# THE EFFECT OF RANDOM THRESHOLD LEVELS ON PCM QUANTIZATION NOISE

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**Summary** It is common with PCM systems to consider a uniform quantization noise (or error) distribution over plus or minus one half the quantization level. This distribution must be modified to describe the effect of uncertainties in the location of the decision thresholds which separate adjacent quantization levels. The uncertainties exist because electronic decision circuits trigger somewhere within a narrow band surrounding an intended voltage instead of at exactly that voltage. In this paper the encoding error distributions, are calculated in terms of the distributions of the threshold locations. It is shown that the variance of the quantization noise is increased by an amount almost exactly equal to the variance of the threshold uncertainty distribution. The effect of the uncertainty bands becomes noticeable only where the uncertainty band is an appreciable part of the quantization interval. Prime examples are systems with small voltage differences between encoding levels such as low-level systems (0 to 50 millivolts) and systems with large number of quantization levels. Because error variance is related to the square of the uncertainty range, a threshold uncertainty range of 1/10 the quantization interval will increase the net encoding variance by approximately $(1/10)^2$ or 1.0%.

**Introduction** For an n bit PCM encoder where the input voltage is uniformly distributed throughout its range, the quantization error (or noise) is uniformly distributed over a quantization interval. Thus the error probability density function (p.d.f.) is

$$p(e) = \begin{cases} 2^n & ; \quad \frac{-1}{2^{n+1}} \leq e \leq \frac{+1}{2^{n+1}} \\ 0 & ; \quad \text{elsewhere.} \end{cases} \tag{1}$$

The input voltage is encoded into a binary sequence representing the voltage at the <u>center</u> of the quantization interval. It is implicit in the above statements that a decision mechanism is used to precisely determine in which interval an incoming voltage resides. In actual practice there is <u>no perfect</u> decision mechanism (or threshold device). Instead, the decision (or threshold) level which is intended to lie exactly at the boundary of two contiguous quantization intervals resides instead within an uncertainty band (±b) surrounding this intended nominal boundary. The result is that the PCM encoding error differs from that given in (1). This paper is concerned with the exact error distribution including threshold uncertainty.

The position of the actual threshold level can be represented by a random variable, T; having a p.d.f. given by

$$p\;(T = x) = \begin{cases} p_T\;(x) & ; \quad -b \leq x \leq +b \\ 0 & ; \quad \text{elsewhere,} \end{cases} \tag{2}$$

where the range T is ±b. The purpose of this paper is to determine the effect of T on the quantization error distribution. The particular case of interest is when the input voltage (random variable, V) is uniformly distributed between its limits $\pm V_M$ and $p_T$ (x) is symmetric about x = 0.

Thus

$$p\;(V = x) = p_V = \begin{cases} \dfrac{1}{2V_M} & ; -V_M \leq x \leq V_M \\ 0 & ; \quad \text{elsewhere.} \end{cases} \tag{3}$$

**Distributions** The problem is here treated by dividing the $2^n$ quantization intervals (or levels) into $2^n$-1 equal width "central" zones. A central zone is bounded by the center of two contiguous quantization intervals. Thus an input voltage at either bound of a central zone will be encoded into a binary sequence with zero error; all other location in the zone will be accompanied by an encoding error. There are also two end zones, each of half the width of the central zones. The situation is illustrated in Figure 1. Note that the terms "interval" or "level" are standard PCM terms meaning the voltage ranges between nominal threshold voltages. The term "zone" refers to the range between the digital encoding voltages. The chance that V lies in a particular central zone is $1/2^n$. The chance that V lies in a particular end zone is $1/2^{n+1}$. The quantization error p.d.f., p (e = x) = $p_e(x)$ is determined by first finding the conditional p.d.f.'s of the error given that V is in a typical central zone, in one end zone, and in the other end zone. These p.d.f.'s are respectively $p_{e/c}(x)$, $P_{e/A}(x)$ , and $P_{e/B}(x)$ . Hence

$$p_e\;(x) = \frac{2^n-1}{2^n}\;p_{e/c}\;(x) + \frac{1}{2n+1}\left[\;p_{e/A}\;(x) + p_{e/B}\;(x)\right] \tag{4}$$

