

PREAMBLE DESIGN FOR SYMBOL TIMING ESTIMATION FROM SOQPSK-TG WAVEFORMS

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ABSTRACT

Data-aided symbol synchronization for bursty communications utilizes a predetermined modulation sequence, i.e., a preamble, preceding the payload. For effective symbol synchronization, this preamble must be designed in accordance with the modulation format. In this paper, we analyze preambles for shaped offset quadrature phase-shift keying (SOQPSK) waveforms. We compare the performance of several preambles by deriving the Cramér-Rao bound (CRB), and identify a desirable one for the Telemetry Group variant of SOQPSK. We also demonstrate, via simulation, that the maximum likelihood estimator with this preamble approaches the CRB at moderate signal-to-noise ratio.

Keywords: SOQPSK, symbol timing, preamble, Cramér-Rao bound, maximum likelihood.

1. INTRODUCTION

Shaped offset quadrature phase-shift keying (SOQPSK) is a continuous phase modulation that is utilized in many bursty communication standards due to its spectral efficiency and constant-envelope property.¹ Variations on the phase pulse used in this modulation have resulted in several versions of SOQPSK. The Telemetry Group version, referred to as SOQPSK-TG, is a standardized partial-response version of SOQPSK,^{2,3} and is the primary focus of this paper.

In wireless bursty communications, a typical burst packet begins with a set of bits, referred to as the *preamble*, to facilitate carrier phase, frequency offset and symbol timing estimation. This is typically followed with another set of bits that facilitate frame synchronization, which is followed by the payload (data). In this paper, we address preamble design for data-aided symbol timing estimation from a SOQPSK-TG modulated waveform. We first derive the Cramér-Rao bound as a function of the preamble sequence, which yields a lower bound on symbol timing mean-square error (MSE) for a given preamble. Using this expression, we compare the performance of several preamble sequences and identify one that yields good performance. We then simulate the maximum likelihood (ML) estimator for this preamble and verify that its MSE approaches the Cramér-Rao bound when the signal-to-noise ratio is sufficiently high.

This paper is organized as follows. In Section 2 we define the SOQPSK notation used in our analysis. In Section 3 we derive the Cramér-Rao bound and determine a sequence

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that is well-suited for symbol-timing estimation. In Section 4 we simulate the ML estimator for this preamble sequence to show that its MSE tracks the Cramér-Rao bound above a signal-to-noise ratio threshold. Finally, Section 5 concludes the paper with a summary of our results.

2. BACKGROUND

SOQPSK is a phase-modulated waveform with desirable spectral properties for transmission over bandwidth-constrained channels.⁴ The continuous-time complex baseband waveform* can be expressed as^{5,6}

$$s(t) = \sqrt{\frac{E_b}{T_b}} \exp\left\{j\pi \sum_{k \in \mathbb{Z}} \alpha_k q(t/T_b - k)\right\}, \quad (1)$$

where T_b denotes the duration of transmission per information bit, E_b is the waveform energy per bit, $q(t)$ is the real-valued and normalized phase function, and $\{\alpha_k \in (-1, 0, 1) | k \in \mathbb{Z}\}$ is a sequence of ternary symbols that is obtained by precoding the binary information bit stream $\{a_k \in (0, 1) | k \in \mathbb{Z}\}$ according to

$$\alpha_k = (-1)^{k-1} (2a_{k-1} - 1)(a_k - a_{k-2}). \quad (2)$$

Here $\{a_k\}$ represents an interleaved QPSK bit stream, i.e., the even time indices refer to the in-phase (I) bits and the odd time indices refer to the quadrature (Q) bits. The precoding operation from information bits to the ternary symbols can be viewed as a four-state trellis, in which the permitted state transitions vary based on the even/odd parity of the bit index k , as shown in Figure 1(a).

