WIDEBAND PCM-FM BIT ERROR PROBABILITY USING DISCRIMINATOR DETECTION

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Summary. - The expression for bit probability of PCM/FM is derived for a receiver with an IF bandwidth equal to or greater than the data rate, limiter-discriminator detection; followed by a post-detection filter with bandwidth equal to the data rate. The optimum deviation ratio is shown to be essentially constant regardless of the IF bandwidth-to-data rate ratio and system performance is shown to degrade when this ratio is greater than unity. Pre-modulation filtering of the transmitted PCM data is shown to have a negligible effect on system performance. The system is experimentally tested and the analytical results are shown to good agreement with experimental data.

Introduction. - The block diagram of a typical PCM/FM telemetry system is shown in Figure 1. The premodulation filter is used to control radio frequency interference by limiting the radiated electromagnetic spectrum. The post-detection filter is used to improve the signal-to-noise ratio SNR at the input to the binary decision device.

The objective of this paper is to derive and illustrate the decision device bit error probability P_e as a function of IF signal-to-noise ratio SNR_{IF} and the ratio of IF bandwidth-to-data rate $b_o\tau$.

Meyerhoff and Meyer[1] derived and illustrated P_e for discriminator detection of PCM/FM assuming the IF filter was gaussian and the modulating waveforms were impulses. They did not consider the effect of a post detection filter. Similarly, Shaft[2] examined the case in which the modulating waveform was a binary square wave, but only for $b_o\tau = 1$. Chen[3] examined the same case as Shaft, but for $b_o\tau \geq 1$. Schilling[4] et. al. examined the case of digital signals demodulated by a discriminator followed by a post-detection filter, and Klapper[5] among others considered the case in which the post-detection filter was an integrate and dump circuit. They both assumed that b_o is chosen according to Carson's rule, i.e., $b_o = 2(1+\beta)/\tau$ and that at low error rate the errors caused by gaussian and spike noises can be calculated separately.

In this paper P_e for discriminator detection is derived and experimentally tested for the general case of $b_o\tau \geq 1$. β_{OPT} and the effect of premodulation filtering are also examined.

Theoretical Analysis. - For a binary square waveform, and a Gaussian low pass filter* with sampling at peak signal, the error probability due to the Gaussian noise is given by [4]

$$P_{eg} = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{\beta^2 E/N_o}{1.515\left[\mathrm{erf}(0.837\,b_o\tau) - 0.945\,b_o\tau\, e^{-0.7(b_o\tau)^2}\right]}} \qquad (1)$$

* The assumption of Gaussian LPF makes the analysis easier. This is a good approximation of the LPF used in the test. The latter is more rectangular and would result in better performance.

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where E/N_0 is the signal energy per bit to noise density ratio and β is the frequency deviation index defined as

$$\beta = \Delta f \, \tau / 2 \tag{2}$$

Here Δf is the frequency separation between the two possible transmitted frequencies. For large $b_o\tau$, (1) becomes,

$$P_{e_g} = \frac{1}{\sqrt{2\pi}} \int_{\beta\sqrt{\frac{1.32E}{N_0}}}^{\infty} e^{-x^2/2} \, dx \tag{3}$$

independent of $b_o\tau$. The error probability due to the spike noise is

$$P_{e_s} = \begin{cases} \frac{\beta}{\pi} e^{-E/(N_0 b_o \tau)} & , \beta < 0.734 \\ \frac{\beta}{\pi} e^{-E/(N_0 b_o \tau)} \sin\left(\frac{\pi}{2}\frac{D}{\tau}\right) & , 0.734 \le \beta \le 4.24 ; \ \frac{D}{\tau} = 0.755\left[\ln 3\sqrt{2}/\beta\right]^{1/2} \\ 0 & , \beta > 4.24 \end{cases} \tag{4}$$

The spike noise plays a more important role as $b_o\tau$ increases as seen from (4). As $b_o\tau \to \infty$, (4) becomes

$$P_{e_s} \begin{cases} \beta/\pi & , \beta < 0.734 \\ \frac{\beta}{\pi} \sin\left(\frac{\pi}{2}\frac{D}{\tau}\right) & , 0.734 \le \beta \le 4.24 \\ 0 & , \beta > 4.24 \end{cases} \tag{5}$$

which is less than 0.5 for all β. The total error probability P_e is obtained by adding (1) and (4) or (3) and (5),

$$P_e = P_{e_g} + P_{e_s} \tag{6}$$

P_e is plotted in Fig. 2 for $b_o\tau$ = 1, 2, 3, 5 & 10 and β = 0.7 as a function of E/N_0. It is remarked that (1) leads to an unreasonably small error rate when $b_o\tau$ = 1 and β = 0.7. However, the error rate of the ideal PCM/FM system[6,7] can be used since for $b_o\tau$ = 1, with low pass filtering, the performance is very close to the ideal system. This error rate which is due to the Gaussian noise only is added the error rate due to the spike noise and the result is shown in Fig. 2. for $b_o\tau = 1$. It was assumed in (1) that the binary square wave is filtered by a pre-modulation filter with a one-sided bandwidth B_p capable of passing the first harmonic, viz. $2B_p = 1/\tau$. Pre-modulation filtering can reduce the spectral occupancy of the modulated signal. The best performance is expected, however, when there is no pre-modulation filter. The binary square wave of peak amplitude 2V can be written in Fourier series,

$$e_m(t) = \frac{4V}{\pi}\left[\sin\frac{\pi}{\tau}t + \frac{1}{3}\sin\frac{3\pi}{\tau}t + \frac{1}{5}\sin\frac{5\pi}{\tau}t + \ldots\ldots\right] \tag{7}$$

If $b_o\tau$ = 1.5 the modulation signal energy is increased by a factor 0.5dB. Without pre-modulation filter, the increase in signal energy is 0.8 dB. By using a binary random sequence, nearly 90% of the signal energy is concentrated within a frequency equal to the bit rate, as compared with only 83% (see Eq. (7)) when a binary square waveform is used.

