AN EXACT SOLUTION OF INJECTION PHASE-LOCKING

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Summary Recent advances of solid-state device technology of generating microwave power from low voltage dc power in one step have renewed interest in the study of injection locking. The solid state millimeter-wave devices have many potential applications such as parametric amplifier pumps, transponder sources, local and self-test oscillators, and high bit rate millimeter-wave repeater systems.

The purpose of this article is to solve the nonlinear differential equation of injection locking. Using the method of Riccati's equation, an exact solution has been obtained which is much simpler and more explicit than that shown by Mackey. This article shows the tracking and acquisition behavior of the loop for different initial phase offsets and for different ratios of initial frequency offset D to loop gain B. It also compares the transient and steady-state responses with the exact solution and the linear approximation solution.

This article concludes that the difference of steady-state responses obtained from the exact solution and the linear approximation solution will be greater if the D/B ratio is greater.

Introduction Since Adler[1] showed that injection phase-locking is achievable for low frequency, many investigations[2-5] have shown that it is also achievable in the microwave frequency region. Recent advances of solid-state device technology of generating microwave power from low voltage dc power in one step have renewed. interest in the study of injection locking. The solid-state millimeter-wave devices of injection locking have many potential applications such as parametric amplifier pumps, transponder sources, local and self-test oscillators, and high bit rate millimeter wave repeater systems[5].

The purpose of this article is to give a thorough study of the nonlinear differential equation of injection locking loops[1,2]

$$\dot{\phi} + B\sin\phi = \Delta\omega_o \tag{1}$$

where

$\dot{\phi}$ = the instantaneous beat frequency

= $d\phi/dt$

$\Delta\omega_o$ = initial frequency offset

B = loop gain constant

= $(\omega_o/2Q)\,(E_i/E_o)$

ω_o = free-running oscillator frequency

Q = figure of merit of the loaded oscillator

E_i, E_o = injection signal and oscillator cavity voltages

For convenience of writing, let the initial frequency offset $\Delta\omega_o \equiv D$, then (1) becomes

$$\dot{\phi} + B \sin\phi = D \tag{2}$$

We first briefly discuss the linear approximation solution of the nonlinear differential equation (2). Then, using the method of Riccati's equation, we show an exact solution of (2). There are three conditions of the solution: overdamped condition when $0 < D < B$, critically damped condition when $D = B$, and oscillatory or underdamped condition when $D > B$. The exact solution we obtained is a function of loop gain B, initial frequency offset D, and initial phase offset ϕ_o , which is much simpler and more explicit than that shown in Reference 2, equations (12) and (13).

Linear Approximation Solution From (2) the first order differential equation indicates that the injection locking loop behaves like the first order phase-locked loop (PLL). It is generally approximated by $\sin\phi \doteq \phi$ for $\phi \ll 1$ rad. Then (2) becomes

$$\phi + B\phi = D \tag{3}$$

The complete solution of (3) is

$$\phi = \left(\phi_o - \frac{D}{B}\right) e^{-Bt} + \frac{D}{B} \tag{4}$$

where

ϕ_o = the initial phase offset

The solution in (4) has been plotted as dashed lines in Figures 1 through 3 for the initial phase offsets to be 10, 60, and 170 degrees. From the solution in (4), several results are already well known for the first order PLL[6,7] . First, the injection locking will be acquired when the initial frequency offset is zero and with any initial phase offset, the steady-state phase error will be zero

$$\phi_{ss} = \phi_o e^{-Bt} = 0 \text{ as } t \to \infty \quad \text{if } D = 0 \tag{5}$$

For nonzero initial frequency offset, the steady-state solution from (4) is

$$\phi_{ss} = D/B \text{ as } t \to \infty \tag{6}$$

which has been plotted in Figure 4. Secondly, in the locking condition $\dot{\phi} = 0$, the steady-state solution obtained from (2) is[6]

$$\phi_{ss} = \sin^{-1} (D/B) \tag{7}$$

The initial frequency offset D should be less than or equal to the loop gain B in order that (7) has a solution. The maximum initial frequency offset which the loop can acquire injection locking is called the maximum pull-in frequency

$$\Delta \omega_m \equiv D_m \leqq B \tag{8}$$

Note that there is a discrepancy between (6) and (7) because of the linear approximation assumption made by many people in order to obtain an easy solution.