Figure 2 shows the definition of error voltage (e) for a typical central zone, e.g. the zone with zero nominal threshold and bounded by encoding levels $\pm a = \pm V_M/2^n$ . As seen in the figure the input voltage V is equal to the encoding level, ±a, plus the error, e, or

$$e = \begin{cases} V-a < 0, & V > T \\ V+a > 0, & V < T. \end{cases} \quad (5)$$

Equation (5) assumes that a >b which, from a practical viewpoint, is the only condition of interest for a PCM encoder.

Then for e < 0,

$$p_{e/c}(x) = p_{e/c}(e = x) = p(V-a = x, V > T) = p(V = x+a, T < x + a)$$

$$= p_V(x + a)\, P_T(T < x + a) = p_V(x + a) \int_{-b}^{x+a} p_T(z)\, dz$$

because V and T are independent.

Likewise for e > 0,

$$p_{e/c}(x) = p(V = x-a, V < T) = p_V(x-a)\, P_T(T > x-a)$$

$$= p_V(x-a) \int_{x-a}^{+b} p_T(z)\, dz.$$

Now if V is assumed uniformly distributed, $p_V(z)$ is the p.d.f. of V given that V lies between -a and +a. Therefore $p_V(z) = 1/2a = 2^{n-1}/V_M$ and

$$p_{e/c}(x) = \begin{cases} \dfrac{2^n}{2V_M} \displaystyle\int_{-b}^{x+a} p_T(z)\, dz; & -(a+b) \lesssim x \lesssim 0, \\ \dfrac{2^n}{2V_n} \displaystyle\int_{x-a}^{+b} p_T(z)\, dz; & 0 \lesssim x \lesssim (a+b). \end{cases} \quad (6)$$

This expression can be further reduced by noting that the limits of integration x + a and x - a become $x + a \geq b$ when $x \geq -(a - b)$ and $x-a \leq -b$ when $x \leq +(a-b)$. Thus for these ranges of x, $p_T(z)$ is integrated over its entire non-zero region and the integrals therefore equal unity. Thus

$$p_{e/c}(x) = \begin{cases} \dfrac{2^n}{2V_M} \displaystyle\int_{-b}^{x+a} p_T(z)\,dz; & -(a+b) \leq x \leq -(a-b), \\ \dfrac{2^n}{2V_M}\;; & -(a-b) \leq x \leq +(a-b), \\ \dfrac{2^n}{2V_M} \displaystyle\int_{x-a}^{+b} p_T(z)\,dz; & +(a-b) \leq x \leq (a+b). \end{cases} \tag{7}$$

The central term of (7) corresponds to values of V outside the threshold uncertainty region. Specifically $x \leq (a-b)$ implies $-a \leq V \leq -b$ and $x \geq -(a-b)$ implies $a \geq V \geq -b$.

The effects of the two end zones must now be considered. The lower end zone (A) is depicted in Figure 3. If V lies in this zone it follows that it lies below the lowest threshold uncertainty region (again assuming $b < a$). Thus the conditional p.d.f. $p_{e/A}(x)$ is independent of the threshold d.b.f. or

$$p_{e/A}(x) = \begin{cases} \dfrac{1}{a} = \dfrac{2^n}{V_M}\;; & -a \leq x \leq 0, \\ 0\;; & \text{elsewhere.} \end{cases} \tag{8a}$$

Likewise

$$p_{e/B}(x) = \begin{cases} \dfrac{1}{a} = \dfrac{2^n}{V_M}\;; & 0 \leq x \leq a, \\ 0\;; & \text{elsewhere.} \end{cases} \tag{8b}$$

