DOPPLER JITTER VERSUS DIGITAL ERROR RATE

By C. L. WEBER

Summary. - In the design of digital communication systems, the error rate is the criterion which is invariably emphasized. In many digital systems, however, there is relative motion between transmitter and receiver which must be controlled by making use of Doppler frequency information. A coherent digital system is herein modeled, in which the trade-off that exists between Doppler measurement capability and subcarrier demodulation error rate is quantitatively presented.

System parameters which effect only one of the two above criteria are initially optimized. The dependence of performance on the remaining system parameters is then analytically and graphically presented.

Introduction. - In digital communication systems, there often is the need to maintain continuous Doppler frequency information for range-rate and range determination as well as the need to maintain the error rate below a specified value. Range-rate and range data contribute to the specification and possibly the eventual control of the trajectory or orbit of the vehicle. In addition to specifying various system parameters, the allocation of the total available transmitted power to the various information bearing subcarrier signals should be carried out based on knowledge of the effect that this choice will have on Doppler tracking capability as well as error rates. The goal herein is to provide this information for a general class of coherent digital communication systems.

The type of coherent digital system to be considered is one which transmits the data signals by phase modulating the rf carrier with bi-phase modulated data subcarriers. The frequencies of the several subcarriers are assumed to have been judiciously choosen so that the spectra of the modulated data are non-overlapping [1, 2].

For definiteness, the demodulation of the subcarriers will be assumed to be carried out employing squaring loops, Costas loops, etc., so that all necessary subcarrier phase and synchronization reference information is obtained directly from the data signal, thereby eliminating the need for separate sync channels and eliminating the need for placing any power in the subcarrier reference. This has been termed single-channel mechanization with a suppressed subcarrier [3].

System Description. - A basic simplified diagram of the general type of system to be considered is depicted in Figure 1. In the transmitter portion of the system, the data signals, $\{s_k(t),\ k = 1, \ldots, K\}$, bi-phase modulate the frequency multiplexed subcarrier waveforms $\{\sqrt{2p_k}\cos \omega_k t,\ k = 1, \ldots, K\}$. The input to the carrier phase modulator, $\theta(t)$, is therefore given by

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$$\theta(t) = \sum_{k=1}^{K} \sqrt{2p_k}\; s_k(t) \cos \omega_k t \qquad (1)$$

where p_k, $k = 1, \ldots, K$ is the average power in the k^{th} bi-phase modulated subcarrier waveform before carrier phase modulation.

The trend in modern digital communication systems is to employ subcarriers with 100% modulation, that is, there is no residual power at the subcarrier frequency for tracking purposes. The reason for this is that with the advent of squaring loops [4-6], Costas loops [7], delay-locked loops, etc., coherent phase reference and bit synchronization information can be obtained directly from the 100% modulated data signal. The $\{s_k(t),\ k = 1, \ldots, K\}$ are thus assumed to consist of a sequence of ± 1's with bit times, $\{T_{bk},\ k = 1, \ldots, K\}$ respectively.

With this modulation scheme assumed, the output of the phase modulator is then given by

$$s(t) = \sqrt{2P}\; \sin(\omega_c t + \theta(t) + \theta_0) \qquad (2)$$

where P is the overall average transmitted power, and θ_0 is some unknown constant reference angle. The received waveform is then given by

$$y(t) = \sqrt{2P}\; \sin\left(\omega_c t + \int^{t} \omega_d(\tau)d\tau + \theta(t) + \theta_0\right) + n(t) \qquad (3)$$

where n(t) is assumed to be white Gaussian noise with one-sided spectral density N_0 watts/hertz, and $\omega_d(\tau)$ represents the Doppler frequency shift due to the relative range-rate between transmitter and receiver.

