# ALAMOUTI SPACE-TIME CODING FOR QPSK WITH DELAY DIFFERENTIAL

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## ABSTRACT

Space-time coding (STC) for QPSK where the transmitted signals are received with the same delay is well known. This paper examines the case where the transmitted signals are received with a non-negligible delay differential when the Alamouti 2x1 STC is used. Such a differential can be caused by a large spacing of the transmit antennas. In this paper, an expression for the received signal with a delay differential is derived and a decoding algorithm for that signal is developed. In addition, the performance of this new algorithm is compared to the standard Alamouti decoding algorithm for various delay differentials.

## KEY WORDS

Space-Time Coding, Delay Differential, QPSK, Aeronautical Telemetry

## INTRODUCTION

In aeronautical telemetry channels, aircraft are commonly equipped with two transmit antennas to alleviate signal outages due to aircraft shadowing. However, this configuration causes severe signal level variation at the receiver when both antennas are in view. In [1], Crummett, et al. proposed using space time coding as a solution to this problem. Their solution involves employing an Alamouti 2x1 space-time code (STC) to overcome the interference due to the two transmit antennas. However, as was noted in that paper, frequently the two transmit antennas are placed on the aircraft in such a way that their spacing can cause the path between one of the transmit antennas and the receive antenna to be significantly longer than the other transmit path. This path length differential in turn results in a delay differential between the two received signals that can be a

significant fraction of a symbol period. Alamouti's STC assumes that the differential propagation delay on the two signal paths is negligible. This paper examines the effect of a non-negligible differential delay on the composite received signal and derives a decoding algorithm designed for the case where the differential delay is less than a symbol. The performance of this new algorithm is simulated and BER versus SNR plots are presented for delay differentials up to $0.5T_s$. In this paper QPSK modulation with full response pulse shaping (specifically a half-sine pulse) is examined. It is expected that the results can be extended to partial response signaling and OQPSK, although the resulting algorithm would be more complicated.

This paper proceeds as follows: A brief overview of the Alamouti 2x1 STC is given in the next section to provide a background for the contributions of this paper. That section is followed by a development of an expression for the received signal when the delay differential is not negligible. Then modifications of the Alamouti decoding algorithm are presented which account for the delay differential. After this algorithm is presented, its performance is compared to the original Alamouti algorithm in the results section and the conclusions of the paper are presented.

## OVERVIEW OF ALAMOUTI 2X1 SPACE TIME CODE

In [2] Alamouti proposed a method of signaling that exploits transmit diversity. This section will provide a brief overview of Alamouti's space-time coding scheme as a preface to the following section. Alamouti's paper showed that the same diversity benefit can be obtained when using transmit diversity as when using receive diversity. Specifically, he showed that a system with two transmit antennas and one receive antenna can perform as well in the presence of multipath as a system with one transmit antenna and two receive antennas. The key to Alamouti's scheme, hereafter referred to as the Alamouti 2x1 STC, is that the symbol streams transmitted on the two antennas are different but related in such a way that the receiver can separate the two overlapping signals. Specifically, if the first antenna transmits the symbol stream

$$\mathbf{s_0} = s_0, s_1, \ldots, s_k, s_{k+1}, s_{k+2}, s_{k+3}, \ldots \tag{1}$$

then the second antenna transmits the symbol stream

$$\mathbf{s_1} = -s_1^*, s_0^*, \ldots, -s_{k+1}^*, s_k^*, -s_{k+3}^*, s_{k+2}^*, \ldots \tag{2}$$

in which the symbols in $\mathbf{s_0}$ are conjugated, pairs of symbols are switched, and odd indexed symbols are negated. These two symbol streams pass through independent channels $h_0$ and $h_1$ and are received simultaneously by the receiver. The receiver passes the received signals through a matched filter and pairs the matched filter output samples for processing. The pairs of samples can

be placed in a column vector and are represented as

$$\begin{bmatrix} x(kT_s) \\ x^*((k+1)T_s) \end{bmatrix} = \begin{bmatrix} h_0 & h_1 \\ h_1^* & -h_0^* \end{bmatrix} \begin{bmatrix} s_k \\ s_{k+1} \end{bmatrix} + \begin{bmatrix} w(kT_s) \\ w((k+1)T_s) \end{bmatrix} \tag{3}$$