The phase function $q(u)$ is often defined as the integral of a phase frequency function $g(u)$, i.e., $q(u)$ is expressed as

$$q(u) \equiv \int_{-\infty}^u g(u) du. \quad (3)$$

For SOQPSK-TG, this phase frequency function is given by²

$$g(u) = A \frac{\cos\left(\frac{\pi\rho B}{2}(u-4)\right) \sin\left(\frac{\pi B}{2}(u-4)\right)}{1 - 4\left(\frac{\rho B}{2}(u-4)\right)^2} \frac{\pi B}{2}(u-4) w((u-4)/2), \quad (4)$$

where $w(u)$ is a windowing function defined as

$$w(u) = \begin{cases} 1, & |u| < \gamma_1 \\ \frac{1}{2} \left[1 + \cos\left(\frac{\pi}{\gamma_2}(u - \gamma_1)\right) \right], & \gamma_1 \leq |u| < \gamma_1 + \gamma_2 \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

The parameters are given by $\gamma_1 = 3/2$, $\gamma_2 = 1/2$, $\rho = 0.7$, $B = 1.25$, and A is chosen such that $\int_{-\infty}^{\infty} g(u) du = 1/2$. This phase frequency function is plotted in Figure 1(b).

*Our convention for the real-valued passband waveform is $\sqrt{2}\Re\{s(t)\}$.

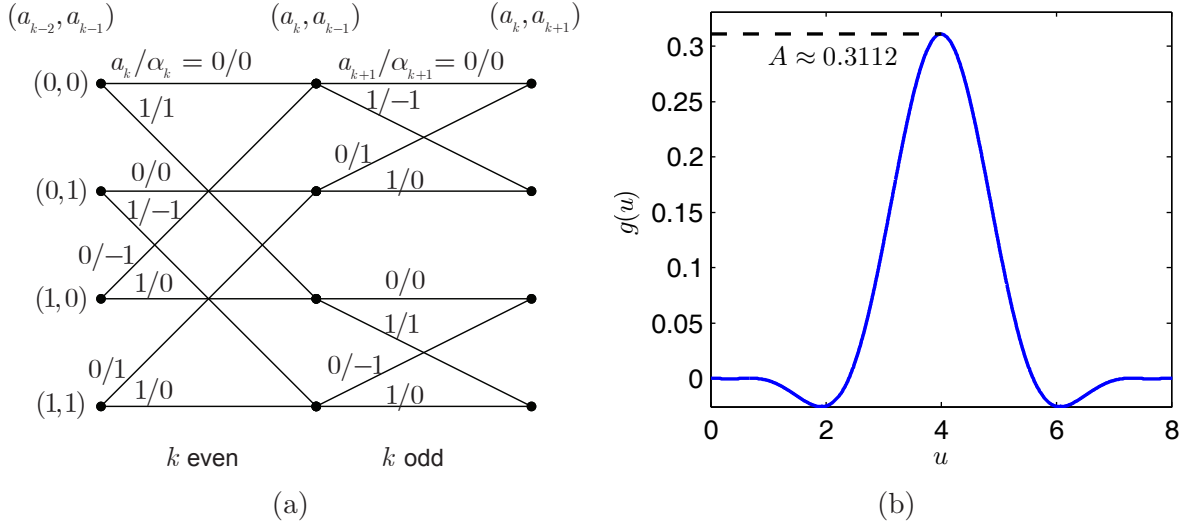


Figure 1. The SOQPSK (a) trellis diagram showing how the information bits $a_k \in \{0,1\}$ are converted to modulation symbols $\alpha_k \in \{-1,0,1\}$, and (b) phase frequency pulse for the Telemetry Group variant (SOQPSK-TG).

