This paper presents an optimum and a sub-optimum, but easily realizable, method for the detection of K binary symbols corrupted by white Gaussian noise with unknown mean.

Basically the optimum procedure requires picking the sequence from a set of $2^K$ sequences which minimizes a certain functional. However this procedure requires a great deal of computation. This computational problem is considerably reduced by the use of an efficient searching procedure developed in this paper.

However a sub-optimum procedure exists and is very simple to instrument with the advantage that decisions are made in a bit-by-bit fashion. This procedure is analyzed and the average error probability is obtained.

I. Introduction

Analog mechanization of the simple integrate-and-dump symbol detector used in binary PSK communication systems may be difficult at high data rates due to the inability of electronically dumping in a time interval short compared to a pulse duration. A possible alternative which circumvents this problem is to digitize the integrator output and to perform the dumping mathematically using a digital computer (1,2). The resulting detector, being a hybrid of analog and digital operations is the subject of the present investigation.

The problem is formulated in terms of a binary antipodal pulse train corrupted by additive white Gaussian noise of unknown mean. The inclusion of the unknown mean accounts for unknown D.C. bias and drift which would normally be encountered in the receiver. The presence of the D.C. term affects the ideal decision rule; changing it from a bit-by-bit decision scheme to one involving an ordering of the detector outputs. Although this is impractical to implement when the number of pulses is large, it reduces in the case of an infinite sequence to a bit-by-bit decision scheme.

A sub-optimal, but easily instrumentable, bit-by-bit scheme is analyzed which includes the effects of a high pass filter (to remove the D.C. bias) and a quantizer.

II. Analysis of idealized model

2.1 Formulation: We assume that the receiver is presented with the random waveform

$$x(t) = s(t) + n(t) + \mu \sqrt{E/T}$$

where

$$s(t) = \sum_{k=1}^{K} a_k \phi(t-kT)$$

$$\phi(t-kT) = \begin{cases} \frac{\sqrt{E/T}}{T} & \text{for} \ (k-1)T \leq t \leq kT \\ 0 & \text{otherwise} \end{cases}$$

Dr. Holmes and Dr. Butman are with the Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California.
\[ a_k = \pm 1 \text{ with a priori probability} \]
\[ p(1) = p(-1) = 1/2 \]

\( n(t) \) is zero-mean white Gaussian noise of two sided spectral density \( \frac{N_0}{2} \text{ watts/Hz} \) and \( \mu \) is an unknown D.C. offset. Because \( \mu \) is unknown we assume, with no loss of generality, that it is a random variable with an underlying probability distribution \( p(\mu) \).

Given the waveform \( x(t) \) the receiver must determine with minimum probability of error the binary sequence \( \tilde{a} = \{a_k\}_{k=1}^K \). In order to accomplish this we consider the \( K \)-dimensional sufficient statistic \( \tilde{x} = \{x_k\}_{k=1}^K \) obtained by projecting (matched filtering) \( x(t) \) onto the signal space spanned by the \( K \) orthogonal basis functions \( \varphi(t-kT) \) \( k = 1, 2, ..., K \), thus,
\[ x_k = \frac{1}{E} \int x(t) \varphi(t-kT) dt \]
\[ = a_k + n_k + \mu \]

where
\[ n_k = \frac{1}{E} \int n(t) \varphi(t-kT) \quad k = 1, 2, ..., K \]
are independent zero-mean Gaussian variables with variance
\[ \sigma_k^2 = E[n_k^2] \]
\[ = \frac{1}{E^2} \int \int \frac{N_0}{2} \delta(t-\tau) \varphi(t-kT) \varphi(\tau-kT) dtd\tau \]
\[ = \frac{N_0}{2E} \]

2.2 The Ideal Observer Decision Rule: The ideal observer decision rule minimizes the probability of error by selecting the binary sequence \( \tilde{a} \) for which the a posteriori probability distribution \( p(\tilde{a}|\tilde{x}) \) is a maximum overall possible binary sequences of the same length. There are at most \( 2^K \) such sequences. If they are equiprobable, it is equivalent from Bayes’ rule,
\[ p(\tilde{a}|\tilde{x}) = \frac{p(\tilde{x}|\tilde{a}) p(\tilde{a})}{p(\tilde{x})} \]
to maximize the conditional probability density \( p(\tilde{x}|\tilde{a}) \) which is
\[ p(\tilde{x}|\tilde{a}) = \int p(\tilde{x}|\tilde{a}, \mu) p(\mu) d\mu \] (1)

where
\[ p(\tilde{x}|\tilde{a}, \mu) = \left( \frac{E}{N_0} \pi \right)^{\frac{K}{2}} \exp \left[ -\frac{E}{N_0} \sum_{k=1}^{K} (x_k - a_k \mu)^2 \right] \] (2)

