STATISTICAL DYNAMICS OF A FIRST-ORDER PHASE-LOCKED LOOP

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Summary The paper deals with a procedure for studying the transient behavior of a first-order phase-locked loop in the presence of noise. This procedure requires the numerical solution of an ordinary differential equation. Excellent agreement has been found between theoretical results and those obtained via computer simulation.

1) Introduction The statistical behavior of phase-locked loops (PLL's) has been the subject of extensive analitical and experimental studies in the last few years. Although yielding important results, however, these studies have been mainly concerned with the steady-state operating conditions of the loop, whereas, owing to mathematical difficulties, few results have been obtained on its transient behavior.

This is unfortunate as a theory on the transient phenomena in a PLL driven by signal plus noise would be of the greatest interest in many practical applications as, for example, in solving noisy acquisition problems or in studying the statistical performance of a PLL driven by an intermittent reference.

To show where these mathematical difficulties are, let us formulate the problem for a first-order PLL.

Up to the time $t=0$ the system operates in stationary conditions with an input which is the sum of a sinusoidal reference plus noise. At $D=0$ the reference power, or the noise level, or both, change abruptly and assume new constant values. The task is to find the transient probability density function $p(\phi,t)$ of the phase-error ϕ of the loop for $t>0$.

As is well known [1] -[4] this function can be found by solving a Fokker-Planck equation with appropriate boundary conditions. Up to the present time, however, any attempt to solve it has failed.

Dominiack and Pickholtz [5] found an interesting numerical solution technique, and different approximate solution methods are indicated in [6],[7] and [8]. The last two papers merit further consideration as careful comparison of the mathematical techniques they use suggests a way for improving previous results.

Both methods are based on the following idea. One guesses that the solution to the Fokker-Planck equation can be conveniently approximated by a given function $f(\phi,q)$ of ϕ, and a certain number of unknown time-dependent parameters $q=(q_1, q_2, .. q_N)$. With this assumption and using the Fokker-Planck equation it is possible, in various ways (see [7] and [8]), to obtain a set of simultaneous ordinary differential equations in the unknown components of q , whose solution provides all that is needed to define $f(\phi,q)$ completely.

In particular, under the hypothesis of zero loop detuning, a raised cosine form has been assumed for $f(\phi,q)$ in [7] (ϕ is taken modulo 2π)

$$f(\phi,q) = 1/2\pi + q\cos\phi \tag{1}$$

whereas in [8] it has been set

$$f(\phi,q) = \exp(\phi^2/2q)/(2\pi q)^{\frac{1}{2}} \tag{2}$$

Now we will determine the range of signal-to-noise ratios within which expressions (1) and (2) give good results. To this end we observe that, due to the change in the input Voltage characteristics, transient phenomena take place in the system beginning from t=0. Sooner or later, however, these phenomena will fade and new stationary operating conditions will he attained. Therefore, for t=0 and for $t=\infty$, (1) and (2) should agree with the well known expression for the stationary probability density function of a first-order PLL. This expression is [4]

$$p_o(\phi) = \exp(\alpha\cos\phi)/2\pi I_o(\alpha) \tag{3}$$

where α is the signal-to-noise ratio in the loop and $I_o(\alpha)$ is the zero-order modified Bessel function of the first kind. Consider now the case $\alpha >>1$ and the opposite one $\alpha <<1$.

For $\alpha >>1$, $I_o(\alpha) \simeq$,$\exp\alpha/(2\pi\alpha)$ and from (3) (see [4])

$$p_o(\phi) = \exp(-\alpha\phi^2/2)/(2\pi/\alpha)^{\frac{1}{2}} \tag{4}$$

is easily found.

Instead, for $\alpha <<1$, $I_o(\alpha) \simeq 1$ so that, within the approximation $\exp(\alpha\cos\phi) \simeq 1+\alpha\cos\phi$, expression (3) becomes

$$p_o(\phi) = 1/2\pi+\alpha\cos\phi/2\pi \tag{5}$$

By comparing (4) and (5) with (1) and (2) we are led to the following conclusions. For low signal-to-noise ratios expression (1) may give good results whereas expression (2) badly approximates the true probability density function.

