FILTERING EFFECTS IN A SPREAD-SPECTRUM COMMUNICATION SYSTEM

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Summary. -Binary antipodal direct-sequence biphase modulation is employed (for the purpose of interference reduction) over a channel disturbed by white noise and an "external" coherent sinusoidal interference. Before these are added, the signal suffers distortion in the form of linear filtering whose effects are to be determined. The receiver is a coherent "rematched filter" (matched to the distorted signal).

The mean and variance of the detection variable are expressed as an output SNR (signal to noise ratio). The variance is the sum of three components: due to noise, external interference, and self interference. Concise formulas for the first two contributions are developed. The third is approximated and found to be quite small in many cases of interest.

Results are applied in the case in which the filter has a bandpass characteristic and external interference is dominant. With fixed signal power entering the filter, there is an optimal cbip rate above which filter distortion effects increase faster than process gain; the optimal cbip rate is approximately equal to the filter noise bandwidth B (Hertz). For an ideal bandpass filter and a single pole bandpass filter, the optimal chip rates are 1.0B and 0.95B, respectively.

System Considered. - The system under consideration is modeled by the block diagram of figure 1. Attention is restricted to cases of narrowband signals, and complex-envelope formulation[1] is employed, according to the convention

$$\text{(real signal)} = 2\,\mathrm{Re}\left[\,\text{(complex envelope)}\; e^{j\omega_c t}\right]$$

where ω_c is the carrier frequency. Transmitted signal modulation is binary-antipodal direct-sequence biphase, which can be written

$$s(t) = \sqrt{\frac{P}{2}}\; m(t) \sum_{k=-\infty}^{\infty} a_k\, \phi_k(t) \qquad (1)$$

where P is the signal power, m(t) is the information modulation, a function which is either +1 or -1 for the duration of each bit, $\{a_k\}$ is a set of statistically independent, binary random variables, taking the values ±1 equiprobably, and cbip function ϕ_k is given by

$$\phi_k(t) = \begin{cases} 1 & , \text{ for } (k-1)\,\tau \leq t < k\,\tau \\ 0 & , \text{ otherwise} \end{cases}$$

in terms of the chip time τ. We focus attention on a single-bit transmission $\pm s'(t)$ given by

$$s'(t) = \sqrt{\frac{P}{2}} \sum_{k=1}^{K} a_k\, \phi_k(t) \qquad (2)$$

where K is the number of chips per bit, and the information transmitted determines whether plus or minus applies.

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As illustrated in figure 1, the signal passes through a filter of transfer function $H(\omega)$ in terms of frequency ω (relative to the carrier frequency ω_c). Realizability is not required. The filtered version of $s(t)$ is denoted $\tilde{s}(t)$, and the filtered version of $s'(t)$ is denoted $\tilde{s}'(t)$.

A coherent sinusoidal interference $i(t)$ of power I

$$i(t) = \sqrt{I/2} \tag{3}$$

and a white noise $n(t)$ are added to the distorted signal emerging from the filter. The noise power per Hertz is denoted N_o.

A coherent "rematched filter" receiver is employed. For each bit transmission, this receiver forms a detection variable λ by correlating the received signal with the reference signal $\tilde{s}'(t)$.

$$\lambda = 2\,\mathrm{Re} \int_{-\infty}^{\infty} [\tilde{s}(t) + i(t) + n(t)]\,[\tilde{s}'(t)]^* \, dt \tag{4}$$

The output decision is based on the sign of λ.

Measure of Performance. - The measure of performance considered is the SNR (signal-to-noise ratio) defined by

$$\mathrm{SNR} = \frac{(\text{mean of } \lambda)^2}{2\,(\text{variance of } \lambda)}$$

when $+s'(t)$ is transmitted. It can readily be verified that the SNR so defined would not change if $-s'(t)$ were transmitted.