We assume that the complex baseband waveform at the receiver consists of the transmitted waveform from (1) with an unknown (but deterministic) timing offset, plus additive white Gaussian noise. Because most receivers are equipped with digital processing, we shall pass to a discrete-time model, by first filtering the received continuous-time waveform with an anti-aliasing filter, and then sampling it at the Nyquist frequency. We shall assume a high sampling rate, such that the anti-aliasing filter has negligible impact on the SOQPSK-TG signal, and therefore, it only limits the bandwidth of the additive noise. With these assumptions, the observed samples are given by

$$y_m = [s(t - \epsilon T_b) + n(t)]|_{t=mT}, \quad (6)$$

for $m = 0, \dots, N - 1$, where T denotes the uniform sampling period, N is the number of samples within the estimation window, ϵ is the timing offset normalized to T_b , and $n(t)$ is a zero-mean, complex-stationary, circulo-symmetric Gaussian random process with a bandlimited power-spectral density[†]

$$S_n(\omega) \equiv \begin{cases} N_0, & \text{for } |\omega| < \pi/T \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

with $N_0 > 0$ denoting the power spectrum of the noise. Thus, the discrete-time samples $n_m \equiv n(mT)$, for $m = 0, \dots, N - 1$, are zero-mean, independent and identically distributed Gaussian random variables, each with variance[‡]

$$\sigma^2 \equiv E_n[|n_m|^2] = N_0/T. \quad (8)$$

[†]The power-spectral density of a zero-mean, stationary complex random process is defined as the Fourier transform of the covariance function $K_n(\tau) = E_n[n^*(t)n(t + \tau)]$. Circulo-symmetric complex random processes have $E[n(t)n(t + \tau)] = 0$ for all τ .

[‡]Our standard notation for indicating the expected value of $f(X)$ with respect to the probability density function of X shall be $E_X[f(X)]$.

For analytic simplicity, we shall henceforth assume that the bit interval is an integer multiple of the sampling period, and we shall denote their ratio as

$$K \equiv T_b/T \in \mathbb{N}^+. \quad (9)$$

3. THE CRAMÉR-RAO BOUND

Using the signal model presented in the previous section, the joint probability density function of $\mathbf{y} = [y_0 \dots y_{N-1}]^\dagger$, where \dagger denotes transpose, is

$$p(\mathbf{y}|\boldsymbol{\alpha}; \epsilon) = \frac{1}{\pi^N \sigma^{2N}} \exp\{-\|\mathbf{y} - \mathbf{s}(\boldsymbol{\alpha}; \epsilon)\|^2/\sigma^2\}, \quad (10)$$

in terms of the vector of signal samples, $\mathbf{s}(\boldsymbol{\alpha}; \epsilon) = [s_0 \dots s_{N-1}]^\dagger$, and the noise variance σ^2 from (8). In (10) we have listed $\boldsymbol{\alpha}$ and ϵ as arguments of the probability density function to emphasize that they are parameters: we have indicated the former as a conditional variable because a probability distribution could be assigned to it, and we have separated ϵ with a semi-colon to emphasize that it is the deterministic unknown quantity.

The Fisher Information for ϵ is defined as⁷

$$F(\boldsymbol{\alpha}; \epsilon) \equiv -E_{\mathbf{y}|\boldsymbol{\alpha}} \left[\frac{\partial^2}{\partial \epsilon^2} \ln p(\mathbf{y}|\boldsymbol{\alpha}; \epsilon) \right]. \quad (11)$$

Twice differentiating the logarithm of (10) and substituting it in (11), we find that

$$F(\boldsymbol{\alpha}; \epsilon) = \frac{2}{\sigma^2} \sum_{m=0}^{N-1} \left| \frac{\partial s_m}{\partial \epsilon} \right|^2 = \frac{2E_b \pi^2}{T_b \sigma^2} \sum_{m=0}^{N-1} \left| \sum_{k \in \mathbb{Z}} \alpha_k g(m/K - k - \epsilon) \right|^2. \quad (12)$$

The Cramér-Rao bound for estimating ϵ , with the knowledge of the preamble $\boldsymbol{\alpha}$, is given by the multiplicative inverse of (12), so,

$$\text{CCRB}(\boldsymbol{\alpha}; \epsilon) = \frac{K}{2\pi^2(E_b/N_0)} \left[\sum_{m=0}^{N-1} \left| \sum_{k \in \mathbb{Z}} \alpha_k g(m/K - k - \epsilon) \right|^2 \right]^{-1}, \quad (13)$$

where we have substituted the expression in (8) for σ^2 . To emphasize that this is a lower bound on the MSE *conditioned* on the knowledge of the preamble, we will refer to it as the conditional CRB (CCRB).

