DIGITAL TRANSMISSION OVER THE RC CHANNEL

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Summary This work evaluates the performance of digital transmission links over a lumped RC channel when one of three static equalization schemes is utilized:

1. Use of a Sidelobe Transmitter
2. Transversal Filter at the Receiver
3. Frequency Compensation of the Transfer Characteristics to form a rectangular Nyquist Channel.

The effect on the probability of error of thermal noise and intersymbol interference is considered. In each of the equalization schemes, the intersymbol interference is reduced to zero, and the waveshape at the transmitter which maximizes the SNR at the receiver decision instant is transmitted. It is found that the performance of the Sidelobe Transmitter and the Transversal Filter Receiver are equal theoretically and that Frequency Compensation provides the poorest results. The comparisons carried out in this work have utilized the RC channel as an example. Detailed generalization of the comparative performance as a function of the impulse response of the channel utilized would be desirable but cannot be made readily. An analysis of the coaxial line is planned for the future. The eigenfunctions and corresponding eigenvalues of the system are required in order to evaluate the optimal performance of the Sidelobe Transmitter and Transversal Filter Receiver. For arbitrary impulse responses, these eigenfunctions are usually complex to obtain. However, one is able to state in general that since the Sidelobe Transmitter or Transversal Filter combined with transmission of the optimum waveshape represent the best linear filter, their performance will not be inferior to the Frequency Compensated rectangular Nyquist channel. This follows since frequency compensation networks fall into the class of linear filters.

1. Sidelobe Transmitter The system configuration within which the Sidelobe Transmitter is employed is shown in Fig. 1. The input to the Sidelobe Transmitter consists of the digitally encoded information to be conveyed to the receiver. The output of the Sidelobe Transmitter is sent through the RC channel of impulse response h(t), and the receiver matched filter with response k(t) before a decision is made at the sampling instant. In order to reduce the intersymbol interference to zero at the sampling instant belonging to other bits, the Sidelobe Transmitter signal for each bit consists of a main lobe and sidelobes. The magnitude of the sidelobes is chosen to cancel the tail of the main lobe.

It can be shown that for a given transmitter energy per bit the optimum shape of the sidelobes is identical to that of the main lobe, $s_1(t)$. Thus, the transmitted signal for each bit s(t) consists of:

$$s(t) = s_1(t) + \sum_i \varepsilon_i s_1(t-t_i) \tag{1}$$

The ϵ_i are chosen such that the matched filter output $p(t_j)$ at the sampling instants is:

$$p(t_j) = \int_{t_{j-1}}^{t_j} d\tau \left[s_1(\tau) + \sum_i \varepsilon_i s_1(\tau - t_i) \right] \ell(t_j - \tau) = 0 \quad \text{all } j \neq 1 \tag{2}$$

The matched filter output $p(t_1)$ is maximized subject to the constraint that the available transmitted energy per bit is fixed:

$$E = \int_{-\infty}^{\infty} |s(\tau)|^2 d\tau \tag{3}$$

The optimum waveshape $s_1(t)$ is obtained as a solution to the eigenfunction equation.

$$\lambda_k s_k(\tau) = \int_0^{t_1} s_k(v) Q(\tau, v) dv \tag{4}$$

where the kernel $Q(\tau, v)$ is obtained from the channel impulse response:

$$Q(\tau, v) = \int_0^{t_1} h(t-v) h(t-\tau) dt \tag{5}$$

The waveshape $s_1(\tau)$ corresponds to the solution to Equation (4) with the largest eigenvalue λ_1 For binary transmission, we send either + s(t) or - s(t).

$$s(t) = \pm \left[s_1(t) + \sum_i \varepsilon_i s_1(t-t_i) \right] \tag{6}$$

For m-ary transmission, we may send rn levels of s(t) with a waveshape in each interval of $s_1(t)$. Alternately we may transmit orthogonal signals with waveshapes $s_1(t)$, $s_2(t)$, $s_3(t)$ etc. These shapes correspond to the eigenvalues λ_1, λ_2 λ_3, etc. The choice of alternatives depends on the magnitude of the sub-optical eigenvalues relative to the optimal eigenvalue λ_1. From Equation (1), we find, since $s_1(t)$ is confined to one bit interval, that the transmitted energy per bit E is:

$$E = E_T \left[1 + \sum_i \varepsilon_i^2 \right] \tag{7}$$

where E_T is the transmitted energy in the "main lobe. "

$$E_T = \int_0^{t_1} \left| s_1(t) \right|^2 dt \tag{8}$$

Hence, the energy in the main lobe interval at the receiver is:

$$Eg = \lambda_1 E_T = \lambda_1 \frac{E}{1 + \sum_i \varepsilon_i^2} \tag{9}$$

where spill-over into the main lobe interval of sidelobe energy is either cancelled or is assumed to be negligibly small. When the system is subjected to additive noise of density η, the SNR at the decision instant is, since the intersymbol interference is zero:

$$SNR = \frac{Eg}{\eta} = \lambda_1 \frac{E}{\eta \left[1 + \sum_i \varepsilon_i^2 \right]} \tag{10}$$

The probability of error is:

$$Pe = \frac{1}{2} \operatorname{erfc} \sqrt{SNR} \tag{11}$$

The reduction λ_1 I is due to two causes: the attenuation of the medium and the dispersion of received energy outside the bit interval corresponding to the main lobe. The factor $\frac{1}{1 + \sum_i \varepsilon_i^2}$ is due to the reduction of intersymbol interference to zero.

