Performance of Soft-Decision Block-Decoded Hybrid-ARQ Error Control

Dr. Michael Rice
Department of Electrical and Computer Engineering
Brigham Young University
Provo, UT 84602

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Abstract

Soft-decision correlation decoding with retransmission requests for block codes is proposed and the resulting performance is analyzed. The correlation decoding rule is modified to allow retransmission requests when the received word is rendered unreliable by the channel noise. The modification is realized by a reduction in the volume in Euclidean space of the decoding region corresponding to each codeword. The performance analysis reveals the typical throughput - reliability trade-off characteristic of error control systems which employ retransmissions. Performance comparisons with hard-decision decoding reveal performance improvements beyond those attainable with hard-decision decoding algorithms. The proposed soft-decision decoding rule permits the use of a simplified codeword searching algorithm which reduces the complexity of the correlation decoder to the point where practical implementation is feasible .

1 Introduction

For data communications systems where receiving erroneous data is worse than receiving no data at all, ARQ and hybrid-ARQ techniques are effective forms of error control [1]-[4] . Automatic-Repeat-Request (ARQ) error control employs *error detection* in conjunction with a *retransmission protocol* to achieve low bit error rates with relatively simple decoder complexity. The throughput of such systems, however, degrades as channel noise increases since more time is spent retransmitting received packets which contain detectable errors. Hybrid-ARQ error control combines *forward-error-correction* with error-detection to reduce the frequency of the retransmissions without severely increasing the probability of decoder error [4].

It is well known that for FEC decoding of block codes, the difference in performance between hard-decision and soft-decision decoding on an AWGN channel is approximately 2 to 3 dB [5]. This fact provides the motivation for investigating the possible performance gains of soft decision decoders over hard decision decoders in a hybrid-ARQ application. In this paper soft-decision block decoding methods are applied to the hybrid-ARQ error control problem where it is shown soft-decision decoding offers significant improvements over hard-decision decoding in the hybrid-ARQ error control application.

2 Development of Soft-Decision Decoding with Retransmission Requests

Assume that the information to be transmitted is in binary form and is encoded using a linear binary (n, k) code with minimum Hamming distance d_{min}. Each bit of the code word is transmitted over an AWGN channel with binary antipodal signaling using a waveform of energy $\mathcal{E}$ Joules. The transmitted waveforms are corrupted by additive white Gaussian noise with one-sided power spectral density N_0. The coherent demodulator employs a matched filter followed by a sampler to produce an (n x l)-dimensional vector of real numbers $\mathbf{r} = [r(T), r(2T), \ldots, r(nT)]'$ where T is the time allotted to the transmission of one channel symbol and $'$ is the transpose operator. Each element of $\boldsymbol{r}$ is, after convenient normalization, a Gaussian random variable with mean ± 1 and variance $N_0/2\mathcal{E}$. The sign on the mean is positive if a "one" was sent and negative if a "zero" was sent.

The optimum FEC soft-decision decoder is a correlation decoder and follows the rule [5]

$$\hat{\boldsymbol{c}} = \boldsymbol{c}_m \quad \text{if} \quad \boldsymbol{r}^T \boldsymbol{c}_m > \boldsymbol{r}^T \boldsymbol{c}_{m'}, \quad \text{for all } m' \neq m \tag{1}$$

where $\boldsymbol{c}_m = [c_{m1}, c_{m2}, \ldots, c_{mn}]$ with c_{mj} 0 {-1, +1} for 1# j \$ n. In n-dimensional Euclidean space where the codewords form 2^k vertices of the n-dimensional hypercube centered at the origin ? with vertices whose coordinates are in the set {- 1, + 1}, correlation decoding has the following geometric interpretation :

$$\hat{\boldsymbol{c}} = \boldsymbol{c}_m \Leftrightarrow \boldsymbol{r} \in R_m \tag{2}$$

here R_m is a hyper-pyramid with vertex at the origin ? and centered on $\mathbf{c}_m$ and is called the *decoding region* for codeword $\boldsymbol{c}_m$.

