GEOMETRIC MODEL FOR TRACKER-TARGET LOOK ANGLES AND LINE OF SIGHT DISTANCE

By
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ABSTRACT

To determine the tracking abilities of a Telemetry (TM) antenna control unit (ACU) requires '*truth data*' to analyze the accuracy of measured, or observed tracking angles. This requires we know the actual angle, i.e., that we know where the target is above the earth. The positional truth is generated from target time-space position information (TSPI), which implicitly places the target's global positioning system (GPS) as the source of observational accuracy. In this paper we present a model to generate local look-angles (LA) and line-of-sight (LoS) distance with respect to (w.r.t.) target global GPS. We ignore inertial navigation system (INS) data in generating relative position at time T; thus we model the target as a global point in time relative to the local tracker's global fixed position in time. This is the first of three companion papers on tracking This is the first of three companion papers on tracking analyses employing Statistically Defensible Test & Evaluation (SDT&E) methods.[a]

KEY TERMS

TM antenna, ACU, Latitude, Longitude, Altitude, Look-Angle, Line-of-Sight, TSPI, GPS, Local Plane, Spherical Trigonometry, Law of Spherical Cosines; Law of Spherical Signs.

INTRODUCTION

The target TSPI file is our truth source. It contains GPS, and often INS data. Our task is to translate target *global latitude* (TLAT), *longitude* (TLNG) and *altitude* (TALT) at *time* (T) to *local* tracking 'look-angles' of *azimuth* (AZ), *elevation* (EL) and *line-of-sight* distance (LoS) at time T. Time correlates target position (SLAT, SLNG, SALT) with a fixed location as tracker site origin. Below we develop a set of global-to-local transformations between tracker and target at time T using derived target angles and target/tracker angle differences:

$$\Delta\phi = \text{TLAT} - \text{SLAT}, \qquad \text{(Ia)}$$

$$\Delta\lambda = \text{TLNG} - \text{SLNG}, \qquad \text{(Ib)}$$

$$\Delta H = \text{TALT} - \text{SALT}. \qquad \text{(Ic)}$$

Our geometric model is merely one of a set of tools for we've developed for estimating tracking efficiency. It serves as our primary analysis space[b]. The purpose of our geometric model is to estimate ACU tracking errors and mode probabilities. Error is not exclusive to an ACU, but rather extends to the scope of the test scenario. Target inertial data is useful for modeling tracking angle errors due to tracker-target antenna alignment. A second paper focuses

[a] Rf. 1 & 2.
[b] Rf. 3.

on modeling tracking error which employs GPS and INS; a third on modeling autotracking mode of an ACU employing receiver gain (AGC) and antenna scanning controls.

LOCAL AZIMTUH AND ELEVATION MODELS:

The flat azimuth model is shown in figure 1. Basically, two local areas on an earth tangent plane (an earth local frame), with each locally flat plane a distance, 'd' apart – the calculated earth distance, which is $d = R \times \cos^{-1}(L)$, where L denotes the earth surface arc between the two points. LoS designates the distance between points at altitude. In the distance equation, the earth radius is approximated as $R \approx 6378137$m, and θ is calculated via projections of latitude, longitude and longitudinal differences via the spherical geometry.

