# THE COMPLEX DIGITAL FILTER AND ITS APPLICATIONS IN DIGITAL SIGNAL PROCESSING

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**Summary** Digital computer simulation of communication systems have been gaining wide acceptance and usage as a tool for analysis. In some cases, when the number of independent parameters is large or the processes are highly nonlinear, it is the only viable technique. In most digital computer simulations, the digital representation of bandpass filters can impose serious synthesis problems when conventional digital filter synthesis techniques are utilized. It is shown that the use of a complex (real and imaginary) technique of digital filter synthesis can eliminate several of the synthesis problems associated with conventional techniques. Three applications of the com lex technique are described in this paper.

The three applications discussed in this paper are listed below with a short description of each.

1. A lowpass-to-bandpass transformation is described that preserves all lowpass characteristics of the filter. For instance the gain and group delay functions remain symmetrical for <u>any</u> center frequency and bandwidth.

2. The synthesis of analytic representations of real signals can be easily achieved by the use of a complex digital filter. An important advantage of analytic signals is that their envelope and phase are instantaneously available.

3. Equalization of bandpass characteristics can be effected at lowpass and then shifted to the proper frequency without any undesirable warping effects.

Complex digital filter synthesis has been described previously, but very little emphasis has been placed on the application of this technique. Advances in the art of miniature high speed digital circuitry will allow the advantages of complex filtering to be realized in actual systems as well as digital simulations.

**Introduction** Digital filtering refers to the process of combining weighted values of input samples (X(nT), where T is the sampling period), and output samples (Y(nT))to produce a sequence of output samples that have certain desirable qualities. A goal in designing a digital filter may be to produce some pre-desired amplitude and phase characteristics. The accurate generation of these characteristics may be important, particularly in simulations of analog systems on digital computers.

Digital-filter theory is fairly advanced and is well documented[1,2,3,4] . In applying the existing theory the designer typically specifies a particular type and complexity of lowpass filter that fit a set of design requirements. With this information he simply generates or looks up in a table the appropriate pole and zero locations of the lowpass filter. By utilizing the appropriate lowpass-to-bandpass transformation[5], bandpass pole locations and/or transfer functions can be specified. At this point the designer specifies an appropriate sampled data transformation and generates his digital filter coefficients.

This paper is principally concerned with the lowpass-to-bandpass transformation and will show how a technique, investigated by Crystal and Ehrman[6], can yield digital filters, both recursive* and non-recursive, which have some very useful properties. First, the lowpass-to-bandpass transformation is an exact one, i.e., all lowpass properties are preserved, the gain and phase functions remain symmetrical for any center frequency and bandwidth. No other transformation possesses this property. For bandlimited sampled signals, the filter output is the analytic representation of its input, i.e. , Y(nT) = X(nT) + j $\hat{X}$(nT), where Y and X denote the output and input respectively, and $\hat{X}$ is the Hilbert transform of X. With knowledge of the analytic, signal one then has available the instantaneous envelope and phase.

This paper is separated into three main sections. First, filtering (both analog and digital) theory is reviewed. Then applications of complex digital filters are described and results presented, and finally some conclusions are reached.

**Review of filtering theory** The transfer function of a lowpass (LP) filter can be represented generally as

$$H(s) = \frac{\sum_{i=0}^{N} a_i s_i^{\,i}}{\sum_{j=0}^{M} b_j s^j} , \quad M \geq N \tag{1}$$

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* A non-recursive digital filter operates on weighted values of input samples only. Recursive filters operate on weighted values of input and/or output samples.

To be physically realizable, all zeros and all poles of H(s) must be complex conjugate pairs. By expanding (1) into partial fractions and utilizing the standard Z-transform technique of Radar and Gold[1] a set of real linear difference equations can be obtained with the form:

$$Y(nT) = \sum_{i=1}^{L} C_{i1}X(nT) + C_{i2}X(nT-T) + C_{i3}Y(nT-T) + C_{i4}Y(nT-2T) \tag{2}$$

where L = half the number of poles (each equation can be uniquely identified with a pair of complex conjugate poles)

$C_{ij}$ = real coefficients calculable from the pole positions and residues found from the partial fraction expansion of (1)

i = 1, 2,.. No. of Poles

j = 1,2,3,4

T = sampling period

Y(nT), X(nT) = output and input sample values, respectively, taken at the time instant, nT

If a bandpass filter is desired then the pole and possibly the zero positions, would be modified according to one of the lowpass to bandpass transformations. The most common transformation is the geometric transformation, which maps each lowpass pole into a pair of complex conjugate bandpass poles. This mapping is shown in equation (3).

$$s'_x = w_o \{s_x/2 \pm [(s_x/2)^2 - 1]\}^{1/2} \tag{3}$$

where $s_x$ = Lowpass pole

$w_o$ = The desired bandpass center frequency

$s'_x$ = The bandpass pole

Another common transformation is the Linear Transform. It is shown in equation (4).

$$s'_x = s_x \pm jw_o \tag{4}$$

Here the lowpass poles are linearly transformed both up and down the jw axis. Since both the linear and geometric transformations map each lowpass pole into a pair of complex conjugate poles, the number of parallel digital filter sections, as exemplified by (2), is double that of the original lowpass filter.