Incomplete decoding exists when there are received sequences which are not decoded into any codeword. The resulting condition is termed a *decoding failure*. In

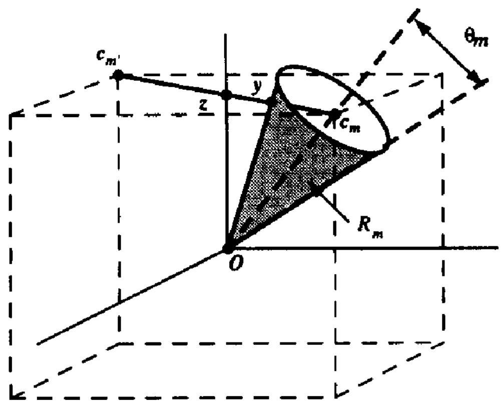

Figure 1: Three-Dimensional Geometric Representation of Hybrid-ARQ Soft-Decision Decoding Region R'_{m}.

hybrid-ARQ error control, the decoder is necessarily an incomplete decoder where a decoding failure is regarded as a request for a retransmission. The modification of correlation decoders to admit retransmission requests is realized by a reduction in the volume of the decoding region for each codeword. Let R' be the new decoding region for codeword c_m as shown in Figure 1. The following notation is used:

*R(**a**, **b**) is the line segment defined by the points (vectors) **a** and **b** in R^n.*

*d(**a**, **b**) is the Euclidean distance between the points **a** and **b** in R^n.*

$\mathrm{d_H}$ ($\boldsymbol{c}_m$, $\boldsymbol{c}_{m'}$) is the Hamming distance between the codewords $\boldsymbol{c}_m$ and $\boldsymbol{c}_{m'}$.

R'_m is the hyper-cone with vertex at the origin ***O*** and centered on $\boldsymbol{c}_m$ and is defined m by $\boldsymbol{1}_m$ which is the angle between the lines R(***O***, $\boldsymbol{c}_m$)and R(***O***, ***y***) where ***y*** is a point on R($\boldsymbol{c}_m$, $\boldsymbol{c}_{m'}$) where $\boldsymbol{c}_m$ and $\boldsymbol{c}_{m'}$ are codewords such that $\mathrm{d_H}(\boldsymbol{c}_m, \boldsymbol{c}_{m'}) = d_{\mathrm{min}}$.

In order for the hyper-cones R' and R'_m to be disjoint, the point ***y*** must lie between the points $\boldsymbol{c}_m$ and ***z*** on R($\boldsymbol{c}_{m'}$, $\boldsymbol{c}_m$) where ***z*** is the bisector of R($\boldsymbol{c}_m$, $\boldsymbol{c}_{m'}$). Define the parameter D as the tractional distance between the points $\boldsymbol{c}_m$ and ***y*** and $\boldsymbol{c}_m$ and ***z***:

$$\rho = \frac{d_{\mathrm{E}}(\boldsymbol{c}_m, \boldsymbol{y})}{d_{\mathrm{E}}(\boldsymbol{c}_m, \boldsymbol{z})} = \frac{d_{\mathrm{E}}(\boldsymbol{c}_m, \boldsymbol{y})}{\sqrt{d_{\mathrm{min}}}} \tag{3}$$

so that $\mathrm{d_E}(\boldsymbol{c}_m, \boldsymbol{y})$ = D%$\mathrm{d_{min}}$. Note that D is restricted to the real interval (0,1] to ensure that the point $\boldsymbol{y}$ is between points $\boldsymbol{z}$ and $\boldsymbol{c}_m$. The cosine of the angle $\mathit{1}_{\mathrm{m}}$ is given by

$$\cos\theta_m = \frac{\boldsymbol{c}_m^T \boldsymbol{y}}{\sqrt{\boldsymbol{c}_m^T \boldsymbol{c}_m}\sqrt{\boldsymbol{y}^T \boldsymbol{y}}} = \frac{n-\rho d_{\min}}{\sqrt{n}\sqrt{(n-\rho d_{\min})-\rho(1-\rho)d_{\min}}}. \tag{4}$$

For the case D = 1, equation (4) represents the largest possible hyper-cone to guarantee disjoint equivolume decoding regions. Thus (4) determines the angle for each decoding region R'_m ,m = 1, 2, . . . , 2^k.