Thus by combining (4), (7), and (8) and recalling that the end zone distributions contribute to the total result only between +a and -a, we obtain

$$p_e(x) = \begin{cases} \dfrac{1}{2V_M}\,(2^n-1)\displaystyle\int_{-b}^{x+a} p_T(z)\,dz , & -(a+b) \leq x \leq -a \\[2ex] \dfrac{1}{2V_M}\left[(2^n-1)\displaystyle\int_{-b}^{x+a} p_T(z)\,dz + 1\right] ; & -a \leq x \leq -(a-b) \\[2ex] \dfrac{1}{2V_M}\left[(2^n-1) \; +1\right] = \dfrac{2^n}{2V_M} ; & -(a-b) \leq x \leq (a-b) \\[2ex] \dfrac{1}{2V_M}\left[(2^n-1)\displaystyle\int_{x-a}^{+b} p_T(z)\,dz + 1\right] ; & (a-b) \leq x \leq a \\[2ex] \dfrac{1}{2V_M}\,(2^n-1)\displaystyle\int_{x-a}^{+b} p_T(z)\,dz ; & a \leq x \leq (a+b). \end{cases} \tag{9}$$

Equation (9) is plotted in Figure 4 for a uniform distribution of the threshold T between ±b, that is for $p_T(z) = 1/2b$ for $-b \leq z \leq b$ and zero elsewhere.

If the threshold distribution is approximated by a normal function of zero mean and standard deviation $\sigma << a = V_M/2^n$ then it can be assumed with very little error that the PCM encoder error is bounded by(a + b)which may be assumed to have a maximum value less than 2a. Then the threshold p.d.f. is given by

$p_T(x) = \phi_T(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp(-x^2/2\sigma^2)$ and if $\Phi_T(u) - \Phi_T(v) = \displaystyle\int_v^u \phi_T(x)\,dx$, Equation (9) becomes (approximately)[1]

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[1] It is again assumed that V is uniformly distributed from $-V_M$ to $+V_M$.

$$p_e(x) = \begin{cases} \frac{1}{2V_M}(2^n-1)\left[\Phi_T(x+a) - \Phi_T(-b)\right] & ; -(a+b) \leq x \leq -a \\ \frac{1}{2V_M}\left\{(2^n-1)\left[\Phi_T(x+a) - \Phi_T(-b)\right] + 1\right\} & ; -a \leq x \leq -(a-b) \\ \frac{2^n}{2V_M} & ; -(a-b) \leq x \leq (a-b) \\ \frac{1}{2V_M}\left\{(2^n-1)\left[\Phi_T(x-a) - \Phi_T(b)\right] + 1\right\} & ; (a-b) \leq x \leq a \\ \frac{1}{2V_M}(2^n-1)\left[\Phi_T(x-a) - \Phi_T(b)\right] & ; a \leq x \leq (a+b). \end{cases} \tag{10}$$

**Moments** It will be seen from Figures 4 and 5, and can be shown analytically, that the p.d.f. of the total error is equivalent to convolution of the p.d.f.'s of the voltage V and threshold T, so long as V is uniformly distributed. Thus for V uniformly distributed it follows that the error random variable (e) may be considered the sum of the random variables V and T. Hence moments are also additive.

Now since V and T commonly have zero means, the error e will also have zero mean. The variance $\overline{e^2}$ is composed of a central zone and end zone contributions, the latter independent of the distribution of T. For uniformly distributed T

$$\overline{e^2} = \frac{2^n-1}{2^n} \cdot \frac{a^2+b^2}{3} + \frac{1}{2^n} \cdot \frac{a^2}{3} = \frac{a^2}{3} + \frac{b^2}{3}\left(\frac{2^n-1}{2^n}\right) \tag{11}$$

where $a^2/3$ and $b^2/3$ are the variances of V (in a central or end zone) and T, respectively, and the coefficients are the probabilities that V lies in one of these zones. For the guassian threshold variable we obtain

$$\overline{e^2} = \frac{2^n-1}{2^n}\left(\frac{a^2}{3} + \sigma^2\right) + \frac{1}{2^n} \cdot \frac{a^2}{3} = \frac{a^2}{3} + \sigma^2\left(\frac{2^n-1}{2^n}\right) \tag{12}$$

It will be noted that for large n, the error variance is approximately the sum of the variances V and T. The right hand most terms of (11) and (12) represent the changes from the usual PCM results.