SNR Calculations. -From equation (4) we write λ as the sum of three components

$$\lambda = \lambda_s + \lambda_i + \lambda_n$$

where

$$\lambda_s = 2\,\mathrm{Re} \int_{\infty}^{\infty} \tilde{s}(t)\,[\tilde{s}'(t)]^* \, dt$$

$$\lambda_i = 2\,\mathrm{Re} \int_{-\infty}^{\infty} i(t)\,[\tilde{s}'(t)]^* \, dt$$

$$\lambda_n = 2\,\mathrm{Re} \int_{-\infty}^{\infty} n(t)\,[\tilde{s}'(t)]^* \, dt$$

By substitution of eq's. (2) and (3), we readily determine that λ_i and λ_n both have zero means. Thus the mean of λ equals the mean of λ_s. From eq's. (1) and (2)

$$\lambda_s = P \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{K} a_\ell a_k \,\mathrm{Re} \int_{-\infty}^{\infty} \tilde{\phi}_\ell(t)\, \tilde{\phi}_k^*(t)\, dt$$

where $\tilde{\phi}_k$ is the filtered version of ϕ_k. We separate λ_s into two terms $\overline{\lambda}$ and $\Delta\lambda$ depending on whether $k = \ell$ or $k \neq \ell$ respectively; that is

$$\overline{\lambda} = P \sum_{k=1}^{K} \int_{-\infty}^{\infty} |\tilde{\phi}_k|^2 \, dt \tag{5}$$

$$\Delta\lambda = P \sum_{k=1}^{K} \sum_{\substack{\ell=-\infty \\ \ell \neq k}}^{\infty} a_\ell a_k \, \mathrm{Re} \int_{-\infty}^{\infty} \tilde{\phi}_\ell(t) \, \tilde{\phi}_k^*(t) \, dt$$

We see by inspection that $\overline{\lambda}$ is deterministic and that $\Delta\lambda$ has zero mean. Therefore $\overline{\lambda}$ is the mean of λ.

The variance σ^2 of λ is, by definition,

$$\sigma^2 = \overline{(\Delta\lambda + \lambda_i + \lambda_n)^2} \tag{6}$$

where the bar denotes averaging. We can readily see that

$$\sigma^2 = \sigma_s^2 + \sigma_i^2 + \sigma_n^2$$

where σ_s^2, σ_i^2, and σ_n^2 are the respective variances of $\Delta\lambda$, λ_i and λ_n; that is, the three cross terms resulting from eq. (6) are all zero. Therefore, we can write

$$\mathrm{SNR} = \frac{1}{\dfrac{1}{\mathrm{SNR}_s} + \dfrac{1}{\mathrm{SNR}_i} + \dfrac{1}{\mathrm{SNR}_n}}$$

where

$$\mathrm{SNR}_s = \frac{(\overline{\lambda})^2}{2\sigma_s^2}$$

$$\mathrm{SNR}_i = \frac{(\overline{\lambda})^2}{2\sigma_i^2}$$

$$\mathrm{SNR}_n = \frac{(\overline{\lambda})^2}{2\sigma_n^2}$$

are separate signal-to-noise ratios accounting for self interference, external interference and noise.

We now evaluate $\overline{\lambda}$. Applying Parseval's rule to the integral in eq. (5), we see that all terms in the sum are equal,

$$\overline{\lambda} = PK \int_{-\infty}^{\infty} |\tilde{\Phi}_1(\omega)|^2 \frac{d\omega}{2\pi} = PK \int_{-\infty}^{\infty} |\Phi_1(\omega)|^2 |H(\omega)|^2 \frac{d\omega}{2\pi}$$

where $\tilde{\Phi}_k$ and Φ_k are the Fourier transforms of $\tilde{\phi}_k$ and ϕ_k respectively. It is now recognized that $\bar{\lambda}$ can be written

$$\bar{\lambda} = \lambda_o \ L_p$$

where λ_o is the value of λ_s obtained with no filter ($H \equiv 1$) and L_p is the "power loss" due to filtering*

