = \frac{1}{2\lambda} \left(\sum \frac{A_q C_q}{q^2 \pi^2} + \lambda_2 \sum \frac{C^2_q}{q^2 \pi^2} \right) . \tag{20}$$

If X_1 is held to be zero, we obtain

$$\lambda_2 = - \frac{\sum \frac{A_q C_q}{q^2 \pi^2}}{\sum \frac{C^2_q}{q^2 \pi^2}} \quad \text{and} \tag{21}$$

One can compute the resulting S/N:

$$S/N = \frac{(3)^{3/2} \tau^{3/2}}{\sqrt{2}\, C\, \pi} \sqrt{\sum \frac{A_k^2}{k^2} - \frac{\left(\sum \frac{A_k C_k}{k^2} \right)^2}{\sum \frac{C_k^2}{k^2}}} \tag{22}$$

If S(t) is square, this corresponds to about a 4-db improvement over the S/N of our reference case.

As another example, consider the case of a double bit decision system.

First recall that the matched filter for distinguishing between any two signals is the difference of the matched filters for each separately. The same K must be used in equations (7) and (19).

This fact implies that in any multiple-decision system a matched filter can be constructed for each signal separately, and the output which is largest is the most likely signal to have been transmitted.

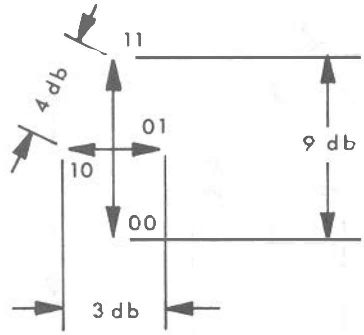
For the double bit decision we have four possible signals

$$\pm S_1(t) = \pm \sum_{k = 1, 3, 5, 7, \ldots} \frac{4}{k\pi} \sin \frac{k \pi t}{2\tau}$$

$$\pm S_2(t) = \pm \sum_{k = 2, 6, 10, 14} \frac{8}{k\pi} \sin \frac{k \pi t}{2\tau} \tag{23}$$

where (1 1) is +S(t)   (0 0) is $-S_1 t$ (1 0) is $+S_2(t)$
and (0 1) is $-S_2(t)$

The matched filters are constructed according to equation (7) and yield about 3 db improvement over the reference.



To compute the resulting bit error rates, note that the cost of calling (1 1) a (1 0) is less than the cost of calling (1 1) a (0 0); similarly, calling (1 0) a (0 1) costs two bits of error. Note also that every decision processes two bits.

Assuming that each message is equally likely to be transmitted and that the confusions are mutually exclusive, and taking into account the symmetry, we have:

$$P_e = \text{Bit Error Rate} = 1/2 \left[2P(S_1/S_2) + P(-S_1/S_1) + P(-S_2/S_2)\right] \quad (24)$$

where $P(S_a/S_b)$ = the probability of deciding S_a was sent given S_b actually was sent. This is symmetrical for the matched case; i.e., $P(S_a/S_b) = P(S_b/S_a)$.

The above system gives slightly better than a 3-db improvement for all reasonable S/N ratio at the input.

But we can improve on this system by extending the matched filter responses to include a bit on each side of the double bit baud and again constraining the matched filters to have zero responses from the additional bits.

These filters are also designed by solving the simple calculus of variations problem: maximize the peak signal under the constraints of zero or small crosstalk and fixed noise energy.

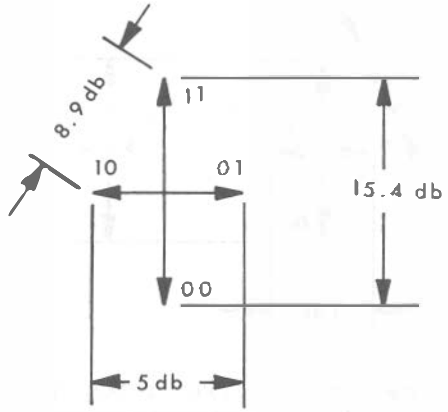
This corresponds to maximizing the quantity J

$$J = \lambda_1 \int_0^{\tau} S(t) \sum A_k \sin \frac{k \pi t}{4 \tau} + \int_{\tau}^{3\tau} S(t) \sum A_k \sin \frac{k \pi t}{4 \tau}$$

$$+ \lambda_2 \sum A_k^2 k^2 . \quad (25)$$

This yields the same type of formula as equation (19) except note the filter matched to the (1 0) signal contains harmonics 2, 6, 10..., only, and the filter matched to (1 1) contains only 1, 3, 5...etc.

If S(t) is square and the crosstalk is held to zero we obtain an improvement in signal-to-noise as compared to our reference case as shown in the vector diagram.



This again is about a 5-db or slightly better improvement over the reference case and 2 db better than the simple double-bit decisions. In the above examples the different signals were orthogonal; this, however, is not a necessary condition.

Similar systems can be worked out for more than two-bit decisions for filtered PCM, for split phase, and for RZ signals, and finally for other than f^2 noise backgrounds.

If we anticipate expanding our matched filters in the same Fourier series, we can handle a variety of noise backgrounds and signals.