The relationships cited above are now applied to the RC channel. We assume here that the sampling instant at the receiver is at time T after initiation of the main pulse. In Appendix A, we derive the delay T_1, past T at which a better decision may be made. The impulse response is:

$$h(t) = e^{-\alpha t} u(t) \tag{12}$$

which yields:

$$Q(\tau) = \frac{e^{-a|v-\tau|} - e^{-2aT} e^{a(v+\tau)}}{2a} \tag{13}$$

From Equation (4):

$$s_1(\tau) = K \sin \omega_1 (t-T) \tag{14}$$

where K is chosen to satisfy E_T of Equation (8). The characteristic frequency ω_1 is given by the first nonzero solution of

$$\omega_1 = - a \tan \omega_1 T \tag{15}$$

The eigenvalue λ_1 is given by:

$$\lambda_1 = \frac{1}{a^2 + \omega_1^2} \tag{16}$$

The magnitude of the sidelobes ϵ_i is given by:

$$\varepsilon_i = \varepsilon_1 \left[e^{-aT} + \varepsilon_1 \right]^{i-1} \tag{17}$$

where the first sidelobe ϵ_1 is:

$$\varepsilon_1 = \frac{-4 \sin^3 \omega_1 T}{2\omega_0 T - \sin 2\omega_1 T} \tag{18}$$

The magnitude of the first sidelobe ϵ_1 is plotted in Fig. 2. The relative magnitude of all sidelobes is shown in Fig. 3. The polarity of the sidelobes ϵ_i alternate, since the magnitude of ϵ_1 exceeds e^{-aT}. The sum of the squares of the sidelobes forms a convergent series since the term in the brackets of Equation (17) has an absolute value less than unity. The total energy E at the transmitter is therefore

$$E = E_T \left[1 + \sum_i \varepsilon_i^2 \right] = E_T \left[1 + \frac{\varepsilon_1^2}{1 - \left[e^{-aT} + \varepsilon_1 \right]^2} \right] \tag{19}$$

From Equation (9) and Equation (18), the usable energy Eg in the bit interval at the receiver is therefore:

$$Eg = \frac{E}{a^2} \frac{1}{\left[1 + \left(\frac{\omega_1}{a}\right)^2\right]\left[1 + \frac{\varepsilon_1^2}{1 - \left[e^{-aT} + \varepsilon_1\right]^2}\right]} \tag{20}$$

The probability of error for binary transmission though the RC channel with zero delay is:

$$Pe = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{\eta a^2} \frac{1}{\left[1 + \left(\frac{\omega_1}{a}\right)^2\right]\left[1 + \frac{\varepsilon_1^2}{1 - \left[e^{-aT} + \varepsilon_1\right]^2}\right]}} \tag{21}$$

2. Transversal Filter Utilization at the Receiver The configuration within which the Transversal Filter at the receiver is utilized is shown in Fig. 4. In the last decade the Transversal Filter has found wide use in digital links as a static as well as a dynamic equalizer. It has at times been used in conjunction with frequency compensation where the Transversal Filter is utilized to provide the adaptive portion of compensation. Here, we confine our analysis to the use of the Transversal Filter at the receiver as a static equalizer.

The operation of the Transversal Filter is such as to produce delayed and weighted replicas of the receiver input, which w en summed yield a signal waveform which cancels the "tails" of the receiver input at future sampling instants of the matched filter. We note that the Transversal Filter of Fig. 4 is a tapped analog delay line.

The impulse response of the system from transmitter output to receiver decision $\ell(t)$ is composed of the convolution of the channel response h(t) with the matched filter response k(t), plus delayed and weighted replicas thereof.

If we require that the tap weights be adjusted to produce zero intersymbol interference at the sampling instants t_j we must have:

$$p(t_j) = \int_{-\infty}^{\infty} s_1(\tau) \left[\ell(t_j - \tau) + \sum_i \varepsilon_i \ell(t_j - [\tau - t_i])\right] d\tau = 0$$

$$j \neq 1 \tag{22}$$

The above relationship is identical to:

$$p(t_j) = \int_{-\infty}^{\infty} \left[s_1(\tau) + \sum_i \varepsilon_i \, s_1(\tau - t_i) \right] \ell(t_j - \tau) \, d\tau = 0 \tag{23}$$