or

$$\mathbf{x_k} = \mathbf{Hs_k} + \mathbf{w_k} \tag{4}$$

where the second sample in $\mathbf{x_k}$ has been conjugated. The vector $\mathbf{x_k}$ can be left multiplied by the conjugate transpose of the matrix $\mathbf{H}$. Because $\mathbf{H}$ is a unitary matrix, this operation results in the following expression:

$$\mathbf{y_k} = \mathbf{s_k} + \mathbf{H^H w}. \tag{5}$$

Thus the matched filter output is transformed into a column vector whose elements are the original symbols plus independent Gaussian noise. Maximum likelihood (ML) decisions can be made on the values of the original symbols individually because of the orthogonal nature of the Alamouti 2x1 STC.

Note that the preceding analysis assumed that the two channels vary slowly with respect to the symbol rate so that the effects of the channel can be considered constant over at least two symbol periods. It is also assumed that the two channels are known and that there is no differential delay in the two channel paths. The next section considers the effects of a differential channel delay on the received signal and the following section develops a decoding algorithm for the Alamouti 2x1 STC that takes a differential channel delay into account.

### RECEIVED SIGNAL MODEL WITH DELAY DIFFERENTIAL

The received signal will be the superposition of the two transmitted signals, $r_0$ and $r_1$ plus additive white Gaussian noise as expressed in (6).

$$r(t) = r_0(t) + r_1(t) + n(t) \tag{6}$$

The received signal $r(t)$ is passed through a matched filter (MF) and the output $x(t)$ is sampled at the symbol rate. Because of linearity, the contribution from each transmitted signal to $x(t)$ can be considered individually. The $k^{th}$ symbol of the first signal, $s_k^0$, passes through channel $h_0$ and arrives at time $t = kT_s + \tau_0$. The output of the MF due to this symbol can be written as

$$x_0(t) = h_0 s_k^0 R_p(t - \tau_0 - kT_s) \tag{7}$$

where $R_p(\tau)$ is the deterministic autocorrelation function of the pulse shape given by

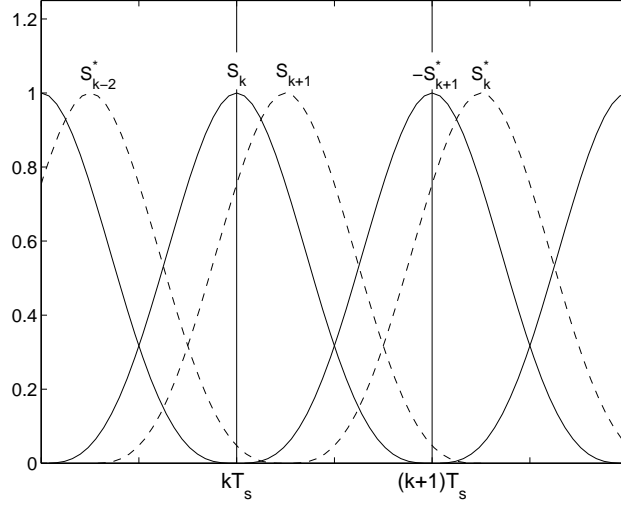$$R_p(\tau) = \int_{-LT_s}^{LT_s} p(t)p(t + \tau)dt. \tag{8}$$

Figure 1: Matched Filter Output with Delay Differential of $.35T_s$

where $p(t)$ is the pulse shape which has support over the interval $-LT_s \leq t \leq LT_s$. Similarly, the $k^{th}$ symbol of the second signal, $s_k^1$, passes through channel $h_1$ and arrives at time $t = kT_s + \tau_1$, resulting in the MF output

$$x_1(t) = h_1 s_k^1 R_p(t - \tau_1 - kT_s). \tag{9}$$

Noise at the receiver input also passes through the MF. If the pulse shape is full response or if it satisfies the Nyquist criterion for zero intersymbol interference (ISI), then the noise sampled at intervals $T_s$ equal to the symbol period will be uncorrelated and can be expressed as $w(t)$. The composite signal out of the MF is then

$$x(t) = h_0 s_k^0 R_p(t - \tau_0 - kT_s) + h_1 s_k^1 R_p(t - \tau_1 - kT_s) + w(t). \tag{10}$$

