Analysis of FM Demodulator Output Noise With Applications to FM Telemetry

Dr. Rajendra Kumar
Professor
The Department of Electrical Engineering
California State University,
Long Beach, California
Tel: 562-985-1556
Email: kumar@csulb.edu

ABSTRACT

This paper presents analysis for evaluating the probability density function (pdf) of the noise at the output of the frequency demodulator. It is shown that the noise is non-Gaussian and that for low to medium signal-to-noise power ratios, its pdf differs very significantly from the Gaussian pdf commonly assumed in simplified analysis. These results are very important for analyzing the performance of the PCM/FM type of modulation schemes used in telemetry systems as illustrated in the paper.

INTRODUCTION

This paper presents an exact analysis for the probability density function (pdf) of the noise at the output of a frequency demodulator. In the literature [2-3], the FM demodulator output noise is assumed to be Gaussian distributed at high signal-to-noise power ratio (SNR). At low SNR, it is analyzed in terms of a Gaussian noise and a sequence of impulse functions (clicks) based on the classical theory propounded by Rice [1]. In such an analysis, assuming relatively low bandwidth of the post demodulator video (LPF) filter, the variance of the total noise power at the filter output is evaluated. The probability distribution of the filter output noise is assumed to be Gaussian. While such an analysis is adequate for the case of analog information signals such as speech, for the case of digital signals, one of the most important characteristics of the noise is its pdf.

This paper presents an exact analysis of the FM demodulator output noise. It is shown that the pdf is given in terms of Hypergeometric function. The derived expression is applicable to all SNRs. At relatively low to medium SNR levels, the pdf of noise differs drastically from the Gaussian pdf with the difference becoming progressively smaller with increasing SNR. At high SNRs (>25 dB), the difference is relatively small. However, since the typical SNR used in digital telemetry is in the 10-15 dB range, the non-Gaussian distribution is very important in evaluating the probability of bit error. Due to the analytical difficulties, the analysis does not take into account the effect of video filter following the FM demodulator. However, this effect is expected to be relatively small when the video filter bandwidth is of the order of the IF bandwidth which has been shown to be optimum from earlier simulation results.

SIGNAL MODEL

The signal at the output of the IF filter or at the input of the frequency demodulator is given by

$$v(t) = A\cos(\omega_{IF}t + \theta_s(t)) + n(t) \qquad (1)$$

where ω_{IF} denotes the IF frequency, $\theta_s(t)$ is the signal modulation and is equal to $D_f \int_{-\infty}^{t} m(\tau)d\tau$ where D_f denotes the frequency modulator sensitivity and m(t) is the message signal. The term n(t) represents band limited 'white' noise with one-sided power spectral density equal to N_0 and assumed to have a Gaussian distribution. The variance of n(t) is equal to N_0B_{IF} where B_{IF} is the IF filter equivalent noise bandwidth. The noise term n(t) may be expressed in terms of the following in-phase and quadrature representation

$$n(t) = x_n(t)\cos(\omega_c t) - y_n(t)\sin(\omega_c t) \qquad (2)$$

where $x_n(t)$ and $y_n(t)$ are baseband 'white' noise processes with (one-sided) power spectral density $2N_0$ and of bandwidth $B_{IF}/2$. The processes $x_n(t)$ and $y_n(t)$ are independent Gaussian and have variance $\sigma^2 = N_0B_{IF}$. The frequency demodulator output may be expressed as the derivative of the phase $\theta_T(t)$ of the received signal v(t) expressed in AM-PM form with $\theta_T(t)$ given by

$$\theta_T = \theta_s(t) + \tan^{-1}\left\{\frac{R_n\sin(\theta_n - \theta_s)}{A + R_n\cos(\theta_n - \theta_s)}\right\} \qquad (3)$$

where $\theta_s(t)$ denotes the desired signal and $R_n(t)$ and $\theta_n(t)$ represent the amplitude and phase of the additive noise n(t) with its complex envelope given by

$$g_n(t) = R_n(t)e^{j\theta_n(t)};\; g_n(t) = x_n(t) + jy_n(t) \qquad (4)$$

The complex envelope of the received signal is similarly given by

$$g_s(t) = A e^{j\theta_s(t)} \qquad (5)$$

In the first instance it is assumed that the modulation index is relatively small ($\theta_s(t)$ is small compared to $\theta_n(t)$). In this case the noise term in (3) may be approximated as

$$\phi_n \cong \tan^{-1}\left\{\frac{R_n\sin(\theta_n)}{A_c + R_n\cos(\theta_n)}\right\} \qquad (6)$$

