DEMODULATION OF FM SIGNALS BY A DISCRETE PHASE LOCKED LOOP

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Summary This paper discusses the discrete-phase locked loop for demodulation of an incoming frequency modulated analog signal imbedded in white noise. The discrete samples admitted into the system adoptively, represent an estimate of the message at the discrete instants of time which can be improved further with a correction factor.

The work is in two parts: (1) even without the noise, perfect demodulation is not possible. To arrive at the best estimate, requirements are specified regarding the ratio of carrier frequency to maximum frequency deviation, taking into consideration the lock range and the Nyquist sample rate. A simulation study is performed to specify the bounds for the estimate to be accurate enough. (2) With the above considerations, the receiver is analyzed with noise. For high carrier to noise ratio, it is shown that the performance is at least as good as an analog discriminator. The only requirement being that the condition specified in (1) is more stringent. Digital computer solutions of the system check the above and provide a qualitative picture of the threshold behavior comparable to analog feedback discriminators.

Introduction The advantages of digital demodulation of analog FM signals are utilization of large scale integrated digital circuitry as well as the digital computer itself. This takes care of the problems associated with analog phase lock loops such as phase comparator inaccuracies, non-linearities associated with voltage controlled oscillator, and the possibility of the above components together with the filter, getting saturated due to noise spikes. Digital circuitry has problems too, like quantization noise, round off, and overflow errors, but normally these are minor compared to the overall gains. So there is an increasing interest in digital and discrete phase locked loops [1-6].

In reference [1], a discrete phase locked loop is proposed and analyzed without noise. As an application, it is shown that by measuring actual time periods of the clock it is possible to demodulate an FM signal. Comparison of its performance or analytical results with noise are not available.

In this paper, the incoming sinusoidal FM signal embedded in white noise is considered. The discrete samples admitted into the system adoptively are utilized to estimate the message. The system is analyzed to arrive at a very simple implementable structure, whose performance is at least as good as an analog discriminator but with better threshold behavior.

Operation of the Loop Figure 1 represents the block diagram of the discrete phase locked loop [1]. The received FM signal embedded in noise is filtered to eliminate the portion of the noise outside the Carson's rule band width. The analog output of the filter is sampled when the clock signals it to do so by sending a sharp pulse or other type of signal. The discrete sample admitted into the loop controls the time period of the clock which is equal to the next sample period in such a way as to decrease the difference between the phase of the incoming signal and the output phase of the clock. The time period of the clock when its input is zero is equal to the time period of the carrier. The samples of the incoming signal at the discrete instants are passed through a low pass filter. The output of the filter is an estimate of the message.

To understand the phase lock operation of the loop, reference should be made to the incoming sinusoidal signal in Fig. 2.[1] Assuming the carrier as above for simplicity, the basic purpose of the clock is to predict the positive going zero crossings and to lock the sampling instants to them. The sampler takes the initial sample e(0) at time t(0). This indicates that the clock is lagging behind by $\sin^{-1}e(0)$ radians. The next time period T(1) of the clock is given by

$$T(1) = T - Ge(0)$$

where T is the period of the clock when its input is zero and equal to the period of the incoming signal. The clock signals the sampler to take the next sample after T(1), i.e., second sample e(1) at time t(1) bringing the sampling instant closer to the zero crossing. Thus, in a few sampling instants the loop catches up with the positive going zero crossings of the incoming signal. The above refers to a phase step operation and the loop tracks it with zero steady state error. Similarly the loop locks to the phase ramp with steady state phase error within the lock range to be specified later.

Analysis [1] First, considering the loop without noise, the incoming signal is:

$$A \sin(\omega_0 t + \theta_i(t)) \qquad (1)$$

[1] For the sake of continuity, the phase lock operation and the analysis without noise are explained here. This loop has been proposed in reference [1].

where

$$\theta_i(t) = d_f \int_0^t a(u)d(u)$$

T = period of the clock with its input equal to zero.

At the κth sample the signal admitted into the system is:

$$A \sin\left[\omega_0 t(k) + \theta_i\{t(k)\}\right] \tag{2}$$

where t(k) is the time at the κth sampling instant. The phase of the incoming signal can be written as

$$\alpha(k) = \omega_0 t(k) + \theta_i(k) \tag{3}$$

A sample of the incoming signal is taken when the clock sends a pulse. This discrete timing information which the clock provides to the sampler constitutes the phase of the former. The clock phase advances by 2π on each sampling instant. On the κth sampling instant the clock phase is given by

$$\beta(k) = 2\pi k \tag{4}$$

It can also be written in the format of the phase of the incoming signal as

$$\beta(k) = \omega_0 t(k) + \theta_0(k) \tag{5}$$

As said before the sampler compares the phases of the incoming signal and that of the clock at the sampling instants. The output of the sampler at the κth sampling instant is given by

$$\begin{aligned} e(k) &= A \sin[\alpha(k) - \beta(k)] \\ &= A \sin[\theta_i(k) - \theta_0(k)] = A \sin \phi(k) \end{aligned} \tag{6}$$

where $\phi(k)$ is the phase error and $\theta_i(k)$ and $\theta_0(k)$ are called input and output phases of the loop respectively.