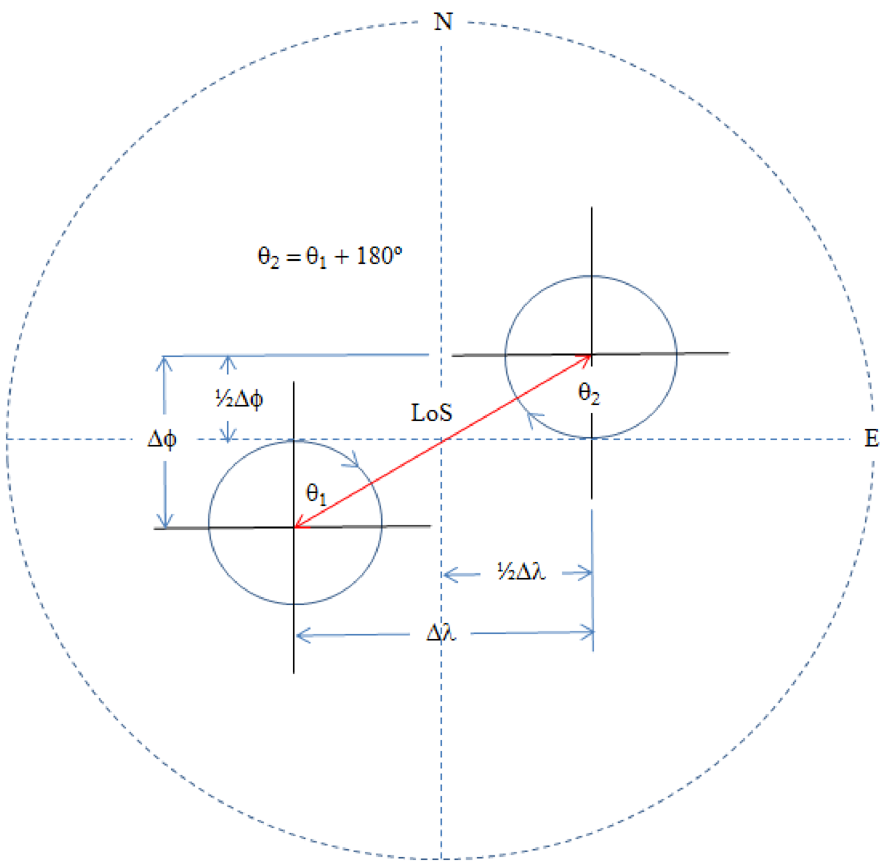


Figure 1 Flat Azimuth Model

Our model formulae are derived from spherical law of cosines (cf. below). There are several estimates of arc length, 'd' on sphere. We employ two (cf. figure 1.1 below):

$$\cos(d) = \sin(\phi_1) \times \sin(\phi_2) + \cos(\phi_2) \times \cos(\phi_1) \times \cos(\Delta\lambda), \quad (1a)$$

$$L = R \times \mathrm{acos}(d). \quad (1b)$$

Substituting $d \Rightarrow L/R$ yields a more accurate close-distance formula:

$$\sin^2(½l) \equiv \mathrm{haversin}(l) = \sin^2(½\Delta\phi) + \cos(\phi_1) \times \cos(\phi_2) \times \sin^2(½\Delta\lambda), \quad (2a)$$

$$L = R \times 2\mathrm{asin}(\sqrt{l}). \quad (2b)$$

The haversine is more accurate for smaller arc distances (angles); where smaller is relative w.r.t. R. Mapping the plane of figure 1 onto a sphere, we get a grid shown in figure 1.1 below. In this figure, from a perspective above the plane ϕ designates latitude, λ longitude and L designates the projection of LoS onto the spherical plane.

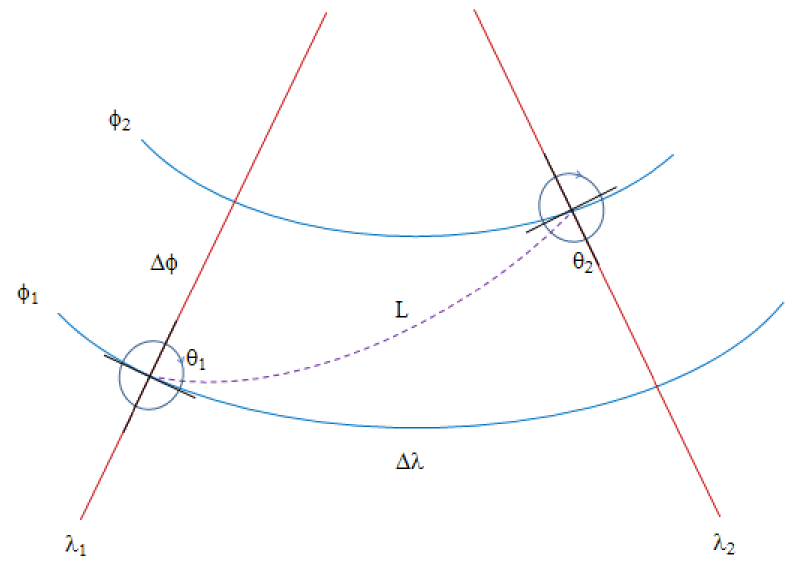


Figure 1.1 Mapping of Plane in Figure 1 onto a Sphere

The azimuthal calculations are based on the law of cosines and sines for a spherical coordinate system:

$$\theta = (180/\pi) \times \operatorname{atan2}(\varphi) \tag{3a}$$

$$\varphi = (\sin(\Delta\lambda) \times \cos(\phi_1), \sin(\phi_1) \times \cos(\phi_2) - \cos(\phi_1) \times \sin(\phi_2) \times \cos(\Delta\lambda)). \tag{3b}$$