Equivalently one may express ϕ_n as

$$\phi_n = \tan^{-1}\left[\frac{y_n}{A_c + x_n}\right] \qquad (7a)$$

$$x_n = R_n\cos(\theta_n);\; y_n = R_n\sin(\theta_n) \qquad (7b)$$

Differentiation of (7a) with respect to the time t yields the following expression for the FM demodulator output noise.

$$\dot{\phi}_n = \frac{(A + x_n)\dot{y}_n - \dot{x}_n y_n}{[(A_c + x_n)^2 + y_n^2]} \tag{8}$$

In the following, an expression for the pdf of $\dot{\phi}_n$ is derived which is valid for the complete range of SNR.

DEMODULATOR OUTPUT NOISE DISTRIBUTION

To derive the requisite pdf, the set of random variables (RV) X_1, X_2, X_3, and X_4 are defined for notational convenience by

$$X_1 = (A + x_n);\ X_2 = y_n;\ X_3 = \dot{x}_n;\ X_4 = \dot{y}_n \tag{9}$$

then the desired RV $\dot{\phi}_n$ may be expressed in terms of RVs X_1, X_2, X_3, and X4 as

$$\dot{\phi}_n = (X_1X_4 - X_2X_3)/(X_1^2 + X_2^2) \tag{10}$$

From the discussion of the in-phase and quadrature representation of n(t) in (2), it follows that X_1 and X_2 are independent and Gaussian distributed with variance $\sigma^2 = N_0B_{IF}$. Therefore X_3 and X_4 are also Gaussian and probabilistically independent. The power spectral density (one-sided) of $\dot{x}_n(t)$ is given by $P_{\dot{x}_n}(f) = (2\pi f)^2 P_{x_n}(f)$ where $P_{x_n}(f)$ is the power spectral density (PSD) of $x_n(t)$, and thus the variance of $\dot{x}_n(t)$ denoted by σ_d^2 is given by

$$\sigma_d^2 = \int_0^\infty 4\pi^2 f^2 P_{x_n}(f)\,df \tag{11}$$

For the case when the IF filter is assumed to be ideal with bandwidth $B_{IF} = 2$ B, σ_d^2 may be evaluated to be

$$\sigma_d^2 = (2\pi B)^2 \sigma^2 / 3 \tag{12}$$

Note however that (11) is more general and applies to any filter shape. Similarly the variance of $\dot{y}_n(t)$ is also given by σ_d^2. It easily follows as is well known that the cross correlation function of $y_n(t)$ and $\dot{y}_n(t)$ denoted by $R_{y_n\dot{y}_n}(\tau) \equiv \overline{y_n(t) \cdot \dot{y}_n(t + \tau)}$ is given by

$$R_{y_n\dot{y}_n}(\tau) = \frac{dR(\tau)}{d\tau} \tag{13}$$

and thus the variables y_n and $\dot{y}_n$ are uncorrelated if $R(\tau)$ has a maximum at τ=0 which is true for most practical filters including the ideal filter case. Because of the Gaussian distribution of the two variables, it follows that they are also statistically independent. Similarly the variables x_n and $\dot{x}_n$ are also independent. In summary, X_1, X_2, X_3, X_4 are statistically independent and Gaussian distributed random variables. In order to evaluate the pdf (probability density function) of $\dot{\phi}_n$, a set of intermediate random variables Y_1, Y_2, Y_3, Y_4 is defined by

$$Y_1 = X_1X_4 - X_2X_3;\; Y_2 = X_1^2 + X_2^2;\; Y_3 = X_1X_4;\; Y_4 = X_1^2 \quad (14)$$