Neglecting frequency shifters, frequency synthesizers, bandpass limiters, etc., the coherent carrier tracking loop generates the reference signal

$$r(t) = \sqrt{2}\; \cos\left(\omega_c t + \int^{t} \omega_d(\tau)d\tau + \hat{\theta}_0(t)\right) \qquad (4)$$

The data bearing waveforms which comprise $\theta(t)$ are assumed to be at frequencies outside the bandwidth of the carrier phase locked loop (PLL). The Doppler frequency is assumed to be varying slowly enough to be within this bandwidth, however, so that the carrier PLL is able to track this signal. Therefore, the output data bearing signal of the carrier tracking loop which goes into the various subcarrier demodulators is given by

$$y_0(t) = s_0(t) + n_0(t) \qquad (5)$$

where

$$s_0(t) = \sqrt{P}\; \sin(\theta(t) + \phi_r(t)),$$

$\phi_r(t) = \theta_0 - \hat{\theta}_0(t)$ is the carrier loop phase error, and the additive noise $n_0(t)$ has the same statistics as n(t) [8].

With the Doppler frequency slowly varying, any cycle slipping can be assumed to be due only to the additive noise. No detuning will be assumed to exist between the received carrier and the voltage controlled oscillator (VCO) rest frequency. Then the approximate steady state mod 2π probability density function of ϕ_r is given by [8, 9]

$$p(\phi_r) \approx \frac{\exp(\alpha_r \cos \phi_r)}{2\pi I_0(\alpha_r)}, \quad -\pi \le \phi_r \le \pi, \tag{6}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero, and

$$\alpha_r \triangleq \frac{P_c}{N_0 B_{L_r}} \tag{7}$$

is the signal-to-noise ratio of the carrier tracking loop. In (7), B_{L_r} is the one-sided noise bandwidth of the carrier PLL based on the linear theory of PLL's [8-10] and P_c is the average power of the received signal at the carrier frequency. The overall signal-to-noise ratio of the received signal is given by

$$\beta_r \triangleq \frac{P}{N_0 B_{L_r}} \tag{8}$$

In order to ultimately specify the trade off between Doppler tracking capability and digital demodulation capability, the distribution of power between the carrier tracking loop and the subcarrier demodulation loops must be specified. To do this, s(t) is represented by the series (see, e.g., Lindsey [5])

$$\begin{aligned} s(t) &= \sqrt{2P}\ \mathrm{Im}\{\exp[j(\omega_c t + \theta(t) + \theta_0)]\} \\ &= \sqrt{2P}\ \mathrm{Im}\{\exp(j\omega_c t+\theta_0) \prod_{k=1}^{K} \sum_{m_k=-\infty}^{\infty} (j)^{m_k} \\ &\qquad J_{m_k}(\sqrt{2p_k})\exp[j\, m_k(\omega_k t + \tfrac{\pi}{2} s_k(t))]\} \end{aligned} \tag{9}$$

where $J_{m_k}(\,\cdot\,)$ is the Bessel function of order m_k, and Im means "the imaginary part of". The signal which enters the carrier tracking loop is

$$\sqrt{2P_c}\,\sin(\omega_c t + \theta_0) + n(t)$$

where the average power in the tracking signal, P_c, is given by the average power of the component of s(t) in (8) at ω_c. This is obtained by setting $m_k = 0$, $k = 1, \ldots, K$, from which we obtain the percent of the total power that enters the carrier tracking loop, namely,

$$\frac{P_c}{P} = \prod_{k=1}^{K} J_0^2(\sqrt{2p_k}) \tag{10}$$

To obtain the power in the subcarriers, $s_0(t)$ is similarly expanded. The average power in the k^{th} subcarrier data signal at the output of the k^{th} extraction filter (see Fig. 1) is obtained by assuming $\phi_r(t)$ is essentially constant over the bit time T_{b_k} of the k^{th} data signal, and setting $m_k = \pm 1$ and $m_{k'} = 0$ for all $k' \neq k$. This average power, P_{s_k}, is given by