A different lowpass-to-bandpass transformation has been developed[6] that will be referred to as the complex transformation. The complex transformation differs from the two transformations previously described in that its application leads to circuit structures which are not physically realizable. This restriction does not exist if the filter is implemented digitally. Mathematically the transformation can be expressed as:

$$s'_x = s_x + jw_o \tag{5}$$

It is immediately apparent that after application of this transformation one ends up with poles that are not complex conjugates. This implies circuit elements, resistors, inductors, capacitors, that are complex. Thus the filter is unrealizable. However, when dealing with filters that have been implemented digitally, the only requirement is that the filter coefficients (i. e., $C_{ij}$ in (2)) be complex. The necessary complex arithmetic can be handled routinely in a digital computer.

**Complex filter theory** In developing the complex filter one starts with a real continuous lowpass filter characteristic. The goal is to shift the center frequency so that the lowpass amplitude and phase characteristic are centered at some arbitrary frequency, $w_o$.

Utilizing the standard Z-transform mapping, $Z = e^{sT}$, and substituting this into (5) the unit delay operator, Z', becomes:

$$Z' = \exp\{(s + jw_o)T\} = \exp\{sT\} \exp\{jw_oT\} = \alpha Z \tag{6}$$

where $\alpha = \exp jw_oT$ . Equation (6) simply states that a lowpass transfer function, in the Z-Domain, can be made to exhibit characteristics that are shifted in frequency by replacing the variable Z with $\alpha$ Z.

**Recursive filters** Using equation (2) for a two-pole (L=1) recursive filter, and transforming the equation to the Z-Domain, gives

$$Y(Z) = C_1X(Z) + C_2Z^{-1}X(Z) + C_3Z^{-1}Y(Z) + C_4Z^{-2}Y(Z) \tag{7}$$

If one desired to shift the filter characteristic that (7) represents, to a frequency $w_o$, then $\alpha_o = \exp\{jw_oT\}$ and

$$Y(Z) = C_1X(Z) + C_2\,\alpha_o^{-1}Z^{-1}X(Z) + C_3\alpha_o^{-1}Z^{-1}Y(Z) + C_4\alpha_o^{-2}Z^{-2}Y(Z) \tag{8}$$

Retransforming (8) back into the time domain one obtains:

$$\tilde{Y}(nT) = K_1X(nT) + \tilde{K}_2X(nT-T) + \tilde{K}_3Y(nT-T) + \tilde{K}_4Y(nT-2T) \tag{9}$$

where $K_1 = C_1$, $\tilde{K}_2 = C_2\,\alpha_o^{-1}$, $\tilde{K}_3 = C_3\,\alpha_o^{-1}$, $\tilde{K}_4 = C_4\alpha_o^{-2}$ and "~" denotes a complex number.

The significance of (9) can be seen by finding the real and imaginary components of Y(nT).

$$
\begin{aligned}
R_e\{Y(nT)\} &= K_1 \cdot R_e\{X(nT) + [R_e\{K_2\} \cdot R_e\{X(nT-T)\} \\
&- I_m\{K_2\} \cdot I_m\{X(nT-T)\}] + [R_e\{K_3\} \cdot R_e\{Y(nT-T)\} \\
&- I_m\{K_3\} \cdot I_m\{Y(nT-T)\}] + [R_e\{K_4\} \cdot R_e\{Y(nT-2T)\} \\
&- I_m\{K_4\} \cdot I_m\{Y(nT-2T)\}]
\end{aligned}
\tag{10a}
$$

$$
\begin{aligned}
I_m\{Y(nT)\} &= K_1 \cdot I_m\{X(nT)\} + [I_m\{K_2\} \cdot R_e\{X(nT-T)\} \\
&+ R_e\{K_2\} \cdot I_m\{X(nT-T)\}] + [I_m\{K_3\} \cdot R_e\{Y(nT-T)\} \\
&+ R_e\{K_3\} \cdot I_m\{Y(nT-T)\}] + [I_m\{K_4\} \cdot R_e\{Y(nT-2T)\} \\
&+ R_e\{K_4\} \cdot I_m\{Y(nT-2T)\}]
\end{aligned}
\tag{10b}
$$

Equations 10a and 10b can be put into diagrammatical form, as shown in Figure 1. The algorithm that Figure I represents encompasses all recursive digital filters. If any of the other transformations (s-to-Z-Domain), such as bilinear or matched-Z, are utilized instead of the Z-transform then Figure 1 will remain the same in form while possibly having a change in coefficient value. There are four possible combinations of input signal type (real or complex) and filter type (shifted or nonshifted). Each case is analyzed for its particular impact on filter structure.

