A NEW BANDLIMITED M-ARY NON-COHERENT TELEMETRY SYSTEM

By M. S. STONE

Summary.-In this paper new M-ary non-coherent signal structures that require less transmission bandwidth than conventional FSK are presented. A method of synthesizing these signal sets by the use of frequencies or tones is given. In order to evaluate system performance, an approximation to the error probability for large signal energy-to-noise spectral density ratio is derived. Using this formula certain signal structures are evaluated for M = 8, 16, and 32. It is shown that for error rates of less than 1×10^{-3}, the signal structures that reduce the transmission bandwidth by 25% are within 1 dB of the corresponding FSK signal set. Thus these signal structures provide an efficient method of communication when the bandwidth is constrained to be less than required for the transmission of FSK.

Introduction.-For some applications the bandwidth required for the transmission of non-coherent M-ary FSK is not available. Therefore, one must use signalling waveforms requiring less bandwidth, but for the sake of efficiency yielding an error rate near that of FSK. In this paper it is shown that such signal structures can be obtained by the methods of Signal Design in which one finds signal sets that locally minimize the error probability for a given allowable dimensionality or bandwidth. The optimal non-coherent signal structure for dimensionality $D=2M-2$ has been given by Weber[1]. Stone[2] has obtained optimal signal structures for $M \geq 6$, $D \leq 2M-K$, $K \leq [2M/3]$ where [] denotes "integer part of."

The first part of this paper is devoted to presenting the system model and resulting general expression for the probability of error. The remaining sections describe a method of synthesizing the above mentioned signal structures and evaluating performance in terms of probability of error.

System Model and Detection Probability[3].-We assume that the transmittable signal set consists of M equipowered, equiprobable narrowband waveforms defined on the interval $[0,T]$ which are of the following form

$$S_j(t,\theta) = A_j(t)\cos[\omega_c t + \phi_j(t) + \theta] \qquad (1)$$

where

$\frac{\omega_c}{2\pi}$ = carrier frequency

$A_j(t)$ = amplitude modulation

$\phi_j(t)$ = phase modulation

θ = random variable, uniformly distributed in $[0, 2\pi]$

A portion of this paper was abstracted from a Ph. D. (E.E.) dissertation[2] submitted to the University of Southern California.

Mr. Stone is employed by System Engineering Laborotory, TRW Systems Group.

The received waveform is then

$$V(t) = VS_j(t,\theta) + N(t) \tag{2}$$

where V is the channel scale factor and N(t) is a sample function from a white Gaussian noise process with one-sided spectral density N_o watts/Hz. Owing to the narrowband assumption, the signal energy is given by

$$E = V^2 \int_o^T S_j^2(t,\theta)\, dt = \frac{V^2}{2}\int_o^T A^2_j(t)dt \qquad j=1,\ldots M \tag{3}$$

The optimum synchronous non-coherent receiver computes from the received signal the following:

$$R_j = \left[\left\{\int_o^T V(t)A_j(t)\cos[\omega_c t + \phi_j(t)]\, dt\right\}^2 + \left\{\int_o^T V(t)A_j(t)\sin[\omega_c t + \phi_j(t)]\, dt\right\}^2\right]^{1/2}$$

$$j=1,\ldots M \tag{4}$$

and using this data decides the k^{th} signal was transmitted if

$$R_k = \max_j R_j \tag{5}$$

With the model discussed above it can be shown that the probability of detection is a function of the following parameters:

1) The signal waveform inner products

$$\alpha_{ij} = \frac{V^2}{2E}\int_o^T A_i(t)A_j(t)\cos[\phi_i(t) - \phi_j(t)]dt$$

$$\beta_{ij} = \frac{V^2}{2E}\int_o^T A_i(t)A_j(t)\sin[\phi_i(t) - \phi_j(t)]dt$$

$$i,j=1,\ldots M \tag{6}$$

and

2) the resulting signal inner product matrix Γ

$$\Gamma = \left[\begin{array}{c|c} A & B^t \\ \hline B & A \end{array}\right] \tag{7}$$

where

$$A = ||\alpha_{ij}||, \quad \alpha_{ii} = 1, \quad \alpha_{ij} = \alpha_{ji}$$

$$B = ||\beta_{ij}||, \quad \beta_{ii} = 0, \quad \beta_{ij} = -\beta_{ji}$$

t denotes matrix transpose.