The output of the sampler is multiplied by a constant G, (refer to Fig. 1) to be defined later. The *j*th clock period is given by

$$T(j) = T - AG \sin \phi(j-1)$$

The corresponding angular frequency is given by

$$\omega(j) = \frac{2\pi}{T - AG \sin \phi(j-1)} \tag{7}$$

From Equation (5)

$$\theta_0(k+1) = \beta(k+1) - \omega_0 t(k+1) \tag{8}$$

But by definition

$$\beta(k+1) = \sum_{j=1}^{k+1} \omega(j)T(j) \tag{9}$$

and

$$t(k+1) = \sum_{j=1}^{k+1} T(j) \tag{10}$$

(Ref: Fig. 2, considering starting time t(0) = 0) and

$$\omega_0 = 2\pi/T \tag{11}$$

Using (9), (10) and (11) in (8), one gets

$$\begin{aligned} \theta_0(k+1) &= \omega_0 AG \sum_{j=1}^{k+1} \sin \phi(j-1) \\ &= \omega_0 AG \sum_{j=0}^{k} \sin \phi(j) \end{aligned} \tag{12}$$

Equation (12) can be rewritten as

$$\theta_0(k+1) = \theta_0(k) + \omega_0 AG \sin \phi(k) \tag{13}$$

Eliminating the output phase in equation (13) results in

$$\phi(k+1) - \phi(k) + \omega_0 AG \sin \phi(k) = \theta_i(k+1) - \theta_i(k) \tag{14}$$

Let it be assumed that the phase error is small. The equation (14) becomes

$$\theta(k+1) + (\omega_0 AG-1)\phi(k) = \theta_i(k+1) - \theta_i(k) \tag{15}$$

In case of phase step input $\theta_i(k+1) = \theta_i(k)$ the solution approaches zero on the next sampling instant, i.e. $\phi(k+1) \rightarrow 0$ for $\omega_0 AG = 1$. Even when the input is not a step, $\omega_0 AG = 1$ will give minimum error. Hence unity is the best value of $\omega_0 AG$ for best tracking performance.

With $\omega_0 AG$ equal to unity, equations (13) and (14) become

$$\theta_0(k+1) = \theta_0(k) + \sin \phi(k)$$

or

$$\theta_0(k+1) = \theta_0(k) + \sin [\theta_i(k) - \theta_0(k)] \tag{16}$$

$$\phi(k+1) - \phi(k) + \sin \phi(k) = \theta_i(k+1) - \theta_i(k) \tag{17}$$

Equations (16) and (17) give the output phase and phase error respectively in terms of the input phase.

From Fig. 2 one can get an expression for the real time at the κth sampling instant as

$$t(k) = t(0) + kT - G \sum_{j=0}^{k-1} e(j) \tag{18}$$

From equations (16) and (18), and using the condition $\omega_0 AG = 1$, one gets

$$t(k) = t(0) + kT - \frac{\theta_0(k)}{\omega_0} \tag{19}$$

Now, $t(0) = 0$ implies

$$t(k) = kT - \frac{\theta_0(k)}{\omega_0} \tag{20}$$

and

$$\theta_0(0) = 0 \tag{21}$$

For a phase step equation (17) reduces to

$$\phi(k+1) = \phi(k) - \sin \phi(k) \tag{22}$$

Showing that the steady state error is zero. And for a frequency step of $(\omega - \omega_0)$, equations (17) and (20) give

$$\phi(k+1) = \phi(k) - \frac{\omega}{\omega_0} \sin \phi(k) = 2\pi \frac{(\omega - \omega_0)}{\omega_0} \tag{23}$$

indicating that the loop tracks the frequency step with a finite steady state error

$$\phi_{ss} = \sin^{-1} \frac{(\omega-\omega_0)}{\omega} 2\pi \tag{24}$$

if and only if

$$\frac{2\pi}{2\pi+1} < \frac{\omega}{\omega_0} < \frac{2\pi}{2\pi-1} \tag{25}$$

The above refers to lock range condition.