Case 1: Input signal is real, no shifting takes place ($w_o = 0$). All cross-terms are zero because the imaginary parts of all the coefficients are zero. Figure 1, for this case, reduces to the top three delay elements and their corresponding four real coefficients (this is termed the real filter). This is the standard configuration for real signals given in (1).

Case 2: Input signal is complex, no shifting takes place ($w_o = 0$). The cross-terms are still zero. Since there are two input signals the filter structure reduces to two identical filters in parallel. The upper section, real filter, (referring to Figure 1) handles the real part of the input signal and the lower section, imaginary filter, controls the imaginary part of the input signal.

Case 3: Input signal is real, shifting takes place ($w_o \neq 0$). The cross-coupling coefficients ($I_m\{K_i\}$, i = 1, 2, 3, 4) are in general non-zero. Even though the imaginary signal input is zero, the imaginary filter will receive inputs (at its summer) from the real filter, and in turn the real filter receives contributions from the imaginary filter. Therefore, the output will contain both real and imaginary components.

Case 4: Input signal is complex, shifting takes place ($w_o \neq 0$). For this case, the complex signal output is made up of direct feedthrough terms and crosscoupling terms. That is, the real output is composed of the real input signal filtered by the real filter and contributions from internal points (cross-coupling), of the imaginary filter. The imaginary signal output is analyzed analogously as the real signal output.

Note, that in actually implementing complex recursive digital filters on general purpose digital computers, one can use the dual-filter implementation shown in Figure 1 or use a single-filter implementation where all quantities, signals, coefficients, and operations are dimensioned and executed as complex quantities. Since both methods are mathematically identical they are computationally identical.

**Non-Recursive Filters** In considering non-re0ursive filters, a general input-output relationship can be expressed as;

$$Y(nT) = \sum_{i=0}^{N} a_i \, x(nT-iT) \qquad (11)$$

Various methods of choosing the $a_i$ to construct desired characteristics can be found in (7) and (8). Concise steps for finding the $a_i$ for arbitrary conditions of gain and/or phase are given in the two references, and the detailed description of these methods are beyond the scope of this paper. Transforming (11) into the Z-Domain, one obtains:

$$Y(Z) = \sum_{i=0}^{N} a_i Z^{-i} X(Z) \qquad (12)$$

Substituting the shifting relationship (6), into one ontains:

$$Y(Z) = \sum_{i=0}^{N} a_i \alpha^{-i} Z^{-i} X(Z) \qquad (13)$$

Retransforming 13 back into the time domain gives

$$\underset{\sim}{Y}(nT) = \sum_{i=0}^{N} \underset{\sim}{K}_i \, X(nT-iT) \qquad (14)$$

An algorithm to implement 15a and 15b is displayed in Figure 2. As for the recursive case, equation 15 encompasses all transversal-type filters. The algorithm can be analyzed analogously to that shown for the complex recursive filter. Specifically:

Case 1: Input signal is real, no shift takes place ($w_o = 0$). All cross-terms are zero because the imaginary parts of all the coefficients are zero. Figure 2, for this case, reduces to the top delay elements, real coefficients, and summer. This is termed the real filter and is the standard configuration for real signals given in (1).

Case 2: Input signal is complex, no shifting takes place ($w_o = 0$). The cross-terms are still zero. Since there are two input signals the filter structure is two identical transversal filters in parallel. The upper section, real filter, (referring to Figure 2) handles the real part of the input signal and the lower section, imaginary filter, controls the imaginary part of the input signal.

Case 3: Input signal is real, shifting takes place ($w_o \neq 0$). The cross-coupling coefficients ($I_m\{K_i\}$, i = 1, N), are in general, non-zero. Even though the imaginary signal input is zero, the imaginary filter will receive inputs (at its summer) from the real filter, and in turn the real filter receives contributions from the imaginary filter. Therefore the output will contain both real and imaginary components.

Case 4: Input signal is complex, shifting takes place ($w_o \neq 0$). For this case, the complex signal output is made up of direct feedthrough terms and cross-coupling terms. That is, the real output is composed of the real input signal filtered by the real filter and contributions from internal (cross-coupling) points of the imaginary filter. The imaginary signal output is analyzed analogously as the real signal output. The tradeoffs involved in implementing the complex nonrecursive digital filter algorithm are exactly the same as that shown for the recursive filter.

**Applications** Three applications of complex digital filters will be discussed in this section;

1. The simulation of narrow-band filtering effects in bandpass simulations
2. Obtaining an analytic signal from a real bandlimited signal
3. Simulating complex non-recursive filters to be used as all-pass phase equalizers.

It will be shown that through the use of complex digital filters in the above mentioned applications (rather than conventional real digital filters), the designer generally has a simpler problem in synthesizing certain types of filters, and he can gain information which is difficult to obtain with real filter design methods.