Now it is necessary to specify \( p(\mu) \). For the sake of reaching a conclusion we will assume that \( \mu \) is Gaussianly distributed with known
variance \( \frac{N_0}{2} \lambda E \) and mean zero. This assumption is equivalent to maximizing our uncertainty (entropy) of \( \mu \) under an average power constraint. Substitution of

\[
p(\mu) = \sqrt{\frac{\lambda E}{N_0 \pi}} \exp \left( -\frac{\lambda E}{N_0} \mu^2 \right)
\]

into (1) and integrating over \( \mu \) gives

\[
p(x|a) = \left( \frac{E}{\pi N_0} \right)^{\frac{K}{2}} \left( \frac{\lambda}{\lambda + K} \right)^{\frac{1}{2}} \exp \left[ -\frac{E}{N_0} (x-a)^T R^{-1} (x-a) \right]
\]

where

\[
R^{-1} = \left( I - \frac{1}{\lambda + K} \Gamma \Gamma^T \right)
\]

is the inverse of the normalized covariance matrix of the noise plus unknown D.C. offset, \( R = (2E/N_0) E \left[ (n+\mu) (n+\mu)^T \right] = I + \lambda^{-1} \Gamma \Gamma^T \), \( \Gamma \) is the \( K \) dimensional vector with components \( \Gamma_k = 1 \) for all \( k \), \( I \) is the \( K \times K \) identity matrix and the superscript \( T \) denotes transpose. It is obvious from the definition of \( \Gamma \) that its Euclidian norm is

\[
||\Gamma||^2 = K
\]

Equation (4) reveals that the optimum decision procedure is to select from the set of possible binary sequences the sequence \( \hat{a} \) which minimizes the quadratic form

\[
Q(x, \hat{a}) = (x-a)^T R^{-1} (x-a)
\]

\[
= \sum_{k=1}^{K} \left( x_k - a_k \right)^2 + \frac{\lambda K}{\lambda + K} \left( \frac{\Gamma x}{||\Gamma||^2} \right)^T \Gamma \left( \frac{\Gamma x}{||\Gamma||^2} \right)
\]

where

\[
x = \frac{1}{K} \sum_{k=1}^{K} x_k
\]

and

\[
a = \frac{1}{K} \sum_{k=1}^{K} a_k
\]

also

\[
Q(x, \hat{a}) = ||x-a||^2 - \frac{K^2}{K+\lambda} (\bar{x} - \bar{a})^2
\]

2.3 Mechanization of the Optimum Decision Rule: A brute force approach for determining \( \hat{a} \) entails the computation of \( Q(x, \hat{a}) \) for each one of the \( 2^K \) possible binary sequences, followed by a search for the minimum. However, a more economical procedure is available because we can determine the opti-
mum sequence in closed form when $\bar{a}$ is held fixed. We observe that $\bar{a}$ can take on one of the $K+1$ values

$$\bar{e}_J = \frac{2J}{K} - 1 \quad J = 0, 1, 2, \ldots, K$$

(12)

where $J$ is the number of 1's in $\bar{a}$. There are $\binom{K}{J}$ sequences with $J$ 1's and hence with $\bar{a} = \bar{a}_J$.

Let $A_J$ denote the set of sequences with $\bar{a} = \bar{a}_J$ and let

$$Q_J = \min_{\bar{a} \in A_J} Q(\bar{x}, \bar{a})$$

(13)

$$= \min_{\bar{a} \in A_J} ||\bar{x} - \bar{a}||^2 - \frac{K^2}{\lambda + K} (\bar{x} - \bar{a}_J)^2$$

Then it is clear that the sequence $\bar{a}_J$ corresponding to $Q_J$ is the one with the $J$ 1's opposite the largest $J$ elements of $x$. Next, since

$$\min_{\bar{a}} Q(\bar{x}, \bar{a}) = \min_J Q_J = Q_J^*$$

(14)

it follows immediately that $\bar{a} = \bar{a}_J$.