Performance as a Frequency Demodulator (i) Without noise:

To reconstruct the message (assumed to be band limited) the sample rate should be at least twice the message band width, i.e.,

$$\omega_0 - d_f\, a(t)_{max} \geq 2W \tag{26}$$

where

$\omega_0 - d_f\, at_{max}$ = min^m sampling frequency
W = one sided message band width

Consider that the discrete phase locked loop under lock condition is tracking the phase of the FM signal with small phase error. This implies that the variations in the incoming frequency are well within the lock range, i.e.,

$$\sin \phi \simeq \phi$$

Substituting this in equation (17),

$$\phi(k+1) \simeq (k+1) - \theta_i(k)$$

i.e.,

$$\sin \phi(k+1) = \theta_i(k+1) - \theta_i(k) \tag{27}$$

since sin of the phase error is available at each sample,

$$\begin{aligned} \sin \phi(k+1) &\simeq \left[\frac{\theta_i(k+1) - \theta_i(k)}{T(k+1)} \right] T(k+1) \\ &\simeq \dot{\theta}_i \left[t(k) + \frac{t(k+1) - t(k)}{2} \right] T(k+1) \\ &\simeq d_f\, a \left[\frac{t(k+1) + t(k)}{2} \right] T(k+1) \end{aligned} \tag{28}$$

as the-sampling is at IF frequency, T(k+1) for all k could be considered small.

$$\max^{m} T(k+1) = \frac{2\pi}{\omega_0 - d_f\, a(t)_{max}} \qquad \text{for all } k$$

$$\min^{m} T(k+1) = \frac{2\pi}{\omega_0 + d_f\, a(t)_{max}} \tag{29}$$

if $\omega_0 >> d_f\, a(t)_{max}$

$T(k+1) \simeq T$

From equation (28)

$$a\left(\frac{t(k)+t(k+1)}{2}\right) \simeq \frac{\sin\phi(k+1)}{Td_f} = \left[\frac{1}{ATd_f}\right] e(k+1) \tag{30}$$

i.e., $\frac{e[t(k+1)]}{ATd_f}$ is an estimate of a $\left[\frac{t(k+1)+t(k)}{2}\right]$ showing a delay of half a sample period on an average. But equation (29) appears to be quite a restriction. To improve this, consider equation (27)

$$\sin\phi(k+1) \simeq d_f\, a\left(\frac{t(k+1)+t(k)}{2}\right)[T - AG\sin\phi(k)]$$

$$\sin\phi(k+1) \simeq Td_f\, a\left(\frac{t(k+1)+t(k)}{2}\right) - \frac{d_f\, a\left(\frac{t(k+1)+t(k)}{2}\right)}{\omega_0}\sin\phi(k) \tag{31}$$

Since ω_0 is at IF frequency and for $\frac{d_f\, a\left(\frac{t(k+1)+t(k)}{2}\right)}{\omega_0}$ reasonably small, we can write

$$\sin\phi(k+1) \simeq Td_f\, a\left(\frac{t(k)+t(k+1)}{2}\right) - \frac{d_f\, a\left(\frac{t(k)+t(k+1)}{2}\right)}{\omega_0}\sin\phi(k+1)$$

states essentially sample rate/message rate is large enough (which is really so) and is needed for practical implementation since sample period is varying and particularly with noise giving rise to random sample periods creating delay implementation difficulties. Equation (31) becomes

$$\sin\phi(k+1)\left[1 + \frac{d_f\, a\left(\frac{t(k+1)+t(k)}{2}\right)}{\omega_0}\right] \simeq Td_f\, a\left(\frac{t(k)+t(k+1)}{2}\right)$$

$$a\left(\frac{t(k)+t(k+1)}{2}\right) \simeq \frac{\sin\phi(k+1)\left[1 + \frac{d_f}{\omega_0}\, a\left(\frac{t(k)+t(k+1)}{2}\right)\right]}{Td_f} \tag{32}$$

Since $\frac{d_f\, a\left(\frac{t(k+1)+t(k)}{2}\right)}{\omega_0} < 1$ for all k, the value of the estimate $a\left(\frac{t(k)+t(k+1)}{2}\right) \simeq \frac{e(k+1)}{ATd_f}$ is always less than the exact value. A reasonably thing to consider is the second order approximation, i.e., in equation (32) the right hand side $a\left(\frac{t(k)+t(k+1)}{2}\right)$ is replaced by equation (30), i.e.,

$$a\left(\frac{t(k)+t(k+1)}{2}\right) \simeq \frac{\sin\phi(k+1)}{Td_f}\left[1 + \frac{d_f}{\omega_0}\frac{\sin\phi(k+1)}{Td_f}\right]$$

$$a\left(\frac{t(k)+t(k+1)}{2}\right) \simeq \frac{\sin\phi(k+1)}{Td_f}\left[1 + \frac{\sin\phi(k+1)}{2\pi}\right] \tag{33}$$

i.e.,

$$\hat{a}\left(\frac{t(k)+t(k+1)}{2}\right) = \frac{e(k+1)}{ATd_f}\left[1 + \frac{e(k+1)}{2\pi A}\right]$$

$$\left[1 + \frac{e(k+1)}{2\pi A}\right] = \text{correction factor}$$

$$\hat{a}\left(\frac{t(k)+t(k+1)}{2}\right) = \left[e(k+1) + \frac{e^2(k+1)}{2\pi A}\right]\frac{1}{ATd_f}$$

Figures 1 and 3 show the block diagrams for the estimate of a(t) without and with correction factor.