The optimum deviation index can be obtained by setting $\partial P_e/\partial\beta = 0$, i.e.

$$\frac{1}{\pi\sqrt{\frac{E}{N_0 b_o\tau}}} e^{-\frac{E}{N_0 b_o \tau}} = \sqrt{\frac{1.32\, b_o\tau}{\left[\mathrm{erf}(0.837 b_o\tau) - 0.945 b_o\tau e^{-0.7(b_o\tau)^2}\right]}} \, e^{-\frac{(1.32\beta)^2}{2\left[\mathrm{erf}(0.837 b_o\tau) - 0.945 b_o\tau e^{-0.7(b_o\tau)^2}\right]}} \tag{8}$$

from which $\beta = \beta_{opt.}$ can be computed for a given $b_o\tau$ and E/N_0 with say, $\beta < 1$. When E/N_0 approaches infinity, and $b_o\tau$ and β remain finite, the left side of (8) is upper bounded by $\frac{1}{\sqrt{\pi}} e^{-E/(b_o\tau N_0)}$. A comparison of exponents leads to $\beta_{OPT} = 0.71$ for $b_o\tau \geq 2$. For $b_o\tau = 1$ and $E/No >> 1$, the optimum deviation is close to that of the ideal system i.e. $\beta_{OPT} = 0.71$ since the spike noise contribution is negligilbly small. The bit error probability is plotted in Fig. 3 for $b_o\tau = 1, 2, 3$, $\beta = 0.7$ and with low-pass filtering.

Without low-pass filtering, the error probability at the optimum frequency deviation is shown in Appendix I as

$$P_e = \frac{1}{2} \exp - \left(\frac{0.61 E}{N_o b_o \tau} \right) \tag{9}$$

which is also plotted in Fig. 3 for $b_o\tau = 1, 2, 3$. With a low-pass filter, there is a signal energy improvement of nearly 1 dB for $b_o\tau = 1$, 1.5 dB for $b_o\tau = 2$ and 2 dB for $b_o\tau = 3$. When $b_o\tau \to \infty$ and $\beta < 0.734$, the signal energy improvement due to the low-pass filter is 2 dB as seen from (3), (4), and (9).

Test Description. - Table 1 lists the major items of test equipment used to construct the PCM/FM telemetry system shown in figure 1. The PCM binary wavetrain generated by the data encoder was selected as alternate "ones" and "zeroes" with a NRZ format. The resulting waveform was a square wave as shown in figure 4 ; it was not possible to generate a random binary sequence with the data encodes used. The difference in measured P_e between the square wave and random sequences was assumed to be small since, as indicated in the previous section, the difference between these two approaches in signal energy concentrated in a bandwidth equal to the data rate is quite small. During tests the pre-modulation filter bandwidth was varied from infinity (no filter) to a bandwidth of one-half the data rate to determine its effect on β and P_e. Similarly, β was varied from approximately 0.4: 1 to over 2: 1 with the FM transmitter (signal generator) to determine β_{OPT} with different pre-modulation filter bandwidths.

The FM receiver used in the tests had two discrete IF bandwidths, 100 KHz and 500 KHz. Since the data encoder used had a maximum data rate capability of 100 kbps, the 100 KHz IF bandwidth was used for tests at $b_o\tau = 1$. This bandwidth was also used for tests at $b_o\tau = 2$ by reducing the data encoder data rate to 50 Kbps. A limitation inherent in the test setup precluded the use of data rates lower than 50 Kbps, hence the 100 KHz IF bandwidth could not be used for tests at $b_o\tau = 3$. These tests were conducted by retuning the 500 KHz IF to 300 KHz and setting the data rate at 100 Kbps.

In all tests the video output from the FM discriminator in the receiver was applied to a bit rate synchronizer which contained a low-pass filter at the input and a sampling detector decision device. The low-pass filter served as the post-detection filter in these tests.