**Simulation of Narrow-band filtering effects in bandpass simulations** When a bandpass transfer function is desired, the general method of digital filter synthesis, as mentioned previously, is to map each lowpass pole into a complex conjugate pair of bandpass poles by utilizing one of the previously mentioned transformations (3), (4), and (5). Application of the geometric transformation to bandpass synthesis yields filters in which the pass-band gain is a replica of the lowpass gain (preservation). The linear and geometric transformations both will preserve the complete lowpass transfer function if the center frequency $w_o$ is >> BW. For this very narrow-band condition the contributions to gain and phase from the complex conjugate set of poles centered around $-w_o$ is insignificant. As the filter becomes more wideband, the contributions from the negative cluster of poles becomes appreciable and as a result the bandpass transfer function (relative to the lowpass transfer function) becomes warped. The geometric transformation warps the bandpass pole locations in such a way as to offset any gain contributions, from the negative cluser, in the pass-band. The linear transformation, on the other hand, does not warp the pole locations but simply shifts them; however, it possesses the property[8] of phase preservation even under relatively wideband conditions. The complex transformation, which does not generate any new poles but simply shifts existing ones, will preserve gain and phase under all conditions of center frequency and bandwidth.

The best way of illustrating the properties of the three transformations is by using an example. Figure 3 is a graph of the complex s-plane. Curve A is a plot of the lowpass poles and their loci for a 10-pole Butterworth filter of bandwidth 100 Hz. Curve B shows the bandpass poles created when the lowpass poles are shifted by means of the geometric transformation to a center frequency of 200 Hz. Curve C displays the linearly transformed poles with the same center frequency. Finally, Curve D, which is the upper portion of Curve C, shows the pole locations after a complex transformation.

The warping of the pole locations is clearly evident in Curve B. The poles are much more closely packed on the origin sides of the clusters for this case. Also, the apparent difference in locations of the linear and geometric poles is another indication of the amount of warping that must be applied in order to preserve pass-band gain in this wideband example.

Figure 4A is a plot of the power gain function of a 6-pole lowpass Butterworth filter with a bandwidth of 100 Hz. Also plotted are the gain characteristics of the linear, geometric, and complex transformed bandpass filter with center frequencies of 350 Hz. The measurements were obtained by synthesizing appropriate digital filters, obtaining the impulse response of each, and then obtaining gain and group delay characteristics by taking the discrete Fourier transform of
the impulse responses.

The most striking point in Figure 4A is curve D, the complex bandpass filter. It is centered at a frequency of 350 Hz, with a bandwidth of 200 Hz, and is monotonically decreasing on both sides in the stop band including the negative frequency region. The curve is also identical in shape and level to the lowpass characteristic (Curve A). The geometric filter exhibits an excellent pass-band gain characteristic but unsymmetrical stop band skirts, while the linear filter displays unsymmetrical pass-band and stop band characteristics. As can be seen, both the linear and geometric filters possess even gain symmetry around the line $f = 0$. Note that the corresponding group delay curves, Figure 4B, reveal the complex filter delay (Curve D) is an exact duplicate, in shape and level, of the lowpass group delay. Also, its delay characteristics are centered on the shifting frequency $f_o$ ($w_o/2$). The geometric filter displays typical distortions of the delay characteristic. In contrast, the linear filter almost exactly preserves the lowpass group delay, which in some applications may be a good reason for using it. Again, both the linear and the geometric filters exhibit even group delay symmetry around the line f = 0.

At this point the main application can be readily discussed. If the designer is given the problem of designing a narrow-band communication system simulation, he will quickly realize that in order to synthesize real digital bandpass filters with lowpass characteristics (narrow-band) the filter center frequencies must be located at a frequency that is much greater than the signalling rates that are to be simulated. This implies that the sampling rate would have to be approximately twice the center frequency (at an absolute minimum). To have a moderate amount of carrier resolution (samples/cycle), the sampling rate would have to be at least four to six times the filter center frequency. If complex filters are utilized then, knowing that the filter transfer function is always narrow-band independent of center frequency, the filters center frequency need only have to be equal to its bandwidth. This implies that a sampling rate that is only four to six times the data bandwidth can be used.

The significant gain that can be realized by utilizing complex digital filters rather than real digital filters for narrow-band simulations is that the sampling rate can be designed to be in the order of one-tenth, or smaller, than that used in real filters. This implies that for identical real time lengths of simulation run, the complex filters can process ten or more times as much simulated data.

**Obtaining the analytic signal from real bandlimited signals** From Figure 4, it was seen that the complex filter indeed shifts the lowpass characteristics in one direction (in frequency). If one designs a complex bandpass filter with the following characteristics:

1. The filter bandwidth is larger (3-5 times) than the signal bandwidth
2. The filter pass-band gain is relatively flat
3. The power gain ratio, power gain at frequency f 1 divided by the power gain at frequency $-f_1$ , is large (50 dB or greater).

then the filter output, for a bandpass input (double-sided spectra), essentially contains the input power that is in the positive frequency domain. The negative frequency power is rejected by the complex filter gain function. Conditions 1 and 2 above insure that the filter contributes a negligible amount of amplitude distortion and, due to the large BT product involved (BT = 3-5), a minor phase distortion. Condition 3, attenuation of the negative frequency components of the input signal, is the most important in that the technique (analytic signal generation) is premised on it. If all three conditions are met, then by definition the real part of the output equals the input signal (there is a factor of one-half due to energy conservation), and the imaginary part of the output equals the Hilbert transform of the input (the same factor of one-half is present here also). The closeness of approximation between the complex output and the true analytic signal representation is directly related to how well the three previous conditions were met.