**Conclusions** The exact PCM encoding error distribution, including threshold uncertainties, is given in Equation (9). Specific uniform and guassian threshold distributions are considered in Equation (10) and Figures 4 and 5. The second moments are given in Equations (11) and (12) where it is seen that the system encoding error variance is increased from its normal PCM value by a weighted threshold variance. The weighting is $(2^n-1)/2^n \approx 1$ for large n. Thus, for practical purposes, it may be assumed that a direct addition of normal encoding and threshold uncertainty variances occurs. These increases can be expected to be small except where the uncertainty range is an appreciable part of the quantization interval. For more or less fixed uncertainty ranges, the larger the number of quantization levels and the smaller the overall system voltage range, the more significant will be the uncertainty range.

For example, an 8 bit, 50 millivolt system has an quantization interval range of 0.195 millivolts. If a uniform threshold uncertainty range is 1/10 this value, or 19.5 microvolts, the total encoding error variance becomes

$$\overline{e^2} = \frac{(0.195/2)^2}{3} + \frac{(0.0195/2)^2}{3}\left(\frac{256-1}{256}\right)$$

$$= \frac{(0.195/2)^2}{3}\left[1 + \frac{1}{100} \cdot \frac{255}{256}\right].$$

The second term in the brackets is the added error caused by the threshold uncertainty; it accounts for a variance of about one percent.