The voltage polarity of a 1 micro-second strobe sample of the post-detection filtered data, with the sample taken at the center of each bit period, was used to determine whether a "one" or "zero" had been received. As shown in figure 4, the operations on the detected data consisted of post-detection filtering the noisy data from the receiver, using a level crossing detector on the post-detection filtered data to produce a binary waveform, and sampling this binary waveform at the bit period center to reconstruct the transmitted wavetrain. By using the level crossing detector, amplitude variations in the post-detection filtered output were eliminated. Noise-caused edge jitter on this constant amplitude waveform resulted in the bit errors measured. Errors were detected in an "Exclusive Or" circuit by comparing the transmitted NRZ waveform with the reconstructed waveform at the decision device output. Synchronism between the bit rate synchronizer clock, from which the sampling strobes were obtained, and the transmitted data train was obtained in separate tests for hard-line synchronization (hard-line between the data encoder and bit synchronizer) and for synchronization achieved by the bit rate synchronizer alone operating on the received noisy data. No measurable difference in P_e vs. SNR_{IF} was observed. Figure 5 compares the P_e vs. SNR_{IF} measured by Avco-MSD and previously reported results by EMR and Aeronutronic[8]. These results were based on $b_o\tau = 1$ and show close agreement between the three tests. Figure 6 is a comparison between the theoretical P_e vs. $b_o\tau$ results derived in this paper and the results of experiments conducted at Avco-MSD. It can be seen that very close agreement was achieved. β used in figure 6 was 0.7 and was based on test results shown in figure 7. In that figure P_e is shown vs. β for the case of $b_o\tau = 3$ and SNR_{IF} held constant at 9db. It can be seen from this figure that the best performance (lowest P_e) was obtained without a premodulation filter ($R=\infty$). With the exception of the "no filter" case, $\beta_{OPT.}$ (β at which P_e was minimized) occured at $\beta \approx 0.7$ as indicated earlier in the analytical results. The dip in P_e occuring at $\beta = 2.$[illegible] for $R=\infty$ is unexplained at this time, but is assumed to be due to harmonic contributions. It can be seen from figure 7 that the use of a promodulation filter with bandwidth 0.5-to-1.5 times the data rate does not appreciably affect P_e. Results similar to those shown in figure 7, but not shown in this paper, were obtained at other values of SNR_{IF} and for $b_o\tau = 2$.

Figure 8 compares system performance with and without a premodulation filter and illustrates the small penalty in system performance resulting from the inclusion of premodulation filter in the system.

Appendix I - Error Probability of Wideband PCM/FM Without Low-Pass Filtering. - We start with the general expression of the error probability for discriminator detection as given by Meyerhoff and Mazer (Eq. 3 of Ref. 1)

$$F(\rho, G) = \frac{e^{-\rho}}{2}\int_0^\infty \frac{e^{y}}{(1+u^2)^{3/2}}\left\{I_o(y) + (2y + G^2\rho)\left[I_o(y) + I_1(y)\right]\right\} du \quad \text{(A1)}$$

where

$$y = \frac{\rho}{2}\left[1 - \frac{(u+G)^2}{1+u^2}\right]$$

Here ρ is the signal-to-noise ratio at the imput of the discriminator with the noise power taken over the IF bandwidth. Let $\rho_o = \frac{E}{N_o b_o \tau}$ be the received signal-to-noise ratio at the imput of the IF filter. Then ρ/ρ_o is the carrier power reduction factor. For Gaussian IF filter and square wave modulating waveform, the expression for G and ρ/ρ_o given by Eqs. (25) and (30) of Ref. 2 are essentially valid for any $b_o\tau$. The error probability can be obtained by substituting G and $\frac{\rho}{\rho_o}$ in (A1). We will show that a very simple relationship exists between G and ρ/ρ_o as given by (A3) below.

Following the notations in Ref. 2, we have*

$$\alpha = a_f \sqrt{2}/(2\sigma_f) = \sqrt{2\pi}\, a_f \tau/(2 b_o \tau) \ , \quad \mu = b_o \tau \sqrt{2\pi}$$

$$h(\tau) \approx 1 \ , \quad h'(\tau) = \sqrt{2\pi}\ \sigma_f \exp(-\mu^2)$$

and since $\mu < 1$, $\nu < 1$ when $b_o \tau \geq 1$,

$$\exp(-\mu^2) \simeq 0 \ , \quad \exp(-\mu^2)\nu \approx 0 \qquad \text{(A2)}$$

Thus G = 2α , where G is the generalized frequency deviation.

Also ρ/ρ_o = exp. $(-1/2\, G^2)$. (A3)

(A2) indicated that to maintain G at its optimum value, the frequency deviation a_f must increase proportionally as $b_o \tau$ increases. (A3) is independent of $b_o \tau$. The optimum G which holds for the $b_o \tau = 1$ case [2], i.e. $G_{OPT.} = 1$, should also hold for $b_o \tau > 1$. For G = 1, the error probability has a closed form solution [1],

$$F(\rho, 1) = \frac{1}{2} \exp(-\rho)$$

Thus the error probability for wideband PCM/FM using discriminator detection with Gaussian IF filter is

$$P_e = F(\rho_o, 1) = \frac{1}{2} \exp - \left(\frac{0.61 E}{N_o b_o \tau} \right) \qquad \text{(A4)}$$

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*In page 137 of Ref. 2, $f_3'(t)$ should be equal to $-af_2(t) + \sqrt{2\pi}\, \sigma_f \exp(-\mu^2)$. This is obviously misprinted in Ref. 2. Thus the expression for G is greatly simplified. This symbol of Ref. 2 is replaced by μ here to avoid confusion.