The point $\boldsymbol{r}$ is contained in the hyper-cone R'_m if the angle $\mathit{1}$ formed by the lines ℓ ($\boldsymbol{O}$, $\boldsymbol{c}_m$) and ℓ ($\boldsymbol{O}$, $\boldsymbol{r}$) is less than $\mathit{1}_m$:

$$\boldsymbol{r} \in R'_m \Leftrightarrow \theta < \theta_m \Leftrightarrow \cos\theta > \cos\theta_m \quad \text{for } \theta, \theta_m \in [0, \pi]. \tag{5}$$

Since the decoding regions are disjoint, there is at most codeword $\boldsymbol{c}_m$ which satisfies

$$\frac{\boldsymbol{c}_m^T \boldsymbol{r}}{\sqrt{\boldsymbol{r}^T \boldsymbol{r}}} > \frac{n-\rho d_{\min}}{\sqrt{(n-\rho d_{\min})-\rho(1-\rho)d_{\min}}} \tag{6}$$

so that $\boldsymbol{r}$ is decoded into $\boldsymbol{c}_m$ when there exists such a codeword which satisfies condition (6) else a retransmission is requested.

Decreasing $\mathit{1}_m$ decreases the volume of the R'_m thereby increasing the volume in R^n which contains the vectors which are not decoded into any codeword. The result is an increase in the probability of retransmission and a reduction in the probability of decoder error. The value of $\mathit{1}_m$ is controlled by the parameter D through equation (4). Note for the case D = 1, R'_m is the largest hyper-cone with vertex at **?** centered on $\boldsymbol{c}_m$ which can be included entirely in the hyper-pyramid R_m. This decoding region corresponds to the incomplete decoding rule suggested by Forney [6].

3 Performance

For convenience, assume the all zero codeword c_0 was sent. By equation (6), the probability of correct reception is given by

$$P(C) = Pr\{\theta < \theta_m\}. \tag{7}$$

While It is most convenient to use the cosine to test the angle formed by the received vector and each codeword, it is most convenient to use the cotangent of this angle to evaluate the performance. The cotangent of $\boldsymbol{1}_m$ is given by:

$$\cot \theta_m = \frac{\boldsymbol{c}_m^T \boldsymbol{P}_{\boldsymbol{c}_m} \boldsymbol{y}}{\sqrt{\boldsymbol{c}_m^T \boldsymbol{c}_m}\sqrt{\boldsymbol{y}^T (\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{c}_m})\boldsymbol{y}}} = \frac{n - \rho d_{\min}}{\rho\sqrt{d_{\min}(n - d_{\min})}} \tag{8}$$

and the cotangent of $\boldsymbol{1}$ is

$$\cot \theta = \frac{\boldsymbol{c}_0^T \boldsymbol{P}_{\boldsymbol{c}_0} \boldsymbol{r}}{\sqrt{\boldsymbol{c}_0^T \boldsymbol{c}_0}\sqrt{\boldsymbol{r}^T (\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{c}_0})\boldsymbol{r}}} \tag{9}$$

where $\boldsymbol{Pc}_m = \boldsymbol{c}_m\,[(c_m{}'\;\; \boldsymbol{c}_m)^{-1}]\boldsymbol{c}'$ is the orthonormal projection operator onto the one-dimensional subspace spanned by $\boldsymbol{c}_m$.

Since each of the $r(iT)$ is a Gaussian random variable with mean -1 and variance $F^2 = N_0/2ℰ$, %(n - 1) cot 1 has a noncentral-t probability density function [7]:

$$f_T(t) = \frac{(n-1)^{\frac{(n-1)}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-2}{2}}} e^{-\frac{n(n-1)^2}{2\sigma^2(t^2+n-1)}} \frac{1}{(t^2+n-1)^{\frac{n}{2}}} \int_0^\infty x^{n-1} e^{-\frac{1}{2}\left(x - \frac{t\sqrt{n}}{\sigma\sqrt{t^2+n-1}}\right)^2} dx \tag{10}$$

Thus, equation (7) is equivalent to

$$P(C) = Pr\left\{\sqrt{n-1}\cot\theta > \sqrt{n-1}\cot\theta_m\right\} = \int_{\sqrt{n-1}\frac{n-\rho d_{\min}}{\rho\sqrt{d_{\min}(n-d_{\min})}}}^{\infty} f_T(t)dt. \tag{11}$$