To evaluate the pdf of the random vector Y=[Y_1 Y_2 Y_3 Y_4]; the set of equations (14) are solved for the value of the random vector $\underline{X} = \underline{x} = [x_1\, x_2\, x_3\, x_4]$ with $\underline{Y} = \underline{y} = [y_1\, y_2\, y_3\, y_4]$. The desired solutions of (14) are given by

$$\underline{x} = \underline{g}^{-1}(\underline{y}) = \begin{bmatrix} \pm\sqrt{y_4} \\ \pm\sqrt{(y_2 - y_4)} \\ \pm(y_3 - y_1)/\sqrt{(y_2 - y_4)} \\ \pm y_3/\sqrt{y_4} \end{bmatrix} \tag{15}$$

In (15) x_1 and x_4 (respectively x_2 and x_3) have same sign. Thus there are four possible solutions for (15). Considering first the solution with all signs positive, the Jacobian J of the set of equations (15) may be shown to be

$$J \equiv \left|\frac{\partial \underline{x}}{\partial \underline{y}}\right| = \frac{-1}{[4y_4(y_2 - y_4)]} \tag{16}$$

In (16), $|A|$ denotes the determinant of any matrix A. As the random variables X_1, X_2, X_3, and X_4 are statistically independent, the joint pdf of the random vector $\underline{X}$ denoted by $f_{\underline{X}}(\underline{x})$ is given by

$$f_{\underline{X}}(\underline{x}) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X3}(x_3) f_{X_4}(x_4) \tag{17}$$

and the component (corresponding to the selected solution from (15)) of the pdf of the random vector $\underline{Y}$ denoted by $f^1_{\underline{Y}}(\underline{y})$ is given by

$$f^1_{\underline{Y}}(\underline{y}) = |J| \cdot f_{X_1}(g_1^{-1}(\underline{y})) \cdot f_{X_{21}}(g_2^{-1}(\underline{y})) \cdot f_{X_3}(g_3^{-1}(y_3)) \cdot f_{X_4}(g_4^{-1}(y_4)) \tag{18}$$

In the subsequent development $f^1_{\underline{Y}}(\underline{y})$ and its integral with respect to components of $\underline{y}$ are referred to as pdf and marginal pdfs respectively. The actual pdf is the sum of such components evaluated for all four solutions in (15). In equation (18), g_1, g_2, g_3, and g_4 represent the components of the vector function $\underline{g}(\underline{y})$. The various functions appearing in (18) are given by

$$f_{X_1}(g_1^{-1}(\underline{y}) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-(\sqrt{y_4} - A)^2/2\sigma^2\right\} \tag{19a}$$

$$f_{X_2}(g_2^{-1}(\underline{y}) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-(\sqrt{y_2 - y_4})^2/2\sigma^2\right\} \tag{19b}$$

$$f_{X_3}(g_3^{-1}(\underline{y}) = \frac{1}{\sqrt{2\pi}\sigma_d}\exp\left\{-(y_3 - y_1)^2/[2(y_2 - y_4)\sigma_d^2]\right\} \tag{19c}$$

$$f_{X_4}(g_4^{-1}(\underline{y}) = \frac{1}{\sqrt{2\pi}\sigma_d}\exp\left\{-y_3^2/[2y_4\sigma_d^2]\right\} \tag{19d}$$

where σ^2 and σ_d^2 denote the variances of random variables X_1 and X_3 (respectively X_2 and X_4). Now the product of the last two terms in (19) may be written in the following form

$$f_{X_3}(g_3^{-1}(\underline{y}))\cdot f_{X_4}(g_4^{-1}(\underline{y})) = \frac{1}{2\pi\sigma_d^2}\exp\left\{-\frac{y_1^2}{2y_2\sigma_d^2}\right\}\cdot\exp\left\{-\frac{(y_3-m)^2}{2\sigma_0^2}\right\} \quad (20)$$

$$m \equiv \frac{y_1 y_4}{y_2}; \quad \sigma_0^2 = (y_4/y_2)(y_2-y_4)\sigma_d^2 \quad (21)$$