The range of values that α_{ij} and β_{ij} can take on is specified by the constraint that all admissible Γ matrices must be non-negative definite. The rank of the Γ matrix is equal to the dimension of the signal set it describes. An approximation to the bandwidth necessary for D non-coherent orthogonal waveforms can be obtained by assuming trigonometric waveforms. For an observation interval of T seconds, since such signals can be placed 1/T Hz apart, the minimum bandwidth required is B = D/T, thus establishing a linear relationship between allowed dimensionality and minimal necessary bandwidth.

3) $\lambda^2 = 2E/N_o = S/N_o \dfrac{2 \log_2 M}{R}$ where R is the system data rate in bits/sec and S/N_o is the transmitted signal power-noise spectral density ratio.

The probability of detection is given by

$$P_D(\lambda;\Gamma) = \frac{1}{M} \exp(-\lambda^2/2) \int_{R_{2M}} \cdots \int I_o\left(\lambda \max_k \sqrt{u_k^2 + v_k^2}\right) G(\underline{W}, \underline{O}, \Gamma) d\underline{W}$$

$$= \frac{1}{M} \exp[-\lambda^2/2] E_{\overline{W}} \, I_o\left[\lambda \max_k \sqrt{u_k^2 + v_k^2}\right] \tag{8}$$

where $E_{\overline{W}}$ means expected value over $\overline{W}$ and R_{2M} denotes 2M dimensional real Euclidean space, I_o is the modified Bessel function of the first kind, and $G(W;O;\Gamma)$ is the Gaussian probability density of 2M random variables described by the 2M dimensional vector

$$\underline{W}^t = (u_1, \cdots u_M, v_1, \cdots v_M)$$

which has zero mean vector O and covariance matrix Γ. The probability of detection expression above is in general impossible to evaluate. Therefore, for the purposes of analysis a more tractable formula is needed. In the appendix it is shown that for large λ^2

$$P_E(\lambda,\Gamma) \cong \frac{1}{M} \sum_{\substack{i=1 \\ i \neq j}}^{M} \sum_{j=1}^{M} H(\lambda, \delta_{ij}) \tag{9}$$

where

$$H(\lambda, \delta_{ij}) = Q\left[\frac{\lambda\sqrt{1 - \sqrt{1 - \delta_{ij}^2}}}{2}, \frac{\lambda\sqrt{1 + \sqrt{1 - \delta_{ij}^2}}}{2}\right] - \frac{1}{2} e^{-\frac{\lambda^2}{4}} I_o\left(\frac{\lambda^2 \delta_{ij}}{4}\right)$$

$$\delta_{ij} = \sqrt{\alpha_{ij}^2 + \beta_{ij}^2}\ , \quad 0 \le \delta_{ij} \le 1$$

$Q(a, b)$ is called Marcum's Q-function[4] and is given by

$$Q(a,b) = \int_b^\infty t\, e^{-(t^2+a^2)/2}\, I_o(at)dt$$

Bandlimited Signal Structures and Their Mechanization. -The optimal signal structure for $D = 2M-2$, is described by a matrix of the form

$$\Gamma_2 = \left|\begin{array}{c|c} \alpha_{r_M} & 0 \\ \hline 0 & \alpha_{r_M} \end{array}\right|$$

where α_{r_M} denotes an $M \times M$ regular simplex matrix

The optimal signal structures for $M \ge 6$, $D = 2M-K$, $K \le [2M/3]$, where $[\,]$ denotes "integer part of" is described by

$$\Gamma_k = \left[\begin{array}{c|c|c} \alpha_{r_{M_1}} & 0 & \\ \hline 0 & \alpha_{r_{M_K}} & 0 \\ \hline & & \alpha_{r_{M_1}} \\ \hline 0 & & 0 \\ & & \alpha_{r_{M_K}} \end{array}\right]$$

where $\sum_{i=1}^{K} M_i = M$ and the M_i are chosen so that

$$M_i = \{2M/K\},\ i = 1, \ldots K-1$$

$$M_K = M - (K-1)\{2M/K\}$$

where $\{\}$ "denotes integer closest to"