We note immediately that Equation (23) is identical to Equation (2). Since the optimum waveshape $s_1(\tau)$ must satisfy Equation (4) with the same kernel $Q(\tau, v)$ as in the case of the Sidelobe Transmitter, it follows that $s_1(\tau)$ and $\ell(\tau)$ of Equation (23) and Equation (2) are identical. Hence, the tap weights ϵ_i of the Transversal Filter are identical to those of the Sidelobe Transmitter. Since all the available energy E at the transmitter is utilized in the main lobe, the energy Eg in the main lobe at the receiver is:

$$Eg = \lambda_1 E \tag{24}$$

The noise at the receiver decision instant is augmented by delayed noise from previous time intervals ("noise aliasing"). The noise power is therefore increased by the sum of squares of the tap gains. At the matched filter output, the SNR is:

$$SNR = \frac{Eg}{\eta \left[1 + \sum_i \varepsilon_i^2 \right]} = \frac{\lambda_1 E}{\eta \left[1 + \sum_i \varepsilon_1^2 \right]} \tag{25}$$

Since Equation (25) is identical to Equation (10) the performance of the Transversal Filter at the receiver system and the Sidelobe Transmitter are identical. We note that no assumption regarding the explicit nature of the response of the transmission medium is needed to arrive at Equation (10) and Equation (25). We conclude, as one may have intuitively expected, that the theoretical equivalence of these systems holds for any arbitrary media.

3. Frequency Compensation When analog information confined to a prescribed bandwidth is transmitted through an arbitrary medium, the frequency characteristics of the channel are usually compensated by means of an analog equalizer network. A commonly used equalizer design produces an overall flat amplitude and linear phase characteristic in the frequency band of interest. This equalization technique is often referred to as Frequency Compensation, or Analog Equalization. When the transmitted information is in analog form and possesses a bandlimited spectrum, it is well-known that such an equalization scheme reduces the signal distortion to zero. When digital information is to be transmitted, the designer has often chosen to utilize Frequency Compensation as well. The block diagram of the system is shown in Fig. 5. Since the equalizer network, when cascaded with the channel produces a flat frequency characteristic within the Nyquist bandwidth associated with the bit rate T, such a system is capable of transmission of pulses up to a rate $\frac{1}{T}$ with zero intersymbol interference. A matched filter is utilized to extract optimally in the presence of noise the available signal energy in the bit interval. The optimum waveshape is sent by the transmitter. We consider the case of additive 11white" noise of density η.

For purposes of analysis, the receiver of Fig. 5 is equivalent to that of Fig. 6, since the equalizer filter is linear. We need therefore to design a matched filter in the presence of "colored noise." [1] The matched filter satisfies optimally:

$$\int_{-T/2}^{T/2} f(\tau)R_n(t-\tau) = s_{\frac{\pi}{T}}(t) \qquad (26)$$

where f(τ) is the correlating function of the matched filter. $R_n(\tau)$ is the autocorrelation function of the colored noise (normalized to unity spectral density coefficient). $s_{\frac{\pi}{T}}(t)$ is the transmitted signal through the bandlimited equalized channel.

If we denote:

$$f_T(\tau) = f(\tau) \qquad -\frac{T}{2} < \tau < \frac{T}{2}$$
$$= 0 \qquad \text{otherwise} \qquad (27)$$

then we must have:

$$F_T(\omega)N(\omega) = s_{\frac{\pi}{T}}(\omega) \qquad -\frac{\pi}{T} < \omega < \frac{\pi}{T} \qquad (28)$$

where N(ω) and $s_{\frac{\pi}{T}}(\omega)$ are the Fourier Transform of $R_n(\tau)$ and $s_{\frac{\pi}{T}}(\tau)$

The matched filter output due to the signal in the bit interval is:

$$p_T(t) = \int_{-T/2}^{T/2} s_{\frac{\pi}{T}}(\tau)f(t-\tau)d\tau = \int_{-\infty}^{\infty} s_{\frac{\pi}{T}}(\tau)f(t-\tau)d\tau \qquad (29)$$

With $P_T(\omega)$ the Fourier Transform of $P_T(t)$ we obtain:

$$P_T(\omega) = s_{\frac{\pi}{T}}(\omega)F_T(\omega) \qquad (30)$$

The output in any arbitrary future interval due to the waveform $s_{\frac{\pi}{T}}(\tau)$ is:

$$p_T(kT+t) = \int_{\left(k-\frac{1}{2}\right)T}^{\left(k+\frac{1}{2}\right)T} d\tau\, s_{\frac{\pi}{T}}(\tau)\, f(kT+t-\tau) = \int_{-\infty}^{\infty} s_{\frac{\pi}{T}}(\tau) f_T(kT+t-\tau)\, d\tau \tag{31}$$

The Fourier Spectrum at the matched filter output is therefore

$$P(\omega) = S_{\frac{\pi}{T}}(\omega) F_T(\omega) \left[1 + \sum_{k=1} e^{jk\omega T}\right] \tag{32}$$