Substitution of (19) and (16) in equation (18) yields the joint pdf of the random vector $\underline{Y}$. The desired joint pdf of the random variables Y_1, Y_2 is obtained by integrating the joint pdf $f_{\underline{Y}}^1(\underline{y})$ with respect to y_3, y_4. Since only the last two product terms in (18) are functions of y_3, the marginal pdf of Y_1, Y_2, Y_4 is given by

$$f_{Y_1,Y_2,Y_4}^1(y_1,y_2,y_4) = \frac{1}{4y_4(y_2-y_4)} f_{X_1}(g_1^{-1}(\underline{y})) f_{X_2}(g_2^{-1}(\underline{y})) \times \int_{y_3=-\infty}^{\infty} f_{X_3}(g_3^{-1}(\underline{y})) f_{X_4}(g_4^{-1}(\underline{y}))\, dy_3 \quad (22)$$

The integral in (22) may be easily evaluated by substitution from (20) and is given by

$$\int_{y_3=-\infty}^{\infty} f_{X_3}(g_3^{-1}(\underline{y}) f_{X_4}(g_4^{-1}(\underline{y})\, dy_3 = \frac{1}{\sqrt{2\pi}\sigma_d}\left\{\frac{y_4(y_2-y_4)}{y_2}\right\}^{1/2}\exp\left\{-\frac{y_1^2}{2y_2\sigma_d^2}\right\} \quad (23)$$

With the substitution for the integral from (23) and the expressions for $f_{X_1}(\,)$ and $f_{X_2}(\,)$ from (19a,b) in (22), the desired marginal pdf of Y_1, Y_2, Y_4 is given by

$$f_{Y_1,Y_2,Y_4}(y_1,y_2,y_4) = \frac{1}{4\sqrt{2\pi}\sigma_d}\exp\left\{-\frac{y_1^2}{2y_2\sigma_d^2}\right\}\frac{1}{2\pi\sigma^2 y_2^{3/2}}\exp\left\{-\frac{(A^2+y_2)}{2\sigma^2}\right\}\times$$
$$[y_4/y_2]^{(-1/2)}[1-(y_4/y_2)]^{(-1/2)}\exp\left\{\frac{A\sqrt{y_4}}{\sigma^2}\right\} \quad (24)$$

Finally the marginal pdf of Y_1, Y_2 is obtained by integrating the right hand side of (24) with respect to y_4. Letting I represent the following integral

$$I = \int_{y_4=0}^{y_2} [y_4/y_2]^{(-1/2)}[1-(y_4/y_2)]^{(-1/2)}\exp\left\{\frac{A\sqrt{y_4}}{\sigma^2}\right\} dy_4 \quad (25)$$

With the change of variables $v = \sqrt{(y_4/y_2)}$, the integral may be rewritten as

$$I = 2y_2\int_{v=0}^{1}(1-v^2)^{-1/2}\exp(\beta v)\,dv; \quad \beta \equiv \frac{A}{\sigma^2}\sqrt{y_2} \quad (26)$$

Using the identity 3.389/1 of [4] reproduced below

$$\int_0^1 x^{2\nu-1}(1-x^2)^{\rho-1}\exp(\mu x)\,dx = \frac{1}{2}B(\nu,\rho)\ {}_1F_2[\nu;(1/2),\nu+\rho;(\mu^2/4)]$$
$$+\frac{\mu}{2}B[(\nu+1/2),\rho]\ {}_1F_2[(\nu+1/2);(3/2),(\nu+\rho+1/2);(\mu^2/4)];$$
$$[\mathrm{Re}\,\rho>0; \mathrm{Re}\,\nu>0] \quad (27)$$

where ${}_1F_2$ represents the Hypergeometric function. The application of identity (27) with $\rho = 1/2$ and $\nu = 1/2$ yields

$$I = B[1/2,1/2]\ {}_1F_2[1/2;\ 1/2,1;\beta^2/4] + \beta B[1,1/2]\ {}_1F_2[1;3/2,3/2;\beta^2/4] \quad (28)$$

where B denotes the beta function (Eulers's integral of first kind) [4]. Representing the ${}_1F_2(\ )$ functions in terms of their series expansions, the integral I may be expressed as