Before introducing noise, a simulation study of the above is essential to check the validity of the above estimates. Tone modulation is considered, an example of which is shown in the Appendix.

The conclusions are:

a. As shown in Figs. 4 and 5, the estimate is always better with the correction factor and is very significant when the ratio of carrier frequency to maximum frequency derivation from the carrier frequency ($\omega_0/d_f a(t)_{max}$) is not large enough. Quantitatively one could consider the estimate good enough with the correction factor for ratios of $\omega_0/d_f a(t)_{max}$ as low as ten, whereas the ratio is almost limited to double the value without correction factor.

b. The estimate is good enough provided the ratio of sample rate to message rate is at least ten, although $\omega_0/d_f a(t)_{max}$ satisfies the conditions in (a).

c. For the same ratios of $\omega_0/d_f a(t)_{max}$ and sample rate to message rate, the performance of the estimator is almost the same.

(ii) In the presence of noise:

Referring to Fig. 1, the input to the band pass filter is

$$A \cos\left[\omega_0 t + d_f \int_0^t a(u)du\right] + n'(t)$$

where n'(t) is white gaussian noise with zero mean and spectral density $n_0/2$.

The output of the B.P.F. is

$$v(t) = A \cos[\omega_0 t + \theta_m(t)] + n(t) \tag{34}$$

where

$$\theta_m(t) = d_f \int_0^t a(u)du$$

and

$$n(t) = x(t) \cos[\omega_0 t + \theta_m(t)] - y(t) \sin[\omega_0 t + \theta_m(t)]$$

where x(t), y(t) are gaussian zero mean with spectral density n_0 for $|f| \leq B$ where 2B is the FM band width and

$$E[x^2(t)] = E[y^2(t)] = E[n^2(t)] = 2Bn_0 = N \tag{35}$$

Equation (34) can be written as

$$v(t) = A'\cos(\omega_0 t + \theta_m(t) + \theta_n(t)] \tag{36}$$

where

$$\theta_n(t) = \tan^{-1} \frac{y}{x+A}$$

$$A' = \sqrt{[A + x(t)]^2 + y^2(t)}$$

(a) High carrier to noise ratio, i.e., for $A^2/2 >> N$

$A' \simeq A$

$\theta_n(t) \simeq y/A$

i.e., the effect of noise, is essentially to add a random phase $\theta_n(t)$ to the incoming signal to the sampler implying $\theta_i(t) = \theta_m(t) + \theta_n(t)$. It has been proven analytically and by simulation results:

$$\frac{\theta_i(k+1)-\theta_i(k)}{T(k+1)} \simeq \left[e(k+1) + \frac{e^2(k+1)}{2\pi A} \right] \frac{1}{AT}$$

$$\frac{\theta_i(k+1)-\theta_i(k)}{T(k+1)} \simeq \frac{\theta_m(k+1)-\theta_n(k)}{T(k+1)} + \frac{\theta_n(k+1)-\theta_n(k)}{T(k+1)}$$

$$\simeq \dot{\theta}_m \left(\frac{t(k+1)+t(k)}{2} \right) + \frac{\theta_n(k+1)-\theta_n(k)}{T(k+1)}$$

$$\simeq d_f \, a \left(\frac{t(k)+t(k+1)}{2} \right) + \frac{\theta_n(k+1)-\theta_n(k)}{T(k+1)}$$

$$a \left(\frac{t(k)+t(k+1)}{2} \right) + \frac{\theta_n(k+1)-\theta_n(k)}{T(k+1)} \cdot \frac{1}{d_f} \simeq \left[e(k+1) + \frac{e^2(k+1)}{2\pi A} \right] \frac{1}{ATd_f}$$

the output noise term = $\dfrac{\theta_n(k+1)-\theta_n(k)}{T(k+1)d_f}$

which is coupled to the signal term through T(k+1). A reasonable way to study the effects of noise term classically is to decouple it from the signal and to see what design criteria it specifies. Consider output noise power for a given T(k+1)

$$E\left\{ \left[\frac{\theta_n(k+1)-\theta_n(k)}{T(k+1)d_f} \right]^2 \middle| T(k+1) \right\} = \frac{1}{T^2(k+1)d_f^2} E[\theta_n^2(k+1)]+E[\theta_n^2(k)]-2E[\theta_n(k+1)\theta_n(k)]\}$$