$$\frac{P_{s_k}}{P} = 2J_1^2(\sqrt{2p_k}) \prod_{\substack{k'=1 \\ k' \neq k}}^{K} J_0^2(\sqrt{2p_{k'}})\,[E(\cos\phi_r)]^2 \tag{11}$$

where [5]

$$E(\cos\phi_r) = \frac{I_1(\alpha_r)}{I_0(\alpha_r)} \tag{12}$$

The input signal to the k^{th} subcarrier demodulator is then given by

$$y_k(t) = \sqrt{2P_{s_k}}\,\cos(\omega_k t + \frac{\pi}{2}\,s_k(t)) + n_k(t) \tag{13}$$

where $n_k(t)$ also has the same statistics as n(t). This follows from the fact that the extraction filters $H_k(j\omega)$, $k = 1, \ldots, K$, in Figure 1 are normally broadband with respect to the bandwidths of the synchronization tracking loop, the phase tracking loop, and the matched filter in the subcarrier demodulator.

In the important special case where all subcarrier channels are afforded the same performance level, we have that

$$p_k = p\,, \quad k = 1, \ldots, K\,,$$

from which (10) and (11) reduce to

$$\frac{P_c}{P} = [J_0(\sqrt{2p})]^{2K} \qquad (14)$$

and

$$\frac{P_{s_k}}{P} = 2J_1^2(\sqrt{2p})[J_0(\sqrt{2p})]^{2K-2}[E(\cos\phi_r)]^2 \qquad (15)$$

respectively. Note that p affects $E(\cos \phi_r)$ as well as the other factors in (14) and (15). Since P_{s_k}/P is dependent on $E(\cos \phi_r)$, we see that the performance of the digital demodulation of the subcarrier signals is not directly dependent upon the occurance of cycle slipping events, but only indirectly to the extent that cycle slipping broadens the steady state mod 2π p.d.f. of the carrier phase error ϕ_r. Cycle slipping, as would be expected, is a fundamental concern in determining Doppler tracking capability. Plots of these power distributions versus p for various values of β_r are shown in Fig. 2 for the case of one subcarrier. Note that the maximum value of P_{s_k}/P is seen to occur for values of p ranging from 0.4 at $\beta_r = 1$ to 1.45 at $\beta_r = 100$. This demonstrates that when designing a bi-phase digital communication system (as well as when designing a PLL) a design point must be picked; that is, an input signal-to-noise ratio, β_r, must be choosen at which the system will be designed to operate optimally.

The performance of the subcarrier demodulators and the Doppler measuring subsystem must next be determined so that their trade-off can be established.

<u>Digital Performance</u>. - The output signal of the k^{th} extraction filter which is the input signal to the k^{th} subcarrier demodulator is given by

$$y_k(t) = \sqrt{2P_s}\cos(\omega_k t + \frac{\pi}{2}s_k(t)) + n_k(t) \qquad (16)$$

where $n_k(t)$ has the same statistics as n(t), and where the assumption is maintained that the total available subcarrier power is shared equally among the K data channels. The data signal $s_k(t)$ fully bi-phase modulates the subcarrier reference signal $\cos \omega_k t$. In order to obtain a simplified description of the trade-off between telemetry performance and Doppler tracking capability, we shall make the realistic assumption that bit synchronization is obtained directly from the data channel as opposed to placing a certain amount of residual power in the subcarrier reference, and in addition that the synchronization jitter is negligible with respect to phase jitter in the carrier tracking loop.

We further take the point of view that the squaring loop, Costas loop, or other method which is employed to obtain coherent phase information is functioning with negligible jitter with respect to that in the carrier loop.

This is realistic, since the noise bandwidth of the subcarrier tracking loop can often be made 10^{-3} to 10^{-6} times narrower than that of the carrier tracking loop.