Viceversa, for high signal-to-noise ratios, (2) may go well, instead (1) must be decidedly rejected.

At this point an obvious question comes to mind: can we find another approximation for $p(\phi,t)$ not subjected to the limitations of (1) and (2)? As a reasonable conjecture we can set

$$f(\phi,q) = \exp(q\cos\phi)/2\pi I_o(q) \tag{6}$$

The present paper deals just with this approximation.

2) Time derivative of the phase-error variance In the next section we will obtain an ordinary differential equation for the unknown time-dependent parameter $q=q(t)$ that appears in (6). The approach we will follow involves a certain expression of the time derivative of the phase-error variance that we will determine hare. To this end, let us first write the Fokker-Planck equation whose solution gives $p(\phi,t)$. Suppose that, for $t>0$, the PLL input voltage is the sum of a carrier $\sqrt{2}\,A\sin(\omega t+\theta)$ plus a sample from a zero-mean white Gaussian noise process with single-sided spectral density N_o. Assuming zero loop detuning, it has been found [1]-[4] that

$$\frac{\partial p}{\partial t} \doteq AK\frac{\partial}{\partial\phi}(p\sin\phi) + \frac{K^2N_o}{4}\frac{\partial^2 p}{\partial\phi^2} \tag{7}$$

where K is the open loop gain. Alternatively, letting

B_L	$= AK/4$	loop bandwidth
α	$= A^2/B_LN_o$	signal-to-noise ratio in the loop
τ	$= 4B_L t$	dimensionless time

eq. (7) may be written in the more convenient form

$$\frac{\partial p}{\partial\tau} = \frac{\partial}{\partial\phi}(p\sin\phi) + \frac{1}{\alpha}\frac{\partial^2 p}{\partial\phi^2} \tag{8}$$

Observe now that, from simmetry, $p(\phi,t)$ is an even function of ϕ so that the expected value of ϕ is zero for any τ. Thus denoting $v=v(\tau)$ the variance of one has

$$v = \int_{-\pi}^{\pi} \phi^2 p(\phi,\tau)d\phi \tag{9}$$

and hence

$$\frac{dv}{d\tau} = \frac{d}{d\tau} \int_{-\pi}^{\pi} \phi^2 p(\phi,\tau)d\phi = \int_{-\pi}^{\pi} \phi^2 \frac{\partial p}{\partial \tau}(\phi,\tau)\, d\phi$$

from which, using (8), we get

$$\frac{dv}{d\tau} = \int_{-\pi}^{\pi} \phi^2 \frac{\partial}{\partial \phi}(p \sin\phi)d\phi + \frac{1}{\alpha}\int_{-\pi}^{\pi} \phi^2 \frac{\partial^2 p}{\partial \phi^2}\, d\phi ,$$

or, integrating by parts the first term on the right hand side,

$$\frac{dv}{d\tau} = -2\int_{-\pi}^{\pi} \phi p \sin\phi d\phi + \frac{1}{\alpha}\int_{-\pi}^{\pi} \phi^2 \frac{\partial^2 p}{\partial \phi^2}\, d\phi \tag{10}$$

This is the required expression for dv/dτ.

3) Differential equation for q(τ) In this section we will determine a differential equation for the time dependent parameter q=q(τ) involved in (6). To this purpose consider equation (10). We observe that in deriving it no approximation has been introduced so that (10) does represent an exact expression for dv/dτ.