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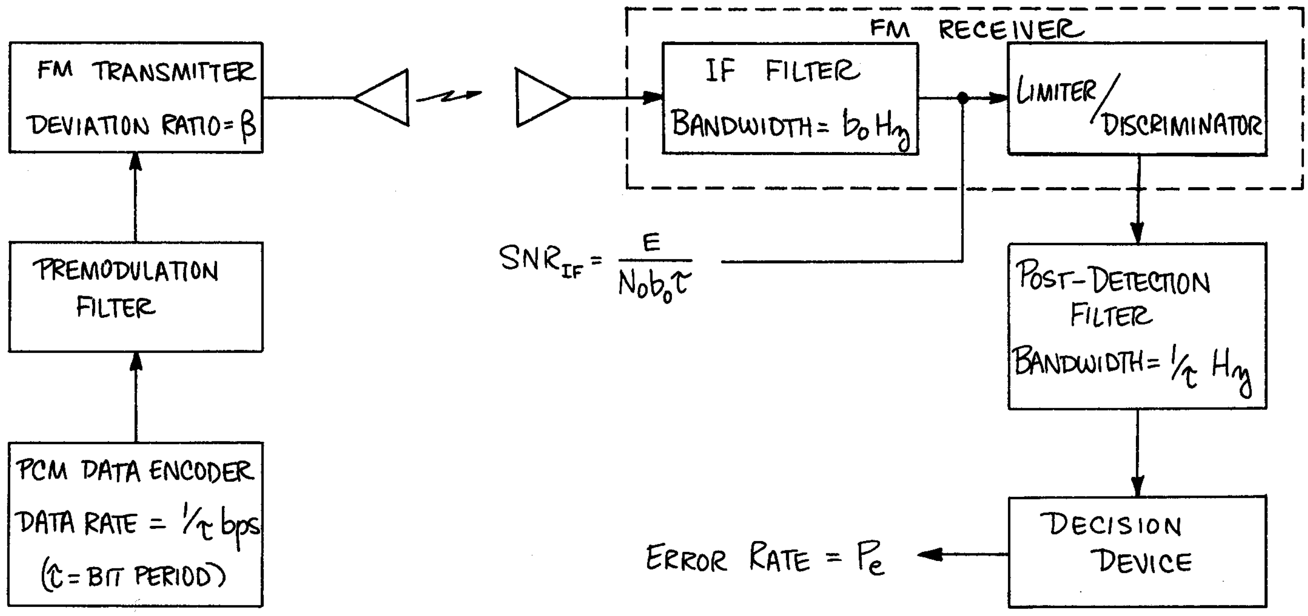