Exact Solution Let us solve the nonlinear differential equation (2) by assuming

$$x = \tan (\phi/2) \tag{9}$$

so that

$$\sin \phi = 2x/(1 + x^2) \tag{10}$$

$$\frac{d\phi}{dx} = 2/(1 + x^2) \tag{11}$$

Substitution of (9), (10), and (11) into (2) gives

$$\dot{x} + Bx = D(1 + x^2)/2 \tag{12}$$

If the initial frequency offset is zero, i.e., D = 0, (12) becomes

$$\dot{x} + Bx = 0 \tag{13}$$

The solution of (13) is

$$x = C e^{-Bt} \tag{14}$$

Suppose at t = 0, the initial phase offset is ϕ_o, then (14) becomes

$$x = \tan (\phi_o/2)\, e^{-Bt} \tag{15}$$

Substituting (15) into (9) gives

$$\phi = 2\tan^{-1}\{\tan(\phi_o/2)\, e^{-Bt}\} \tag{15}$$

When t = 0, the solution (16) gives $\phi = \phi_o$ as it should. Using the initial phase offset ϕ_o as a parameter under the condition of zero initial frequency offset, the transient response of ϕ asymptotically approaches zero as shown in Figure 5, except $\phi_o = \pm\pi$. If the initial phase offset ϕ_o is assumed to be some constant not equal to $\pm\pi$ and the loop gain B is used as a parameter, the transient response of ϕ as shown in Figure 6 also asymptotically approaches zero except B = 0 or B = ∞, for which cases either $\phi = \phi_o$ (constant) or $\phi = 0$ instantaneously. Therefore, the steady-state solution for D = 0 from (16) is

$$\phi_{ss} = 0 \text{ as } t \rightarrow \infty \text{ if } \phi_o \neq \pm\pi \tag{17}$$

The probability of $\phi_o = \pm\pi$ is zero, on the assumption that the probability density of the initial phase offset is uniformly distributed

$$p(\phi_o) = \frac{1}{2\pi}, \quad -\pi \le \phi_o \le \pi \tag{18}$$

There is no delta function, $\delta(\phi_o)$, existing at $\phi_o = \pm\pi$.

However, if the initial frequency offset D is not equal to zero, the solution of the nonlinear differential equation (2) or (12) is not easily obtainable. After several transformations, (2) is transformed into the standard Riccati's equation [8, 9] as shown in Appendix A.

According to the relative values of D and B there are three solutions: overdamped, critically damped, and underdamped cases. Let us discuss the acquisition and tracking behavior for these three conditions as follows

<u>Case 1. 0 < D < B. Overdamped Condition</u> The complete solution from (A-10) is

$$\phi = 2\tan^{-1}\left\{\left(\tan\frac{\phi_o}{2} - \frac{B - \sqrt{B^2 - D^2}}{D}\right)e^{-Bt} + \frac{B - \sqrt{B^2 - D^2}}{D}\right\} \tag{19}$$

When t = 0, the initial phase offset is ϕ_o , as it should be. The steady-state phase error is really the solution of interest

$$\phi_{ss} = 2\tan^{-1}\left\{\frac{B - \sqrt{B^2 - D^2}}{D}\right\} \tag{20}$$

The response of (19) has been plotted as solid lines in Figures 1 through 3 for the initial phase offset to be 10, 60, and 170 degrees. The exact solution of ϕ_{ss} has been plotted in

Figure 4. It should be noted that (20) and (7) are equivalent for $0 \leq D/B \leq 1$ and $0 \leq \phi_{ss} \leq 90°$.