Unfortunately we are unable to use it as we ignore p(ϕ,τ). Nevertheless an approximate evaluation of dv/dτ can he obtained if we substitute p(ϕ,τ) in (10) with (6). In doing so, the right hand side of (10) becomes a function of q whose exact expression will be determined shortly. Before proceeding in this direction, however, we will turn our attention to the left hand side of (10). If we use (6) again in evaluating the integral in (9), we realize that the variance v can be expressed as a function of q, i.e., v = g(q) and hance we can write:

$$\frac{dv}{d\tau} = \frac{dg}{dq}\frac{dq}{d\tau} \tag{11}$$

Thus, equating this approximate expression of dv/dτ to the previous one we obtain a first-order ordinary differential equation whose solution gives q=q(τ).

Now we will elaborate the ideas outlined above. Let us begin with the substitution of (6) in the right hand side of (10). From 6 one has

$$\frac{\partial f}{\partial \phi} = -q\ f \sin\phi$$

so that

$$-2\int_{-\pi}^{\pi} \phi f \sin\phi\ d\phi = \frac{2}{q}\int_{-\pi}^{\pi} \phi \frac{\partial f}{\partial \phi}\ d\phi$$

from which, integrating by parts and bearing in mind that

$$\int_{-\pi}^{\pi} f d\phi = 1$$

one deduces

$$-2\int_{-\pi}^{\pi} \phi f \sin\phi\ d\phi = \frac{2}{q}\left[\exp(-q)/I_o(q)-1\right]. \tag{12}$$

Next we consider the second term in the right hand side of (10). Again, integrating by parts one gets

$$\frac{1}{\alpha}\int_{-\pi}^{\pi} \phi^2 \frac{\partial^2 f}{\partial \phi^2}\ d\phi = \frac{2}{\alpha}\left[1-\exp(-q)/I_o(q)\right]. \tag{13}$$

The last step consists of evaluating g(q) that appears in (11). It has been found [4]

$$g(q) = \pi^2/3+4\sum_{n=1}^{\infty} \frac{(-1)^n I_n(q)}{n^2\ I_o(q)} \tag{14}$$

where I_n (x) is the n-order modified Bessel function of the first kind and argument x.

Thus, collecting the above results, we obtain eventually

$$\frac{dg}{dq}\ \frac{dq}{d\tau} = 2(1/q-1/\alpha)\left[\exp(-q)/I_o(q)-1\right]. \tag{15}$$

4) Numerical solutions and simulation results As an analytical solution to (15) is definitely hopeless, one is forced to resort to numerical techniques. Perhaps the best way of appreaching the problem is to transform (15) into an equation in the unknown v=v(τ). This is promptly done as the left side of (15) is just dv/dτ whereas the right side depends on v by means of q = g (v), the inverse function of v=g(q), whose values are easy to compute.

One obtains

$$\frac{dv}{d\tau} = 2(1/q-1/\alpha)\left[\exp(-q)/I_o(q)-1\right] \qquad (16)$$

with

$$q = g^{-1}(v) .$$

This equation has been numerically solved in two extreme cases. In the first one it has been assumed that, for $\tau = 0$, the phase-error value is completely unknown so that the corresponding probability density function is constant on the range $-\pi<\phi<\pi$ and hence

$$v(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi^2 d\phi = \pi^2/3$$

In the second case, instead, it has been supposed $\phi = 0$ for $\tau = 0$ so that v(0)=0.

The corresponding solutions of (16) are shown in figs. 1 and 2 respectively for some values of α .

The theory outlined above has been tested using a computer simutation of the PLL. In figs. 3 to 9 some theoretical and simulation results are compared. In particular in fig. 3 the variance v is plotted versus $4B_Lt$ for $v(0)= \pi^2/3$ and for v(0) = 0 when $\alpha=3$. As can be seen, the agreement is excellent. Figs. 4, 5 and 6 show the probability density function of ϕ for different values of $4B_Lt$ when $\alpha=3$ and $v(0)=\pi^2/3$. Figs. from 7 to 9 show the same things when v(0)=0.