When $\tau_0 \neq \tau_1$ (which is the case when there is a non-negligible delay differential between the two transmitted signals) it is not possible to have symbol-spaced samples at the output of the matched filter that are sampled at the optimum time for both transmitted signals. So instead the signal with the shortest path delay is sampled at its optimum sampling time and the delay of the other signal is then accounted for. If channel 1 is delayed relative to channel 0 and $x(t)$ is sampled at $t = kT_s + \tau_0$, then the sampled signal is

$$x(kT_s + \tau_0) = h_0 s_k^0 R_p(0) + h_1 s_k^1 R_p(\tau_0 - \tau_1) + w(kT_s + \tau_0). \tag{11}$$

Assuming that $p(t)$ is normalized so that $R_p(0) = 1$ and setting $\tau = \tau_1 - \tau_0$, (11) becomes

$$x(kT_s + \tau_0) = x_k = h_0 s_k^0 + h_1 s_k^1 R_p(-\tau) + w_k \tag{12}$$

Note that (12) is the sampled output of the matched filter due to a single symbol on each transmit channel. If the pulse shape is full response or if it satisfies the Nyquist criterion for zero ISI then that symbol will be the only one that makes a contribution to the output sample on channel 0 (the channel to which the samples are synchronized). However, regardless of whether the pulse is full response or satisfies the Nyquist criterion, more than one symbol on channel 1 will contribute to the output sample because the sample is not aligned with the optimum sample time for that channel. Figure (1) illustrates this principle. The figure shows the response of a matched filter to a sequence of input symbols when the pulse shape is a half-sine pulse. For this illustration only the real part of the filter output is shown, the channels $h_0$ and $h_1$ are set to $1.0$, and the responses of the MF due to the various symbols are overlaid rather than summed. The solid pulses represent symbols on channel 0 and the dashed pulses represent symbols on channel 1. Channel 1 is delayed by $\tau = .35T_s$ relative to channel 0. The solid vertical lines show the optimum sampling times for channel 0. At these times, it can be seen that the corresponding symbols on channel 1 haven't fully entered the match filter and so their contribution to the composite signal hasn't reached it's maximum value yet. In addition, the response of the matched filter to the preceding symbol is not zero. Because in this example full response signaling is used, no other symbols on channel 1 contribute to the MF output at that sampling time.

As a result of the preceding discussion, a complete expression for the sampled matched filter output is

$$x_k = h_0 s_k^0 + h_1 s_k^1 R_p(-\tau) + h_1 s_{k-1}^1 R_p(T_s - \tau) + w_k. \tag{13}$$

Taking into account the Alamouti 2x1 STC (where $s_k^0 = s_k$, $s_{k+1}^0 = -s_{k+1}^*$, $s_k^1 = s_{k+1}$, and $s_{k+1}^1 = s_k^*$), this expression can be expanded and written for samples at $t = kT_s + \tau_0$ and $t = (k+1)T_s + \tau_0$ as

$$x_k = h_0 s_k + h_1 s_{k+1} R_p(-\tau) + h_1 s_{k-2}^* R_p(T_s - \tau) + w_k \tag{14}$$

$$x_{k+1} = -h_0 s_{k+1}^* + h_1 s_k^* R_p(-\tau) + h_1 s_{k+1} R_p(T_s - \tau) + w_{k+1}. \tag{15}$$

When channel 0 is delayed relative to channel 1, the symbol timing is aligned to channel 1 and a similar expression is obtained. The MF output samples at $t = kT_s + \tau_1$ and $t = (k+1)T_s + \tau_1$ are given by

$$x_k = h_0 s_k R_p(-\tau) + h_1 s_{k+1} - h_0 s_{k-1}^* R_p(T_s - \tau) + w_k \tag{16}$$

$$x_{k+1} = -h_0 s_{k+1}^* R_p(-\tau) + h_0 s_k^* + h_1 s_k R_p(T_s - \tau) + w_{k+1}. \tag{17}$$