Our treatment thus far applies to arbitrary $\boldsymbol{\alpha}$. However, in order to derive an analytic Cramér-Rao bound expression, we shall assume that we observe a finite window of an infinitely repeating periodic sequence, such that transient effects that may arise due to the finite duration of the preamble are not present. Let us denote the period of the preamble with L , such that $\alpha_k = \alpha_{k+nL}$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. For simplicity, let us also assume that the preamble corresponds to $M \in \mathbb{N}^+$ full periods of this sequence, such that

$$N = KLM, \quad (14)$$

where *all* terms are positive integers.

In our subsequent derivation we shall find it useful to define the continuous-time function

$$h(u - \epsilon) \equiv \sum_{k \in \mathbb{Z}} \alpha_k g(u - \epsilon - k), \quad (15)$$

for $0 \leq u < L$, and its sampled version

$$z[m; \epsilon] \equiv h(m/K - \epsilon), \quad (16)$$

for $m = 0, \dots, LK - 1$. Using Parseval's relation,⁸ the Fisher Information from (12) can be expressed as

$$F(\boldsymbol{\alpha}; \epsilon) = \frac{2\pi^2(E_b/N_0)}{K} \frac{M}{LK} \sum_{k=0}^{LK-1} |Z[k; \epsilon]|^2, \quad (17)$$

where $Z[k; \epsilon]$ represents the *Discrete Fourier Transform* (DFT) of $z[m; \epsilon]$, i.e.,

$$Z[k; \epsilon] \equiv \sum_{m=0}^{LK-1} z[m; \epsilon] e^{-j \frac{2\pi}{LK} mk}, \quad (18)$$

for $k = 0, \dots, LK - 1$. If $h(u)$ is approximately bandlimited to $\pi K/2$, sampling does not give rise to aliasing, and we have $Z[k; \epsilon] \approx KH(2\pi k/L) e^{-j2\pi k\epsilon/L}$, where $H(\omega) \equiv \int_0^L h(u) e^{-j\omega u} du$ denotes the (continuous-time) Fourier transform of $h(u)$. Substituting this expression back into (17) yields our near-final $F(\boldsymbol{\alpha}; \epsilon)$ expression:

$$F(\boldsymbol{\alpha}; \epsilon) = 2\pi^2(E_b/N_0) LM \varepsilon(\boldsymbol{\alpha}), \quad (19)$$

where

$$\varepsilon(\boldsymbol{\alpha}) \equiv \frac{1}{L} \int_0^L |h(u)|^2 du \quad (20)$$

is the average energy of the overall phase frequency function generated by the preamble $\boldsymbol{\alpha}$. Thus, the Cramér-Rao bound achieved with a preamble of period L is given by

$$\text{CCRB}(\boldsymbol{\alpha}; \epsilon) = \frac{1}{2\pi^2 \varepsilon(\boldsymbol{\alpha})} \times \frac{K}{N} \times \frac{1}{E_b/N_0}. \quad (21)$$