5) Application We now show an application of the theory developed in the preceeding sections. Let us consider a PLL driven by a signal s(t) plus noise n(t). The noise term is as defined in section 2) whereas $s(t)=m(t)\sqrt{2A}\sin(\omega t+\theta)$, with m(t) a periodic sequence of pulses of height 1, width ΔT and period T.

In other words, the reference s(t) is no longer a continuous sinusoid but rather a sequence of synchronizing bursts. We aim at determining the time dependence of the-phase-error variance v. To this purpose let us label t=0 the beginning of an arbitrary burst and suppose we know v(0). Then, the values assumed by v(t) in the interval $0<t<\Delta T$ can he obtained from the results of section 4).

Starting from t= ΔT , it is s(t)=0 so that (7) becomes

$$\frac{\partial p}{\partial t} = \frac{K^2 N_o}{4} \frac{\partial^2 p}{\partial \phi^2} \tag{17}$$

A simple expression for v(t) can be readily determined with the aid of (17) if we suppose that p(ϕ,t) as a function of ϕ, takes significant values only on $-\phi_0<\phi<\phi_0$ with $\phi_0 <\pi$. In fact, multiplying both sides of (17) by ϕ^2 and integrating on the interval $-\pi<\phi<\pi$, yields

$$\frac{dv}{dt} \quad \frac{K^2 N_o}{4} \int_{-\pi}^{\pi} \phi^2 \frac{\partial^2 p}{\partial \phi^2} \, d\phi$$

from which integrating by parts and assuming

$$p(\pi,t) = 0 \, , \quad \partial p(\phi,t)/\partial\phi \Big|_{\phi=\pi} = 0$$

for any ΔT<t<T, one gets

$$\frac{dv}{dt} = K^2 N_o/2$$

and hence

$$v(t)=v(\Delta T)\exp\left[-(t-\Delta T)K^2 N_o/2\right] \quad . \tag{18}$$

The above says that if v(0) were known, we could compute v(t) for any 0<t<T and hence, as v(t) has period T, we would know v(t) completely. Therefore, all that we need is to compute v(0). This can be done, for example, using a cut and try procedure, bearing in mind that it must be v(0)=v(T). Thus, we start with an arbitrary value of v(0) and find the corresponding v(T). In general it will be v(0)≠ v(T). Then we correct v(O) ... and so on.

6) Conclusions An approximate procedure for studying the transient behavior of a first-order PLL in the presence of noise has been indicated. This procedure requires the numerical solution of an ordinary differential equation which describes the time-dependence of the phase-error of the loop.

Excellent agreement has been found between theoretical results and those of a computer simulation of the PLL. The present theory has some advantages over the previous ones. In particular, in regard to those considered in [7] and [8] it has no limitations on the signal-to-noise ratios, whereas, in comparison to the one outlined in [5] it requires the solution of an ordinary rather than a partial differential equation.

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Fig.1 Solutions of eq.(16) for $v(0)=\pi^2/3$ with α as a parameter

Fig.2 Solutions of eq.(16) for $v(0)=0$ with α as a parameter

Fig.3 Variance of the phase-error versus $4B_Lt$ for $\alpha=3$ and two different initial consitions