Figure 1. PCM/FM Telemetry System

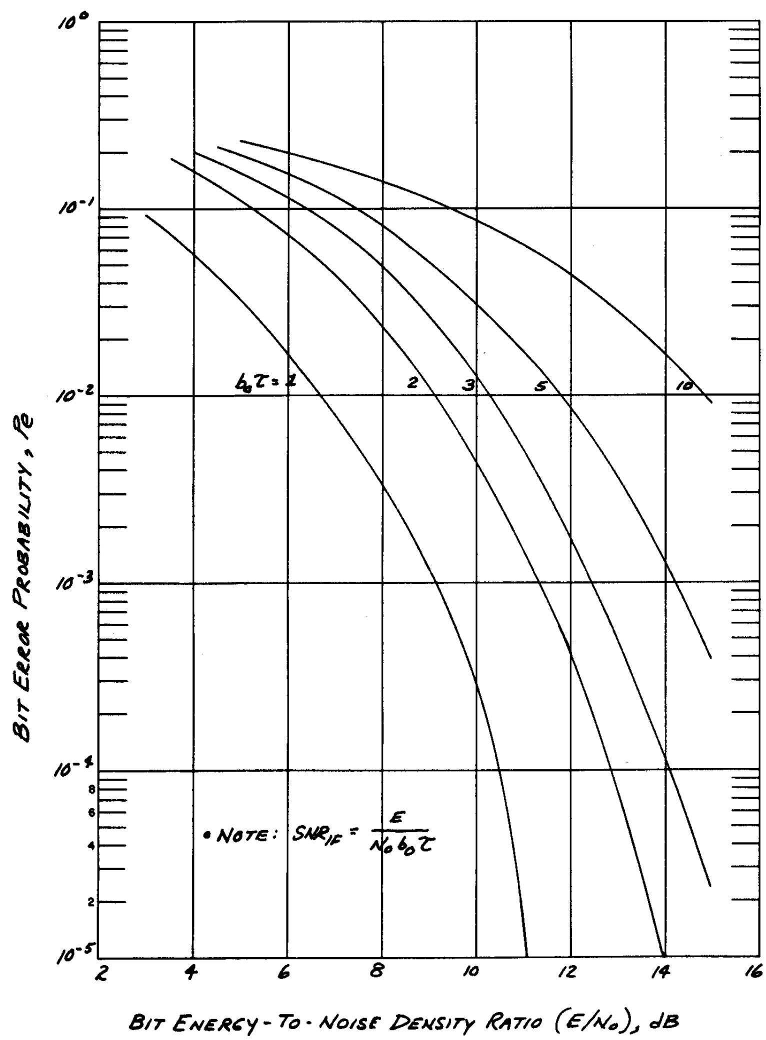

Figure 2. P_e For Wideband PCM/FM With Discriminator Detection

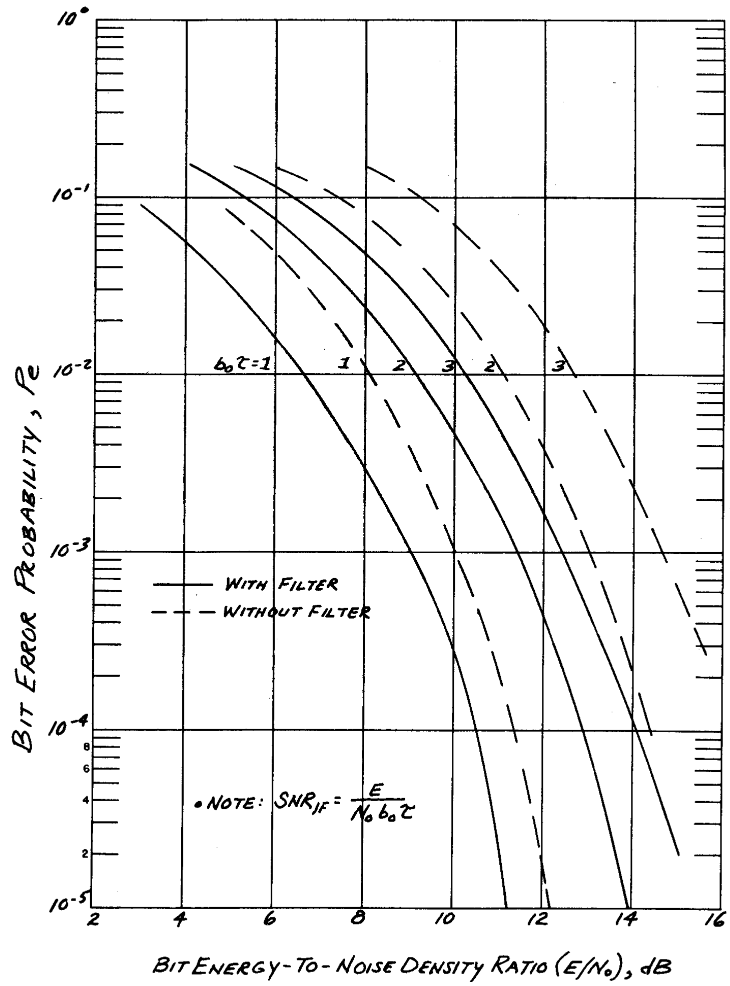

Figure 3. P_e With And Without Post-Detection Filter

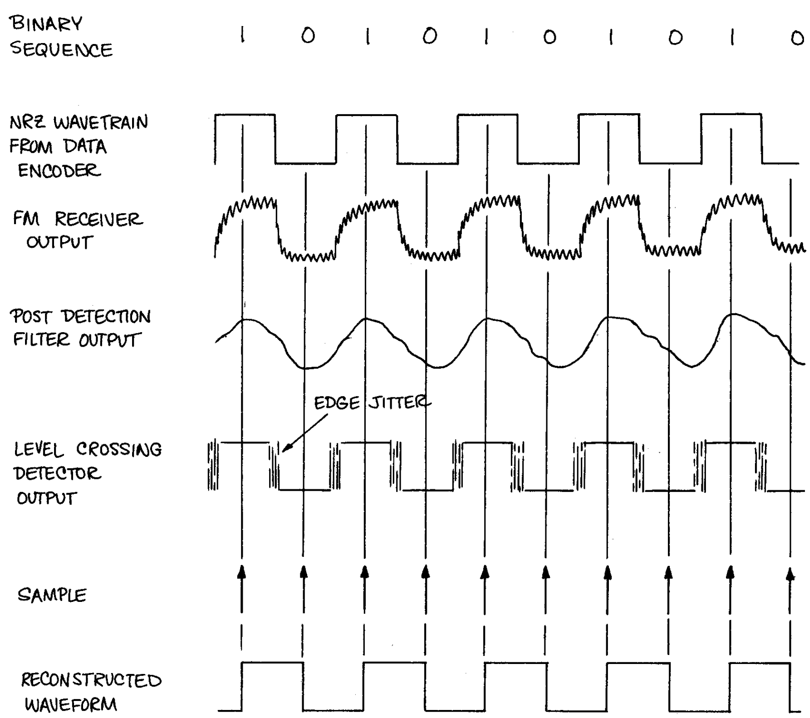

Figure 4. Test Waveforms