It follows that $P(\omega)$ is bandlimited to $\frac{\pi}{T}$. We now utilize the constraint on $P(\omega)$ derived by Gibby & Smith, [2] maintain:

$$p(kT) = 0 \qquad k \neq 0 \tag{33}$$

From Equation (6) in the reference:*

$$\frac{1}{T} P(\omega) = p(0) \tag{34}$$

Substitution of Equation (32) yields:

$$\frac{1}{T} \frac{S_{\frac{\pi}{T}}(\omega) F_T(\omega)}{1-e^{j\omega T}} = p(0) \tag{35}$$

Hence, use of Equation (28) results in:

$$\frac{1}{T} \frac{\left|S_{\frac{\pi}{T}}(\omega)\right|^2}{N(\omega)(1-e^{j\omega T})} = p(0) \qquad -\frac{\pi}{T} < \omega < \frac{\pi}{T} \tag{36}$$

Thus,

$$p(0) \cdot \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} N(\omega)\, (1-e^{j\omega T})\, d\omega = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left|S_{\frac{\pi}{T}}(\omega)\right|^2 d\omega \tag{37}$$

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*Gibby & Smith use a definition of the inverse Fourier Transform without a multiplying factor of $1/2\pi$. This factor is restored here.

The right hand side of Equation (37) represents the portion of the transmitted energy within the Nyquist channel. When the transmitter pulses are confined to a time interval T, only a portion B^2 of the total transmitted energy E is available at the receiver:

$$B^2 E = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} |S_{\frac{\pi}{T}}(\omega)|^2 d\omega \quad (38)$$

The maximum B is obtained by selecting $s_{\frac{\pi}{T}}(t)$, from the set of Prolate Speroidal functions. At the Nyquist rate, the best one can do is a B_2 of .78. For the example of the RC channel, the "colored" noise spectrum is:

$$N(\omega) = \alpha^2 \left(1 + \left(\frac{\omega}{\alpha}\right)^2\right) \qquad 0 < \omega < \frac{\pi}{T} \quad (39)$$

The matched filter output signal p(0) at the decision instant, is, after integration of $N(\omega)$:

$$p(0) = \frac{.78}{\alpha^2 \left[1 + \frac{1}{(\alpha T)^2} \left(\frac{\pi^2}{3} + 2\right)\right]} \quad (40)$$

Hence, the probability of error becomes:

$$Pe = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{\eta \alpha^2} \frac{.78}{1 + \frac{1}{(\alpha T)^2} \left(\frac{\pi^2}{3} + 2\right)}}$$

4. Conclusion We have compared the performance of the Sidelobe Transmitter, Transversal Filter at the Receiver and Frequency Compensation schemes as static equalizers for the RC Channel. The theoretical performance of the first two schemes is identical, and the probability of error as shown in Fig. 7 is lower than that of the Frequency Compensation technique. This follows qualitatively since the effect of the added energy in the Sidelobes of the Sidelobe Transmitter equals the noise aliasing of the Transversal Filter. Frequency compensation schemes involve both band limiting in the equalizer network and time limiting in the matched filter.

Implementation The Sidelobe Transmitter of Fig. 1 consists of a digital delay line with tap weights ϵ_i in accordance with Equation (2). The summed tap weights modulate the output of a shaping network. The shaping network produces a waveshape in each bit interval in accordance with a solution to Equation (4). The timing for both the waveshapes and the digital delay line is obtained from the clock.

The circuit appears to lend itself readily to IC techniques. The digital delay line consists of an m-level shift register. The tap weights summer and modulator consist of resistive networks, with the possible use of emitter followers for isolation. The waveshaping circuit could be synthesized with the aid of a diode-resistor matrix. The clock is a standard digital building block.

The Transversal Filter at the Receiver requires an analog rather than a digital delay line to achieve a performance identical to that of the Sidelobe Transmitter. Use of a digital delay line may be accomplished through use of either: (1) an analog to digital converter (with the attendant quantization noise and complexity) or (2) Decision before introduction of the signal into the delay line. The matched filter is placed in front of the digital delay line and the output of the transversal filter is added to the matched filter output in order to reduce the intersymbol interference at the decision point to zero. However, since the receiver input is corrupted by noise, erroneous decisions would cause the wrong polarity of tails to be introduced into the decision point and cause even greater intersymbol interference then if there had been no transversal filter at all. Thus, this scheme is prone to propagation of errors.

Frequency Compensation requires analog networks. Linearization of amplitude and phase characteristics results in time delay. However, each transmitted bit is confined to exactly one time interval. In the Sidelobe Transmitter, the transmission of sidelobes, results in spreading the effect of a bit over several time slots. When the maximum of the received wave occurs several bit intervals after the arrival of the signal front, (this is not the case in the RC channel), a time delay is also required.

REFERENCES

1. S. Rosenstark and L. Kurz - "Design of Binary Bandlimited Signals Imbedded in Colored Gaussian Noise and Interference. IEEE Transactions on Info. Theory Vol. intersymbol IT-14 No. 2 March, 1965.

2. R. A. Gibby and J. W. Smith - "Some Extensions of Nyquist's Telegraph Transmission Theory. " BSTJ Vol. 44 pp. 1487-1 10, 1965.