The process of determining the largest $J$ components in $\bar{x}$ for $J=0, 1, 2, \ldots, K$ implies that we transform $\bar{x}$ into $\bar{y}$ where the components of $\bar{y}$ are the components of $\bar{x}$ arranged in decreasing order, thus, $y_1 \geq y_2, \ldots, \geq y_K$. The transformation is non-singular and measure preserving because $||\bar{x}|| = ||\bar{y}||$. Also $\bar{x} = \bar{y}$. Let $U$ be the $K \times K$ matrix representing this transformation, then it should become evident that the columns of $U$ are a permutation of the columns of the identity matrix $I$, and $U \bar{y}^T = \bar{y}^T U = I$.

Now since

$$\bar{y} = U \bar{x}$$

(15)

and defining

$$\bar{b} = U \bar{a}$$

(16)

we see that

$$Q(\bar{y}, \bar{b}) = Q(\bar{x}, \bar{a})$$

(17)

Since $\bar{b} = \bar{a}$ we see that $\bar{a}$ is in $A_J$ if and only if $\bar{b}$ is in $A_J$.

Therefore, we have

$$Q_J = \min_{\bar{b} \in A_J} Q(\bar{y}, \bar{b})$$

(18)

$$= \sum_{k=1}^{J} (y_k - 1)^2 + \sum_{k=J+1}^{K} (y_k + 1)^2 - \frac{K^2}{\lambda + K} (y^* + 1 - \frac{2J}{K})^2$$

(19)
because the ordering \( y_1 \geq y_2 \geq \ldots \geq y_K \) and the constraint that \( b \) is in \( A_j \) imply that the optimum choice for \( b_j \) is \( b_k = 1 \) for \( k \leq J \), otherwise \( b_k = -1 \).

From (19) it follows that \( Q_J \) satisfies the recursion formula

\[
Q_J = Q_{J-1} - 4 \left[ y_J + \frac{2J-K(y+1)-1}{K+\lambda} \right]
\]

where the initial condition is

\[
Q_0 = \sum_{k=1}^{K} (y_k+1)^2 - \frac{K^2}{K+\lambda} (\bar{y}+1)^2
\]

The index \( J^* \) for which \( Q_{J^*} \leq Q_J \) for all \( J \neq J^* \) determines that \( \hat{b} = b_{J^*} \), that is

\[
\hat{b}_k = \begin{cases} 
1 & \text{for } k \leq J^* \\
-1 & \text{for } k > J^*
\end{cases}
\]

and hence

\[
a = \hat{U}^T \hat{v}
\]

A schematic diagram of the operations involved in implementing the optimum decision rule is given in Figure 1. The equipment complexity is of order \( K \log K \), mainly to carry out the ordering on \( x \). That is, most of the equipment is for generating the transformation \( U \) (which is a function of \( x \)).

2.4 Sub-Optimum Decoding: The equipment complexity in the optimum receiver becomes intolerably large as \( K \) increases. A sub-optimum procedure which is much simpler can be derived by examining the situation as \( K \) approaches infinity. We have, using the Law of Large Numbers, that

\[
\lim_{K \to \infty} \bar{x} = \mu
\]

and

\[
\lim_{K \to \infty} \bar{a} = 0
\]

Thus, for large values of \( K \)

\[
Q(x, a) \approx \|x - \mu \|^2 + \mu^2 K
\]

and the best choice of \( \hat{a} \) is in this case

\[
\hat{a}_k = \text{sgn} (x_k - \mu)
\]

hence

\[
\hat{a}_k = \text{sgn} (x_k - \bar{x})
\]

The above decision procedure is optimum in the limit as \( K \to \infty \) and may be approximated by means of a high-pass filter, in tandem with the matched filter as described in Section 3.
III. Analysis of an easily instrumentable detector

3.1 Analysis: A different mechanization of the usual matched filter detector is employed in this section for symbol detection (1,2). The approximate matched filtering is accomplished with a simple RC low pass filter in conjunction with a computer to form a digital (or mathematical) rather than an electronic, dump, as illustrated in Figure 2.