Suppose we fix the number of symbols, N/K , and the signal-to-noise ratio E_b/N_0 . Then, via (21), $\text{CCRB}(\boldsymbol{\alpha}; \epsilon)$ is minimized when $\varepsilon(\boldsymbol{\alpha})$ is maximum. In Appendix A, we show that

$$\arg \max_{\boldsymbol{\alpha}} \varepsilon(\boldsymbol{\alpha}) = \{(1, \dots, 1), (-1, \dots, -1)\}, \quad (22)$$

and the maximum achieved by these two preambles is $\max_{\boldsymbol{\alpha}} \varepsilon(\boldsymbol{\alpha}) = 0.25$. Note, however, that both of these preambles result in a pure tone waveform, i.e., $s(\pm(1, \dots, 1), t) \propto e^{\pm j\pi t/2}$. So, symbol timing reduces to estimating the phase of the pure tone. This reveals a significant shortcoming of these preambles: if there is any uncertainty in the absolute phase introduced by the channel, it is impossible to extract timing information. In other words, these preambles are not robust under carrier phase uncertainty, and therefore, we shall *only*

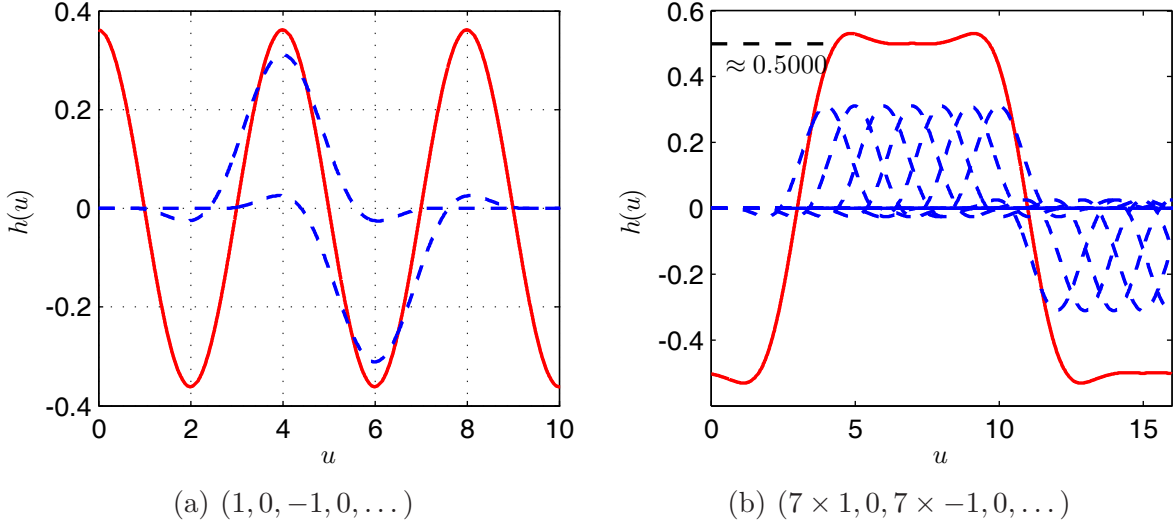


Figure 2. (Color online) The phase frequency function $h(u)$ (solid) and the individual contributions $\alpha_k g(u - k)$ (dashed) resulting from two preambles. The partial-response property of SOQPSK-TG causes destructive interference of the antipodal main lobes in (a), which results in a smaller phase modulation magnitude than that given by the preamble in (b).

consider their performance as a lower bound to the performance achievable with practical preambles.

In Appendix B we show that the modified Cramér-Rao bound (MCRB) and the conditional Cramér-Rao bound have the mathematical relation

$$1/\text{MCRB}(\epsilon) = E_{\alpha} [1/\text{CCRB}(\alpha; \epsilon)] . \quad (23)$$

It immediately follows that there *must* exist at least one preamble for which $\text{CCRB}(\alpha; \epsilon) < \text{MCRB}$ holds. Because the MCRB is a lower bound on non-data-aided symbol synchronization performance, this inequality gives a sufficient condition for preambles to outperform non-data-aided timing. In particular, using the MCRB expression in (31), we find that the set of preambles satisfying

$$0.09881 < \varepsilon(\alpha) < 0.25 \quad (24)$$

are suitable for SOQPSK-TG symbol synchronization, because they are *guaranteed* to outperform non-data-aided synchronization schemes.