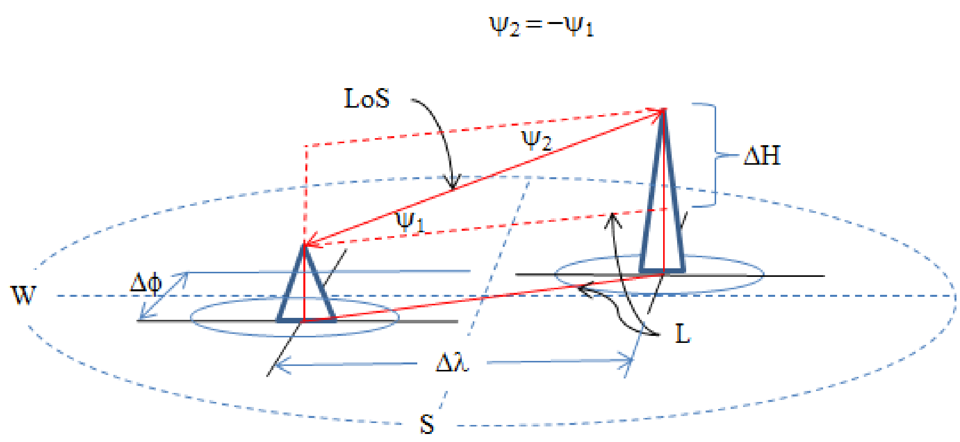


Figure 2 Planar Geometric Elevation Model

A flat or planar geometric elevation model, shown in figure 2 is based on the difference in altitudes and the LoS or L:

$\psi = (180/\pi) \times \mathrm{asin}(\Delta H/LoS) = \mathrm{atan}(\Delta H/L)$. (4)

The arctangent is preferred as $\Delta H \geq LoS$ is possible during flight. The elevation angle is the angle of altitude, ψ. The LoS and azimuth angle formulae are derived below. First we prove the *theoretical* 'truth source', i.e., the laws of spherical trigonometry.

Law of Spherical Cosines (**LSC**):

The Law of Spherical Cosines is one of two fundamental formulae used as our navigation truth, so we shall prove it and its companion, the Law of Spherical Sines. Of course, this *theoretical* 'truth source' compares with *observed* 'truth source' that is GPS TSPI. Using the triangle in figure 3 the LSC, or the so-called cosine rule of sides is represented by the formula:

$\cos(c) = \cos(a)\cos(b) - \sin(a)\sin(b)\cos(C)$. (5)

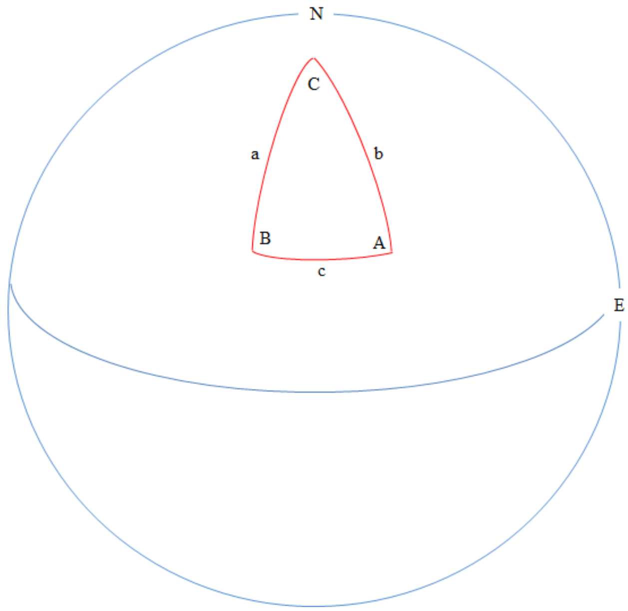


Figure 3 Spherical Reference Points

LSC PROOF:

This proof is attributed to Romuald Ireneus (1997). We use the unit sphere, as the radius will drop out of the calculations. For unit vectors **u**, **v**, **w** from center of the sphere to the corners of the triangles, the inner-product of one vector on the other yields the cosine angle subtended by arc length: $s = 1 \times \theta$. The cosine on the sphere is still the inner-product of the subtending units.