Therefore, the subcarrier telemetry demodulation will be assumed to perform like a perfectly coherent system which has perfect synchronization information. The probability of a bit error for this is given by (see, e.g., [11])

$$P_e = \mathrm{erfc}(\sqrt{2R_s}) \tag{17}$$

where erfc is the complementary error function, defined by

$$\mathrm{erfc}(\rho) \triangleq \int_{\rho}^{\infty} \frac{\exp(-\frac{1}{2}u^2)}{\sqrt{2\pi}}\, du$$

and R_s is signal-to-noise ratio per bit which is defined as the ratio of signal energy per bit to noise spectral density:

$$R_s \triangleq \frac{P_s T_b}{N_0} \tag{18}$$

In (18) the bit time, T_b, has been assumed the same for all digital subcarrier channels. In terms of the parameters which describe the carrier tracking loop, R_s can be expressed as

$$R_s = \beta_r \frac{P_s}{P} \frac{1}{\delta_r} \tag{19}$$

where

$$\delta_r \triangleq \frac{1}{T_b B_{L_r}}$$

is defined as the reciprocal of the product of the bit time and the noise bandwidth of the carrier loop.

With this description of the digital subcarrier demodulation performance, we now consider Doppler measurement capability.

<u>Doppler Measurement</u>. - The signal from which Doppler frequency information is to be obtained, is the carrier VCO output, r(t), which has the representation

$$r(t) = \sqrt{2}\cos(\omega_c t + \int^{t} \omega_d(\tau)d\tau + \hat{\theta}(t))$$

The carrier frequency is removed by mixing with a noncoherent carrier reference, $\sqrt{2}\,\sin(\omega_c t)$, in which case the output of the Doppler mixer becomes (neglecting the double frequency term)

$$c(t) = \cos \Delta(t) \tag{20}$$

where

$$\Delta(t) = \int^{t} \omega_d(\tau)d\tau + \phi_r(t) + \theta_1 \tag{21}$$

The signal c(t) is the input waveform to the Doppler measurement device.

Whatever scheme is employed, we shall assume it ideally obtains the instantaneous frequency of c(t), namely $\dot{\Delta}(t)$. Denoting this signal by d(t), we have

$$d(t) \triangleq \dot{\Delta}(t) = \omega_d(t) + \dot{\phi}_r(t) \tag{22}$$

The measurement disturbances in (22) are seen to be $\dot{\phi}_r(t)$. In this initial approach to provide a trade-off between Doppler and error rate, we shall assume $\omega_d(t)$ is sufficiently slowly varying so that it can be assumed constant. In general, d(t) would be filtered to provide the best estimate, $\hat{\omega}_d(t)$, of $\omega_d(t)$ from d(t).

The fundamental task in acquiring knowledge of Doppler measurement capability is in obtaining necessary statistical information about $\dot{\phi}_r(t)$, namely the first and second moments. The difficulty centers around the fact that, in order to obtain a tractable mathematical model of a PLL, the assumption is generally made that the additive noise is white. In most choices of loop filters, this leads to the conclusion that $\dot{\phi}_r(t)$ is also white. This problem can be partially overcome with the following approach. Let us model the total phase error $\phi_r(t)$ as

$$\phi_r(t) = \int^{t} 2\pi\, N(\tau)d\tau + \phi_m(t) \tag{23}$$

where $N(\tau)$ is a stochastic process consisting of a sequence of pulses which are each of unit area and of short time duration. A pulse is positive whenever the loop slips a cycle to the right and negative whenever the loop slips to the left. The process $\phi_m(t)$ is the mod 2π phase process [8]. When $\phi_r(t)$ is modeled as in (23), the cycle-slipping part and the phase jitter part between cycle slips are additive. At low signal-to-noise ratios the predominate contribution to overall phase error will be due to cycle slips, while at large signal-to-noise ratios, cycle-slips will occur very rarely, and the predominate variation in $\phi_r(t)$ will be due to the mod 2π phase jitter $\phi_m(t)$.