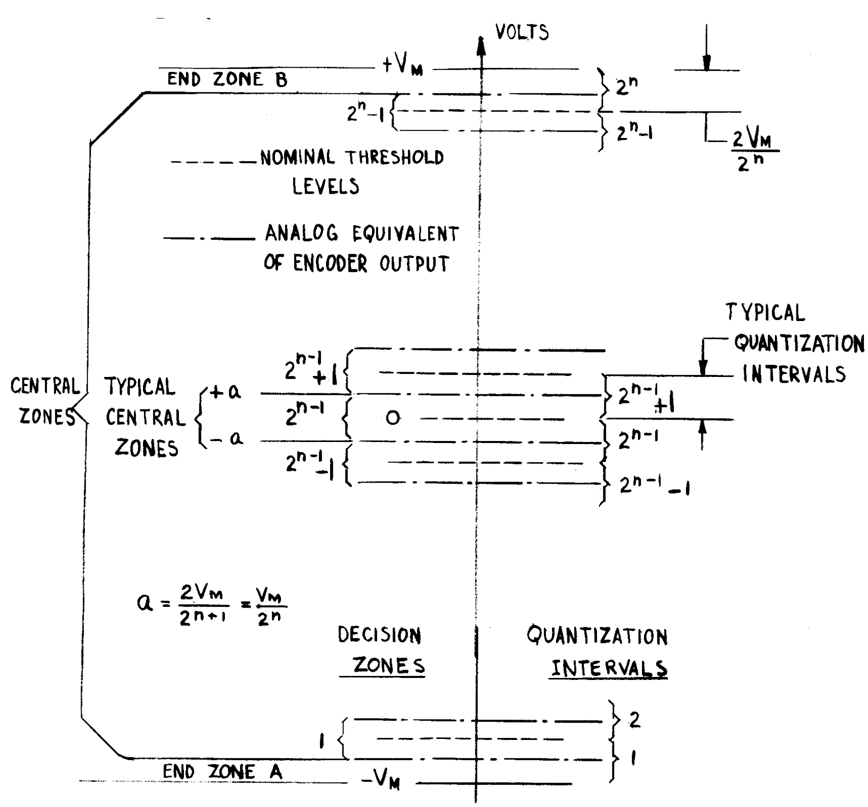


**Figure 1 - Quantization Intervaland Decision Zones**

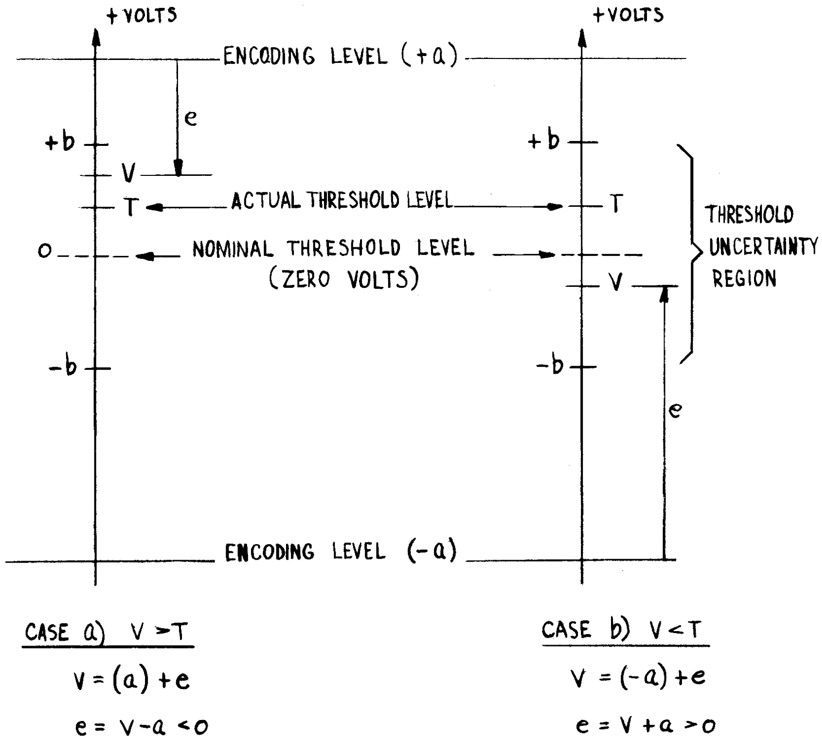


**Figure 2 - Central Zone Errors**

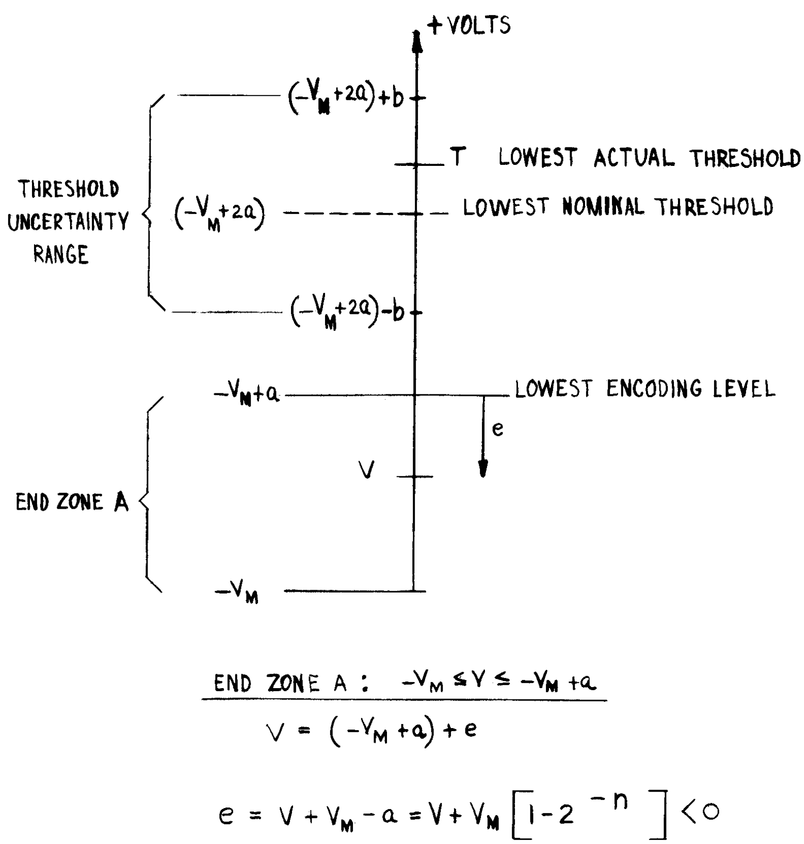