APPENDIX A

We wish to derive the optimum time T_1 at which time the receiver should begin integrating the received energy.

$$Eg = \int_{T_1}^{T_1+T} dt \int_0^T h(t, u) s(u) du \int_0^T h(t, v) s(v) dv \qquad \text{(A-1)}$$

We wish to maximize Eg subject to the constraint.

$$E_T = \int_0^T \int_0^T s(u) s(v) du dv = E_T = \text{const} \qquad \text{(A-2)}$$

and

$$h(t) = e^{-\alpha t} u(t) \qquad \text{(A-3)}$$

We note that:

$$0 < u < T$$

$$0 < v < T$$

We rewrite Eq. (A-1) in the form:

$$Eg = \int_0^T dt \int_0^T h(t, u) s(u) du \int_0^T h(t, v) s(v) dv$$

$$+ \int_T^{T+T_1} dt \int_0^T h(t, u) s(u) du \int_0^T h(t, v) s(v) dv$$

$$- \int_0^{T_1} dt \int_0^T h(t, u) s(u) du \int_0^T h(t, v) s(v) dv \qquad \text{(A-4)}$$

Thus,

$$\lambda s(u) = \int_0^T s(v) [Q_1(u, v) + Q_2(u, v) - Q_3(u, v)] dv \qquad \text{(A-5)}$$

where:

$$Q_1 = \int_0^T h(t, u)h(t, v)dt$$

$$Q_2 = \int_T^{T_1+T} h(t, u)h(t, v)dt \qquad \text{(A-6)}$$

$$Q_3 = \int_0^{T_1} h(t, u)h(t, v)dt$$

We proceed to evaluate Q_1, Q_2 and Q_3. Q_1 was found in the text to be:

$$Q_1(u, v) = \frac{e^{-a|u-v|} - e^{-2aT} e^{a(u+v)}}{2a}$$

$$0 < t < T \qquad \text{(A-7)}$$

previously found

$$Q_2(u, v) = \int_T^{T+T_1} e^{-a(t-u)} v(t-u) e^{-a(t-v)} v(t-v) dt$$

$$u, v < T \qquad \text{(A-8)}$$

hence,

$$Q_2(u, v) = \frac{e^{a(u+v)} e^{-2aT}}{2a} \left(1 - e^{-2aT_1}\right) \qquad \text{(A-9)}$$

and

$$Q_3(u, v) = \frac{e^{-a|u-v|} - e^{-2aT_1} e^{a(u+v)}}{2a} \qquad \max(u, v) < T_1$$

$$= 0 \qquad \max(u, v) > T_1 \qquad \text{(A-10)}$$

Substitution of (A-8), (A-9) and (A-10) into (A-5) yields the following transcendental equation:

$$\tan^{-1}\frac{\omega}{a} + \tan^{-1}\frac{\omega}{a}\tanh\, aT\cdot\frac{T_1}{T} = -\frac{\omega}{a}\, aT\left(1-\frac{T_1}{T}\right) \qquad \text{(A-11)}$$

For a fixed value of aT, one solves (A-11) for values of $\frac{T_1}{T}$ ranging from zero to unity. For each of these values of $\frac{T_1}{T}$, one finds the lowest $\frac{\omega}{a}$ which satisfies the equation. The best T_1 corresponds to the lowest ω.

The shape of the transmitted signal in the interval 0 to T is an exponential in the interval up to T_1, and is sinusoidal from T_1 to T.

This follows from (A-5). The eigenvalue λ is given as before by:

$$\lambda = \frac{1}{1+\left(\frac{\omega}{a}\right)^2} \qquad \text{(A-12)}$$

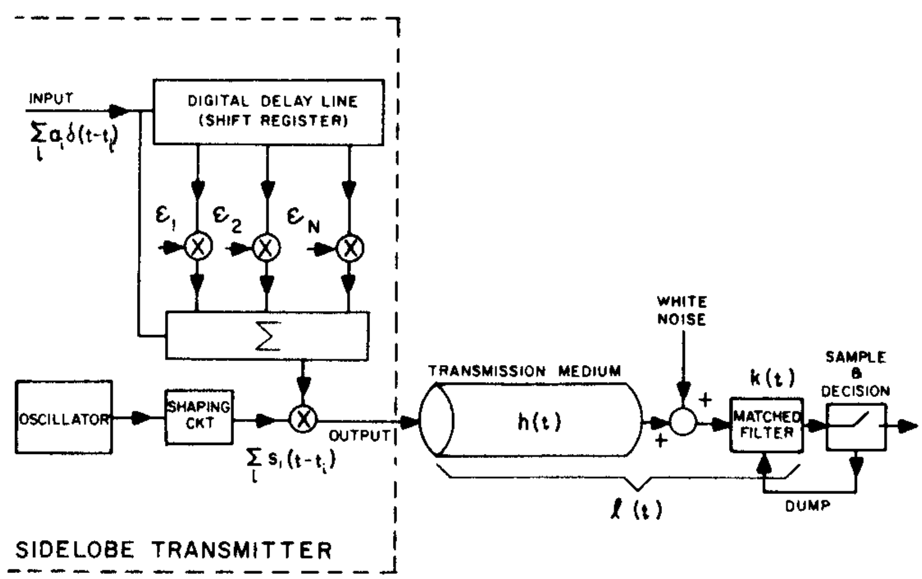

FIG. 1 SYSTEM CONFIGURATION USING SIDELOBE TRANSMITTOR

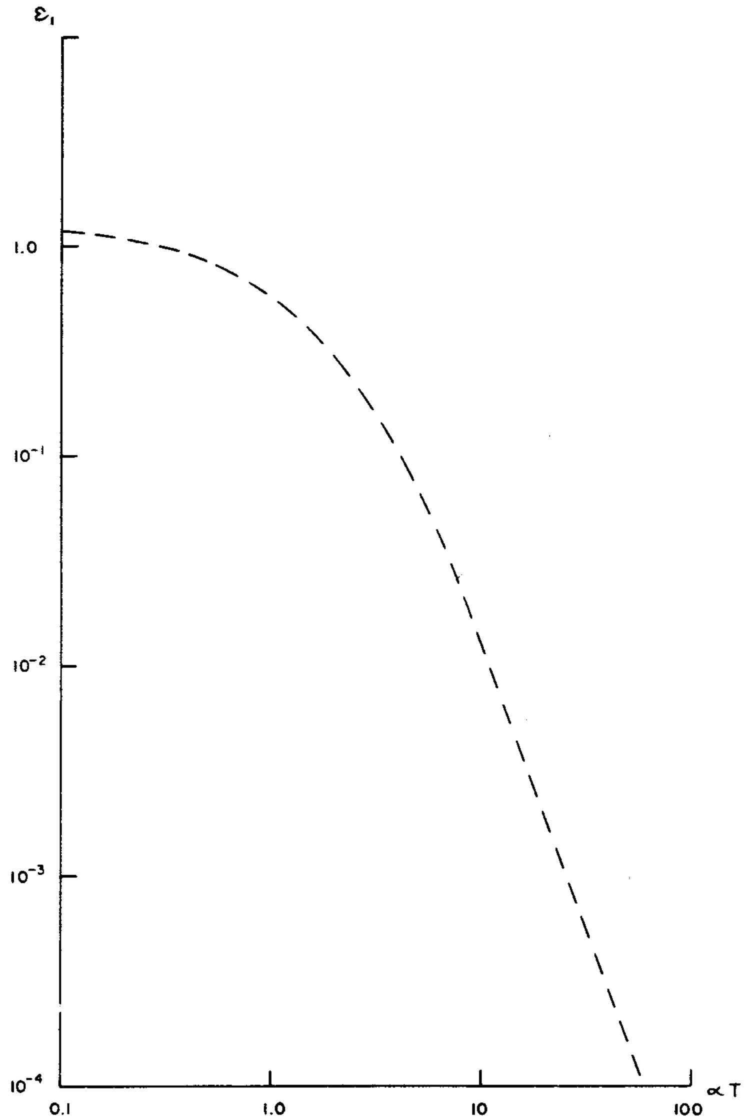

FIG. 2 RC CHANNEL RELATIVE MAGNITUDE OF FIRST SIDELOBE ε_1 vs αT

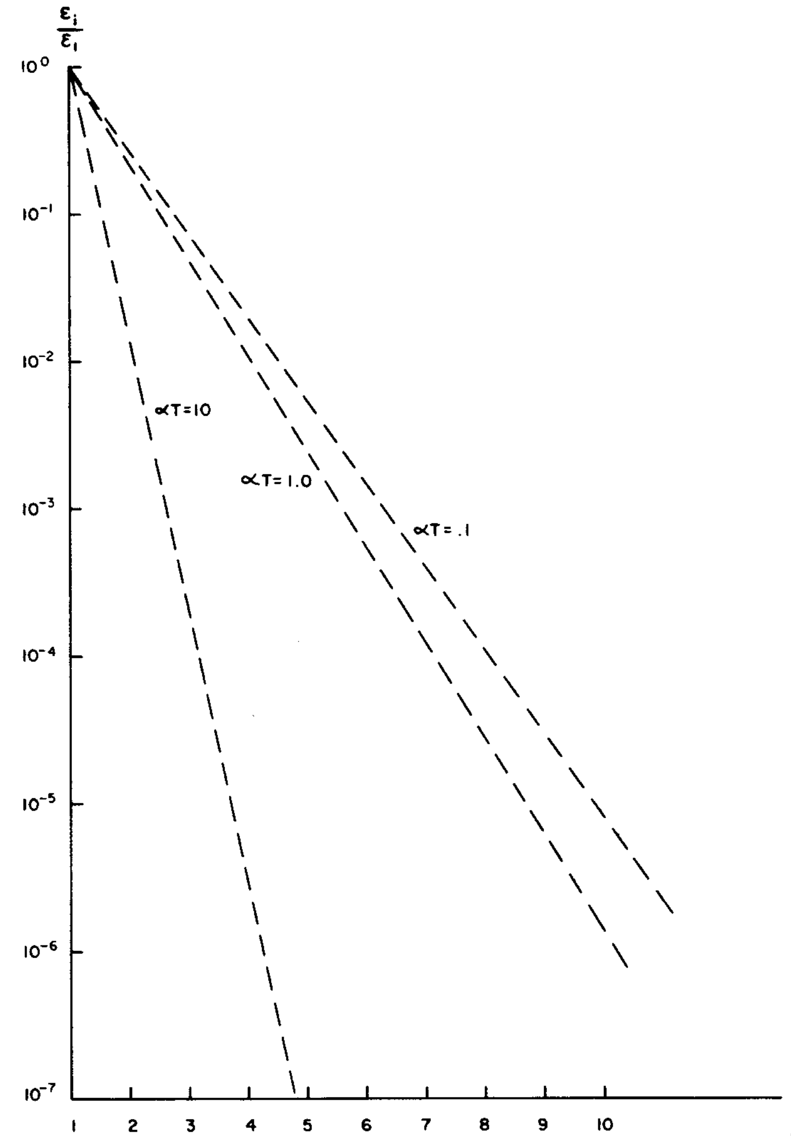

FIG. 3 RELATIVE SIDELOBE MAGNITUDE $\frac{\varepsilon_i}{\varepsilon_1}$ WITH αT AS PARAMETER