As one can see from Equation (1), realizing the transmitted signal set $\{S_j(t,\theta)\}$ is equivalent to realizing $\{A_j(t)\cos\theta_j(t)\}$ and $\{A_j(t)\sin\theta_j(t)\}$. The signal structures described by an arbitrary Γ_k matrix can be synthesized by the use of M - (K/2) frequencies by letting

$$A_k(t) = 1, \quad o \le t \le T; \quad k=1,\ldots M$$

$$\phi_\ell(t) = 2\pi\ell t/T, \quad \ell=1,\ldots M-(K/2) \tag{10}$$

and writing

$$\cos\phi_j(t) = \sum_{\ell=1}^{M-\frac{K}{2}} s_{j\ell}\cos\frac{2\pi\ell t}{T} \qquad j=1,\ldots M$$

$$\sin\phi_j(t) = \sum_{\ell=1}^{M-\frac{K}{2}} s_{j\ell}\sin\frac{2\pi\ell t}{T} \qquad j=1,\ldots M \tag{11}$$

where the $\{s_{j\ell}\}\ \ell, j=1,\ldots M$ are chosen so that

$$\sum_{\ell=1}^{M-\frac{K}{2}}\sum_{p=1} s_{j\ell}\, s_{rp} = a_{jr} \tag{12}$$

Using Equations (11) and (12) in (6) then yields the desired result.

Performance Analysis.-For M = 8, 16, and 32 Figure 1 presents curves of probability of error versus S/N_oR for FSK and for various bandlimited signal structures. The curves for the bandlimited signal structures we obtained using Equation (9). For the range of error rates shown this formula is a valid approximation to the probability of error. The curves for the FSK signal structure were obtained from Lindsey[5]. For each M, the Γ_k signal structure that yields a bandwidth reduction of approximately 10% and that yielding a reduction of 25% are shown. The calculations show that these signal structures require approximately 0.5 dB and 1.0 dB increase in transmitted signal power, respectively, to give the same error rate as the corresponding FSK system.

Conclusions.-We have described a new class of signalling waveforms requiring less transmission bandwidth than non-coherent FSK. A method of synthesizing these waveforms by the use of tones was given. An approximate expression for the error rate was derived and used to evaluate the performance of the signal structures yielding approximately 10% and 25% bandwidth reduction for M = 8, 16, and 32. The above analysis showed that these waveforms provide an efficient method of communication when the bandwidth necessary for the transmission of FSK is not available.

Appendix: Derivation of Equation (9).-Suppose signal $S_k(t)$ was transmitted. An error in correct decoding occurs if at least one R_j, $i\neq k$ is greater than R_k. et Ξ_{ik} denote the event $\{R_i > R_k/S_k(t) \text{ sent}\}$. Then the probability of error, given that $S_k(t)$ was transmitted is given by

$$P(\text{error}/S_k(t)) = \text{Prob}(\Xi_{1k}\cup\Xi_{2k}\cdots\cup\Xi_{k-1,k}\cup\Xi_{k+1,k}\cdots\cup\Xi_{m,k}) \tag{A-1}$$

Using the Union Bound yields

$$P(\text{error}/S_k(t)) \leq \sum_{\substack{i=1 \\ i\neq k}}^{M} \text{Prob}(\Xi_{ik}) \qquad \text{(A-2)}$$

This bound is valid, of course, for all λ^2 but at large λ^2 it becomes an approximation to the probability of error, given $S_k(t)$ was transmitted. One may see this by noting that the probability of any finite number of intersections of the Ξ_{ik} approaches zero as $\lambda \to \infty$. That is

$$\text{Prob}\{\cap \Xi_{ik}\} \to 0 \quad \text{as } \lambda \to \infty \qquad \text{(A-3)}$$

Hence as $\lambda \to \infty$, the probability of the union of a finite number of Ξ_{ik} is approximately equal to the sum of the probabilities of the Ξ_{ik}. $\text{Prob}(\Xi_{ik}) = H(\lambda, \delta_{ij})$ has been evaluated by Helstrom[6] and is as stated above. This then leads to

$$P_e \cong \frac{1}{M} \sum_{\substack{i \\ i\neq k}}^{M} \sum_{k}^{M} \text{Prob}(\Xi_{ik}) = \frac{1}{M} \sum_{\substack{i \\ i\neq k}}^{M} \sum_{k}^{M} H(\lambda, \delta_{ik}) \qquad \text{(A-4)}$$

References

1. C. L. Weber, "A Contribution to the Signal Design Problem for Incoherent-Phase Communication Systems," IEEE Trans. Information Theory, Vol. IT-14, pp. 306-311, March 1968.