Empirical data taken on PLL [12] lead one to believe that the cycle slipping events in disjoint intervals are statistically independent. This, plus the fact that N(t) is a jump process, is sufficient to conclude [13] that it is a generalized Poisson process.

The cumulative number of cycle slips, N(t), actually can be expressed as

$$N(t) = N_+(t) - N_-(t) \tag{24}$$

where $N_+(t)$ consists of the positive pulses in N(t) and thus represents cycle slips in the positive $\phi_r(t)$ direction, and similarly for $N_-(t)$ the negative $\phi_r(t)$ direction. If there is no detuning in the carrier VCO, then the expected number of cycle slips to the right will equal that to the left. Hence

$$E(\dot{N}(t)) = 0$$

The individual processes, $N_+(t)$ and $N_-(t)$ are generalized Poisson processes, and will be assumed to be statistically independent. Therefore, since the steps are always taken to be of unit size,

$$\sigma^2_{N_+} = \sigma^2_{N_-} = E(N_+) = E(N_-)$$

which corresponds to the expected number of steps to the right or left, or equivalently to the expected number of cycles slipped to the right or left respectively. Combining

$$\sigma^2_N = \sigma^2_{N_+} + \sigma^2_{N_-} = E(N_+) + E(N_-) \tag{25}$$

The expected number of cycle slips to the right or left per unit time has been shown to be [8]

$$E(N_+) = E(N_-) = \frac{B_{L_r}}{\pi^2 \alpha_r I_0^2(\alpha_r)} \tag{26}$$

This is the exact result for first order PLL's and an approximation for higher order loops. Since

$$\dot{\phi}_r(t) = 2\pi\, N(t) + \dot{\phi}_m(t) \tag{27}$$

we can therefore write

$$\sigma^2_{\dot{\phi}_r} = 4\pi^2\, \sigma^2_N + \sigma^2_{\dot{\phi}_m} = \frac{8B_{L_r}}{\alpha_r I_0^2(\alpha_r)} + \sigma^2_{\dot{\phi}_m} \tag{28}$$

where the additional assumption has been made that N(t) is statistically independent of $\dot{\phi}_m(t)$.

The remaining task is to determine $\sigma^2_{\dot{\phi}_m}$. As previously indicated, models of PLL's have exclusively assumed the additive disturbance to be white and Gaussian, with the conclusion that $\dot{\phi}$ is also white. Since Doppler measurement is inherently concerned with cycle slipping, all of the various linear theories of PLL's breakdown when attempting to determine Doppler measurement capability. One way, however, in which realistic statistical information can be obtained about $\dot{\phi}_m(t)$ is to assume the loop filter of the PLL is of the form

$$F(s) = \frac{1}{\tau s + 1}$$

With this filter, the Fokker-Planck equation for the p.d.f. of the Markov vector process $(\phi(t), \dot{\phi}(t))$ can be solved [14] in the steady state, from which the variance $\dot{\phi}$, $\sigma^2_{\dot{\phi}}$, is given by

$$\sigma^2_{\dot{\phi}} = \frac{4B_L}{\alpha_r} \qquad (29)$$

At high SNR, the principle disturbance in $\dot{\phi}_r$ is due to $\dot{\phi}_m$, since cycle-slips occur rarely at high SNR. Therefore, at high SNR

$$\sigma^2_{\dot{\phi}_m} \approx \sigma^2_{\dot{\phi}} = \frac{4B_{L_r}}{\alpha_r} \qquad (30)$$

This representation has an additional assumption imbedded; namely, this result is approximately independent of the structure of the loop filter. These results, namely the first term in (28) for the cycle slipping, and (30) for phase jitter are plotted in Figure 3. Cycle-slipping decreases exponentially, while phase jitter between cycle-slips decreases only as $(\alpha_r)^{-1}$. Thus, we can qualitatively determine when $\sigma^2_{\dot{\phi}_m}$ becomes the predominant disturbance in Doppler measurement. The cumulative effect of cycle-slipping and phase-jitter is also shown in Fig. 3 as the dotted line.