The noisy input process, \( x(t) \), which is composed of a series of binary pulses of amplitude \( \frac{E}{T} \) together with white noise is fed into the symbol detector. The unit gain amplifiers are employed to provide isolation stages. The low pass RC combination in conjunction with the computer forms the estimate of the sign of the symbols every "T" seconds. The computer forms the statistic

\[
Z(kT) = Y(kT) - Y(kT - T) \exp \left[ -\frac{T}{R_1C_1} \right]
\]

where \( Y(kT) \) is the output of the filter combination at time \( kT \). The sole purpose of the high pass RC filter (of Fig. 2) is to "remove" any D.C. bias or slow drifts that might arise in the amplifiers or in previous circuitry. With the high pass filter removed it is easy to show that the output at time \( kT \), for a sequences of pulses, is given by

\[
Y(kT) = a_k \sqrt{\frac{E}{T}} \left[ 1 - \rho_1 \right] + \rho_1 Y(kT - T)
\]  

where \( \rho_1 = \exp \left[ -\frac{T}{R_1C_1} \right] \). Hence "dumping" is effected by subtracting the initial condition \( \rho_1 Y(kT - T) \), leaving just the output due to the present symbol. It is easy to show (without the high pass filter in the circuit and without the unknown mean) that as \( R_1C_1 \) approaches infinity the output signal-to-noise ratio approaches the same value as the known optimum "integrate and dump" system. Also, the effect of the high pass filter is negligible if \( R_2C_2 > R_1C_1 \).

In order to compute the probability of error we must assume something about the statistics of the data sequence. Hence, as in Section 2, we assume the "reasonable case" when the input symbols are independent from symbol to symbol, i.e.

\[
p(a_k) = \prod_{i=1}^{k} p(a_k)
\]

where \( a_k = (a_k, a_{k-1}, \ldots, a_1) \). In this analysis it is assumed that the D.C. bias term \( \mu \) has been removed by the high pass filter by turning the receiver on sufficiently long before the reception of the signals.

Define \( Z_k \), the decision statistic generated by the computer, as

\[
Z_k = Y_k - \rho_0 Y_{k-1}
\]

where \( Y_k = Y(kT) \) and \( \rho_0 = \exp \left[ -\frac{T}{R_0C_0} \right] \). In general, due to component tolerances, \( \rho_1 \neq \rho_0 \). Hence, to consider the stationary case we can write
\[ Y_k = \int_{-\infty}^{kT} h(kT-\tau) \left\{ \sum_{j=-\infty}^{kT} a_j \phi(\tau-kT) + n(\tau) \right\} d\tau \]  

(34)

where \( h(\tau) \) is the impulse response of the low pass-high pass filter combination. The probability of error can easily be computed for a particular sequence of \( a_k \), then the result averaged over all the random variables of the set. We note that given \( a_k \), \( Y_k \) is a Gaussian random variable and therefore so is \( Z_k \). Since \( Z_k \) is a Gaussian random variable we need merely find the conditional mean and variance to find the conditional probability of error. Now

\[ E[Z_k|a_k] = E[Y_k|a_k] - \rho_0 E[Y_{k-1}|a_k] \]  

(35)

and from (34)

\[ E[Y_k|a_k] = \int_{-\infty}^{kT} h(kT-u) \sum_{j=-\infty}^{kT} s_j \phi(u-jT) \]  

(36)

where \( E[.] \) is the expectation operator.

It is easy to show that, for the low pass-high pass filter of Figure 1, the impulse response is given by

\[ h(u) = \frac{\alpha_1 \alpha_2}{\alpha_2 - \alpha_1} \exp(-\alpha_2 u) - \frac{\alpha_2^2}{\alpha_2 - \alpha_1} \exp(-\alpha_1 u), \quad u \geq 0 \]  

(37)

where \( \alpha_1 = 1/R_1 C \).