$\cos(a) = \mathbf{u} \bullet \mathbf{v}$; $\cos(b) = \mathbf{u} \bullet \mathbf{w}$; $\cos(c) = \mathbf{w} \bullet \mathbf{v}$. (6)

We define tangents perpendicular to each unit:

$\mathbf{t}_a = \mathbf{v} - \mathbf{u}(\mathbf{u} \bullet \mathbf{v})/\| \mathbf{u} - \mathbf{v}(\mathbf{u} \bullet \mathbf{v}) \| = (\mathbf{v} - \mathbf{u}\cos(a))/\sin(a)$, (7a)

$\mathbf{t}_b = = (\mathbf{w} - \mathbf{u}\cos(b))/\sin(b)$. (7b)

Therefore, the 'law of arcs' is:

$\cos(C) = \mathbf{t}_a \bullet \mathbf{t}_b = (\mathbf{v} - \mathbf{u}\cos(b))(\mathbf{w} - \mathbf{u}\cos(b))/\sin(a)\sin(b)$ (8a)

$= (\cos(c) - \cos(a)\cos(b))/\sin(a)\sin(b) \rightarrow$

$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C).$ (8b)

Similarly, 'the law of angles' is:

$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cos(a).$ (8c)

QED.

Law of Spherical Sines (**LSS**):

Again using the triangle from figure 3 above, the LSS or so-called rule of sines is represented by the formula:

$\sin(A)/\sin(a) = \cos(B)/\cos(b) = \sin(C)/\sin(c).$ (9)

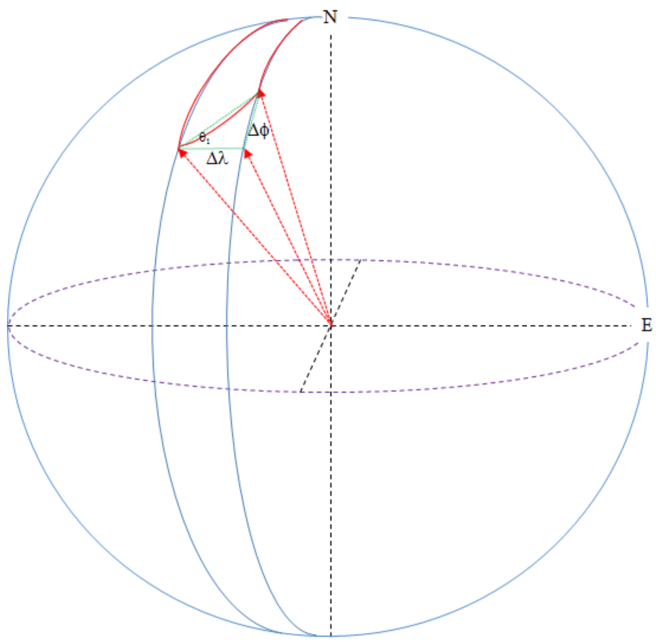


Figure 4 Spherical Model Frame

LSS PROOF:

From Pythagoras' law we know that for any angle (arc length), A

$\sin^2(A) = 1 - \cos^2(A).$ (10a)

Using the LSC from above:

$\sin^2(A) = 1 - (\cos(a) - \cos(b)\cos(c))/\sin(b)\sin(c))^2.$ (10b)

A little manipulation yields:

$\sin(A)/\sin(a) = \cos(a,b,c)/\sin(a,b,c)$ (10c) where

$$\cos(a,b,c) \equiv [1 - \cos^2(a) - \cos^2(b) - \cos^2(c) + 2\cos(a)\cos(b)\cos(c))]^{½} \quad (10d)$$
$$\sin(a,b,c) \equiv \sin(a)\sin(b)\sin(c). \quad (10e)$$

A permutation on pairs (A,a), (B,b), (C,c) proves the rule. This proof is attributed to Isaac Todhunter circa 1863. We will use these spherical trigonometric laws to construct the azimuth formulae.

AZIMUTH AND LOS PROOFS:

For instructional insight we derive azimuth in two ways: the first using vectors, the second via the spherical trigonometry laws derived above. Our analyses calculations employ the trigonometric formulae, without taking account of the earth's elliptical eccentricity – our relatively small distance w.r.t. R between target and tracker permit this. The spherical model frame is shown in figure 4. The error due to ignoring eccentricity is insignificant for the precision of our calculations. To derive azimuth via vectors we create unit vectors on the sphere referenced to a cartesian coordinate system, e.g.:

$$\mathbf{e}_k = (\cos(\phi_k)\cos(\lambda_k), \sin(\phi_k)\cos(\lambda_k), \sin(\phi_k)) \Leftrightarrow (\mathbf{e}_{1(k)}, \mathbf{e}_{2(k)}, \mathbf{e}_{3(k)}). \quad (11a)$$