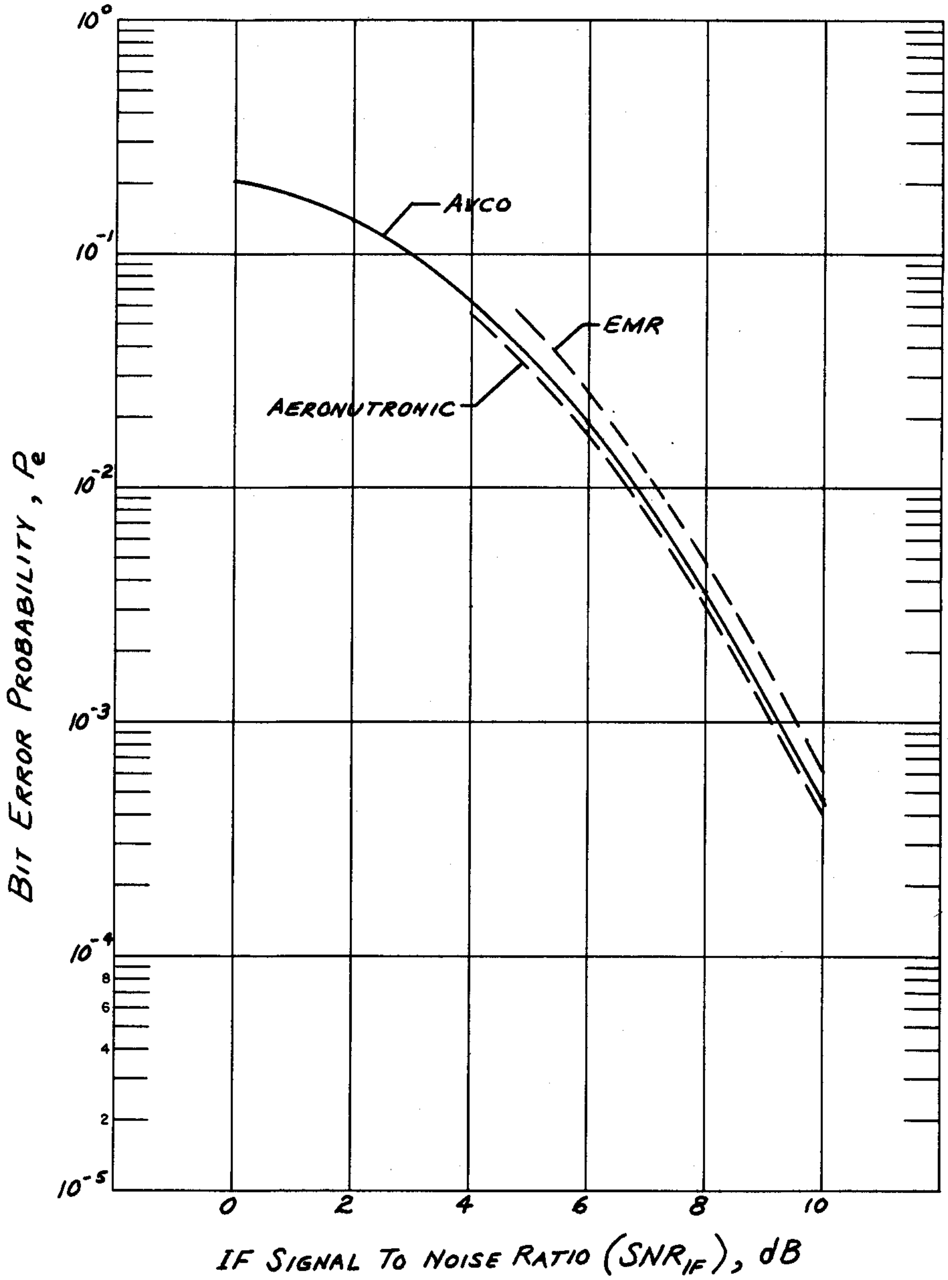

Figure 5. Comparison of Measured P_e For $b_o\tau = 1$

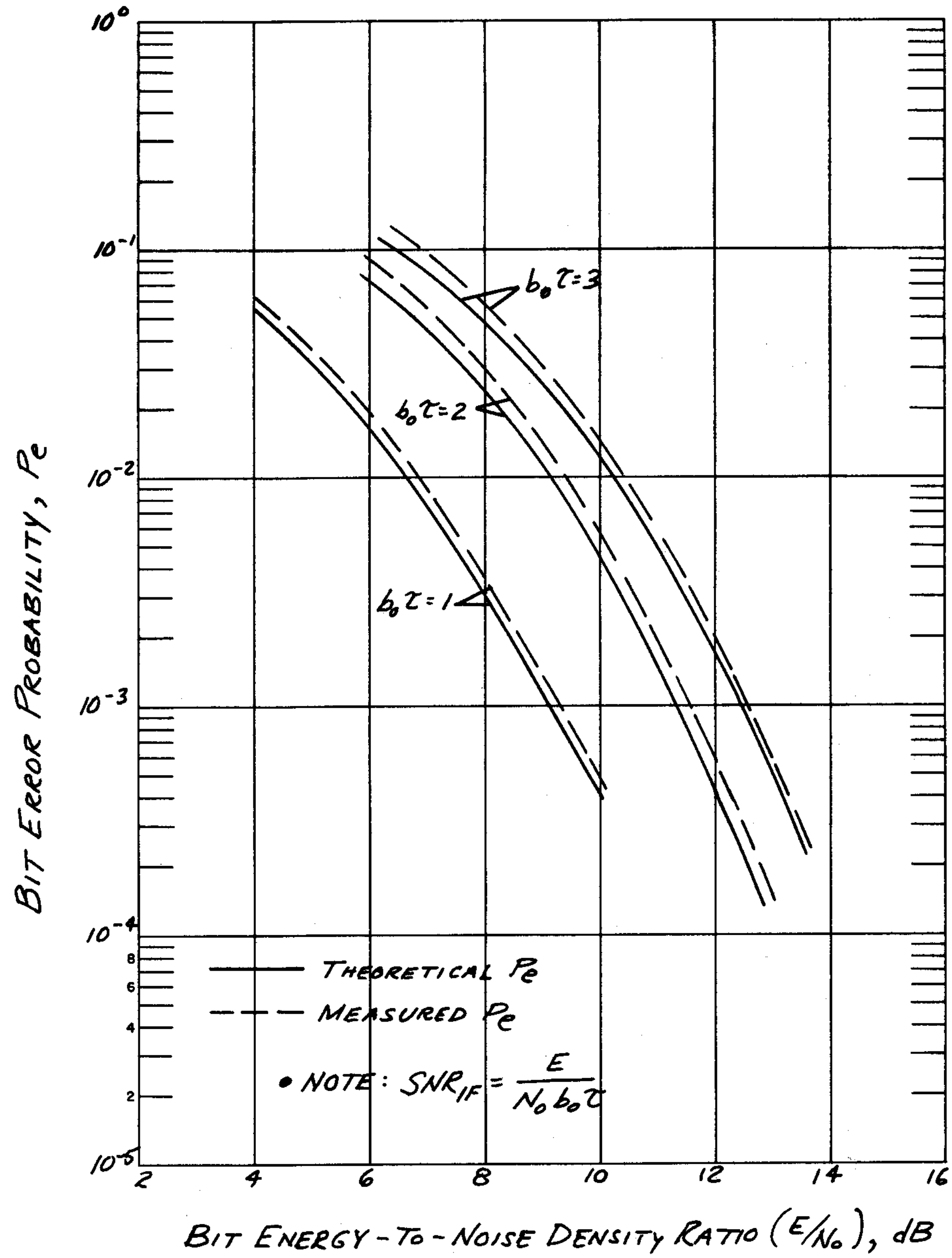

Figure 6. Comparson Of Theoretical And Measured P_e

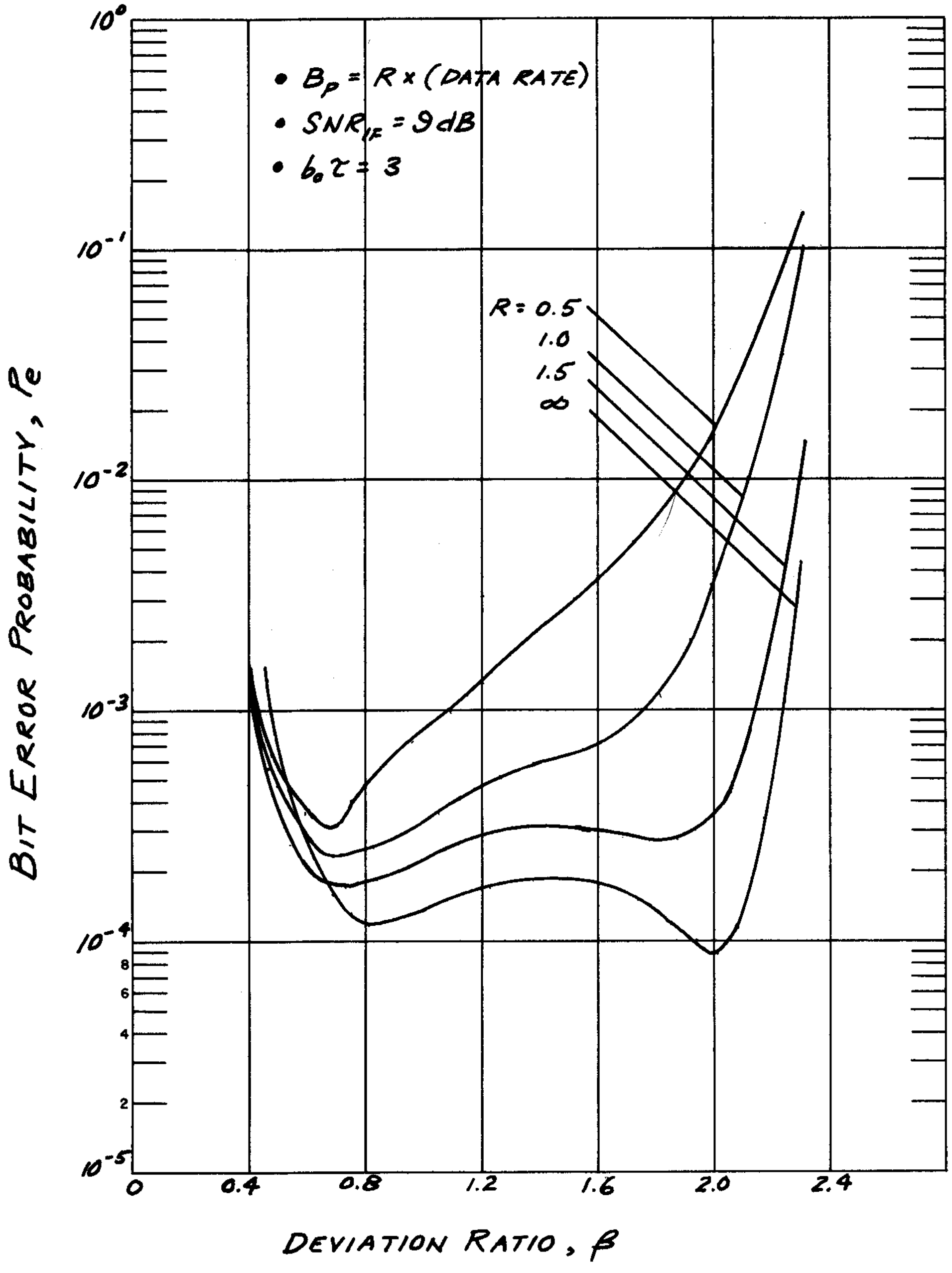

Figure 7. Effect of β On P_e

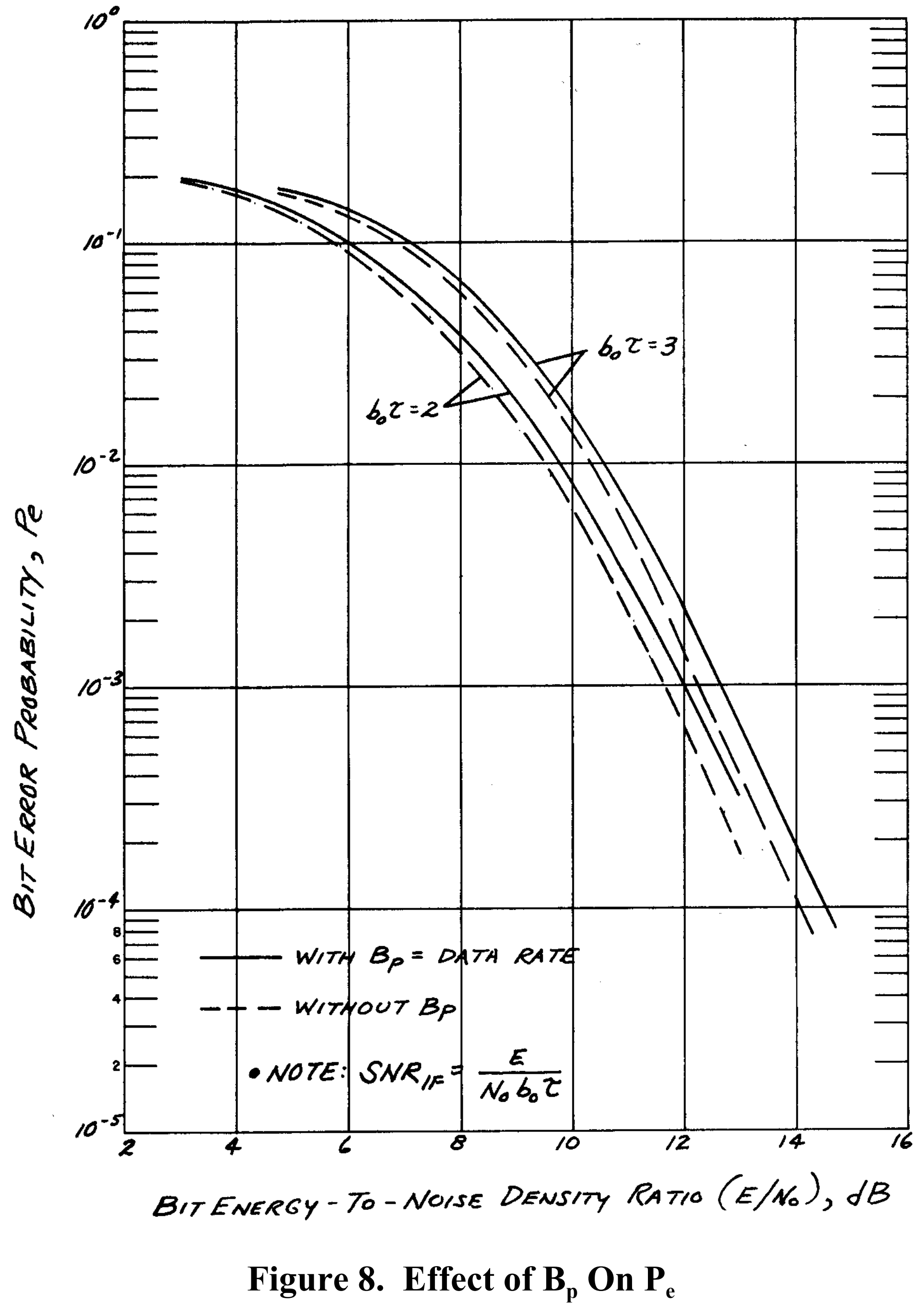

Figure 8. Effect of B_p On P_e

TABLE 1 TEST EQUIPMENT

PCM Data Encoder	Consolidated System Corp. PCM Simulator	NRZ Wavetrain
Premodulation Filter	Allison Labs. Variable Band Pass Filter Model No. 2 CR	M Derived, Double K Section 33 dB/Octave Roll Off
FM Transmitter	Boonton FM Signal Generator Model No. 202F	
FM Receiver	Nems Clarke Model No. 1412	b_o= 100 KHz or 500 KHz,
Post Detection Filter	Telemetrix Inc. Bit Rate Synchronizer Model No. 6203A	6 Pole Linear Phase LPF, 36 dB/Octave Roll Off, Cut Off= Data Rate
Decision Device	Telemetrix Inc. Bit Rate Synchronizer Model No. 6203A	Sampling Detector, 1 μ s Strobe at center of NRZ bit period