Evaluating (36) in conjunction with (37) the conditional mean is found to be given by

\[ E[Z_k|a_k] = \sqrt{E} \left\{ \frac{\alpha_1}{\alpha_1 - \alpha_2} a_k (\rho_2 - \rho_1) + \frac{\alpha_2^2}{\alpha_1 - \alpha_2} (1 - \rho_1) \sum_{j=1}^{\infty} (\rho_2^j - \rho_0 \rho_2^{j-1})a_{k-j} \right\} \]  

(38)

Defining the second term minus the third term in (38) as \( e_k \), the residual error, we have

\[ E[Z_k|a_k] = \sqrt{E} \frac{\alpha_1}{\alpha_1 - \alpha_2} a_k (\rho_2 - \rho_1) + e_k \]  

Consider the conditional variance term, \( \sigma^2 \), using the conditional expectation \( E \).

\[ \sigma^2 = E \| (Y_k - \rho_0 Y_{k-1} - E_{a_k} [Y_k - \rho_0 Y_{k-1}|a_k]) \|^2 \]  

(39)

\[ = E \left[ \left( \int_{-\infty}^{kT} h(kT-u) n(u) du - \rho_0 \int_{-\infty}^{(k-1)T} h[(k-1)T-v]n(v) dv \right)^2 \right] \]
Which, after some algebra, yields the following

\[
\sigma^2 = \frac{N_0}{4} \frac{\alpha_1^2}{\alpha_1 + \alpha_2} \left[ 1 + \rho_0^2 - 2\rho_0 \left( \frac{\rho_2^2 - (\rho_1 + \rho_2)\alpha_1\alpha_2 + \rho_1^2}{(\alpha_1 - \alpha_2)^2} \right) \right]
\]

herefore the conditioned random variable \( Z_k | a_k \) is Gaussian with mean

\[
\frac{\sqrt{E/T}}{\alpha_1 - \alpha_2} \alpha_k (\rho_2 - \rho_1) + \epsilon_k \]

and variance given by Equation (40).

Now the average probability of error is obtained by averaging over all the sequences, i.e.,

\[
P_E = \int \ldots \int P[E|a_k] P(a_k) \, da_k
\]

To evaluate the first integral note that \( P(a_k) = \frac{1}{2} \delta(1-a_k) + \frac{1}{2} \delta(1+a_k) \) and

\[
P[E|a_{k-1}] = \int_{a_k} P[E|a_k] P(a_k) \, da_k
\]

which yields

\[
P[E|a_{k-1}] = \frac{1}{2} \text{Pr}[Z_k > 0 | a_k = -1, a_{k-1}] + \frac{1}{2} \text{Pr}[Z_k < 0 | a_k = 1, a_{k-1}]
\]

For simplification in notation we shall not write the conditioned set \( a_{k-1} \) explicitly henceforth. Since \( Z_k \) is conditionally Gaussian we can show that

\[
\text{Pr}[Z_k > 0 | a_k = -1] = \int_0^\infty \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ -\frac{1}{2} \frac{(Z + \gamma - \epsilon_k)^2}{\sigma^2} \right] \, dZ
\]

and

\[
\text{Pr}[Z_k < 0 | a_k = 1] = \int_0^\infty \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ -\frac{1}{2} \frac{(Z + \gamma + \epsilon_k)^2}{\sigma^2} \right] \, dZ
\]

where for convenience we have let \( \gamma \) be

\[
\gamma = \frac{\sqrt{E/T}}{\alpha_1 - \alpha_2} (\rho_2 - \rho_1).
\]
\[ \varepsilon_k = \sqrt{\frac{E}{T}} \frac{\alpha_1}{\alpha_1 - \alpha_2} (1 - \rho_1) \sum_{j=1}^{\infty} (\rho_1^j - \rho_1 \rho_0^{j-1}) a_{k-j} \]
\[ - \sqrt{\frac{E}{T}} \frac{\alpha_1}{\alpha_1 - \alpha_2} (1 - \rho_2) \sum_{j=1}^{\infty} (\rho_2^j - \rho_2 \rho_0^{j-1}) e_{k-j} \]  \hspace{1cm} (45)

where of course the \( a_k \)'s are random variables taking on values 1 and -1 with probability one half, we see that the mean is
\[ E[\varepsilon_k] = 0 \] since \( E[a_k] = 0 \).