At the output of low pass filter the spectral density of $\theta_n(t) = n_0/A^2$ for $|f| \le f_m$

$$R_{\theta_n}(\tau) = F^{-1}S_{\theta_n}(f) = \frac{n_0}{A^2\pi\tau}\sin 2\pi f_m\tau$$

$$E\left\{\left[\frac{\theta_n(k+1)-\theta_n(k)}{T(k+1)d_f}\right]^2\middle|T(k+1)\right\} = \frac{2}{T^2(k+1)d_f^2}\left\{\frac{2n_0f_m}{A^2} - R_{\theta_n}[T(k+1)]\right\}$$

$$= \frac{2}{T^2(k+1)d_f^2}\left\{\frac{2n_0f_m}{A^2} - \frac{n_0}{A^2\pi T(k+1)}\sin 2\pi f_m T(k+1)\right\}$$

$$= \frac{2n_0}{[AT(k+1)d_f]^2}\left\{2f_m - \frac{1}{T(k+1)\pi}\left[\frac{[2\pi f_m T(k+1)]}{1!} - \frac{[2\pi f_m(k+1)]^3}{3!}\right.\right.$$

$$\left.\left. + \frac{[2\pi f_m T(k+1)]^5}{5!} - \cdots\right]\right.$$

(i.e., by writing the sin term in power series). Simplifying the above we get

$$E\left\{\left[\frac{\theta_n(k+1)-\theta_n(k)}{T(k+1)d_f}\right]^2\middle|T(k+1)\right\} = \frac{8\pi^2n_0f_m^3}{3d_f^2A^2}\left[1 - \left\{\frac{\pi f_m T(k+1)}{\sqrt{5}}\right\}^2 + \cdots\right] \quad (37)$$

The first term refers to the output noise power of a conventional discriminator. The conditional output noise power is less than this. The actual output noise power may be less compared to the discriminator type. To make sure of the above point the best thing is to make

$$\frac{\pi f_m T(k+1)}{\sqrt{5}} << 1$$

so that all the other terms in equation (37) could be neglected except the first term. That is,

$$\frac{f(k+1)}{f_m} >> \frac{\pi}{\sqrt{5}} \quad (38)$$

where f(k+1) is the instantaneous frequency at t(k+1). Under high carrier to noise ratio, the lowest frequency of the incoming signal is $\simeq f_0$-B. Equation (38) reduces to

$$f_0 - B >> \frac{\pi}{\sqrt{5}} f_m \quad (39)$$

That is, output noise power is $N_0 = \frac{8}{3}\pi^2\frac{n_0f_m^3}{d_f^2A^2}$ and is independent of T(k+1)

for all k. The above shows that the output noise has parabolic spectral density. This leads

to the well known result

$$\left[\frac{\text{signal power}}{\text{noise power}}\right]_{\text{output}} = 3\beta^2 \left(\frac{B}{f_m}\right)\left[\frac{\text{carrier power}}{\text{noise power}}\right]_{\text{at IF}}$$

for $A^2/2 >> N$ where β is the modulation index.

To achieve the above performance for a given d_f and f_m choose that f_0 to satisfy equation (39). This essentially takes care of the conditions specified in the previous sections such as the ratio of sample rate to message rate, nyquist sample frequency and lock range condition. An example of tone modulation is shown in the Appendix.

The above is simulated with high carrier to noise ratio. The results are: 1. Figure 6 shows the performance with noise which could be compared to Fig. 2 without noise.

2. At very high signal to noise ratio, the effect of noise is negligible and the error variance is dominated by the errors introduced even without noise in estimating, round off, and truncation.

This results in a constant error variance and as the modulation index is increased, the error variance increases as shown in Fig. 7, since $\omega_0/d_f a(t)_{max}$ gets reduced.

3. At fairly high signal to noise ratio the error variance reduces linearly as the carrier to noise power ratio increases for a fixed modulation index.

The error variance reduces as the modulation index is increased almost in a quadratic manner for a fixed carrier to noise power ratio. These are shown in Fig. 7 and verifies the theoretical results for output signal to noise ratio and compares favorably to analog phase locked loop.

(b) Low carrier to noise ratio:

An attempt is made to come up with an analysis similar to Rice's click analysis, but the problems like adaptive sampling and the non-linear nature of the problem with perturbed arguments of time makes it just not possible to arrive at analytical results. But, it is intuitively clear that the threshold behavior should be better compared to conventional discriminator types since this is a feedback type of system.