$$L_p = \frac{\int_{-\infty}^{\infty} |\Phi_1(\omega)|^2 |H(\omega)|^2 \frac{d\omega}{2\pi}}{\int_{-\infty}^{\infty} |\Phi_1(\omega)|^2 \frac{d\omega}{2\pi}} = \int_{-\infty}^{\infty} S_s(\omega) |H(\omega)|^2 \frac{d\omega}{2\pi} \quad (7)$$

in terms of the normalized signal power density spectrum

$$S_s(\omega) = \tau \left[\frac{\sin(\tau\omega/2)}{(\tau\omega/2)} \right]^2$$

which has been normalized such that $L_p = 1$ when $H \equiv 1$. The last equality in eq. (7) follows from the fact that the chip energy density spectrum $|\Phi_1(\omega)|^2$ has the same shape as the long term power density spectrum of $s(t)$. Furthermore, we evaluate λ_o as

$$\lambda_o = 2 \operatorname{Re} \int_{-\infty}^{\infty} | s'(t) |^2 dt = PK\tau$$

which equals the transmitted energy per bit E. Thus,

$$\bar{\lambda} = EL_p$$

(a) Self Interference. - Eq. (8) is rewritten as

$$\Delta\lambda = P \sum_{k=1}^{K} \sum_{\substack{\ell=-\infty \\ \ell \neq k}}^{\infty} a_k a_\ell \alpha_{k-\ell}$$

where

$$\alpha_{k-\ell} = \operatorname{Re} \int_{-\infty}^{\infty} \tilde{\phi}_\ell(t) \ \tilde{\phi}_k^*(t) \, dt$$

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*The name "power loss" is suggested by the case of passive filtering obeying $|H(\omega)| \leq 1$ at all ω, which implies $L_p \leq 1$. In other cases, L_p may exceed 1, but the results derived here still apply.

We note the symmetry

$$\alpha_{-k} = \alpha_{k}, \text{ for all } k$$

In this notation, the variance σ_s^2 becomes

$$\sigma_s^2 = P^2 \sum_{k=1}^{K} \sum_{k'=1}^{K} \sum_{\substack{\ell=-\infty \\ \ell \neq k}}^{\infty} \sum_{\substack{\ell'=-\infty \\ \ell' = k'}}^{\infty} \overline{a_\ell a_{\ell'} a_k a_{k'}} \; \alpha_{k-\ell} \, \alpha_{k'-\ell'}$$

Omitting terms which are zero and employing symmetry we obtain

$$\sigma_s^2 = 4 P^2 K \sum_{k=1}^{\infty} \alpha_k^2$$

A general evaluation of SNR_s follows directly

$$SNR_s = \frac{K \tau^2 L_p^2}{8 \sum_{k=1}^{\infty} \alpha_k^2} \tag{8}$$

While this is a simple formula, we will go on to develop a far simpler approximation to it, applicable to cases in which α_1 dominates all of the other interchip interference factors, α_2, α_3, ..., a condition to be expected when filtering is not severe.

Note that the quantity A defined by

$$A = \sum_{k=-\infty}^{\infty} \alpha_k$$

has the following concise evaluation: By definition

$$A = \sum_{k=-\infty}^{\infty} \operatorname{Re} \int_{-\infty}^{\infty} \tilde{\phi}_1(t) \, \tilde{\phi}^*_{k+1}(t)\, dt = \operatorname{Re} \int_{-\infty}^{\infty} \tilde{\phi}_1(t) \sum_{k=-\infty}^{\infty} [\phi_{k+1}(t) * h(t)]^* \, dt$$

Using linearily of the filter

$$\sum_{k=-\infty}^{\infty} [\phi_{k+1}(t) * h(t)] = [\sum_{k=-\infty}^{\infty} \phi_{k+1}(t)] * h(t) = 1 * h(t) = H(0)$$

so this factor can be removed from the integral. The remaining integral is

$$\int_{-\infty}^{\infty} \tilde{\phi}_1(t)\, dt = \tilde{\Phi}_1(0) = \Phi_1(0)\, H(0) = \tau H(0)$$