In Table 1, we have listed various preamble sequences and their corresponding $\varepsilon(\alpha)$. We see from this table that the periodically repeating $(1, 0, -1, 0, \dots)$ preamble sequence, which is the preamble used in the full-response SOQPSK MIL-STD 188-181 modulation standard,⁹ is not suitable for the partial-response SOQPSK-TG modulation, because its ε is below the lower bound in (24). The reason for the poor performance with this preamble can be understood from its SOQPSK-TG phase frequency function

$$h(u) = \sum_{k \in \mathbb{Z}} g(u - 4k) - g(u - 4k - 2) , \quad (25)$$

which is plotted in Figure 2(a). Due to the partial response property of SOQPSK-TG, alternating between a ‘+1’ symbol and a ‘-1’ symbol results in significant destructive interference

L (period)	preamble	$\varepsilon(\boldsymbol{\alpha})$
4	1, 0, -1, 0, ...	0.06489
4	1, 1, 0, 0, ...	0.09492
6	1, 1, 0, -1, -1, 0, ...	0.1428
8	1, 1, 1, 0, -1, -1, -1, 0, ...	0.1733
8	$(6 \times 1), (2 \times 0), \dots$	0.1744
16	$(7 \times 1), 0, (7 \times -1), 0, \dots$	0.2115
16	$(5 \times 1), (3 \times 0), (5 \times -1), (3 \times 0), \dots$	0.1449
64	$(31 \times 1), 0, (31 \times -1), (3 \times 0), \dots$	0.2404
1	1, ...	0.25

Table 1. Phase frequency energy per symbol, $\varepsilon(\boldsymbol{\alpha})$, for various preambles, $\boldsymbol{\alpha}$.

between consecutive antipodal phase-frequency pulses. As seen from the table, preambles that repeat several ‘+1’ symbols prior to reversing the polarity, i.e., preambles with the form $(m \times 1, 0, m \times -1, 0, \dots)$ for $m = 2, 3, \dots$ have higher $\varepsilon(\boldsymbol{\alpha})$. The constructive interference between phase frequency pulses for the $m = 7$ (period 16) preamble is shown in Figure 2(b). The ε of this preamble is only 0.73 dB away from the upper bound, and it is more than 5 dB higher than the ε of the $(1, 0, -1, 0, \dots)$ preamble. In addition, its short period—compared to preambles with larger m —can be beneficial in practical scenarios in which the receiver may miss a fraction of the symbols at the beginning of the preamble. Thus, we shall adopt the $m = 7$ preamble for the simulation that follows in the next section.

4. MAXIMUM LIKELIHOOD ESTIMATION

It is well known that the conditional Cramér-Rao bound derived in (21) is achievable by the maximum likelihood (ML) estimator at high enough signal-to-noise ratio.⁷ To identify the regime in which the ML estimator performance approaches this bound, we have performed simulations for the $m = 7$ preamble from Table 1, with $N/K = 128$ ternary symbols.

The ML estimator for a known preamble is a minimum distance estimator, i.e.,

$$\hat{\epsilon}_{\text{ML}} = \arg \min_{\epsilon \in [0, 16)} \sum_{m=0}^{N-1} |y_m - s_m|^2, \quad (26)$$

where s_m are the components of the length- N vector $s(\boldsymbol{\alpha}; \epsilon)$ defined in (10). Noting that $|y_m|^2$ and $|s_m|^2$ are independent of ϵ , we find that the ML estimator can be expressed as

$$\hat{\epsilon}_{\text{ML}} = \arg \max_{\epsilon \in [0, 16)} \Re \left\{ \sum_{m=0}^{N-1} y_m^* s_m \right\}. \quad (27)$$

i.e., the complex correlation of the received symbol vector \mathbf{y} with that of $s(\boldsymbol{\alpha}; \epsilon)$, for different ϵ , is a sufficient statistic for the ML estimator.