I found this approach in an earthquake seismology class note[c], and believe the derivation and an alternative enlightening, so include both here for completeness. For two coordinates we have an arc length (on unit sphere) given by the inner-product:

$$\cos(\Delta) = \mathbf{e}_1 \bullet \mathbf{e}_2. \quad (11b)$$

Thus the arccosine will provide angle. Simple! Although mathematically simple, when angles are small, round-off errors emerge and grow as $\cos(\Delta) \rightarrow 1$. So, we derive an equivalent alternative formula that employs the sine and cosine to create the tangent; and though more complicated, is numerically more accurate for computer computation. Figure 5 shows a way to construct the sine and cosine of the bisected angle:

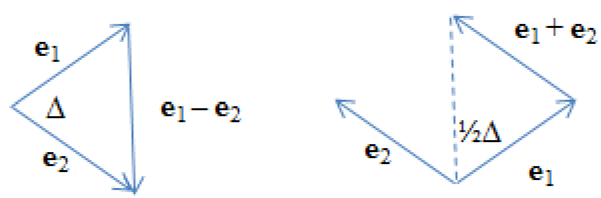


Figure 5a Sine and Cosine of Bisected Angle

We derive the so-called *haversine* representation of LoS from the vectors of figure 5a:

$$\sin(½\Delta) = ½|\,\mathbf{e}_1 - \mathbf{e}_2|,\ \cos(½\Delta) = ½|\,\mathbf{e}_1 + \mathbf{e}_2| \rightarrow \sin(\Delta) = 2\sin(½\Delta)\cos(½\Delta). \quad (11c)$$

So,

$$\Delta = \mathrm{atan2}(\sin(\Delta),\cos(\Delta)), \quad (11d)$$

[c] Rf. 4 & 8.

Where, from 11b and the sine-cosine relations we have:

$$\cos(\Delta) = \mathbf{e}_1 \bullet \mathbf{e}_2,\ \sin(\Delta) = \tfrac{1}{2}|\,\mathbf{e}_1 - \mathbf{e}_2| \times |\,\mathbf{e}_1 + \mathbf{e}_2|. \qquad (11e)$$

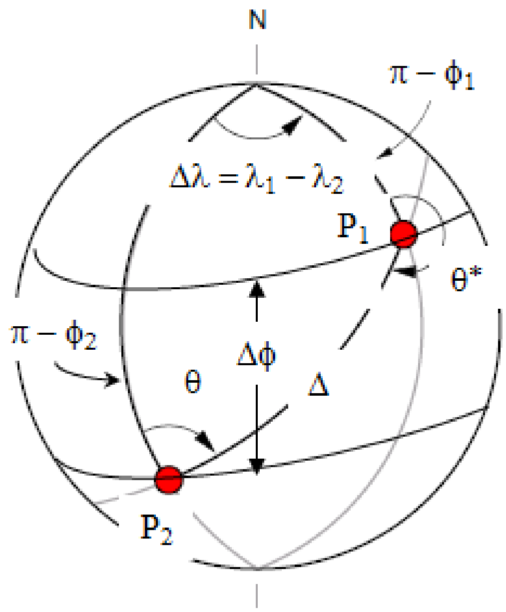


Figure 5b Grid for Spherical Trigonometry Laws

Now we make the simple derivation using the spherical trigonometry laws. Using the grid of figure 5b[d] yields:

$$\sin(\theta)/\sin(\tfrac{1}{2}\pi - \phi_1) = \sin(\Delta\lambda)/\sin(\Delta) = \sin(2\pi - \theta^*)/\sin(\tfrac{1}{2}\pi - \phi_2), \qquad (12a)$$

$$\cos(\pi - \phi_1) = \cos(\Delta)\cos(\pi - \phi_2) + \sin(\Delta_{12})\sin(\pi - \phi_2)\cos(\theta). \qquad (12b)$$

These are rewritten as

$$\cos(\phi_2)\sin(\Delta)\sin(\theta) = \cos(\phi_1)\cos(\phi_2)\sin(\Delta\lambda), \qquad (12c)$$

$$\cos(\phi_2)\sin(\Delta)\cos(\theta) = \sin(\phi_1) - \sin(\phi_2)\cos(\Delta). \qquad (12d)$$

Therefore,

$$\theta = (180/\pi) \times \mathrm{atan2}(\sin(\theta),\cos(\theta)) = \mathrm{atan2}(\varphi), \qquad (13a)$$

$$\varphi = \cos(\phi_1)\cos(\phi_2)\sin(\Delta\lambda)/(\sin(\phi_1) - \sin(\phi_2)\cos(\Delta\lambda)). \qquad (13b)$$

Azimuth is the angle of (13b) which is equivalent to (14a) below. Reversing the arctangent arguments supplements perspective, i.e., we move the look-angles from tracker-to-target to target-to-tracker.
QED.