Thus we obtain

$$A = \tau \left| H(0) \right|^2 \tag{9}$$

If $\alpha_2, \alpha_3, \ldots$ are neglected we obtain the approximations

$$A = \alpha_o + 2\,\alpha_1$$

$$\sum_{k=1}^{\infty} \alpha_k^2 = \alpha_1^{\,2} = \left(\frac{A - \alpha_o}{2} \right)^2 \tag{10}$$

where α_o is evaluated as simply

$$\alpha_o = \int_{-\infty}^{\infty} \left| \tilde{\Phi}_1(t) \right|^2 dt = \frac{\overline{\lambda}}{KP} = \tau L_p \tag{11}$$

When eq's. (9), (10), and (11) are substituted in eq. (8), the result is the desired approximation to SNR_s.

$$SNR_s = \frac{K L_p^{\,2}}{2 \left[\left| H(0) \right|^2 - L_p \right]^2} \tag{12}$$

(b) External Interference. - The component λ_i can be rewritten as

$$\lambda_i = \sqrt{PI} \sum_{k=1}^{K} a_k \, \beta_k$$

where β_k is the deterministic quantity

$$\beta_k = \mathrm{Re} \int_{-\infty}^{\infty} \tilde{\phi}_k^{\,*}(t)\, dt = \tau \, \mathrm{Re}\, H(0)$$

which is the same for all k. Thus the variance of λ_i is

$$\sigma_i^2 = PIK\,\tau^2\, \mathrm{Re}^2\, H(0)$$

from which we readily obtain SNR_i.

$$SNR_i = \left[\frac{KP}{2I} \right] \left[\frac{L_p}{\mathrm{Re}\;\; H(0)} \right]^2 \tag{13}$$

The first factor $KP/2I$ may be identified as the SNR_i that would be achieved if there were no filtering. The second factor is thus a loss or gain due to the filter H.

(c) Noise. - The noise component λ_n can be rewritten as

$$\lambda_n = \sqrt{2P} \sum_{k=1}^{K} a_k \, \gamma_k$$

where γ_k is a random quantity defined by

$$\gamma_k = \mathrm{Re} \int_{-\infty}^{\infty} n(t) \, \tilde{\phi}_k^* (t) \, dt$$

The noise variance becomes

$$\sigma_n^2 = 2P \sum_{k=1}^{K} \overline{\gamma_k^2}$$

Routine evaluation of $\overline{\gamma_k^2}$ leads to

$$\gamma_k^2 = \frac{N_o EL_p}{4PK}$$

which is the same for all k. Thus we have solved for σ_n^2 and hence SNR_n.

$$SNR_n = \frac{EL_p}{N_o} \tag{14}$$

<u>Discussion of Problem and Results.</u> - In the problem considered here, the filter H can represent for example a gradual non-uniformity of channel frequency response, a predictable multipath characteristic, or simply an electrical bandpass filter. We note that our results include the effects of inter-chip and inter-bit interferences which result from a blurring of the signal details in passing through the filter.

In many cases, the probability distribution of λ is approximately Gaussian, due to a central limit theorem phenomenon. If λ were assumed to be exactly Gaussian, then its SNR as defined here would completely determine error probability.

When I = 0, one would conclude by inspection that EL_p/N_o is an upper bound to SNR., based on the fact that $\ _p$ is the physical power loss in filtering and that inter-bit interference could presumably not improve SNR. Our results confirm this conclusion and indicate the amount of degradation beyond EL_p/N_o, expressed in the form of the self interference SNR_s.