Utilization of a Butterworth filter of appropriate bandwidth and center frequency will satisfy the three previous conditions extremely well. It was found that by synthesizing a complex 6-pole Butterworth filter of center frequency 350 Hz with a 3-dB bandwidth of 400 Hz, negative frequency suppression was between 50dB for the lower frequencies (350-200 = 150 Hz) and 80-dB for the higher frequencies (350 + 200 = 550 Hz). For a signal whose null bandwidth was 100 Hz the gain deviation from flatness for this bandwidth was a maximum of 0.05 dB .

Having the analytic signal presents the designer with a large amount of available information that is not readily available when real digital filters are used. For instance, the modulus of the output is the instantaneous envelope of the signal, and the angle of the output components is the instantaneous phase of the signal. If the complex output $(\underset{\sim}{Y}(nT))$ is multiplied by a time varying scalar, then amplitude modulation will result. When $(\underset{\sim}{Y}(nT))$ is multiplied by a complex number that represents $e^{-j\emptyset(t)}$ then phase modulation, where $\emptyset(t)$ is the angular modulation, is the result. If a phase-modulated or frequency-modulated signal is input to the filter then by simply bandpass limiting then finding the envelope of the derivative of the limited output, one has detected the modulating signal. If the envelope of $(\underset{\sim}{Y}(nT))$ is used to calculate a phase angle, $\emptyset_o$ , which is fed back as $e^{-i\emptyset o}$ and multiplied by $(\underset{\sim}{Y}(nT))$ , then a nonlinearity such as AM/PM conversion can be observed and studied. Some of the above applications are in the process of being implemented while the rest have already been implemented. As far as the analytic signal generation is concerned, accurate results have been obtained.

**The simulation of complex transversal phase equalizers** A standard method[9] of phase or group delay equalization for Chebyschev or Butterworth filters is to synthesize an all-pass filter that possesses a pass-band group delay function that is the inverse of that shown in Figure 4B, Curve A. A good approximation to the inverse is a lobe of the cosine function. Therefore the desired filter has a group delay that is given by:

$$\tau (w) = A \cos (\alpha w) \qquad (16)$$

then

$$\emptyset(w) = -\beta \sin (\acute{a}w) \qquad (17)$$

where $\beta = A/\alpha$. It is necessary to note that $\tau(w)$ and $\emptyset(w)$ exist at all frequencies; they are not bandlimited. The desired filters transfer function is

$$H(jw) = e^{j\beta} \sin(\alpha w) \qquad (18)$$

Equation (18) can be manipulated into the form of sums of even and odd order harmonics weighted by Bessel functions. When transformed into the time domain, the filters' impulse response takes the form of equation (11), namely,

$$Y(nT) = \sum_{i=0}^{N} a_i x(nT-iT)$$

where $a_i$ = a function of Bessel functions.

If the complex transformation is applied to this filter then any particular lobe of the group delay cosine function is shifted to any desired frequency. in order to equalize a particular filter, one would measure the peak-to-valley group delay and bandwidth of the recursive lowpass filter in question. From these measurements, β and α are calculated. At this point a first-order phase equalization of the lowpass filter has been accomplished. Next, both the recursive and non-recursive filters are shifted up to the desired center frequency. The filter is now equalized at bandpass. An illustration of this is shown in Figure 5, which displays a lowpass Chebyschev group delay characteristic and a characteristic composed of the series connection of the same filter and a first-order equalizer (Transversal delay line). The effects of equalization are readily apparent. B6th the filter and equalizer characteristics were shifted up to the desired center frequency (350 Hz). A plot of the composite bandpass characteristics Is shown in Figure 6. It can be seen that the bandpass group delay and gain are identical in shape and magnitude to the lowpass -characteristic.

Another related use can be found for the complex transversal filter. The nonshifted equalizer characteristic, which is cosinusoidal in frequency becomes sinusoidal if shifted by a frequency that is one-quarter or three-quarters of the period at f = 0. If the period is made large, then effectively the group delay characteristic at zero frequency is linear. By adjusting the magnitude, β, any practical slope, positive or negative, can be realized. The characteristics would then be shifted up to the desired center frequency where the effects of a linear group delay, or parabolic phase, can be investigated. This process can be extended indefinitely. A parabolic-shaped piece of cosinusoid. could be selected and appropriately shifted. This would yield a cubic phase characteristic. This technique has been applied to a problem where specifications on the maximum allowable magnitude of the parabolic phase component (linear group delay) had to be determined for a specific channel configuration.