$$I = \sum_{k=0}^{\infty} c_k \beta^k = \sum_{k=0}^{\infty} c_k \left(\frac{A}{\sigma^2}\right)^k (\sqrt{y_2})^k \tag{29}$$

where the coefficients c_k may be evaluated in terms of the coefficients in the series expansions of the ${}_1F_2(\ )$ functions appearing in (28), such an evaluation is carried out in the subsequent development. Therefore, the marginal pdf of the random variables Y_1, Y_2 is given by

$$f^1_{Y_1,Y_2}(y_1,y_2) = \frac{1}{4\sqrt{2\pi}\sigma_d} \exp\left\{-\frac{y_1^2}{2y_2\sigma_d^2}\right\} \frac{1}{2\pi\sigma^2 y_2^{1/2}} \exp\left\{-\frac{A^2}{2\sigma^2}\right\} \times \exp\left\{-\frac{y_2}{2\sigma^2}\right\} \sum_{k=0}^{\infty} c_k \beta^k \tag{30}$$

Now the pdf of the desired random variable $Z \equiv Y_1/Y_2$ is given by [5]

$$f^1_Z(z) = \int_{w=-\infty}^{\infty} |w|\ f^1_{Y_1,Y_2}(zw,w)\,dw \tag{31}$$

Substitution of the joint pdf of Y_1, Y_2 from (30) into (31) results in the following expression for the pdf of Z

$$f^1_Z(z) = \frac{1}{4(2\pi)^{3/2}\sigma_d\sigma^2} \exp[-A^2/(2\sigma^2)] \sum_{k=0}^{\infty} c_k (A/\sigma^2)^k$$
$$\times \int_{w=0}^{\infty} (\sqrt{w})^{k+1} \exp\left\{-\left[1+(z\sigma/\sigma_d)^2\right]\frac{w}{2\sigma^2}\right\} dw \tag{32}$$

Using the identity 3.351 in [4] reproduced in equation (33) below

$$\int_0^{\infty} x^{\nu-1} \exp(-\mu x)\,dx = \frac{1}{\mu^\nu}\Gamma(\nu);\ [\mathrm{Re}\,\nu > 0; \mathrm{Re}\,\mu > 0] \tag{33}$$

where $\Gamma(\ )$ denotes the gamma function. The integral in (32) denoted by I_2 is given by

$$I_2 = \left[\frac{2\sigma^2}{1+(z\sigma/\sigma_d)^2}\right]^{(k+3)/2} \Gamma[(k+3)/2] \tag{34}$$

Substituting for I_2 from (34) into (32), the desired pdf $f_Z(z)$ may be expressed in the following form

$$f^1_Z(z) = \frac{\sigma/\sigma_d}{4(\pi)^{3/2}} \left\{\frac{\exp(-A^2/(2\sigma^2))}{[1+(z\sigma/\sigma_d)^2]^{3/2}}\right\} \sum_{k=0}^{\infty} c_k\, \Gamma[(k+3)/2] \left[\frac{2(A/\sigma)^2}{1+(z\sigma/\sigma_d)^2}\right]^{k/2} \tag{35}$$

Form the definition of the coefficients c_k given implicitly by equations (28)-(29) and using the series expansion for the Hypergeometric functions [4], the expression for c_k is given by

$$c_{2j} = B[(1/2),(1/2)]\frac{(1/2)_j}{(1/2)_j \cdot (1)_j}\frac{1}{j!}(\frac{1}{4})^j;\ j=1,2,\ldots \tag{36a}$$

$$c_{2j+1} = B[1,(1/2)]\frac{(1)_j}{(3/2)_j \cdot (3/2)_j}\frac{1}{j!}(\frac{1}{4})^j;\ j=1,2,\ldots \tag{36b}$$

$$c_0 = B[(1/2),(1/2);\ c_1 = B[1,(1/2); \tag{36c}$$

$$(\alpha)_j \equiv (\alpha)(\alpha+1)\cdots(\alpha+j-1);\ j \geq 1;\ (\alpha)_0 = 1 \text{ for any } \alpha \tag{37}$$