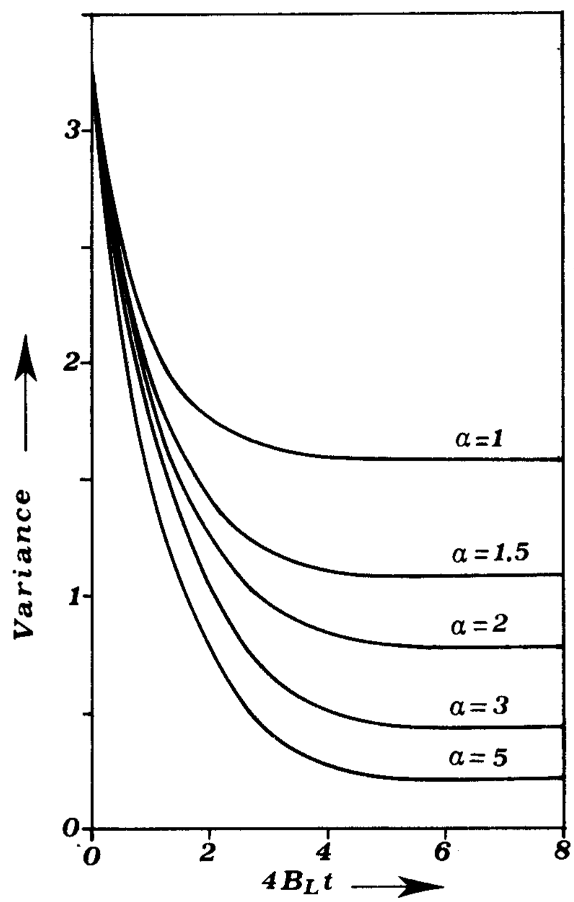


Fig. 1

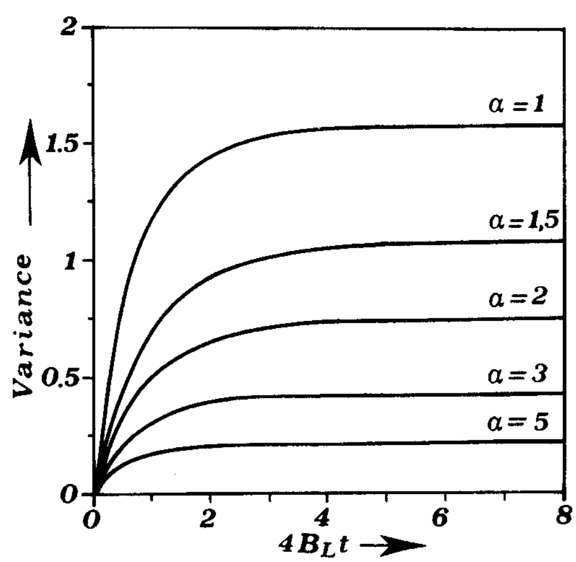


Fig. 2

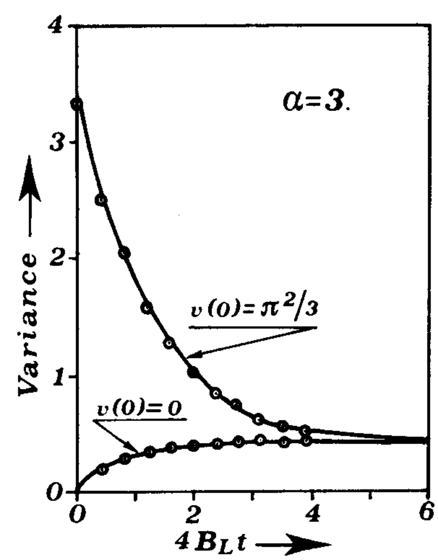
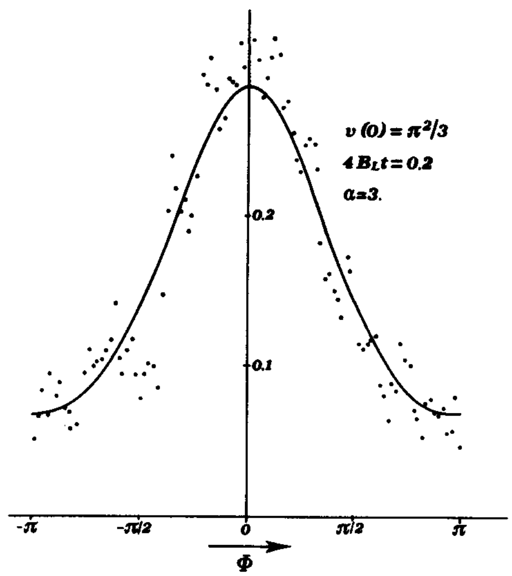
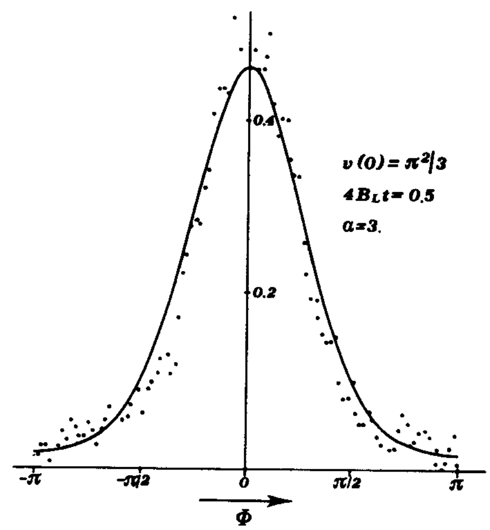
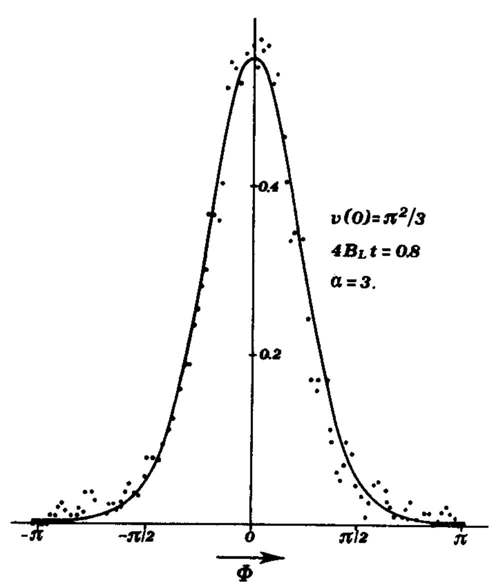