**Conclusions** When using "real" digital filters for bandpass simulations, the outputs of the filters consist solely of discrete samples that represent the multiplication of the input frequency spectrum with the double-sided filter transfer function. "Complex" filter outputs are basically the same except that the filter transfer function, which is a shifted version of the lowpass function, is essentially one-sided (for bandlimited functions). The complex output consists of real and imaginary parts that are Hilbert transform pairs. Utilizing this fact, both envelope and phase are instantly available along with a filter transfer function that is exactly equal to the lowpass transfer function which has been shifted. The question of which transform is "best" cannot be answered. It depends on what one is attempting to simulate. If the problem is to simulate a narrow-band system with a wideband computer simulation, then the complex transformation should be used because the filter characteristics are narrow-band. But, if a wideband simulation is used to model a wideband system, then the actual transformation that the filter circuit designer will implement should be used.

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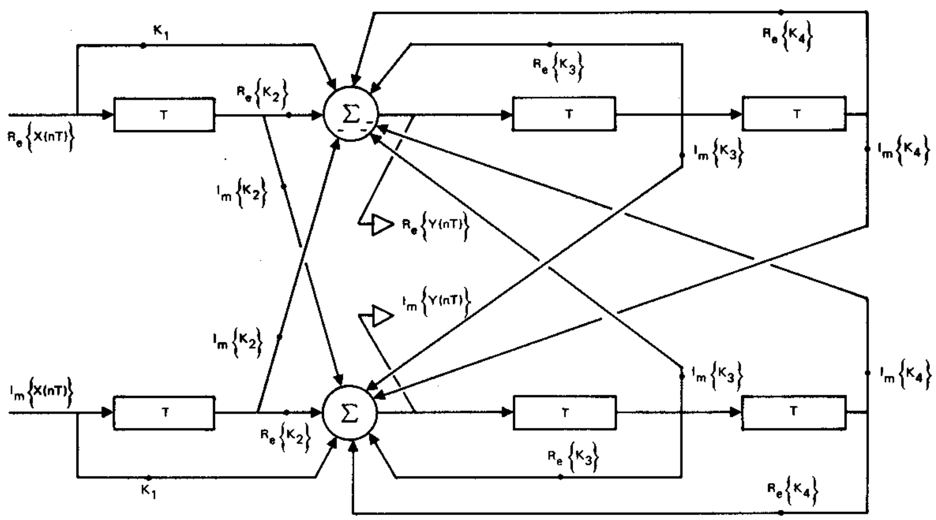