Therefore, substituting the expansion for the gamma function, one obtains the following expression for the summand in (35)

$$c_k\Gamma[(k+3)/2]\, w^{k/2} = B[(1/2),(1/2)]\Gamma[3/2]\,\frac{(1/2)_j \cdot (3/2)_j}{(1/2)_j \cdot (1)_j}\frac{1}{j!}(\frac{w}{4})^j;\ k = 2j \tag{38a}$$

$$c_k\Gamma[(k+3)/2]\, w^{k/2} = B[1,(1/2)]\sqrt{w}\ \ \frac{(1)_i \cdot (2)_i}{(3/2)_i \cdot (3/2)_i}\frac{1}{i!}(\frac{w}{4})^i;\ k = 2i+1 \tag{38b}$$

where in equation (38) w has been defined as

$$w \equiv \left[\frac{2(A/\sigma)^2}{1+(z\sigma/\sigma_d)^2}\right] \tag{39}$$

From equation (38) it follows that

$$\sum_{k=0}^{\infty} c_k\, \Gamma[(k+3)/2] w^{k/2} = B[(1/2),(1/2)]\cdot\Gamma[3/2]\ {}_2F_2[(1/2),(3/2);(1/2),1;(w/4)]$$
$$+ B[1,(1/2)]\sqrt{w}\ {}_2F_2[1,2;\ (3/2),(3/2);(w/4)] \tag{40}$$

Finally the substitution of (40) into (35) results in the following expression for $f_Z(z)$

$$f_Z^1(z) = \frac{\sigma/\sigma_d}{4(\pi)^{3/2}}\left\{\frac{\exp[-A^2/(2\sigma^2)]}{[1+(z\sigma/\sigma_d)^2]^{3/2}}\right\}\times$$
$$\Big\{B\ [(1/2),(1/2)]\cdot\Gamma[3/2]\ {}_2F_2[(1/2),(3/2);(1/2),1;\tilde{w}]$$
$$+\ 2B[1,(1/2)]\sqrt{\tilde{w}}\ {}_2F_2[1,2;\ (3/2),(3/2);\tilde{w}]\Big\}; \tag{41a}$$

$$\tilde{w} \equiv \left[\frac{[A^2/(2\sigma^2)]}{1+(z\sigma/\sigma_d)^2}\right] \tag{41b}$$

It may be shown in a manner similar to the derivation of $f_Z^1(z)$ that the component of the pdf $f_Z(z)$ corresponding to the negative sign for x_1 and x_4 in equation (15) is obtained by replacing the positive sign associated with the second term in (41a) by negative sign. The solution corresponding to the other two solutions are identical to the first two solutions. Hence the pdf $f_Z(z)$ is equal to four times the first term in (41a), i.e.,

$$f_Z(z) = \frac{\sigma/\sigma_d}{2(\pi)^{3/2}}\left\{\frac{\exp[-A^2/(2\sigma^2)]}{[1+(z\sigma/\sigma_d)^2]^{3/2}}\right\}\times\ B\ \ [(1/2),(1/2)]\cdot\Gamma[1/2]\ {}_1F_1[(3/2);\ 1;\tilde{w}] \tag{42}$$

where using the definition of the hypergeometric functions, the ${}_2F_2(\,)$ function has been simplified to the ${}_1F_1\,(\,)$ function and the $\Gamma(3/2)$ has been replaced by $(1/2)\,\Gamma(1/2)$. Equation (42) with (41b) represent the final desired result. Finally substituting the values of the beta and gamma functions : $B[(1/2),(1/2)]=\pi$ and $\Gamma[1/2]=\sqrt{\pi}$, the expression for the pdf $f_Z(z)$ simplifies to

$$f_Z(z)=\frac{\sigma/\sigma_d}{2}\left\{\frac{\exp[-A^2/(2\sigma^2)]}{[1+(z\sigma/\sigma_d)^2]^{3/2}}\right\}{}_1F_1[(3/2);1;\tilde{w}];\quad \tilde{w}\equiv\left[\frac{[A^2/(2\sigma^2)]}{1+(z\sigma/\sigma_d)^2}\right] \quad (43)$$