Carrying out the algebra and noting that \( E[a_j a_i] = \delta_{ji} \) we obtain for the variance
\[ \sigma^2 = \frac{(\frac{E}{T}) \alpha_1^2}{(\alpha_1 - \alpha_2)^2} \left[ \frac{1 - \rho_1}{1 + \rho_1} (\rho_1 - \rho_0)^2 + \frac{(1 - \rho_1)(1 - \rho_2)(\rho_1 - \rho_0)(\rho_2 - \rho_0)^2}{1 - \rho_1 \rho_2} \right. \]
\[ + \left. \frac{1 - \rho_2}{1 + \rho_2} (\rho_2 - \rho_0)^2 \right] \]  \hspace{1cm} (46)

Hence within the approximation for \( \varepsilon_k \) we have for the average probability of error
\[ P_E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} \left\{ \frac{1}{2} \exp \left[ - \frac{(z+\gamma - \varepsilon_k')^2}{2\sigma^2} \right] + \frac{1}{2} \exp \left[ - \frac{(z+\gamma + \varepsilon_k')^2}{2\sigma^2} \right] \right\} \]
\[ \cdot \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[ - \frac{\varepsilon_k'^2}{2\sigma^2} \right] d\varepsilon_k'dz \]  \hspace{1cm} (47)

But since \( \varepsilon_k \) is a symmetric random variable with zero mean we see that \( \varepsilon_k \) and \( -\varepsilon_k \) have the same statistics. Therefore the average error probability can be written as
\[ P_E = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[ -\frac{(z+\gamma - \varepsilon_k)^2}{2\sigma^2} \right] \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[ -\frac{\varepsilon_k^2}{2\sigma^2} \right] d\varepsilon \]  \hspace{1cm} (48)

Furthermore, it is not hard to show by direct calculation that
\[ P_E = \text{erfc} \left[ \frac{\gamma}{\sqrt{\frac{\gamma^2}{\sigma^2} + \sigma_k^2}} \right] \]  \hspace{1cm} (49)

where
\[ \text{erfc}(x) = \int_{x}^{\infty} \frac{1}{\sqrt{\pi}} \exp \left[ -\frac{t^2}{2} \right] dt \]
Consequently, the average probability of error is given by

\[
P_E = \text{erfc} \left[ \frac{\sqrt{\frac{E}{T}}}{\frac{\alpha_1}{\alpha_1 - \alpha_2}} \left( \rho_2 - \rho_1 \right) \right] \frac{\alpha_1^2}{4 \alpha_1 + \alpha_2} \left( 1 + \rho_0^2 - 2\rho_0 F_1 \right) + \frac{\alpha_1^2}{\alpha_1 - \alpha_2} \left( 1 - \rho_1 \right) \frac{\alpha_1^2 (\rho_1 - \rho_0)^2 + F_2}{\alpha_1^2 + \rho_1 \rho_2^2} \]

where

\[
F_1 = \frac{\rho_2 \alpha_2 - (\rho_1 + \rho_2) \alpha_1 \alpha_2 + \rho_1 \alpha_1^2}{(\alpha_1 - \alpha_2)^2} \]

and

\[
F_2 = \frac{(1-\rho_1)(1-\rho_2)(\rho_1 - \rho_0)(\rho_2 - \rho_0)}{1 - \rho_1 \rho_2} + \frac{1 - \rho_2}{1 + \rho_2} (\rho_2 - \rho_0)^2 \]

Some special cases of interest can be obtained from this general result. First let \( \alpha_2 = 0 \) which is equivalent to removing the high pass filter from the detector. We obtain

\[
P_E = \text{erfc} \left[ \frac{\sqrt{\frac{E}{T}}}{\frac{\alpha_1}{\alpha_1 - \alpha_2}} \left( \rho_2 - \rho_1 \right) \right] \frac{\alpha_1^2}{4 \alpha_1 + \alpha_2} \left( 1 - \rho_1 \right) \frac{\alpha_1^2 (\rho_1 - \rho_0)^2 + F_2}{\alpha_1^2 + \rho_1 \rho_2^2} \]

From this result we see as \( N_0 \alpha_1 T \to \infty \) we obtain an irreducible error due to the error in the algorithm \((\rho_0 \neq \rho_1)\):

\[
P_E = \text{erfc} \left[ \frac{\sqrt{\frac{E}{T}}}{\alpha_1 - \alpha_2} \left( \rho_2 - \rho_1 \right) \right] \frac{\alpha_1^2}{4 \alpha_1 + \alpha_2} \left( 1 + \rho_0^2 - 2\rho_0 F_1 \right) + \frac{\alpha_1^2}{\alpha_1 - \alpha_2} \left( 1 - \rho_1 \right) \frac{\alpha_1^2 (\rho_1 - \rho_0)^2 + F_2}{\alpha_1^2 + \rho_1 \rho_2^2} \]