If the initial frequency offset D is equal to B, then ϕ_{ss} is 90 degrees. That is, when D approaches B, the steady-state phase error approaches 90 degrees as expected. However, the derivative of ϕ_{ss} with respect to D B is

$$\frac{d\phi_{ss}}{d(D/B)} = \frac{2\left[1 - \sqrt{1-(D/B)^2}\right]}{\sqrt{1-(D/B)^2}\left\{(D/B)^2 + \left(1 - \sqrt{1-(D/B)^2}\right)^2\right\}} \tag{21}$$

From Figure 4 we see that the slope of ϕ_{ss} at D/B = 1 is

$$\frac{d\phi_{ss}}{d(D/B)} = \infty \quad \text{when } D/B = 1 \tag{22}$$

There is no real solution when D/B > 1, which agrees with Case 3.

<u>Case 2. D = B. Critically Damped Condition</u> The complete solution from A-13 is

$$\phi = 2\tan^{-1}\left\{\left(\tan\frac{\phi_o}{2} - 1\right) e^{-Bt} + 1\right\} \tag{23}$$

The steady-state phase error as $t \to \infty$ is ϕ_{ss} = 90 degrees. The response of (23) has been shown in Figures 1 through 3 in which ϕ_{ss} is 90 degrees, a critical acquisition and tracking condition.

<u>Case 3. D > B. Oscillatory or Underdamped Condition</u> The initial frequency offset is greater than the loop gain constant B, which is the maximum pull-in range. There is no real solution under the oscillatory condition. The acquisition of phase-locking is not possible.

Conclusion From the solutions shown above, this article concludes with the acquisition and tracking behaviors for any initial phase offset $\phi_o \neq \pm n\pi$ as follows:

1) <u>D = 0.</u> The injection locking loop acquires phase lock and tracks the phase with zero steady-state phase error as shown in Figures 1 through 6.

2) <u>0 < D < B.</u> The injection locking loop acquires phase lock and tracks the phase with a steady-state error

$$\phi_{ss} = 2\tan^{-1}\left\{\frac{B - \sqrt{B^2 - D^2}}{D}\right\} \tag{24}$$

as shown in Figures 1 through 4.

3) D = B. The injection locking loop acquires phase lock and tracks the phase with a steady-state phase error ϕ_{ss} = 90 degrees as shown in Figures 1 through 4.

4) D > B. Acquisition can never be achieved.

5) The higher the loop gain is, the faster the acquisition will be. As shown in Figure 6, loop gain B = 0, the acquisition will never be achieved; if loop gain B = ∞, the acquisition will be achieved instantaneously.

APPENDIX A.
THE SOLUTION USING RICCATI'S EQUATION

The steady-state solution of (12) for D ≠ 0 can be obtained by letting

$$y = -Dx/2 \quad \text{and} \quad \dot{y} = -D\dot{x}/2 \tag{A-1}$$

where the dots represent time derivatives. Substituting (A-1) into (12) gives the standard form of Riccati's equation [8, 9].

$$\dot{y} + y^2 + By + D^2/4 = 0 \tag{A-2}$$

Letting $\dot{u} = uy$, (A-2) becomes

$$\ddot{u} + B\dot{u} + (D^2/4)u = 0 \tag{A-3}$$

The solution of (A.-3) is

$$u = C_1 e^{m_1 t} + C_2 e^{m_2 t} \tag{A-4}$$

where C_1 and C_2 are integration constants, and m_1 and m_2 are

$$m_1 = \left(-B + \sqrt{B^2 - D^2}\right)/2 \tag{A-5}$$

$$m_2 = \left(-B - \sqrt{B^2 - D^2}\right)/2 \tag{A-6}$$

In general, there are three cases which are depending on the relative values of B and D.