More conveniently defining the normalized random variable $\Psi=Z/(2\pi B)$ in view of (12), (frequency fluctuation normalized by the filter bandwidth in rad/sec), the pdf of the RV Ψ is given by

$$f_\Psi(\psi)=\frac{\sigma/\sigma_f}{2}\left\{\frac{\exp[-A^2/(2\sigma^2)]}{[1+(\psi\sigma/\sigma_f)^2]^{3/2}}\right\}{}_1F_1[(3/2);1;\tilde{w}];\quad \tilde{w}\equiv\left[\frac{[A^2/(2\sigma^2)]}{1+(\psi\sigma/\sigma_f)^2}\right] \quad (44)$$

where $\sigma_f=\sigma_d/(2\pi B)$ in equation (44) and thus σ_f^2 represents the variance of the normalized frequency fluctuations. Note that under the high SNR condition and ideal filter shape usually treated in the literature, $\sigma_f^2=\sigma^2/(3A^2)\equiv\sigma_l^2$.

Comparison with Gaussian pdf:

Figure 1-2 plot the pdf of the demodulator output noise as computed from (44) for the input SNR = $(A^2/2\,\sigma^2)$ equal to 12 and 20 dB respectively plotted versus ψ normalized by σ_l where $\sigma_l^2=\sigma^2/(3A^2)$ is the variance of ψ predicted on the basis of linear Gaussian assumption. As may be observed from these figures, the pdf as computed from equation (44) differs very markedly from its value predicted from the linear theory for SNR up to 15 dB with lower SNR resulting in higher difference. For an SNR equal to 20 dB the difference is relatively small. Table 1 below shows the rms value of the noise σ_f as computed from (44) and its value σ_l as predicted from linear approximation for various values of SNR.

Table1. Comparison of theory with Gaussian approximation

SNR (dB)	σ_l	σ_f	$20*\log(\sigma_f/\sigma_l)$
6	.2046	.305	3.47
8	.1625	.1895	1.34
12	.1025	.1062	.308
15	.0726	.0738	.142
20	.0408	.0410	.042
25	.0230	.0230	0

While the linear theory gives good approximation for the noise variance for SNR greater than or equal to 12 dB, the results in terms of pdf differ significantly at these SNRs. For example, at SNR of 12 dB, at $x=\psi/\sigma_l$ equal to 5, the value of pdf predicted from linear approximation is 1.5×10^{-5} compared to 1.1×10^{-4} predicted from the results of this paper. Thus the non-Gaussian nature of the noise is of high significance in determining the probability of bit error in digital communication even when the correct variance can be evaluated by an independent means.

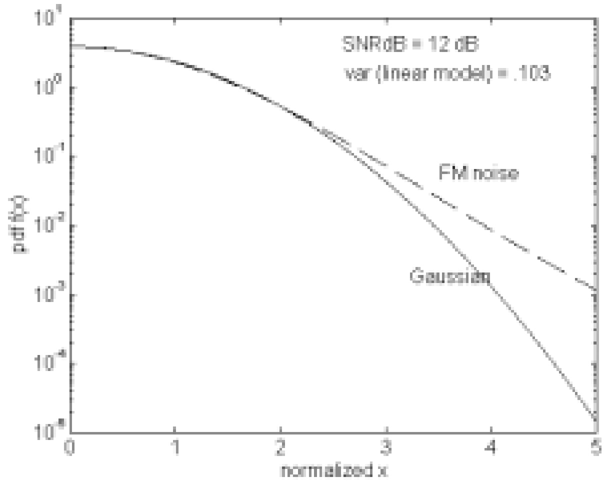

Figure1. FM demod output noise pdf (SNR = 12 dB)