First the noise energy at the output of each harmonic filter can be measured (each harmonic is the output of n-port network, see appendix), and a cost n_k^2 which is weighted by its coefficient can be assigned. Note that the noise outputs of these harmonics are independent random variables for noise inputs which are not pathological. Secondly, the projection of the signal can be measured, as well as the crosstalk. The resulting calculus of variations problem is easily formulated

$$\text{Peak signal} = \sum A_k C_k$$

$$\text{Noise} = \sum n_k^2 A_k^2$$

$$\text{Crosstalk} = \sum X_k A_k$$

So we maximize $J = \lambda_1 \sum n_k^2 A_k^2 + \lambda_2 \sum X_k A_k + \sum A_k C_k$ (26)

where A_k = coefficient of the k^{th} harmonic

C_k = projection of the input signal on the k^{th} harmonic

n_k^2 = noise energy on the k^{th} harmonic

X_k = crosstalk projection on the k^{th} harmonic

We obtain the design equation

$$A_k = \frac{1}{n_k^2}\left(C_k + \lambda 2 \; X_k\right) \qquad (27)$$

$\lambda 2$ can be evaluated as before by multiplying by X_k and summing over k

$$\sum A_k X_k = \sum \frac{C_k X_k}{n_k^2} + \lambda_2 \sum \frac{X_k^2}{n_k^2} \qquad (28)$$

for zero crosstalk we have

$$\lambda 2 = - \frac{\sum \frac{C_k X_k}{n_k^2}}{\sum \frac{X_k^2}{n_k^2}} \qquad (29)$$

Conclusions.- We have shownhow to improve bit error rates for a variety of PCM signals in the presence of nonwhite noise. The techniques imply slightly more complicated detection schemes than are normally encountered, but result in a superior system. We have also shown how the pertinent filters can be built using a pair of passive networks and summing amplifiers.

Appendix. -The impulse responses of our matched filters were all expanded in a Fourier series. It behooves us to describe methods of approximating these transfer functions with lumped elements. Such methods are described in Guillemin's book, "Synthesis of Passive Networks," John Wiley & Sons, pp 708-726. We shall give here a method of approximating even and odd half-period harmonics.

Note that the Laplace transform of an odd harmonic can be written:

$$\frac{k\pi/\tau}{S^2 + \frac{k^2\pi^2}{\tau 2}}\left(1 + e^{-\tau S}\right) = F_k(S) \qquad (1a)$$

$$f_k(t) = \sin \frac{k\pi t}{\tau} \leq 0 \leq t\tau$$

while that for the even harmonic:

$$\frac{k\pi/\tau}{S^2 + \frac{k^2\pi^2}{\tau 2}}\left(1 - e^{-\tau S}\right) = F_k(S) \qquad (1b)$$

$$f_k(t) = 0 \; \tau < t \text{ or } t < 0$$

These both can be written in a form where the network approximation becomes evident

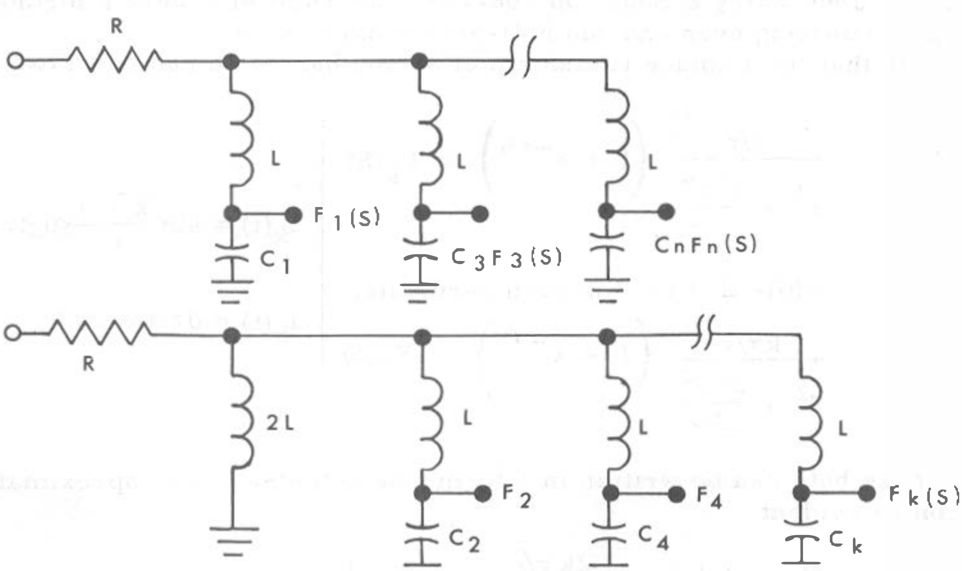
$$F_k \text{ odd } (s) = \frac{2k\pi/\tau}{S^2 + \frac{k^2\pi^2}{\tau^2}} \; \frac{1}{1 + \tanh \frac{\tau S}{2}} = \qquad (2a)$$

$$\frac{\dfrac{2k\,\pi/\tau}{S^2 + \dfrac{k^2\pi^2}{\tau^2}}}{1 + \displaystyle\sum_{q\ \mathrm{odd}} \frac{\frac{4}{\tau}S}{S^2 + \dfrac{q^2\pi^2}{\tau^2}}}$$

$$F_k \text{ even } (S) = \frac{2k\,\pi/\tau}{S^2 + \dfrac{k^2\pi^2}{\tau^2}} \qquad \frac{1}{1 + \coth\dfrac{\tau S}{2}} = \qquad (2b)$$

$$\frac{S^2 + \dfrac{k^2\pi^2\rho}{\tau^2}}{1 + \dfrac{2}{\tau S} + \displaystyle\sum_{q\ \mathrm{even}} \frac{\frac{4}{\tau}S}{S + \dfrac{q^2\pi^2}{\tau^2}}}$$

Each can be synthesized within again constraint and any degree of accuracy desired as a voltage transfer function with two networks as shown below.



$$L = \frac{\tau R}{4}$$

$$C_q = \frac{4\tau}{q^2 \pi^2 R}$$

The outputs of these networks can be summed to give any matched filter.