2. M. S. Stone, "Signal Design for the Non-coherent Gaussian Channel," Ph. D.(E.E.)Dissertation, University of Southern California, Los Angeles, California, June 1969.

3. C. L. Weber, "Elements of Detection and Signal Design," McGraw-Hill, 1968.

4. J. I. Marcum, "A Statistical Theory of Target Detection by Pulsed Radar," IRE Trans. Information Theory, Vol. IT-6, pp. 145-267, April 1960.

5. W. C. Lindsey, "Coded Non-coherent Communications," IEEE Trans. Space Electronics Telemetry, Vol. SET-11, pp. 6-13, March 1965.

6. C. W. Helstrom, "The Resolution of Signals in White Gaussian Noise," Proc. IRE, Vol. 43, pp. 1111-1118, September 1955.

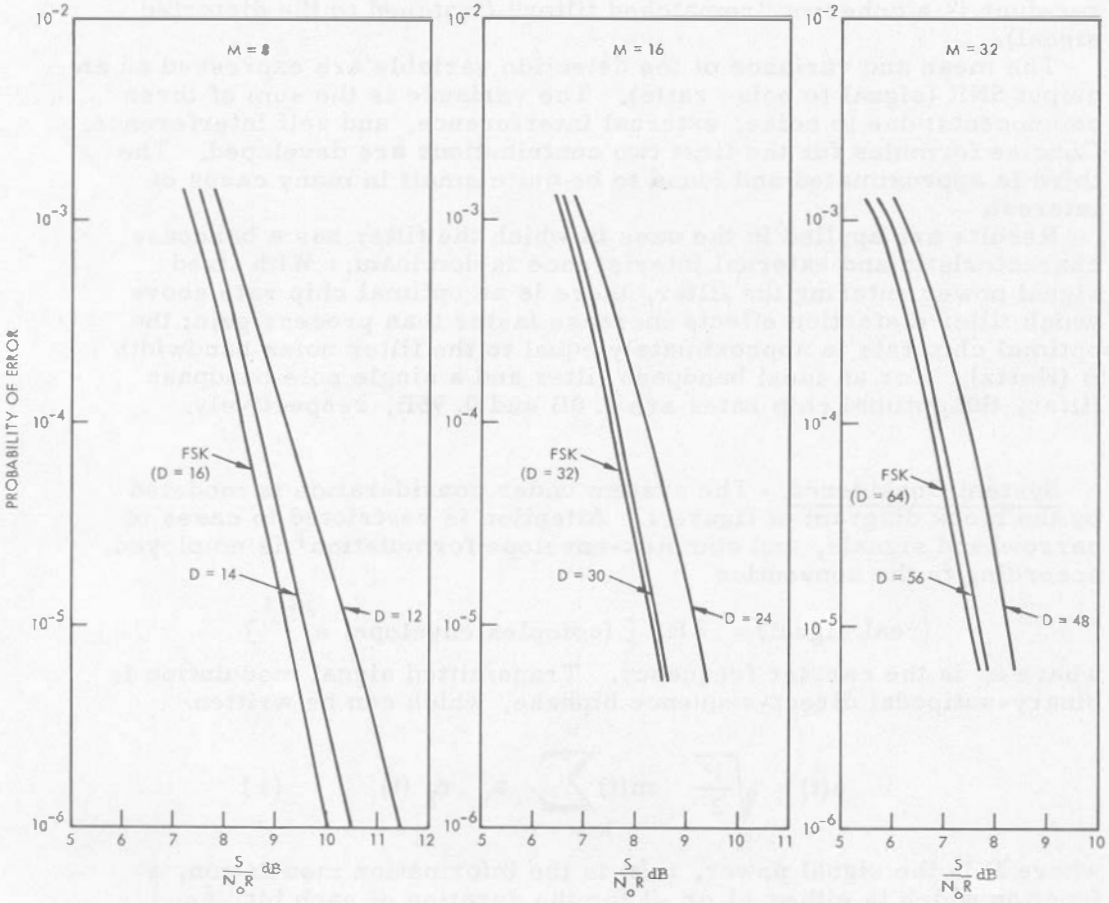

Figure 1. Probability of Error vs. S/N_oR for M = 8, 16, and 32