Case 1. $0 < D < B$. Overdamped Condition Both m_1 and m_2 are negative real numbers. From u and its derivative, we obtain the solution of y; therefore, of x

$$x = -\frac{2}{D} y = -\frac{2}{D} \frac{m_1 C_1 + m_2 C_2 e^{(m_2 - m_1)t}}{C_1 + C_2 e^{(m_2 - m_1)t}} \tag{A-7}$$

The value of $(m_2 - m_1)$ is negative since $0 < D < B$. Therefore, the steady-state solution as $t \to \infty$ is

$$x_{ss} = x(t = \infty) = \frac{B - \sqrt{B^2 - D^2}}{D} \tag{A-8}$$

Assuming the initial phase offset to be ϕ_o at $t = 0$, the complete solution of (12) for $0 < D < B$ is

$$x = \left(\tan\frac{\phi_o}{2} - \frac{B - \sqrt{B^2 - D^2}}{D}\right) e^{-Bt} + \frac{B - \sqrt{B^2 - D^2}}{D} \tag{A-9}$$

The complete solution of ϕ is obtained by substituting x into (9)

$$\phi = 2\tan^{-1}(x)$$

$$= 2\tan^{-1}\left\{\left(\tan\frac{\phi_o}{2} - \frac{B - \sqrt{B^2 - D^2}}{D}\right) e^{-Bt} + \frac{B - \sqrt{B^2 - D^2}}{D}\right\} \tag{A-10}$$

The steady-state solution is

$$\phi_{ss} = \phi(t = \infty) = 2\tan^{-1}\left\{\frac{B - \sqrt{B^2 - D^2}}{D}\right\} \tag{A-11}$$

<u>Case 2. $D = B$. Critically Damped Condition</u> Since m_1 and m_2 are two equal negative roots, the complete solution of x is

$$x = x_{ts} + x_{ss} = C e^{-Bt} + 1 = \left(\tan\frac{\phi_o}{2} - 1\right) e^{-Bt} + 1 \tag{A-12}$$

The complete solution of ϕ is obtained by substituting (A-12) into (9)

$$\phi = 2\tan^{-1}\left\{\left(\tan\frac{\phi_o}{2} - 1\right) e^{-Bt} + 1\right\} \tag{A-13}$$

The steady-state solution of ϕ is 90 degrees. Of course, this result can be obtained directly from (A-11) by substituting D = B.

Case 3. D > B. Underdarnped or Oscillatory Condition m_1 and m_2 are two complex conjugate roots. It is an oscillatory condition. This is another indication showing that the locking condition cannot be achieved.

APPENDIX B
MACKEY'S SOLUTION

The solution from Equations (12) and (13), Reference 2, is given below for comparison:

$$\phi = 2\tan^{-1}\left\{\frac{1}{k} - \frac{\sqrt{1-k^2}}{k}\left[\frac{e^{\sqrt{1-k^2}\,Bt}\left(\frac{2}{k} - \frac{1-\sqrt{1-k^2}}{k} - \tan\frac{\phi_0}{2}\right) + \left(\frac{1-\sqrt{1-k^2}}{k} - \tan\frac{\phi_0}{2}\right)}{e^{\sqrt{1-k^2}\,Bt}\left(\frac{2}{k} - \frac{1-\sqrt{1-k^2}}{k} - \tan\frac{\phi_0}{2}\right) - \left(\frac{1-\sqrt{1-k^2}}{k} - \tan\frac{\phi_0}{2}\right)}\right]\right\} \tag{M12}$$

where $k = \Delta\omega_o/B$.

Realization that $k = \sin\phi_\infty$ where ϕ_∞ is the steady-state phase shift allows (M12) to be written as follows:

$$\phi = 2\tan^{-1}\left\{\csc\phi_\infty - (\cot\phi_\infty)\left[\frac{e^{Bt\cos\phi_\infty}\left(\cot\frac{\phi_\infty}{2} - \tan\frac{\phi_0}{2}\right) + \left(\tan\frac{\phi_\infty}{2} - \tan\frac{\phi_0}{2}\right)}{e^{Bt\cos\phi_\infty}\left(\cot\frac{\phi_\infty}{2} - \tan\frac{\phi_0}{2}\right) - \left(\tan\frac{\phi_\infty}{2} - \tan\frac{\phi_0}{2}\right)}\right]\right\} \tag{M13}$$