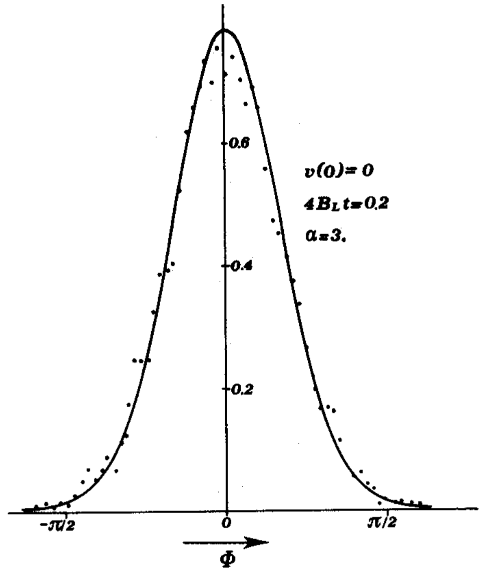
Fig. 3

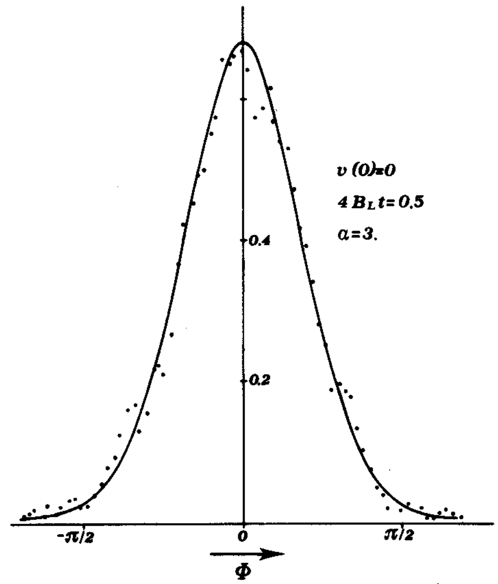




Figs.4-6 Phase-error distributions with the solutions of eq. (16) superimposed







Figs.7-9 Phase-error distributions with the solutions of eq. (16) superimposed