Note that although (53) was obtained by approximating \( \epsilon \) with \( \epsilon' \), if \( \rho_1 = \rho_0 \) then the error expression (equation 53) reduces to a known exact result namely

\[
P_E = \text{erfc} \left[ \frac{\sqrt{\frac{E}{T}}}{\alpha_1 - \alpha_2} \left( \rho_2 - \rho_1 \right) \right] \frac{\alpha_1^2}{4 \alpha_1 + \alpha_2} \left( 1 + \rho_0^2 - 2\rho_0 F_1 \right) + \frac{\alpha_1^2}{\alpha_1 - \alpha_2} \left( 1 - \rho_1 \right) \frac{\alpha_1^2 (\rho_1 - \rho_0)^2 + F_2}{\alpha_1^2 + \rho_1 \rho_2^2} \]

A lower bound to the probability of error can easily be obtained from (43) directly:

\[
P_E \geq \text{erfc} \left[ \frac{\sqrt{\frac{E}{T}}}{\frac{\alpha_1}{\alpha_1 - \alpha_2}} \left( \rho_2 - \rho_1 \right) \right] \frac{\alpha_1^2}{4 \alpha_1 + \alpha_2} \left( 1 + \rho_0^2 - 2\rho_0 (\rho_2 \alpha_2 - (\rho_1 + \rho_2) \alpha_1 \alpha_2 + \rho_1 \alpha_1^2) \right) \]

\[
\sqrt{\frac{E}{T}} \frac{\alpha_1}{\alpha_1 - \alpha_2} \left( \rho_2 - \rho_1 \right) \]

\[
\frac{\alpha_1^2}{4 \alpha_1 + \alpha_2} \left[ 1 + \rho_0^2 - 2\rho_0 \left( \frac{\rho_2 \alpha_2 - (\rho_1 + \rho_2) \alpha_1 \alpha_2 + \rho_1 \alpha_1^2}{(\alpha_1 - \alpha_2)^2} \right) \right] \]
And this lower bound is always less than the approximate result of equation (50) for $\sqrt{E/T} > 0$

3.2 Effect of the Quantization Noise: Up to this point we have tacitly assumed that all signals fed to the computer are in analogue form. However digital computers require a binary representation of the signals. A quantizer (A/D converter) can be used to effect this transformation. Consequently in this section we consider the effect on performance of placing a quantizer at the output of the high pass filter (of figure 2) feeding to the computer.

To analyze the effect of the quantizer on the digital dump system we take the viewpoint that the quantized output is composed of the true output plus quantization noise. In order to proceed to an expression for the probability of error, the distribution of the quantization noise must be specified. Of course the distribution in the quantum levels are determined by the distribution of the incoming waveform if we neglect the internal noise of the quantizer. However in most cases this distribution is very difficult to determine and therefore usually, in most analyses, the author will hypothesize a distribution in the gap. In practice small gap quantizers (e.g. 14 bit A/D converters) have been observed (3) to have quantization (conversion) noise with distributions resembling a quantized-Gaussian distribution with a standard deviation of several s. This distribution is primarily due to the internal noise of the quantizer. In large gap quantizers (e.g. 8 level) a uniform distribution is probably a more accurate model. In this paper we shall consider only a small gap quantizer. Therefore, we hypothesize that the quantizer output can be expressed as

$$X(kT) = Y(kT) + u(kT)$$

where $u(kT)$ is the quantizer noise at time $t = kT$ and that $u(kT)$ is a normally distributed random variable satisfying

$$E[u(kT)] = 0, \quad E[u(kT)u(jT)] = \beta^2 s^2 \delta_{jk}$$

[In effect we are approximating a "discrete Gaussian" distribution with a continuous Gaussian distribution.] Here $\beta$ is a constant of the system, $s$ is the quantum level spacing, and $\delta_{jk}$ is the kronecker delta. It will be further assumed that there is a sufficient number of levels $L$ so that

$$P\left[Y(kT) \geq LS\right] < P_E$$

where $P_E$ is the average probability of error of a bit.