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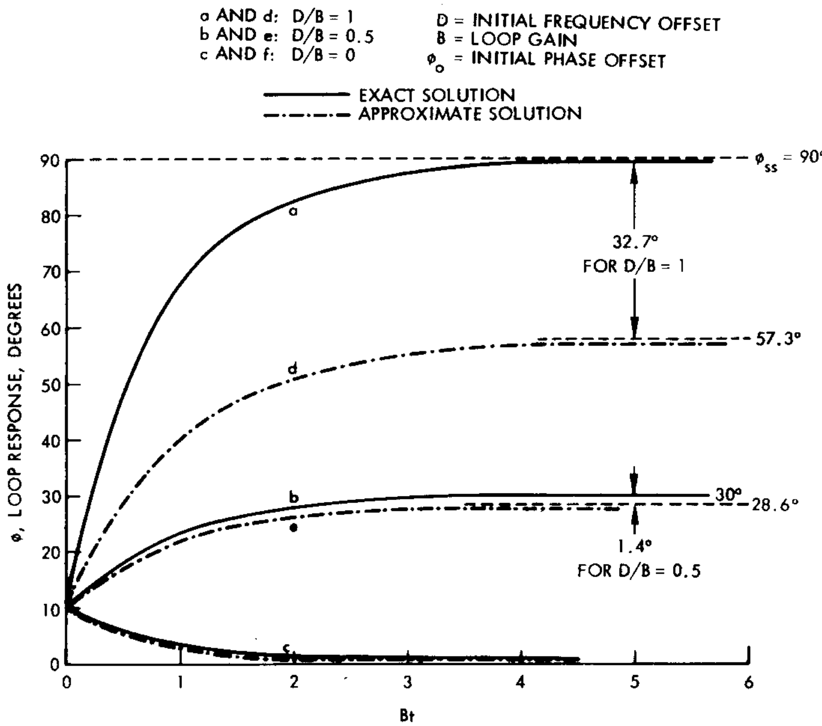