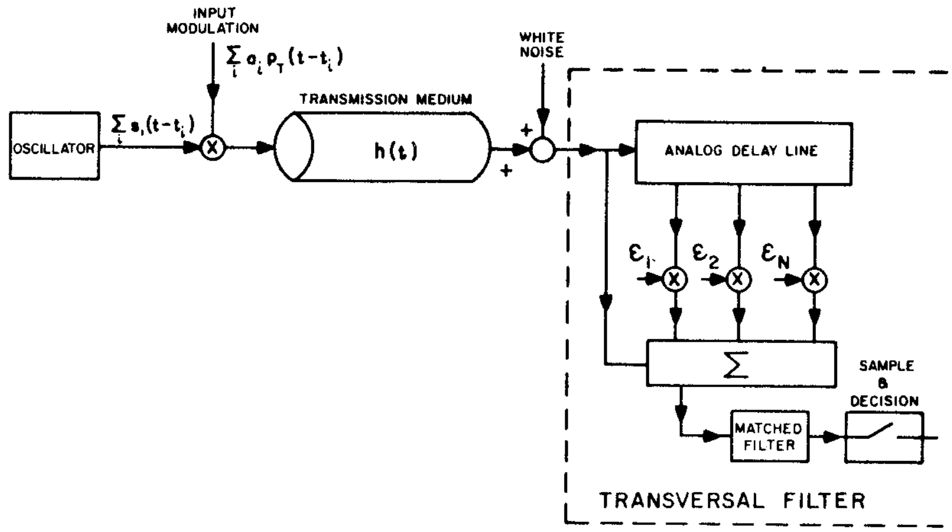

FIG. 4

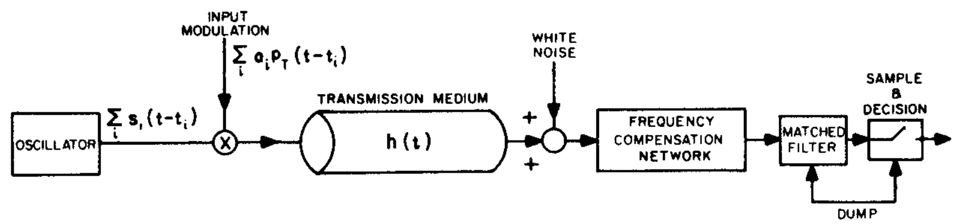

FIG. 5 FREQUENCY COMPENSATION NETWORK CONFIGURATION