A decoding error occurs whenever the received vector is contained in a decoding region corresponding to a codeword other than the one originally transmitted. Suppose $\boldsymbol{c}_0$ was sent, then

$$P(E) = \Pr\left\{\boldsymbol{r} \in R'_m \text{ for all } m \neq 0\right\} \tag{12}$$

which may be bounded using the union bound on the pair wise error probabilities:

$$P(E) \leq \sum_{d=d_{\min}}^{n} A_d P_{2,d}(E) \tag{13}$$

where $\{A_d\}n/d=0$ is the weight distribution of the code and $P_{2,d}(E)$ is the probability that the **r** is decoded into $\mathbf{c}_m$ when c_0 was sent where d_H ($\mathbf{c}_0$, $\mathbf{c}_m$) = d. The pair wise error probability may be bounded as follows: Let N_d be the angle between ℓ (*?*, $\mathbf{c}_0$) and ℓ (*?*, $\mathbf{c}_m$) which is given by
A necessary condition for an error is

$$E' : \quad \phi_d - \theta_m \leq \theta \leq \phi_d + \theta_m \tag{15}$$

$$\phi_d = \cos^{-1} \frac{\mathbf{c}_0^T \mathbf{c}_m}{\sqrt{\mathbf{c}_0^T \mathbf{c}_0}\sqrt{\mathbf{c}_m^T \mathbf{c}_m}} = \cos^{-1} \frac{n-2d}{n}. \tag{14}$$

the probability of which is

$$P(E') = Pr\left\{\phi_d - \theta_m \leq \phi_d + \theta_m\right\} = \int_{\sqrt{n-1}\cot(\phi_d+\theta_m)}^{\sqrt{n-1}\cot(\phi_d-\theta_m)} f_T(t)dt. \tag{16}$$

$P(E')$ is the probability that **r** is contained in the annular region defined by two concentric cones centered on the line ℓ (*?*, $\mathbf{c}_0$) where the inner cone defined by the angle N_d - 2_m and the outer is defined by the angleN_d + 2_m. Thus, the pair wise error probability may be bounded by the ratio of volume of the decoding region R'_m to the volume of this annular region times the probability that **r** is contained in the annular region [8]:

$$P_{2,d}(E) \leq \frac{\int_0^{\theta_m} \sin^{n-2} x dx}{\int_{\phi_d-\theta_m}^{\phi_d+\theta_m} \sin^{n-2} x dx} \int_{\sqrt{n-1}\cot(\phi_d+\theta_m)}^{\sqrt{n-1}\cot(\phi_d-\theta_m)} f_Z(z)dz \tag{17}$$

From these equations, the probability of retransmission may be computed as

$$P(R) = 1 - P(C) - P(E). \tag{18}$$

The performance of hybrid-ARQ error control is measured by two parameters: the throughput and the probability of undetected error [3, 4]. The throughput is defined as the expected value of the number information bits accepted per transmitted bit and is a measure of the efficiency of the error control system. The actual expression depends on details of the retransmission protocol Throughput decreases as $P(R)$

increases and is bounded from above by the code rate k/n. An undetected error occurs when a received word is accepted and that received word is different from the originally transmitted word. The probability of undetected error is given by [3, 4]

$$P(U) = \frac{P(E)}{1 - P(R)}. \tag{19}$$

4 Results and Conclusion

Throughput and reliability curves for the (23,12) Golay code are shown in Figures 2 and 3, respectively. For comparative purposes, these plots include the performance curves of traditional hard-decision hybrid-ARQ decoding. Figure 3 shows that for $P(U)$ fixed at 10^{-10}, the soft-decision curve corresponding to $p = 0.8$ is roughly 3 dB inside the hard-decision curve corresponding to $t' = 2$ and approximately 2 dB inside the curve corresponding to $t' = 1$. The curves of Figure 2 show that the normalized throughput of the hard-decision decoder is greater than that of the softdecision decoder under the same conditions. However , the normalized throughput for both decoding algorithms approaches unity as the bit signal-to-noise ratio increases. Thus, at high signal-to-noise ratios, the performance of the two decoding algorithms is equivalent as measured by throughput but different as measured by the probability of undetected error. As an example, it is seen from Figure 3 that at high signal-to-noise ratios, the $P(U)$ curve corresponding to $p = 0.8$ is 2 dB inside the hard-decision $P(U)$ curve corresponding to $t' = 2$ and 1 dB inside the $t' = 1$ curve. Under the same conditions, the normalized throughput curves of Figure 2 show the throughput to be the same. A further reduction in p results in an even greater coding gain without severely affecting the throughput. This illustrates that at high signal-to-noise ratios, the soft-decision decoding algorithm is capable of outperforming the hard-decision decoder by 2 dB or more.