The digital computer solution for tone modulation provides the following qualitative information. Cycle skipping has been observed for carrier to noise power ratios less than or equal to one. The loop goes out of lock ultimately as the noise is increased. Figure 8 shows the carrier to noise power versus inverse phase error variance for the modulation

index between 0.75 to 2.0 and indicates that the threshold is at carrier to noise ratio equal to one. These support the intuitive observations.

Conclusions The analytical results supported by digital computer solutions indicate that it is indeed possible to achieve a performance at least as good as a discriminator for high carrier to noise ratio and better threshold behavior for low carrier to noise ratio with the discussed discrete phase locked loop for FM demodulation with the specified restrictions on IF frequency for the given modulation index and the bandwidth of the message. The digital structure is very simple to implement and should find wide applications particularly in the narrow band cases. It is needless to say that the loop could be used efficiently for synchronization purposes.

Appendix (a) Without noise:

$$\text{Incoming signals} = \sin\left[\omega_0 t + d_f \int_0^t a(u)\,du\right]$$

$a(t) = \cos \omega_m t \quad , \quad D_f = \beta\omega_m$

The maximum value of the incoming signal, i.e., A, is normalized to 1 since the interest is in $e(k)/A = \sin \phi(k)$

$$\frac{\text{sample rate}}{\text{message rate}} = \frac{\omega_0}{\omega_m} \quad , \quad \frac{\text{carrier frequency}}{\substack{\text{max}^m \text{ frequency duration} \\ \text{from carrier frequency}}} = \frac{\omega_0}{d_f a(t)_{max}} = \frac{\omega_0}{\beta\omega_m}$$

Lock range condition:

$$\frac{2\pi}{2\pi+1} < \frac{\omega_0+\beta\omega_m \cos \omega_m t}{\omega_0} < \frac{2\pi}{2\pi-1}$$

reduces to $\omega_0/\beta\omega_m > (2\pi\text{-}1)$ since $\beta\omega_m \neq 0$. condition for nyquist sample rate

$$\omega_0 - \beta\omega_m > 2\omega_m \quad , \quad \frac{\omega_0}{\beta\omega_m} > \left(1 + \frac{2}{\beta}\right)$$

for analytical results:

$$\theta_0(0) = 0 \quad , \quad t(0) = 0 \qquad \text{for } k = 0 \quad , \quad t(k) = kT - \frac{\theta_0(k)}{\omega_0}$$

$$\theta_i(k) = \beta \sin \omega_m t(k) \quad , \quad e(k) = \sin[\theta_i(k) - \theta_0(k)] = \sin \phi(k)$$

$$\theta(k) = \theta_0(k-1) + e(k-1) \quad , \quad \hat{\theta}_i(k) = \theta_0(k+1)$$

$$a\left(\frac{t(k)+t(k-1)}{2}\right) = \cos \omega_m \left[\frac{t(k)+t(k-1)}{2}\right]$$

$$\hat{a}\left(\frac{t(k)+t(k-1)}{2}\right) = \frac{e(k)}{2\pi\beta\left(\frac{\omega_m}{\omega_0}\right)} \text{ (without correction)} \qquad \text{OR} = \frac{e(k)}{2\pi\beta\left(\frac{\omega_m}{\omega_0}\right)}\left[1 + \frac{e(k)}{2\pi}\right] \text{ (with correction)}$$

For simulation:

$$e(k) = \sin[\omega_m t(k) + \theta_i\{t(k)\}] \quad , \quad t(0) = 0 \text{ at } k = 0 \text{ and } \theta_0(0) = 0$$
$$t(k) = t(k-1) + T - e(k-1) \quad , \quad T = 2\pi/\omega_0$$

and the rest is as for analytical results.

(b) With noise for simulation:

$$e(k) = \sin\left[\omega_m t(k) + \theta_i\left(t(k)\right)\right] + n(k)$$

equation (39) requires that $\omega_0 - \beta\omega_m >> \pi/\sqrt{5} \cdot \omega_m$, i.e., $\omega_0/\beta\omega_m >> (1+\pi/\sqrt{5}\ \beta)$.

The rest is similar to simulation without noise.

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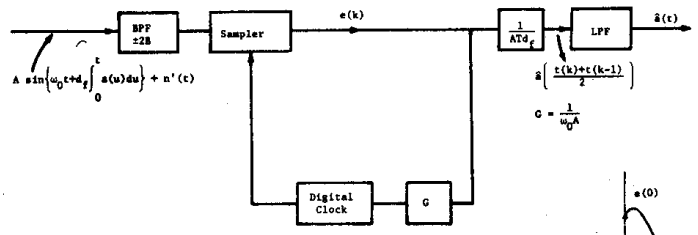

Fig. 1. Block Diagram of the FM Demodulator (without correction factor).