Figure 1. Comparison of Solutions for $\phi_o = 10^\circ$

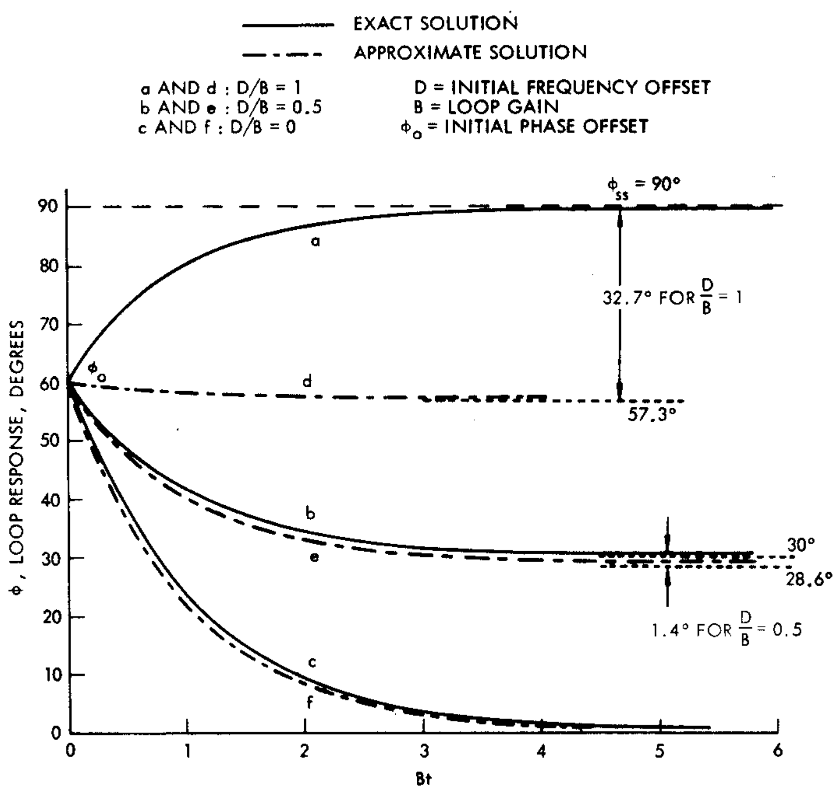

Figure 2. Comparison of Solutions for $\phi_o = 60^\circ$

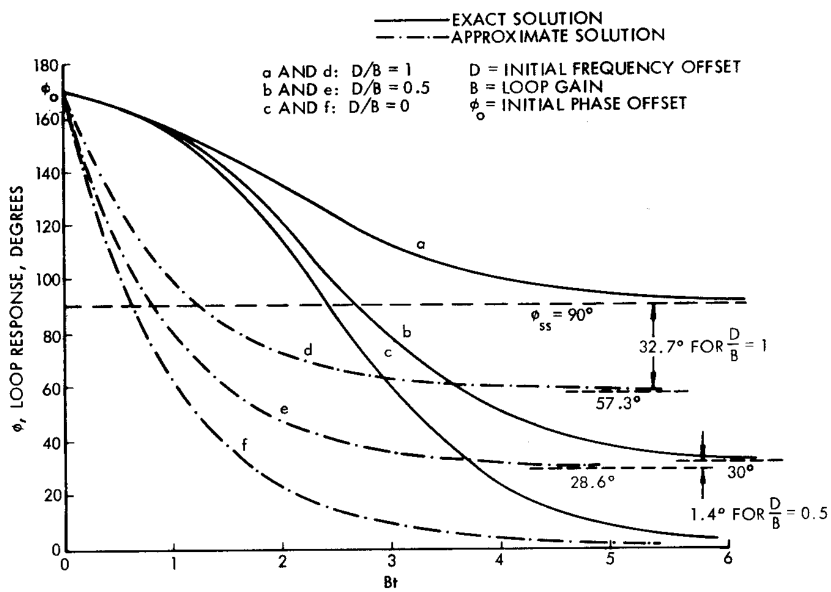

Figure 3. Comparison of Solutions for $\phi_o = 170°$