Note that the analogous expressions to (14) through (17) when there is no delay differential are

$$x_k = h_0 s_k + h_1 s_{k+1} + w_k \tag{18}$$

$$x_{k+1} = -h_0 s_{k+1}^* + h_1 s_k^* + w_{k+1}. \tag{19}$$

The differences between (14) – (17) and (18) – (19) are that the symbols on the delayed channel are scaled by $R_p(-\tau)$ and there is an additional term due to the preceding symbol on the delayed channel. These differences require a modification to the decoding algorithm proposed by Alamouti.

## MODIFICATIONS TO ALAMOUTI 2X1 DECODING ALGORITHM TO ACCOUNT FOR DELAY DIFFERENTIAL

In this section a decoding algorithm for the Alamouti 2x1 STC in the presence of a delay differential is derived. First the algorithm will be developed for the case where channel 1 is delayed relative to channel 0 and then it will be extended to the opposite case.

Equations (14) and (15) can be expressed in matrix form if (15) is first conjugated. The resulting expression is

$$
\begin{bmatrix} x_k \\ x_{k+1}^* \end{bmatrix} = \begin{bmatrix} h_0 & h_1 R_p(-\tau) \\ h_1^* R_p(-\tau) & -h_0^* \end{bmatrix} \begin{bmatrix} s_k \\ s_{k+1} \end{bmatrix}
$$
$$
+ \begin{bmatrix} h_1 R_p(T_s - \tau) & 0 \\ 0 & h_1 R_p(T_s - \tau) \end{bmatrix} \begin{bmatrix} s_{k-2}^* \\ s_{k+1}^* \end{bmatrix} + \begin{bmatrix} w_k \\ w_{k+1} \end{bmatrix} \tag{20}
$$

or

$$\mathbf{x}(k) = \mathbf{H}\mathbf{s_k} + \mathbf{G}\mathbf{s'_k} + \mathbf{w}(k). \tag{21}$$

Multiplying both sides of (21) by $\mathbf{H}^H$ gives

$$\mathbf{y}(k) = \mathbf{H}^H \mathbf{x}(k) = \mathbf{H}^H \mathbf{H}\mathbf{s_k} + \mathbf{H}^H \mathbf{G}\mathbf{s'_k} + \mathbf{H}^H \mathbf{w}(k). \tag{22}$$

The term $\mathbf{H}^H \mathbf{H}$ is

$$
\mathbf{H}^H \mathbf{H} = \begin{bmatrix} h_0^* & h_1 R_p(-\tau) \\ h_1^* R_p(-\tau) & -h_0 \end{bmatrix} \begin{bmatrix} h_0 & h_1 R_p(-\tau) \\ h_1^* R_p(-\tau) & -h_0^* \end{bmatrix} \tag{23}
$$
$$
= \alpha \mathbf{I} \tag{24}
$$

where $\alpha = |h_0|^2 + |h_1|^2 R_p^2(-\tau)$. The term $\mathbf{H}^H \mathbf{G}$ is

$$\mathbf{H}^H \mathbf{G} = \begin{bmatrix} h_0^* & h_1 R_p(-\tau) \\ h_1^* R_p(-\tau) & -h_0 \end{bmatrix} \begin{bmatrix} h_1 R_p(T_s - \tau) & 0 \\ 0 & h_1 R_p(T_s - \tau) \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} h_0^* h_1 R_p(T_s - \tau) & |h_1|^2 R_p(-\tau) R_p(T_s - \tau) \\ |h_1|^2 R_p(-\tau) R_p(T_s - \tau) & -h_0 h_1^* R_p(T_s - \tau) \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} \gamma & \beta \\ \beta & -\gamma^* \end{bmatrix}. \tag{27}$$

Inserting these results into (22) gives

$$\begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} = \alpha \begin{bmatrix} s_k \\ s_{k+1} \end{bmatrix} + \begin{bmatrix} \gamma & \beta \\ \beta & -\gamma^* \end{bmatrix} \begin{bmatrix} s_{k-2}^* \\ s_{k+1}^* \end{bmatrix} + \begin{bmatrix} v_k \\ v_{k+1} \end{bmatrix} \tag{28}$$