**Figure 3 - End Zone Errors**

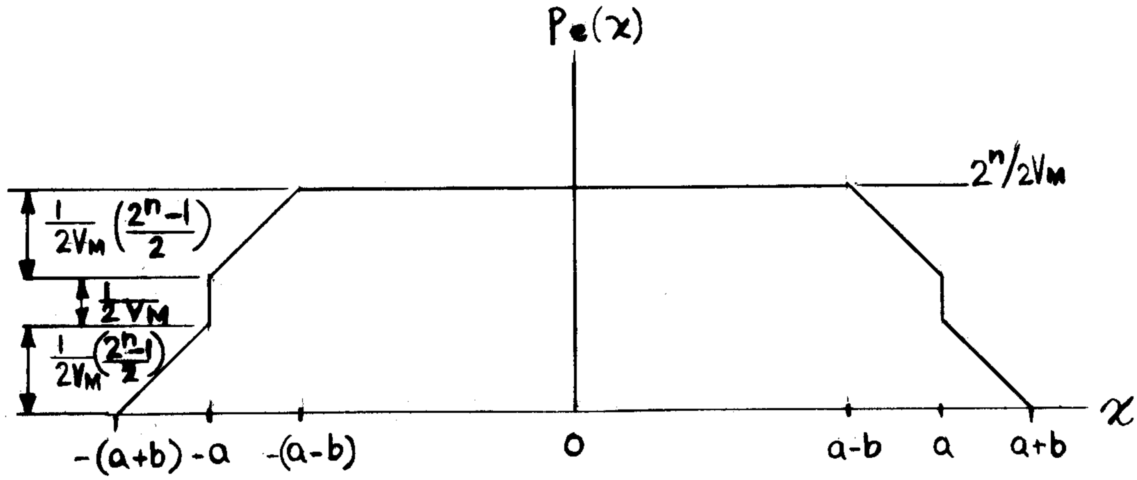


**Figure 4 - Error Distribution for Uniform Voltage and Threshold Distributions**

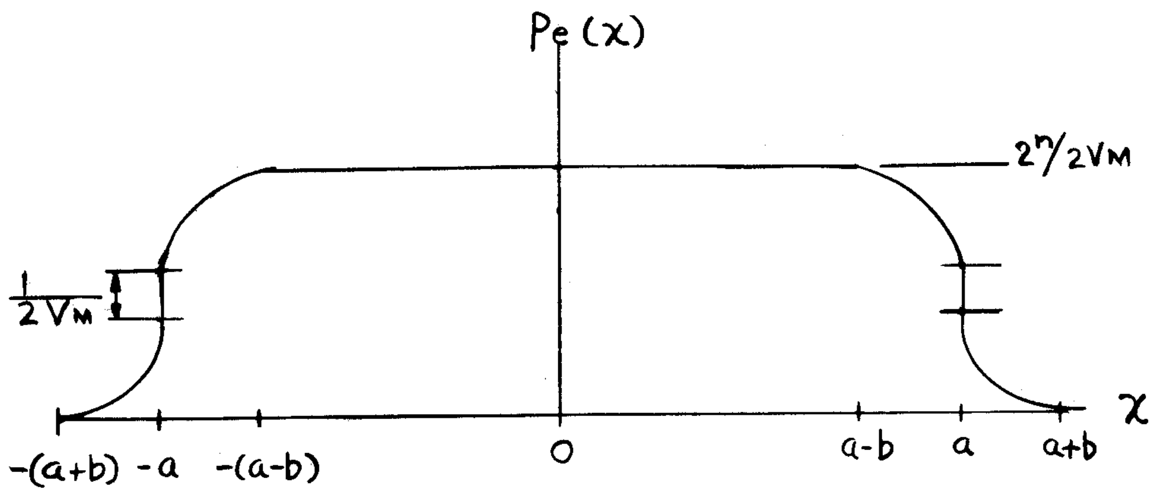


**Figure 5 - Error Distribution for Uniform Voltage and Guassian Threshold Distributions**