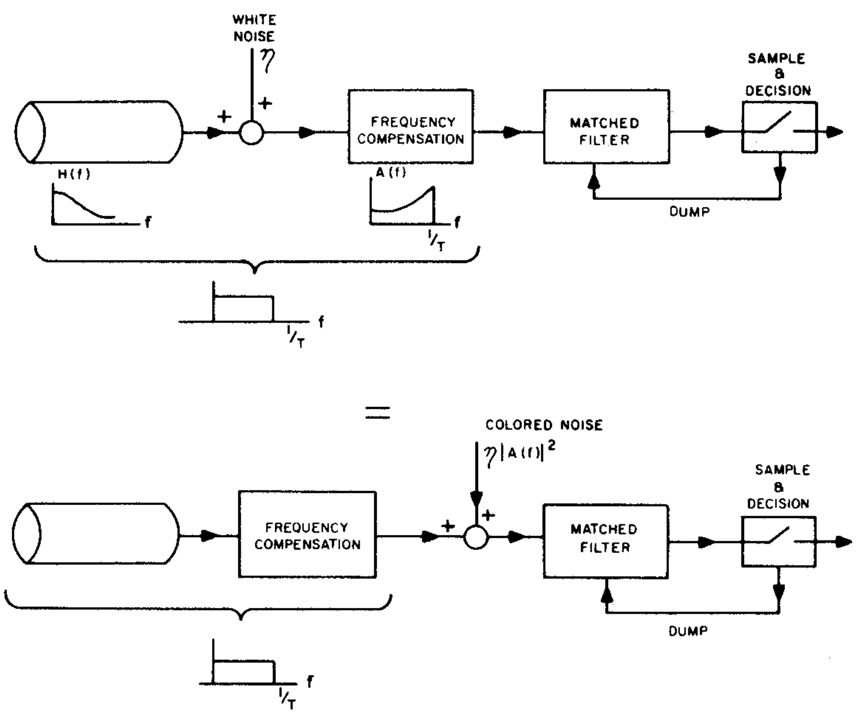

FIG. 6

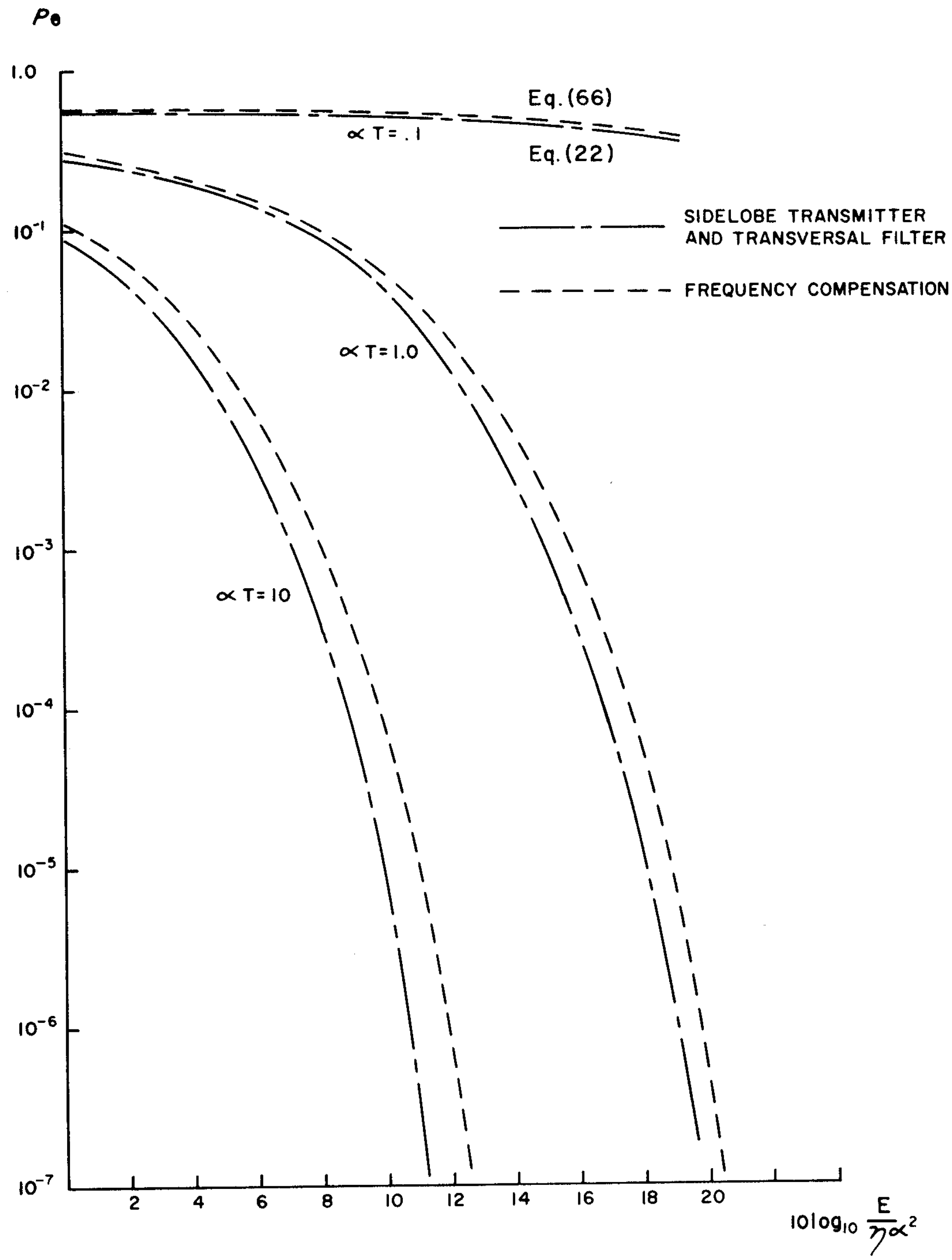

FIG. 7 Pe vs $\frac{E}{\eta\alpha}2$