For the haversine note that

$$\mathrm{haversin}(\theta) \equiv \sin^2(\theta/2);\ \cos(\theta) = 1 - 2\sin^2(\theta/2) = 1 - 2\mathrm{haversin}(\theta). \qquad (14a)$$

[d] Ibid.

Now substitute this form of $\cos(\theta)$ and employ the identity:

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b), \tag{14b}$$

into equation (8b) above to obtain the law of haversines:

$$h \equiv \text{haversin}(L/R) = \text{haversin}(\Delta\phi) + \sin(a)\sin(b)\cos(\Delta\lambda), \tag{14c}$$

$$L = 2R\text{asin}(\sqrt{h}). \tag{14d}$$

Where $½\Delta\phi = a - b$ and $½\Delta\lambda = C$, the latitude and longitude differences of figure 4.
QED.

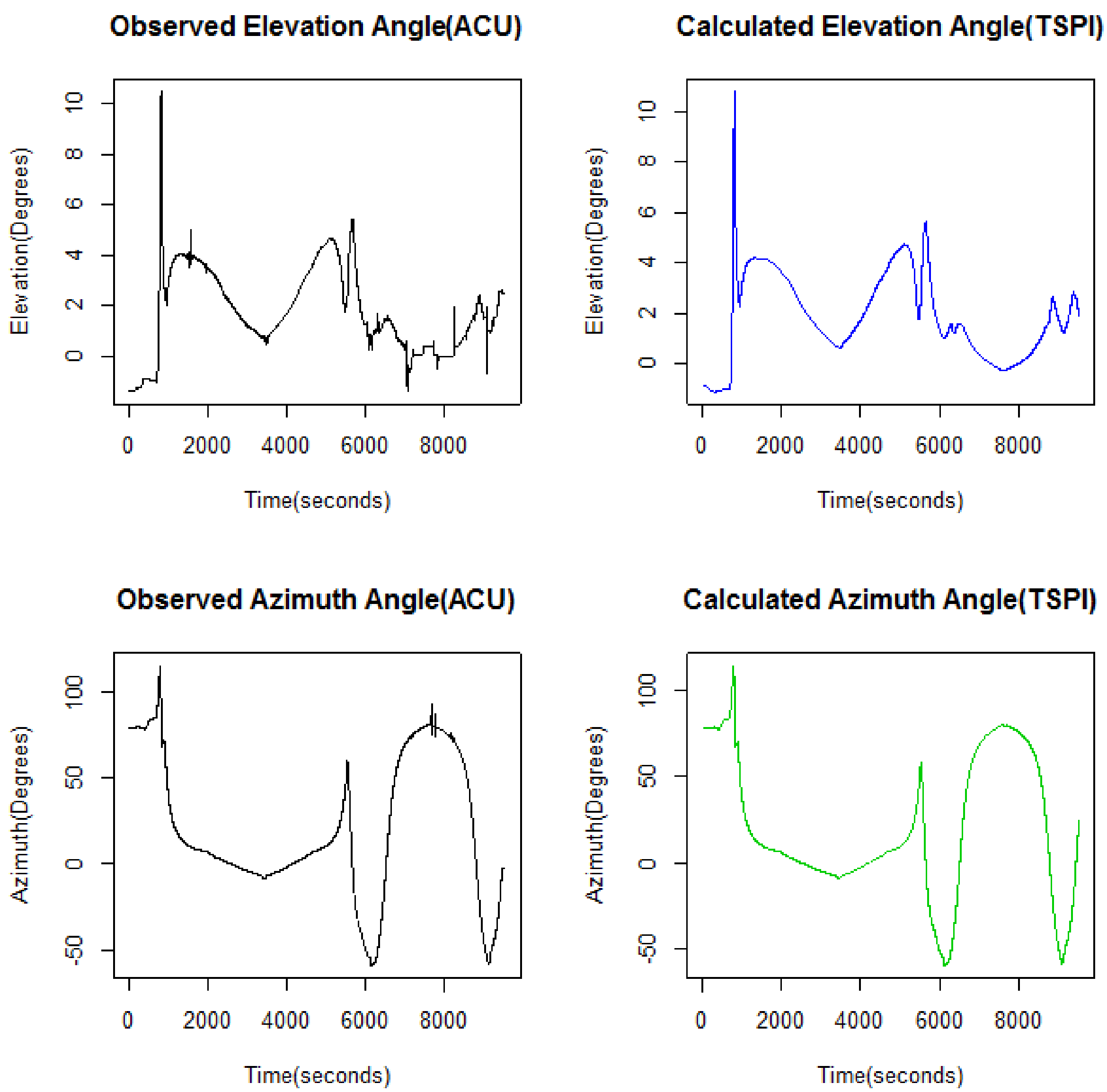
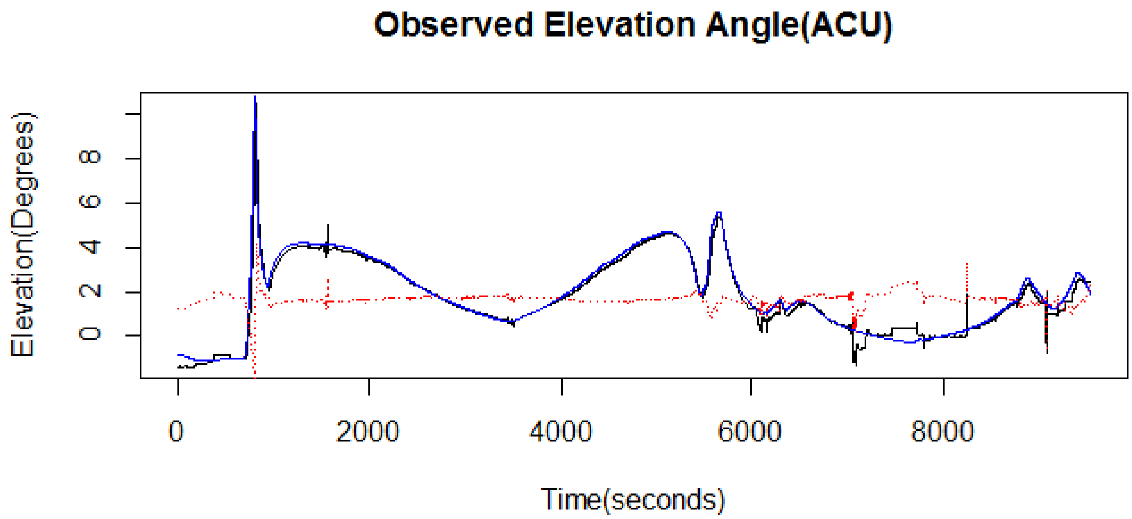