Equation (28) can be written as the two following equations:

$$y_k = \alpha s_k + \gamma s_{k-2}^* + \beta s_{k+1}^* + v_k \tag{29}$$

$$y_{k+1} = \alpha s_{k+1} + \beta s_{k-2}^* - \gamma^* s_{k+1}^* + v_{k+1} \tag{30}$$

At this point there are two equations in which three symbols, $s_{k-2}$, $s_k$, and $s_{k+1}$ are present. One option for estimating $s_k$, and $s_{k+1}$ from the samples $y_k$ and $y_{k+1}$ is to assume that the symbol $s_{k-2}$ is known from the previous decision block and so (30) is only a function of one unknown, $s_{k+1}$, plus white noise. The maximum likelihood (ML) estimate of $s_{k+1}$ under this assumption is given by the expression

$$\hat{s}_{k+1} = \underset{\hat{s}_{k+1} \in \mathcal{S}}{\operatorname{argmin}} \left\{ \left| y_{k+1} - \left( \alpha \hat{s}_{k+1} + \beta \hat{s}_{k-2}^* - \gamma^* \hat{s}_{k+1}^* \right) \right|^2 \right\} \tag{31}$$

$$\tag{32}$$

Then $\hat{s}_{k+1}$ can be used in (29) to find the following ML estimate of $s_k$

$$\hat{s}_k = \underset{\hat{s}_k \in \mathcal{S}}{\operatorname{argmin}} \left\{ \left| y_k - \left( \alpha \hat{s}_k + \gamma \hat{s}_{k-2}^* + \beta \hat{s}_{k+1}^* \right) \right|^2 \right\} \tag{33}$$

When channel 0 is delayed relative to channel 1 then the receiver synchronizes to channel 1. Then the received signal is similar in form to the case when channel 1 is delayed but the location of the symbols changes due to the structure of the Alamouti 2x1 STC and, consequently, (28) has to be modified. After following the same procedure as for the case when channel 1 is delayed relative to channel 0, (28) becomes in this case

$$\begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix} = \alpha \begin{bmatrix} s_k \\ s_{k+1} \end{bmatrix} + \begin{bmatrix} -\beta & \gamma \\ -\gamma^* & -\beta \end{bmatrix} \begin{bmatrix} s_{k-1}^* \\ s_k^* \end{bmatrix} + \begin{bmatrix} v_k \\ v_{k+1} \end{bmatrix} \tag{34}$$
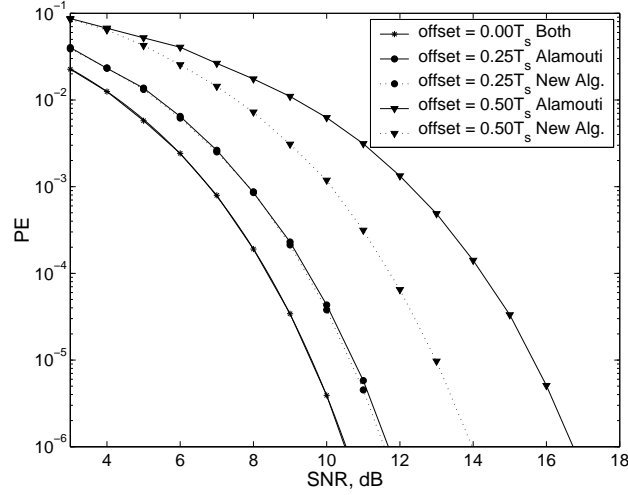
Figure 2: Bit Error rates as a function of $E_b/N_0$ for three differential delays.

where $\alpha$, $\beta$, and $\gamma$ are the same as before. Equation (34) can also be written as two equations as

$$y_k = \alpha s_k - \beta s_{k-1}^* + \gamma s_k^* + v_k \tag{35}$$

$$y_{k+1} = \alpha s_{k+1} - \gamma^* s_{k-1}^* - \beta s_k^* + v_{k+1}. \tag{36}$$