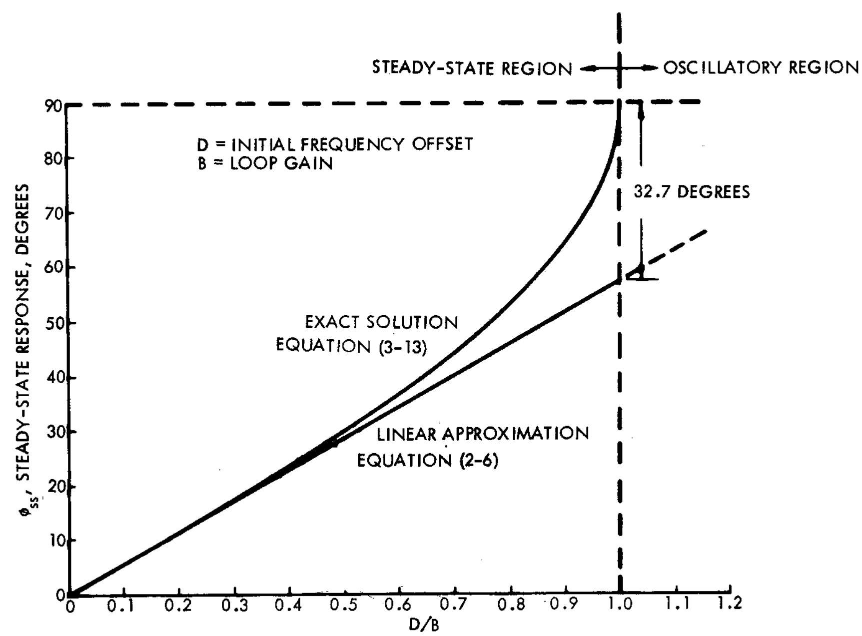

Figure 4. Steady-State Phase Error

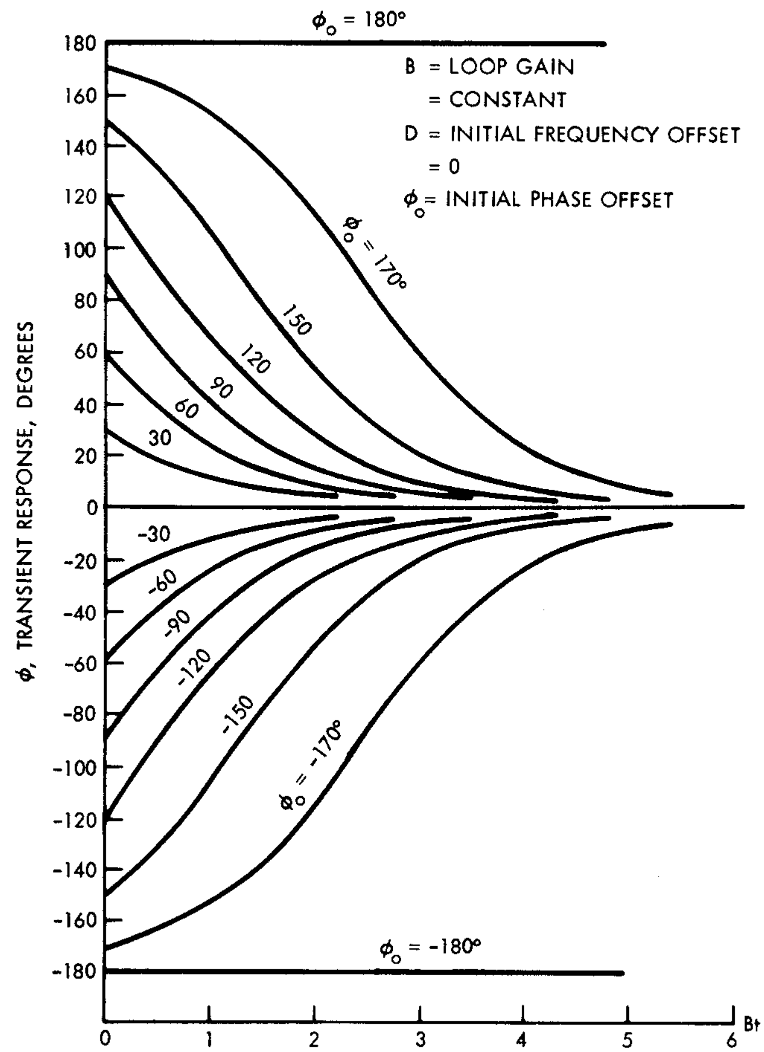

Figure 5. Transient Response for Various ϕ_o

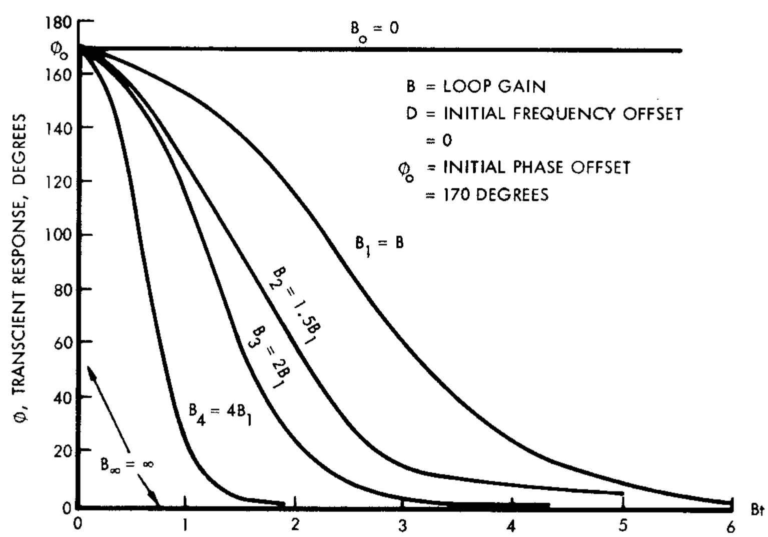

Figure 6. Transient Response for Various B