Figure 6a Observed and Modeled Angles



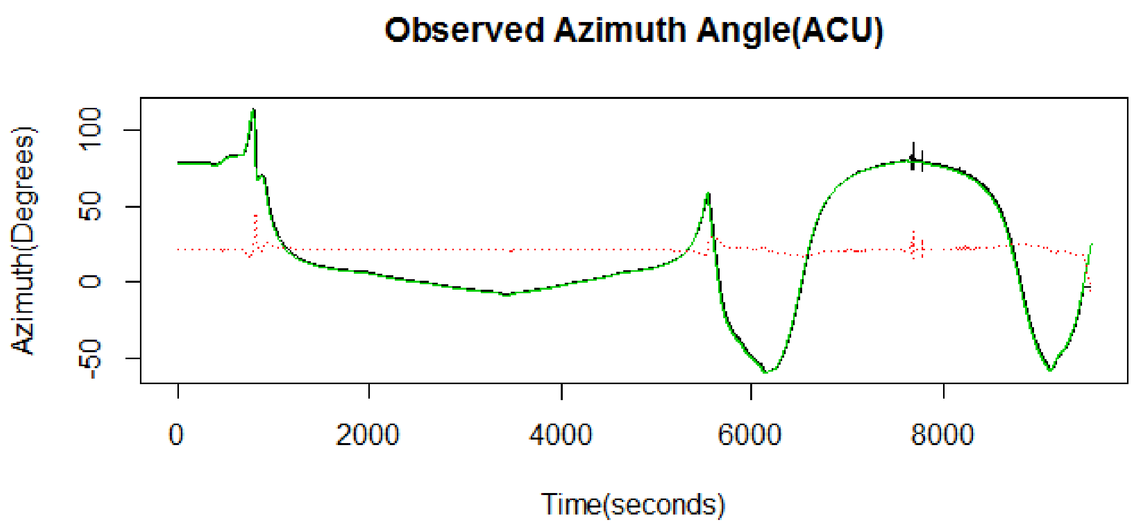


Figure 6b Observed and Modeled Angles with Error (red)