The simplicity of the resulting expressions, eq's. (12), (13), and (14), is remarkable. Filtering effects are expressed solely in terms of L_p and H(0); and since this latter parameter may be considered a scale factor, the entire dependence on the <u>shape</u> of the filter transfer function is given by the single figure $\ _p$. Furthermore, SNR_s, SNR_i, SNR_n, and SNR are all monotonic functions of L_p (for all $L_p \le |H(0)|^2$), and as a result L_p is a particularly useful measure of signal-filter interaction.*

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* The figure $\ _p$ has been used as a measure of bandwidth in signal design, for example in reference 2.

Evidently self interference, according to our approximation, is so small that it can often be neglected. For example, the ratio

$$\frac{SNR_i}{SNR_s} = \frac{P}{I}\left[\frac{|H(0)|^2 - L_p}{\operatorname{Re} H(0)}\right]^2$$

is much less than unity in many cases of practical interest. Appearance of the factor K in eq. (12) indicates that self interference, like external interference, is reduced by the "process gain" of the spread-spectrum modulation.

Application. - As an application of these results, we consider the special case in which H is a passive bandpass filter satisfying

$$H(0) = 1$$

$$|H(\omega)| \le 1, \text{ for all } \omega$$

Under these conditions, eq's. (12) and (13) become simply

$$SNR_s = \frac{KL_p^2}{2(1 - L_p)^2} \qquad (15)$$

$$SNR_i = \frac{KPL_p^2}{2I} \qquad (16)$$

Furthermore, we consider cases dominated by external interference, and restrict attention to SNR_i.

Consider the effect of chip rate $1/\tau$ on eq. (16). At low chip rate $L_p \approx 1$ so that

$$SNR_i \approx \left(\frac{E}{2I}\right)\frac{1}{\tau}, \quad \text{weak filtering} \qquad (17)$$

Performance increases in proportion to chip rate, to be expected in a spead-spectrum system. At the other extreme, high chip rate, only the center of the signal spectrum is passed by the filter.

$$L_p \approx \frac{S_s(0)}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \tau B$$

where B is the filter noise bandwidth defined by

$$B = \int_{-\infty}^{\infty} |H(\omega)|^2 \frac{d\omega}{2\pi}$$

thus

$$SNR_i \approx \left(\frac{EB^2}{2I}\right)\tau, \quad \text{severe filtering} \qquad (18)$$

Here performance decreases inversely with chip rate. Thus if the transmitted power (entering the filter) is fixed, there is an optimal chip rate between these two extremes which maximizes SNRi. The optimum point occurs near where the asymptotes (17) and (18) intersect, namely where

$$\frac{1}{\tau} = B$$

that is, a chip rate equal to the filter noise bandwidth in Hertz. The precise location of the optimum depends on the detailed shape of the filter transfer function.[†]

Figure 2 is a graph of SNR_i with fixed power P for two specific filters: an ideal bandpass filter, defined by

$$H(\omega) = \begin{cases} 1 & , \text{ for } |\omega| \le \pi B \\ 0 & , \text{ otherwise} \end{cases}$$

and a single tuned bandpass filter obeying[*]

$$|H(\omega)|^2 = \frac{4B^2}{4B^2 + \omega^2}$$

The optimal chip rate to be employed through the ideal bandpass filter is 1.0B and the filtering loss at this point is $L_p^2 = -2.22$ db. For a single tuned filter, there is a much broader maximum with the optimal chip rate falling at 0.95 B, and a filtering loss at this point of $L_p^2 = -4.7$db.

If the power leaving the filter, PL_p, is fixed, an appropriate constraint in some situations, the performance at high chip rate follows the asymptote

$$SNR_i \approx \frac{EL_p B}{2I} \qquad (19)$$

which is independent of chip rate. Thus there is generally no optimal chip rate, but rather SNR_i approaches a constant value at high chip rate. The low chip rate performance is still given by eq. (17). The two asymptotes (17) and (19) intersect at

$$\frac{1}{\tau} = B$$

so that there is relatively little to be gained by increasing chip rate beyond B. Figure 3 is a graph of SNR_i under these conditions for the same two specific filters.