The throughput curves for the modified correlation decoder shown in Figure 2 illustrate the dependence of throughput on the bit signal-to-noise ratio and the parameter p. The probability of undetected error, plotted in Figure 3, illustrates a similar functional dependence. The curves in both figures illustrate the throughput - reliability trade-off typical of error control schemes using retransmissions. In the soft-decision case, both the throughput and the undetected error probability increase with p for fixed SNR. This is due to the increase in the volume of the decoding region R'_m. The soft-decision hybrid- ARQ correlation decoder allows improved performance over that attainable by the hard-decision decoder and allows more flexibility in performance characteristics than its hard-decision counterpart.

The modified decoding rule also permits the use of a low complexity decoder which approximates the ideal decoder quite well. This low-complexity decoder is presented in [9].

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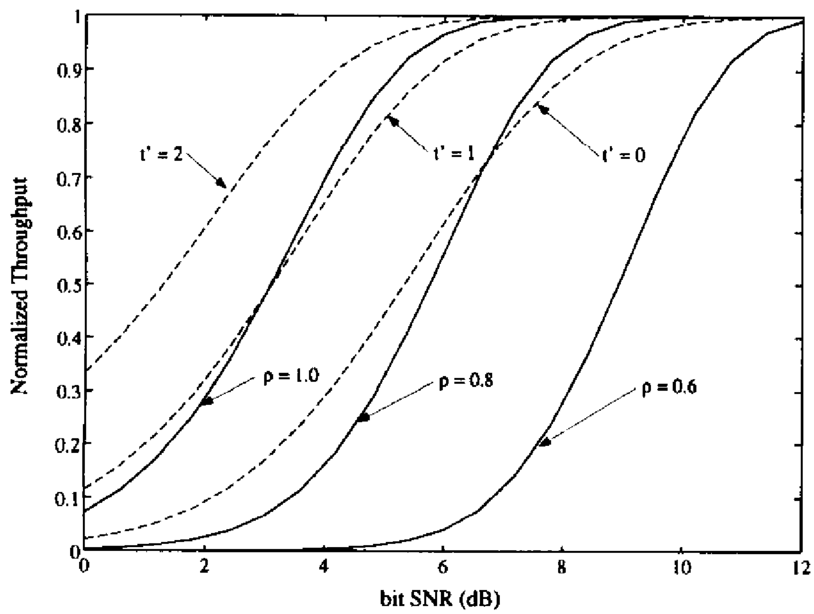

Figure 2: Normalized Throughput for the (23,12) Golay Code for Various Values of *p* and *t′* (solid lines = soft-decision decoding, dashed lines = hard-decision decoding).

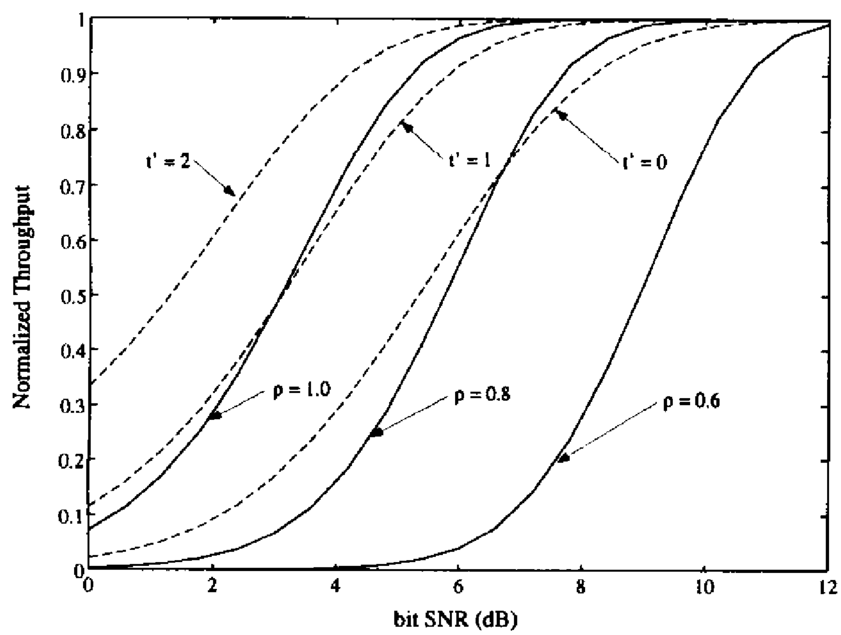

Figure 3: Probability of Undetected Error for the (23,12) Golay Code for Various Values of *p* and *t′* (solid lines = soft-decision decoding, dashed lines = hard-decision decoding) .