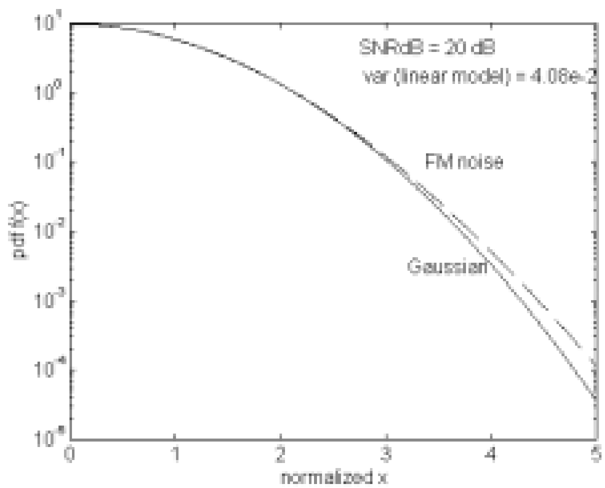

Figure2. FM demod output noise pdf (SNR = 20 dB)

Probability of Bit Error:

To evaluate the impact of non-Gaussian noise distribution on the digital signal bit error probability P_e, the probability of error is computed both with Gaussian and non-Gaussian distribution given by (44). For the case of bipolar NRZ signaling, the sampled signal at the FM demodulator output takes values $\pm V$ for some voltage V. The sampled output SNR equal to $(V/\sigma_f)^2$ is dependent upon the input SNR, the modulation index and several other factors. In this paper to evaluate the impact of non-Gaussian noise distribution, the probability of error is computed for a given output SNR both for the Gaussian and non-

Gaussian distribution of noise both with the same variance. The non-Gaussian noise distribution is parameterized by the input SNR. Figures 4 compares the results for the non_Gaussian noise distribution corresponding to input SNR of 12 dB. The figure shows in very clear terms, the difference in P_e resulting from the non-Gaussian distribution.

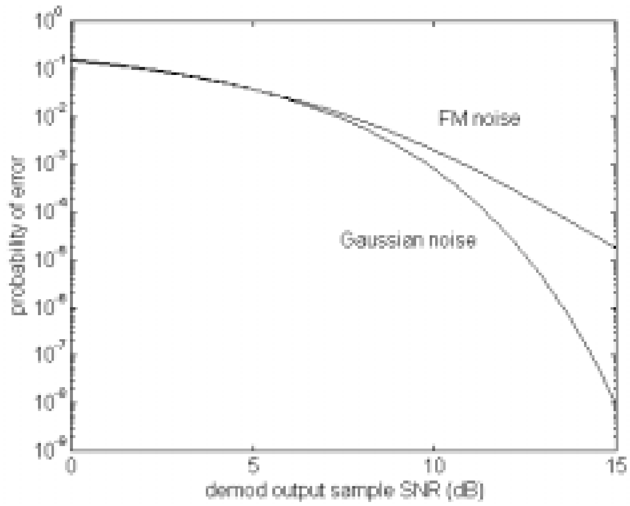

Figure 3. Probability of bit error for NRZ signal (SNR_i = 12 dB)

CONCLUSIONS

The paper has presented an exact analysis of the noise at the output of FM demodulator. It has been shown that for low to medium SNRs the pdf of the FM demodulator output noise differs significantly from the Gaussian pdf. A detection example has been presented to illustrate possible impact of the non-Gaussian noise on the probability of detection error. The derivation of the paper assumed low modulation index and does not include the effect of the post demodulation (video) filter on the probability distribution (the effect on the variance is implicitly accounted for in the detection example). For the digital modulation schemes, earlier simulation studies show that the best performance is achieved when the video filter bandwidth is of the order of the IF bandwidth. In such cases the impact of the video filter is expected to be relatively small. A semi-analytical approach can be used to evaluate the impact of video filter under more diverse conditions.

REFERENCES

[1] S.O. Rice, Mathematical Analysis of Random Noise, in selected papers on Noise and Stochastic Processes, ed. J.L. Doob and L.S. Ornstein, Dover Publications, 1954.

[2] Couch, L.W., Digital and Analog Communication Systems, Macmillan Publishing, 2000.

[3] Taub, H. and D.L. Schilling, Principles of Communication Systems, McGraw-Hill, 1986

[4] Gradshteyn, I.S. and I.M. Ryzhik, Tables of Integrals, Series, and Products, Academic Press, 1980.

[5] Papoulis, A., Probability, Random Variables, and Stochastic Processes, McGraw-Hill, 1991.