**Figure 1. Complex Recursive Digital Filter Algorithm**

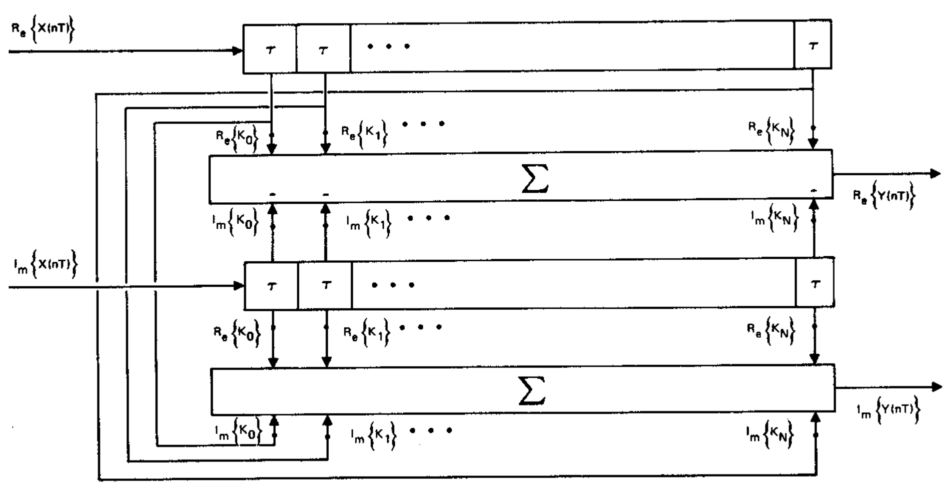


**Figure 2. Complex Nonrecursive Digital Filter Algorithm**

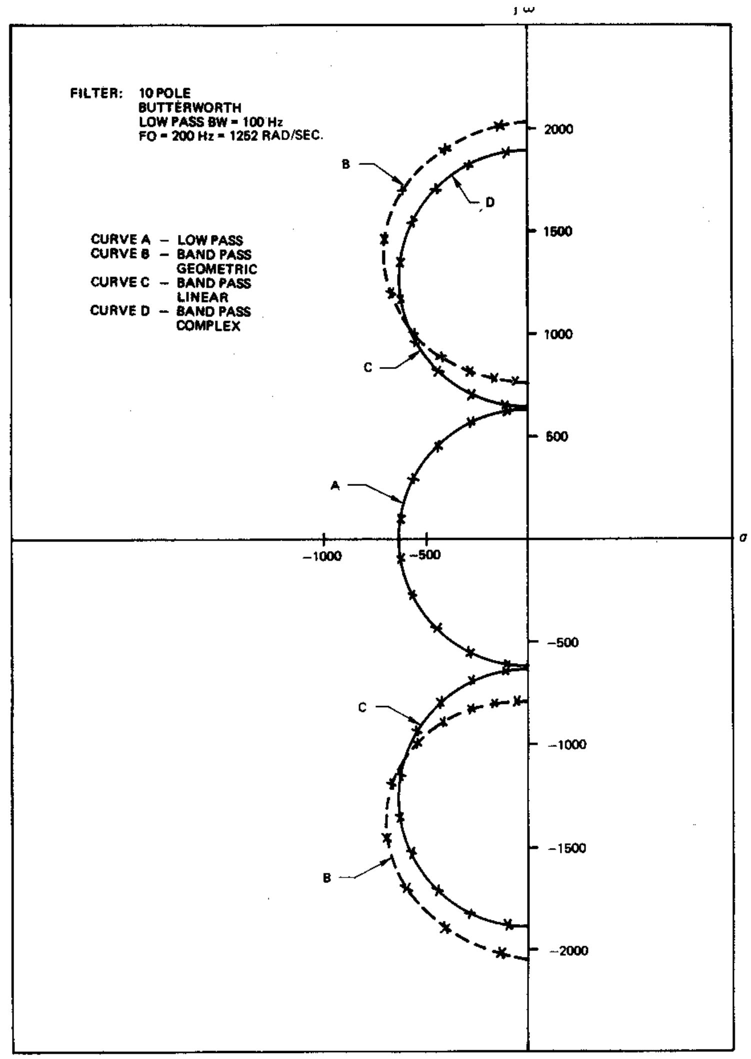


**Figure 3. Pole Position Loci Plot**

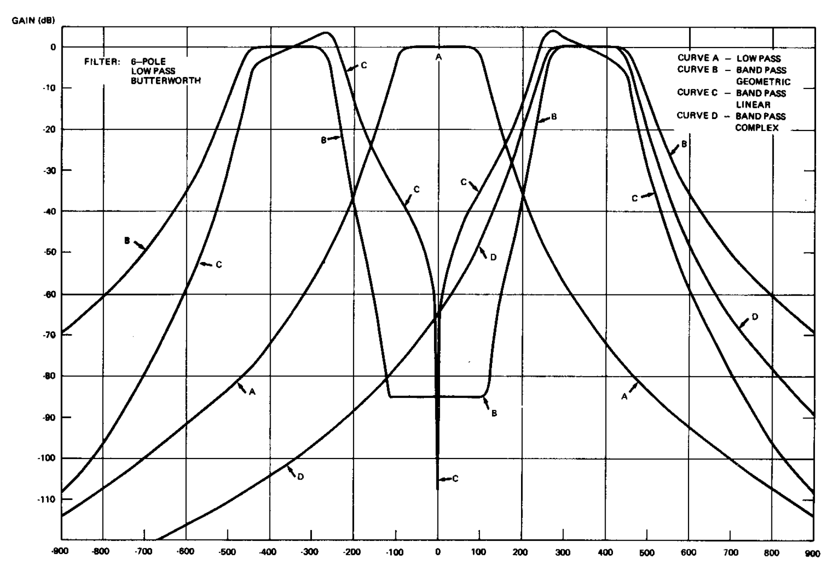


**Figure 4a. Gain Plots**

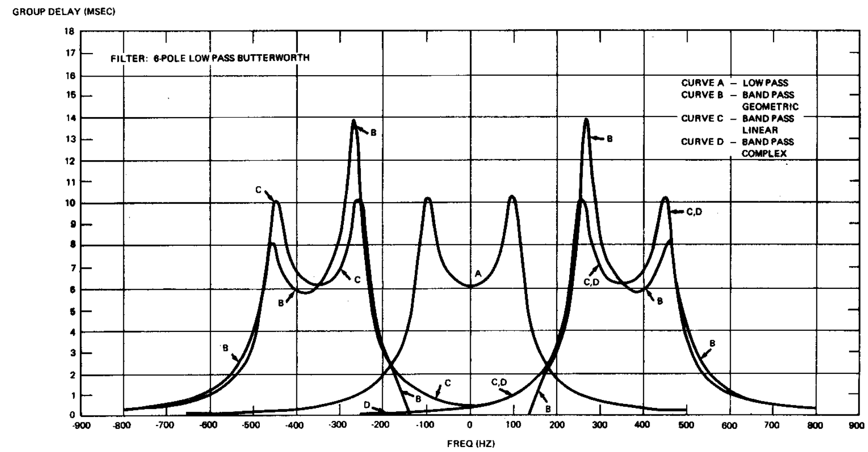