In this case symbol $s_{k-1}$ is known from the previous decision block so (35) only depends on the value of $s_k$. The ML estimate of this symbol is given by

$$\hat{s}_k = \operatorname*{argmin}_{\hat{s}_k \in \mathcal{S}} \left\{ \left| y_k - \left( \alpha \hat{s}_k - \beta \hat{s}_{k-1}^* + \gamma \hat{s}_k^* \right) \right|^2 \right\} \tag{37}$$

$$\tag{38}$$

Then $\hat{s}_k$ can be used in (36) to find the following ML estimate of $s_{k+1}$

$$\hat{s}_{k+1} = \operatorname*{argmin}_{\hat{s}_{k+1} \in \mathcal{S}} \left\{ \left| y_{k+1} - \left( \alpha \hat{s}_{k+1} - \gamma^* \hat{s}_{k-1}^* - \beta \hat{s}_k^* \right) \right|^2 \right\} \tag{39}$$

The low complexity of this algorithm is attractive. However, it is sub optimal because estimates of some symbols are used in the estimation of other symbols and errors in the first estimates will affect the probability of making an error in the following estimates. In spite of the sub optimal nature of this new algorithm, it does offer a performance improvement over the original Alamouti decoding algorithm when a differential delay is present as will be shown in the following section.

## RESULTS

Figure 2 compares the performance of the new decoding algorithm with that of the unmodified Alamouti decoding algorithm as a function of delay differential. The bit error rates at differential

delays of $\tau = 0.0T_s$, $\tau = 0.25T_s$, and $\tau = 0.5T_s$ are displayed. The theoretical BER curve for QPSK is also included. As can be seen, new algorithm matches the theoretical performance for QPSK when there is no differential delay. This is expected because new algorithm is equivalent to the Alamouti decoding algorithm when there is no delay differential and the Alamouti algorithm performs the same as uncoded single channel QPSK on the AWGN channel. For a differential delay of $\tau = 0.25T_s$ the new algorithm is about 0.1 dB better than the original Alamouti decoding algorithm and about 1 dB worse than the theoretical performance for QPSK at $BER = 10^{-5}$. For a differential delay of $\tau = 0.5T_s$ the modified algorithm is about 3.4 dB worse than theoretical QPSK and about 2.7 dB better than the original Alamouti algorithm. As can be seen, the new algorithm performs better than the original Alamouti algorithm, and the improvement in performance increases with increasing delay differential.

Figure 2 also shows that the presence of a differential delay degrades performance for both of the algorithms examined in this paper. This result is due to the fact that in (30) the term accounting for the second symbol on channel 1 hasn't completely entered the matched filter at the sampling time as illustrated in Figure 1. Consequently not all of the energy in this symbol is used to estimate the value of the symbol. This degradation increases with increasing differential delays. The algorithm developed in this paper compensates for the interference due to preceding symbols on channel 1 being delayed into the samples used in the Alamouti decoding but it does not compensate for the sub optimal sampling of the delayed channel. This fact explains why the algorithm presented in this paper performed only a little better than the original Alamouti algorithm when $\tau = 0.25T_s$. In this case for the pulse shape used in the simulations, there is very little interference from preceding pulses but there is a significant decrease in the matched filter output due to the channel 1 pulse (about 1.2 dB). It is expected that an algorithm which samples both channels at the optimum sampling time would perform better than either of the algorithms presented in this paper. Investigations are underway to develop such an algorithm.

### CONCLUSION

This paper has shown that large differential delays between the two transmit paths can cause performance degradation on systems employing the Alamouti 2x1 STC. This performance degradation can be reduced by modifying the Alamouti decoding algorithm to account for the differential delay. A new algorithm has been developed in this paper for decoding Alamouti 2x1 STC signals in the presence of a delay differential. Simulations show that this new algorithm offers a performance improvement over the original Alamouti decoding algorithm when a differential delay is present. For aeronautical telemetry systems employing STC to mitigate co-channel interference due to dual antenna use, this algorithm promises to improve performance when the antenna spacing is large enough to cause a significant delay differential.

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