<u>Doppler Error Versus Error Rate.</u> - With these results, the tradeoff between Doppler measurement capability and error rate can be established. For the case of one subcarrier and $\delta_r = 3$, the representation of probability of error versus Doppler measurement error, normalized by the r-f loop bandwidth, is displayed in Figure 4. In the vicinity of the minimum value of P_E, for a given β_r, the Doppler variance can be decreased somewhat by varying p, without detracting appreciably from the error rate. As indicated earlier, however, the choice of p to minimize error rate is quite dependent on SNR.

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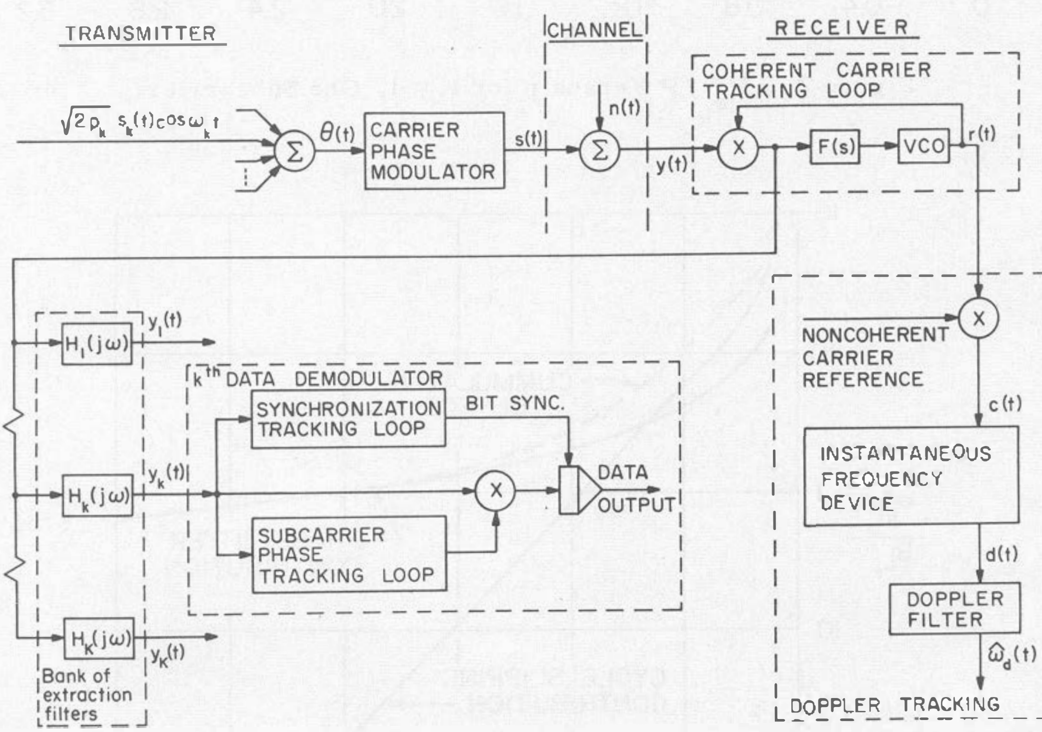

Figure 1. General System Configuration.