Figures 6a and 6b show observed (track measured) and theoretical (calculated from TSPI) azimuth and elevation angles. The sample times are seconds from test start time, $T = T_0 \equiv 0$. It is clear that between 6k to 8k seconds that tracking error increased, significantly w.r.t. to the rest of the profile. From the whole angle tracking profile we conclude that overall the measured was quite accurate w.r.t. the calculated, or theoretical tracking profile. Figure 6b shows the error superimposed on the angle means, which is a numerical verification of our conclusion. A sample of observed and calculated angles is shown in figure 7.

```
> Az.El.Angles[1000:1010,]
         Time Az.Actual El.Actual   Az.Calc  El.Calc    Az.Err   El.Err
1000 13:50:27    40.832     2.819 37.32431 3.131660 -3.507695 0.312660
1001 13:50:28    40.623     2.835 37.12126 3.141152 -3.501738 0.306152
1002 13:50:29    40.398     2.846 36.95551 3.154514 -3.442488 0.308514
1003 13:50:30    40.216     2.846 36.77259 3.169780 -3.443412 0.323780
1004 13:50:31    40.008     2.846 36.57514 3.182340 -3.432862 0.336340
1005 13:50:32    39.788     2.852 36.41386 3.195245 -3.374143 0.343245
1006 13:50:33    39.579     2.879 36.21960 3.207480 -3.359401 0.328480
1007 13:50:34    39.376     2.879 36.04359 3.215207 -3.332410 0.336207
1008 13:50:35    39.189     2.879 35.88712 3.227684 -3.301878 0.348684
1009 13:50:36    38.991     2.879 35.69793 3.236094 -3.293072 0.357094
1010 13:50:37    38.799     2.912 35.54035 3.248258 -3.258650 0.336258

> Az.El.Angles[2000:2010,]
         Time Az.Actual El.Actual   Az.Calc  El.Calc    Az.Err   El.Err
2000 14:07:07     6.768     3.483 5.813365 3.597284 -0.954635 0.114284
2001 14:07:08     6.741     3.472 5.797938 3.595472 -0.943062 0.123472
2002 14:07:09     6.730     3.472 5.779260 3.593769 -0.950740 0.121769
2003 14:07:10     6.708     3.472 5.763937 3.591998 -0.944063 0.119998
2004 14:07:11     6.697     3.472 5.744796 3.589927 -0.952204 0.117927
2005 14:07:12     6.675     3.472 5.729862 3.588120 -0.945138 0.116120
2006 14:07:13     6.664     3.472 5.711254 3.586382 -0.952746 0.114382
2007 14:07:14     6.648     3.472 5.696048 3.584624 -0.951952 0.112624
2008 14:07:15     6.631     3.472 5.680862 3.582863 -0.950138 0.110863
2009 14:07:16     6.609     3.456 5.662248 3.581094 -0.946752 0.125094
2010 14:07:17     6.593     3.451 5.647508 3.579004 -0.945492 0.128004
>
```

Figure 7 Observed and Calculated Angles

SUMMARY

To determine tracking accuracy we need both a measured data and theoretical truth source to compare observed angles with expected angles. The equations used to compare measured with expected angles are:

Line of Site Distance:

$LoS = 2R \times \text{asin}(\sqrt{h}),\ R \approx 6378137\text{m}$ (2a)

$h = \text{haversin}(\Delta\phi) + \sin(\phi_1)\sin(\phi_1)\cos(\Delta\lambda)$ (2b)

$\Delta\phi = TLAT - SLAT$ (Ia)

Azimuth Angle:

$\theta = (180/\pi) \times \text{atan2}(\varphi)$ (3a)

$\varphi = \cos(\phi_1)\cos(\phi_2)\sin(\Delta\lambda)/(\sin(\phi_1) - \sin(\phi_2)\cos(\Delta\lambda))$ (3b)

$\Delta\lambda = TLNG - SLNG$ (Ib)

Elevation Angle:

$\psi = (180/\pi) \times \text{atan}(\Delta H/L)$ (4)

$\Delta H = TALT - SALT$ (Ic)

Equations 15 were given in the introduction and show the relations between modeled, observed and calculated angles that comprise geometric variables.

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