**Figure 4b. Group Delay Plots**

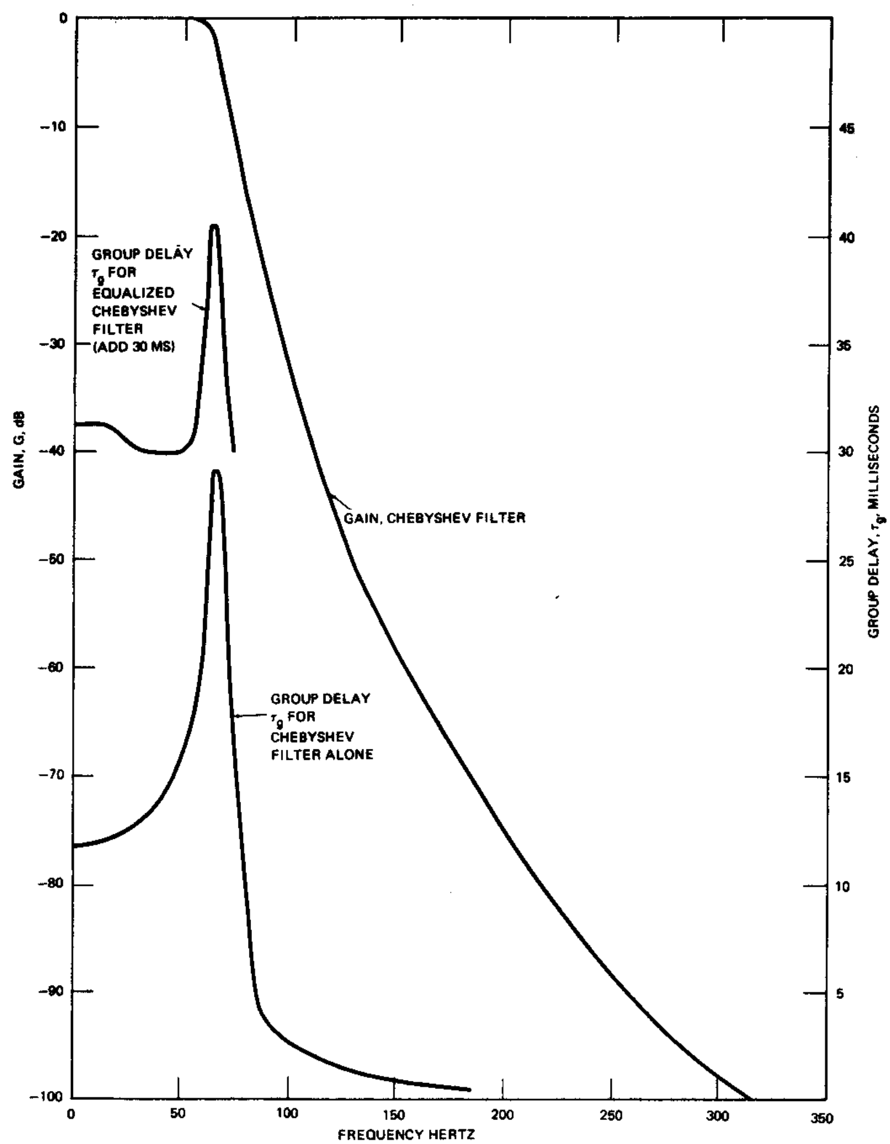


**Figure 5. Lowpass Gain and Group Delay (Equalized and Unequalized)**

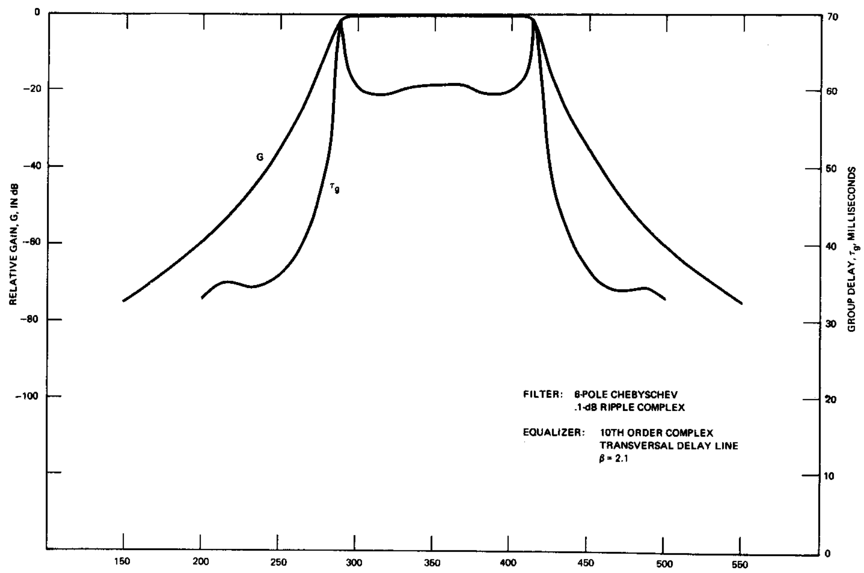


**Figure 6. Bandpass Gain and Equalized Group Delay**

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