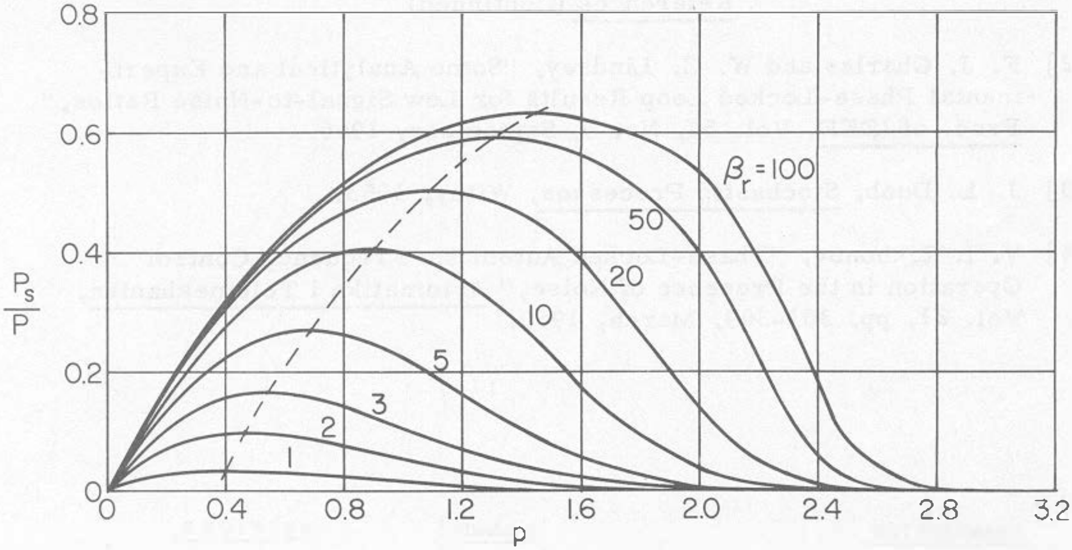

Figure 2. P_{s_k}/P Versus p for K = 1, One Subcarrier.

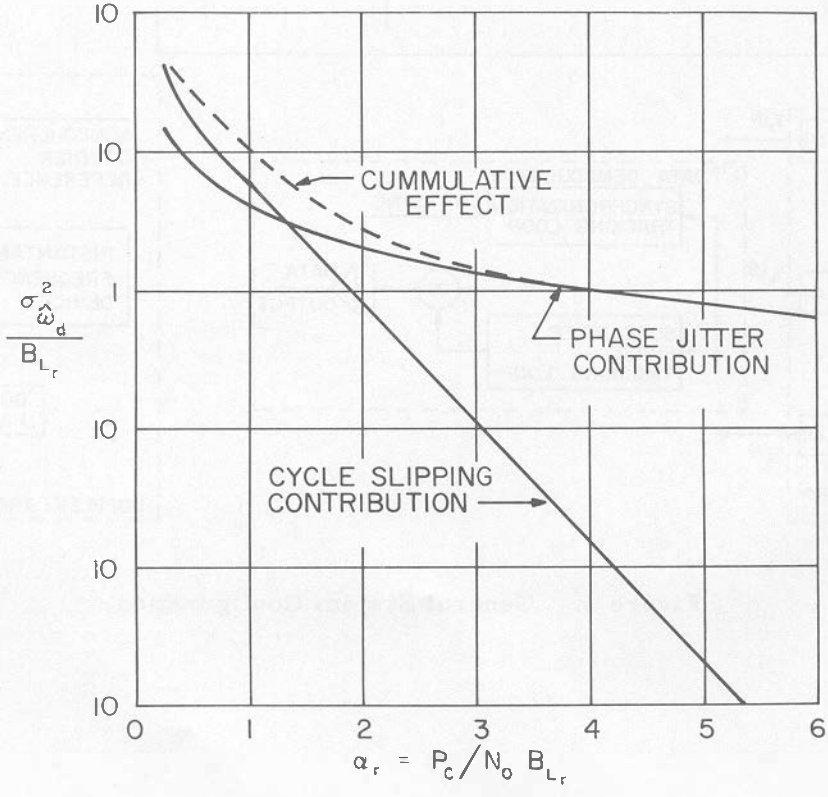

Figure 3. Doppler Error Versus Carrier SNR.