References

1. C. W. Helstrom, Statistical Theory of Signal Detection, Pergamon Press, Oxford, England, 1960, pp. 13-17, 50-55.

2. A. H. Nuttall and W. F. Floyd, "Minimum Bandwidth of M Signals with Specified Code or Correlation Matrix," IEEE Trans. on Information Theory, Vol. IT-15, pp. 38-48, January, 1969.

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† L_p values were obtained by numerical integration of eq. (7).

* The 3-db bandwidth of this filter is $2B/\pi$.

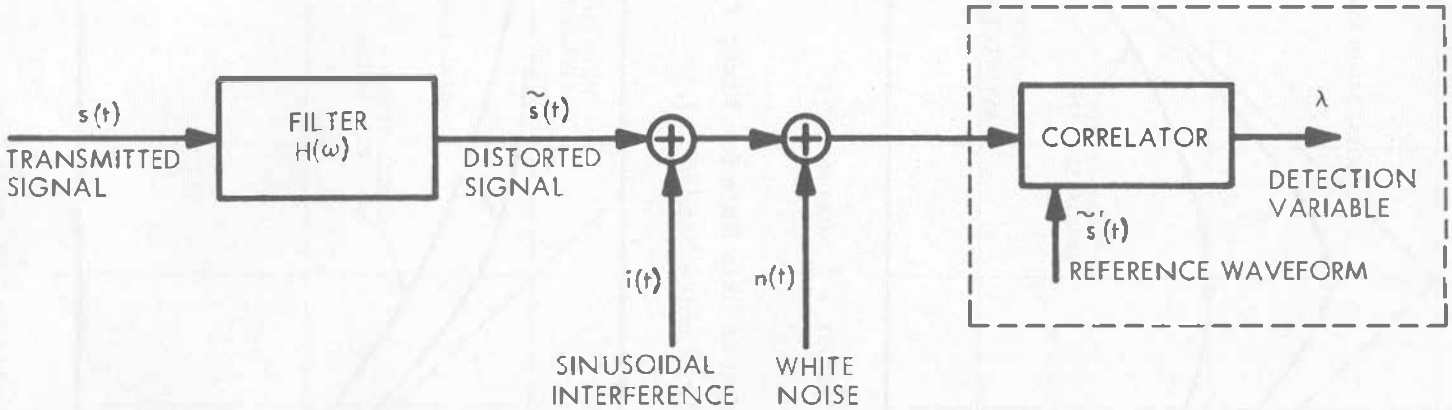

FIGURE 1. Model of System Considered

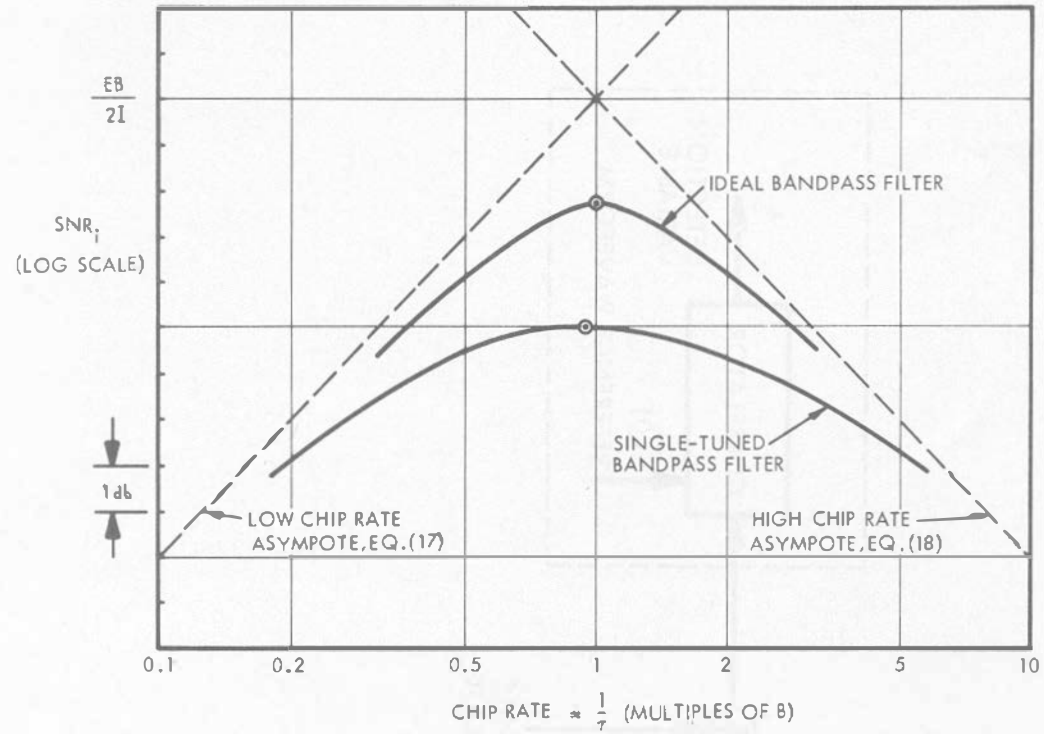

FIGURE 2. SNR$_i$ as a function of Chip Rate for Fixed Transmitted Power P (B = filter noise bandwidth).

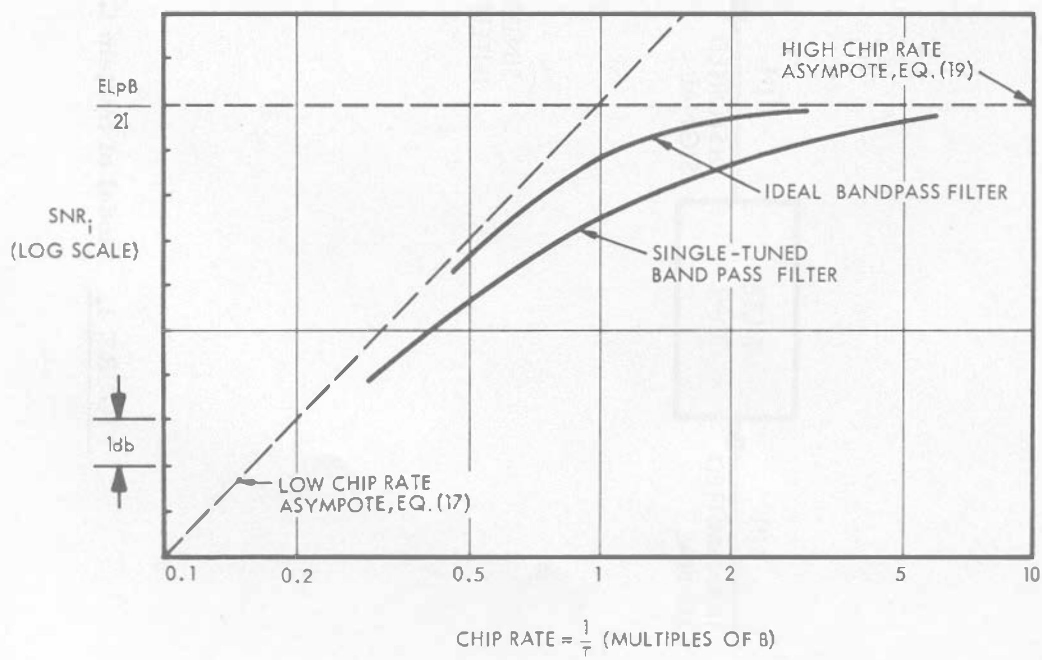

FIGURE 3. SNR$_i$ as a function of Chip Rate for Fixed Power at Filter output PL_p (B = filter noise bandwidth).