Figure 3 plots the mean-square error of this estimator, in comparison to the conditional CRB and the modified CRB. The ML estimator tracks the conditional CRB closely when the

signal-to-noise ratio E_b/N_0 is above -7 dB. In this regime the ML estimator performance is approximately 3.3 dB better than the modified CRB. Below the -7 dB threshold, the performance rapidly degrades towards that of uniformly guessing the value of $\epsilon \in [0, 16)$. This is a well-known behavior observed in nonlinear estimators.

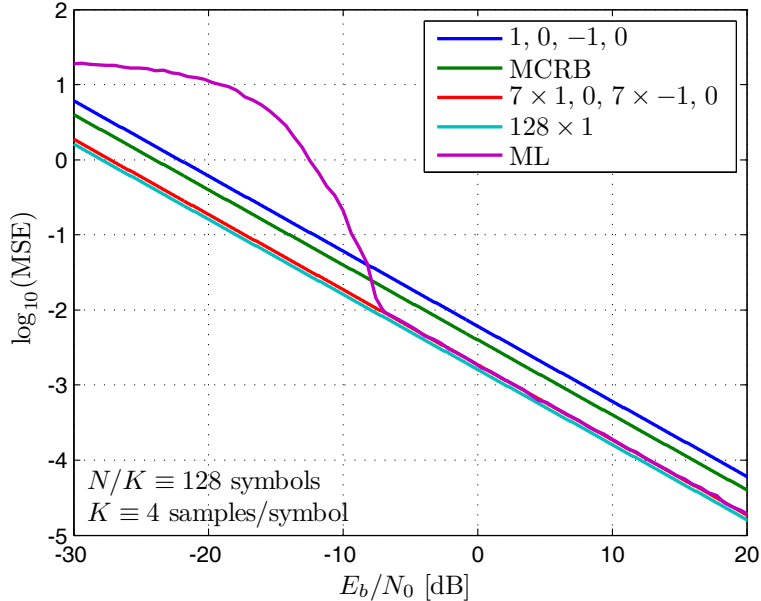


Figure 3. (Color online) The conditional CRB for symbol timing estimation with the 128-symbol $(7 \times 1, 0, 7 \times -1, 0, \dots)$ preamble is plotted along with the simulated mean-square error attained by the ML estimator for this preamble. The modified CRB (MCRB), the conditional CRB for the 128×1 preamble, and the conditional CRB for the 128-symbol $(1, 0, -1, 0, \dots)$ preamble are plotted for reference.

5. CONCLUSIONS

In this paper we have quantified the symbol synchronization performance with different preamble sequences for SOQPSK-TG modulated waveforms, in terms of the (conditional) Cramér-Rao bound achieved with each preamble. We have shown that the relative performance of different preambles depends on their phase-frequency energy per symbol, $\varepsilon(\boldsymbol{\alpha})$. We have determined an upper bound to this quantity by maximizing ε over all preamble sequences, and we have derived a lower bound based on the modified CRB that is sufficient to ensure that the preamble can outperform non-data-aided symbol synchronization.

Our comparison of ε for various preambles of equal length has lead us to conclude that the $(7 \times 1, 0, 7 \times -1, 0, \dots)$ preamble is well-suited for symbol timing estimation: it yields a phase frequency energy per symbol that is close to the maximum, and it has a relatively short period, making it desirable for practical implementation. It is worthwhile to note however, that our analysis assumes perfect knowledge of phase and frequency offset introduced by the channel. To quantify the degradation due to any uncertainty in these parameters, our analysis should be generalized to incorporate these additional parameters.

To provide an independent verification of our theoretical analysis, we have also simulated mean-square error performance of the ML estimator for the period-16 preamble mentioned above, and we have compared it against the Cramér-Rao bound. The simulations have demonstrated that the ML estimator performs near-optimally for $E_b/N_0 > -7$ dB. Below this threshold the estimation performance rapidly degrades, which is a well-known phenomenon for nonlinear estimators.