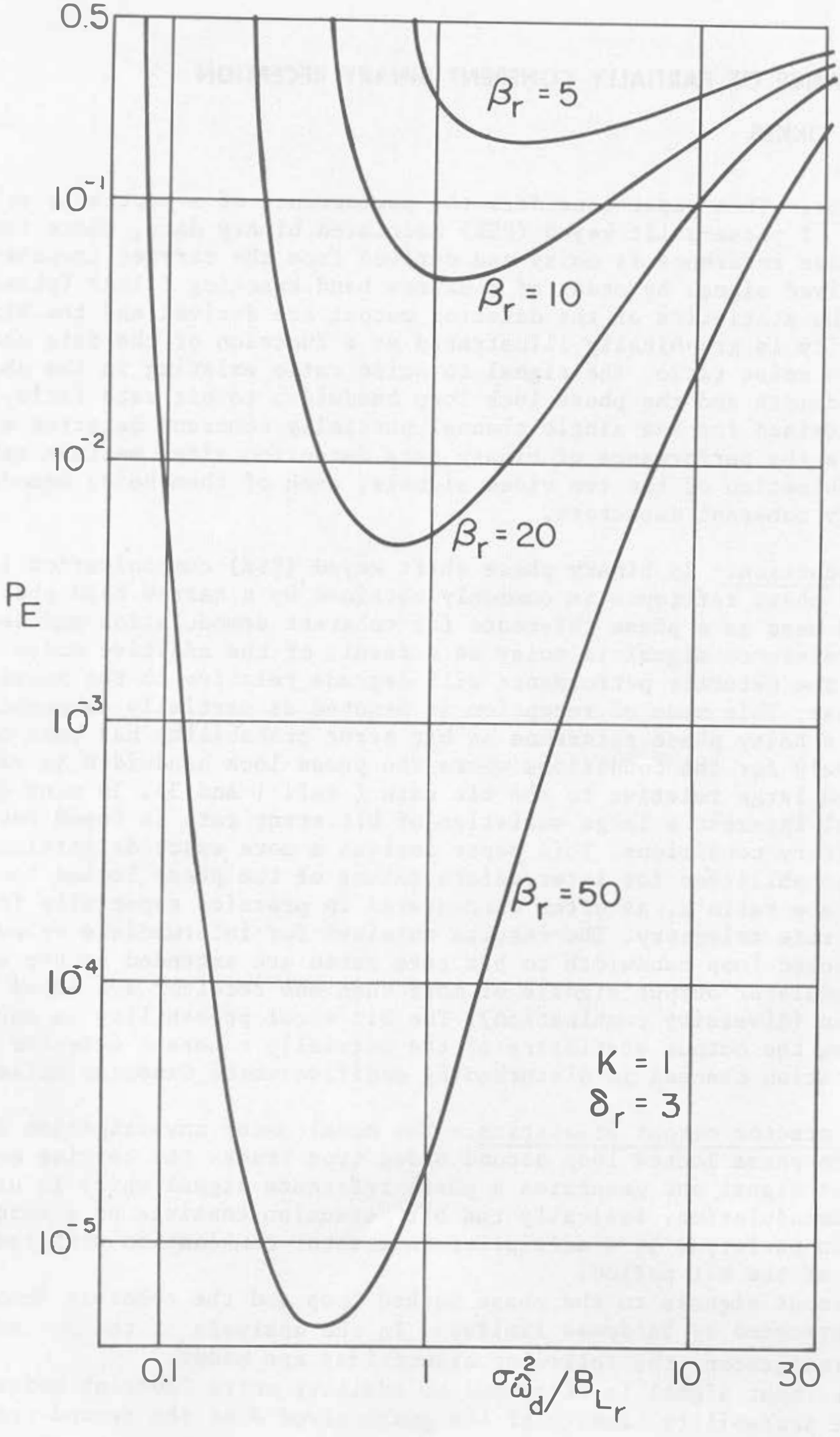

Figure 4. Error Rate Versus Doppler Error For Various SNR.