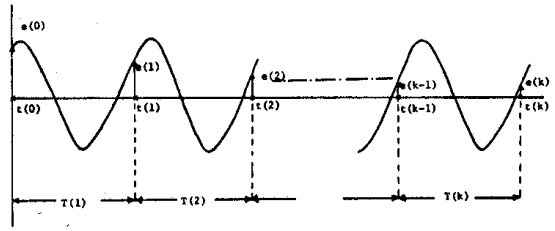

Fig. 2. Incoming Waveform.

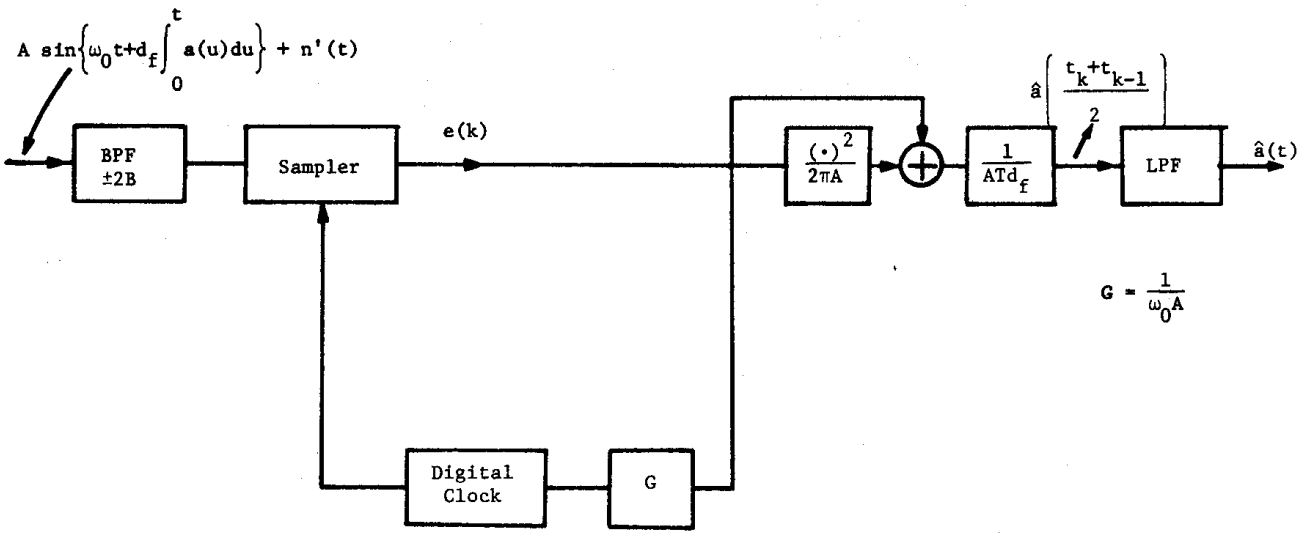

Fig. 3. Block Diagram of the FM Demodulator (with correction factor).

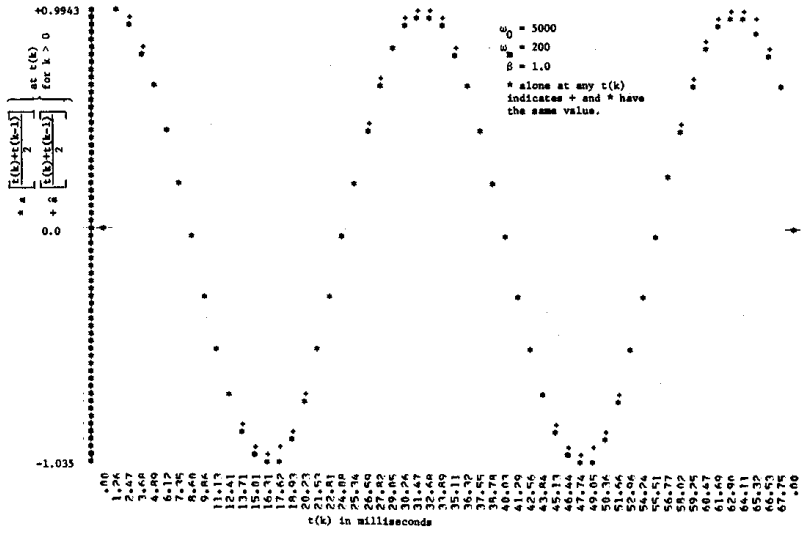

Fig. 4. Without Correction Factor (no noise).