APPENDIX A. MAXIMIZING THE FISHER INFORMATION

Expanding the square on the right-hand side of (12) and interchanging the order of the summations, we obtain

$$F(\boldsymbol{\alpha}; \epsilon) = \frac{2\pi^2 E_b}{N_0 K} \sum_{k,\ell=0}^{N/K-1} \alpha_k \alpha_\ell \left(\sum_{m \in \mathbb{Z}} g(m/K - k - \epsilon) g(m/K - \ell - \epsilon) \right), \quad (28)$$

where we have assumed, for analytic simplicity, that $\{\alpha_k\}$ repeats with period N/K . Let us focus on the kernel

$$g_{k,\ell} = \sum_{m \in \mathbb{Z}} g(m/K - k - \epsilon) g(m/K - \ell - \epsilon). \quad (29)$$

When $K \gg 1$, the sum will have a weak dependence on ϵ , so we can make the approximation

$$g_{k,\ell} \approx g'_{k-\ell} \equiv K \int_{-\infty}^{\infty} g(u) g(u - \ell + k) du. \quad (30)$$

Numerical evaluation of (30) shows that $g'_{k-\ell} \geq 0$ for all k and ℓ . Thus

$$F(\boldsymbol{\alpha}; \epsilon) = \sum_{k,\ell=0}^{N/K-1} \alpha_k g'_{k-\ell} \alpha_\ell \leq \sum_{k,\ell=0}^{N/K-1} g'_{k-\ell}, \quad (31)$$

which is satisfied with equality if and only if α_k for $k = 0, \dots, N/K - 1$ have the same sign, i.e. the sequence of ternary symbols are all equal to 1, or they are all equal to -1 .

APPENDIX B. THE MODIFIED CRAMÉR-RAO BOUND

For non-data-aided symbol synchronization, in which the ternary symbol sequence $\boldsymbol{\alpha}$ is not known a priori, a lower bound on the mean-square error is given by the *modified* Cramér-Rao bound,¹⁰

$$\text{MCRB}(\epsilon) = \frac{1}{E_{\boldsymbol{\alpha}}[F(\boldsymbol{\alpha}; \epsilon)]}, \quad (32)$$

where $F(\boldsymbol{\alpha}; \epsilon)$ is given in (12). If we assume that the ternary symbol sequence is generated from independent and identically distributed bits a_k , via the relation given in (2), the correlation function for the ternary symbols becomes⁴

$$E_{\alpha_k, \alpha_\ell}[\alpha_k \alpha_\ell] = \begin{cases} \frac{1}{2} & k = \ell \\ \frac{1}{4} & |k - \ell| = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (33)$$

Expanding the square on the right hand side of (12) and using this correlation function for the $\{\alpha_k\}$ symbols, we obtain

$$E_{\alpha}[F(\boldsymbol{\alpha}; \epsilon)] = \frac{E_b \pi^2}{N_0 K} \sum_{m=0}^{N-1} \sum_{k \in \mathbb{Z}} g^2\left(\frac{m}{K} - k - \epsilon\right) + g\left(\frac{m}{K} - k - \epsilon\right) g\left(\frac{m}{K} - k + 1 - \epsilon\right) \quad (34)$$

$$\approx \pi^2 \frac{E_b N}{N_0 K} \int_{m=0}^1 \left[\sum_{k \in \mathbb{Z}} g^2(u - k) + g(u - k)g(u - (k - 1)) \right] du, \quad (35)$$

where the second line approximates the sum over m with an integral, making the assumption $K \gg 1$. Numerically evaluating the integral we obtain the modified CRB as

$$\text{MCRB}(\epsilon) = \frac{1}{2\pi^2 B} \times \frac{K}{N} \times \frac{1}{E_b/N_0}, \quad (36)$$

where $B \approx 0.09881$.

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