The computer forms the decision statistic

$$Z(kT) = X(kT) - X(kT-T) \exp \left(-\frac{T}{R_0C_0}\right)$$

which after using (56) the above equation becomes

$$Z_k = Y_k - \rho_0 Y_{k-1} + u_k - \rho_0 u_{k-1}$$

In equation (58) we have simplified the notation, i.e. $Z(kT) = Z_k$ and $\rho_0 = \exp \left(-\frac{T}{R_0C_0}\right)$. With the assumption that $u_k$ is an independent Gaussian random variable the modification of the previous analysis to include quantization is greatly simplified. Again, fixing the sequence $a_k$, it is easy
to show $Z_k | a_k$ is a Gaussian random variable with its mean and variance given by

$$E\left[ Z_k | a_k \right] = \frac{\alpha_1}{\alpha_1 - \alpha_2} a_k (p_2 - p_1) + \epsilon_k$$

$$V\left[ Z_k | a_k \right] = \frac{N_0}{4} \alpha_1^2 \alpha_2 \left[ 1 + p_0^2 - 2p_0 \left( \frac{\rho_1 \rho_2 - (p_1 + p_2) \rho_1 \rho_2 \rho_1}{(\alpha_1 - \alpha_2)^2} \right) \right] + \sigma^2 \left( 1 + p_0^2 \right)$$

where $\epsilon_k$ is the second and third terms of (38). And $\sigma^2$ represents the first term of (59) with $\sigma^2_q$ representing the second term which exhibits the conditional variance and consequently the effect of the quantizer. It is easy to show, using the same approximation for $\epsilon_k$ as before, that the probability of error can be expressed as

$$P_E = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}(\sigma^2 + \sigma^2_q)} \exp \left[ -\frac{1}{2} \frac{(Z + \gamma + \epsilon_k)^2}{\sigma^2 + \sigma^2_q} \right] \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[ -\frac{\epsilon_k^2}{2\sigma^2_{\epsilon}} \right] d\epsilon_k$$

After using (49) and some algebra we get the final result

$$P_E = \text{erfc} \left[ \frac{\gamma}{\sqrt{\sigma^2 + \sigma^2_{\epsilon} + \sigma^2_s}} \right]$$

where $\sigma^2_s = \sigma^2_s \gamma$ and $\gamma, \sigma^2$, and $\sigma^2_{\epsilon}$ are given by equations 40, 44 and 46 respectively. Clearly as $\sigma^2_{\epsilon} \to 0$ the result is identical with the probability of error expression derived assuming no quantization (equation 50).

### IV. Conclusions

The optimum decision procedure for detecting a sequence of $K$ binary pulses imbedded in additive white Gaussian noise with unknown mean has been derived and shown to involve an ordering of the received noisy data. Since ordering of the $K$th pulse involves $\log_2 K$ operations it is clearly impractical for large values of $K$. However, for very large values of $K$ decoding on a bit-by-bit basis is feasible and is in fact asymptotically optimum as $K$ tends to infinity. This is because the unknown mean (or D.C. bias) will, by the Law of Large numbers and the ergodic hypothesis, be equal to the time average of the received signal. Therefore, the mean becomes "better known" as $K$ increases and can be subtracted out. This, in fact, is the function of the high-pass filter which approximates the averaging process needed to remove the bias. Similarly, the low pass filter approximates an ideal integrator as closely as desired, and also allows the employment of the digital dump algorithm to mathematically dump the matched filter.
The main result of section 3 is the expression for the probability of error for the digital dump system. Figure 3 illustrates the effect of the high pass circuit on the system and figure 4 depicts the probability of error versus $E/N_0$, "typical" parameter values.

References


1. Transform \( x_1, x_2, \ldots, x_K \) into ordered sequence \( y_1 \geq y_2 \geq \ldots \geq y_K \)
2. Store the transformation \( U \) from \( y = Ux \)
3. Compute \( Q_0, Q_1, \ldots, Q_K \)
4. Find \( Q_{j*} = \min_j Q_j \)
5. Set \( \hat{b}_k = 1 \) for \( k \leq j^* \) and \( \hat{b}_k = -1 \) for \( k > j^* \)
6. Generate \( \hat{a} = U\hat{b} \)

Fig. 1 - Optimum Receiver

Fig. 2 - Digital Dump Matched Filter
Fig. 3 - Probability of Error vs the Ratio $\frac{R_C 2}{R_C 1}$
Fig. 4 - Probability of Error vs the Ratio $A^2/N_0$