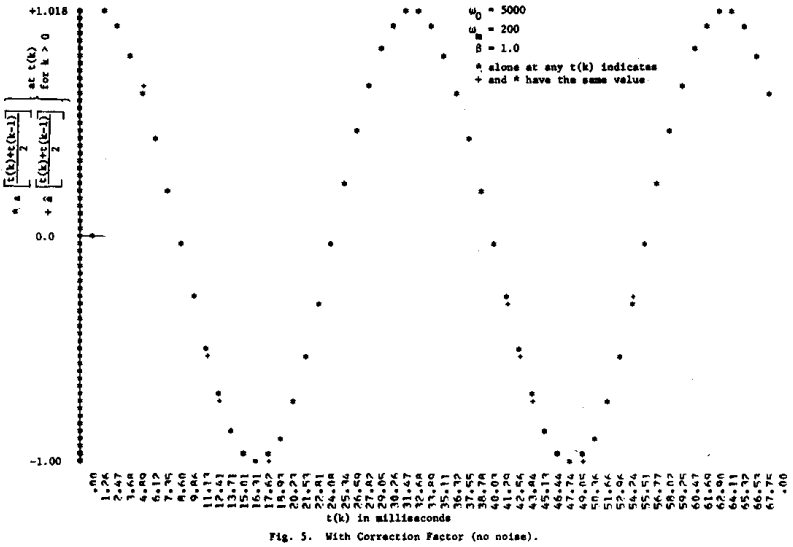

Fig. 5. With Correction Factor (no noise).

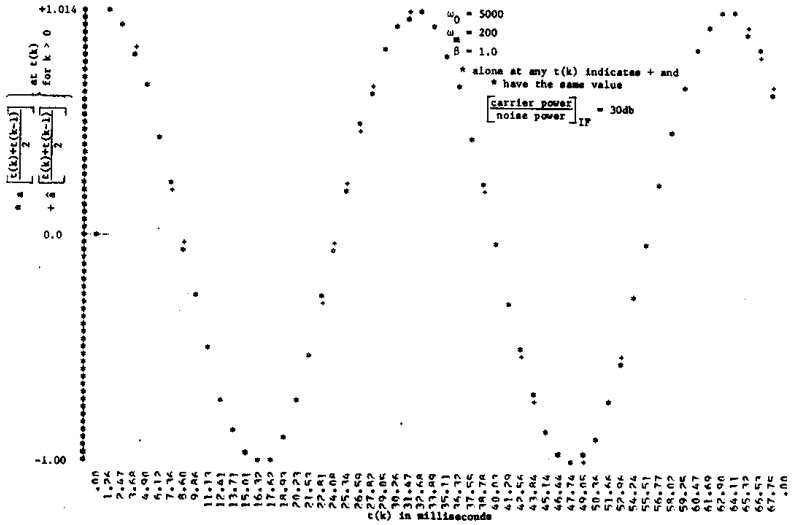

Fig. 6. With Noise and Correction Factor.

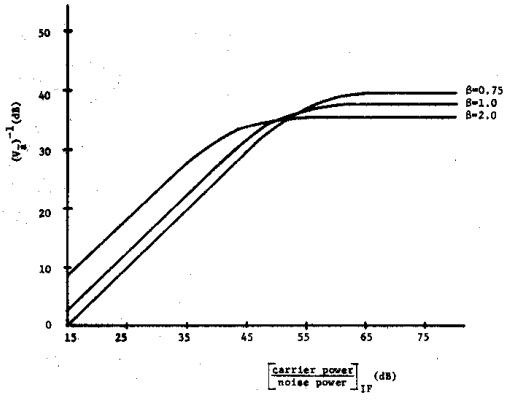

Fig. 7. Inverse of the Message Error Variance as a Function of Carrier to Noise Ratio.

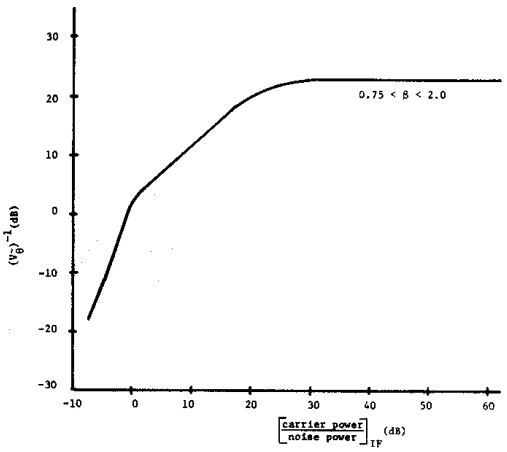

Fig. 8. Inverse of the Phase Error Variance as a Function of Carrier to Noise Ratio

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