A MODEL FOR SEQUENTIAL DECODING OVERFLOW DUE TO A NOISY CARRIER REFERENCE[1]

JAMES W. LAYLAND
Communications Systems Research Section
Jet Propulsion Laboratory

Summary. An approximate analysis of the effect of a noisy carrier reference on the performance of sequential decoding is presented. The analysis uses previously developed techniques for analyzing noisy reference performance for medium-rate uncoded communications adapted to sequential decoding for data rates of 8 to 2048 bits/s. In estimating the 10^{-4} deletion probability thresholds for Helios, the model agrees with experimental data to within the experimental tolerances.

I. Introduction. Convolutional encoding with sequential decoding is a very powerful technique for communicating at low error probability with deep space probes. It has been used successfully with all recent Pioneer spacecraft and will be used with Helios. Most, if not all, of the performance data for this coding technique have been developed without regard to the effects of noisy reference signals in carrier or subcarrier tracking loops. These effects must be known with fair accuracy for the optimal design of telemetry links with sequential decoding.

II. Sequential Decoding - The Computation Problem. Convolutional codes which are sequentially decoded typically have a large enough constraint length so that the undetected error probability out of the decoder is negligible compared to the probability that a block cannot be successfully decoded in the time allowed. Thus, the limiting factor for sequential decoding is the probability that large amounts of computation are required to decode a frame of the code, rather than the probability of error. Experimental and theoretical work has shown that the distribution of the number of computations c_1 needed by the decoder to penetrate 1 bit deeper into the convolutional code tree has a Pareto distribution

$$\Pr\{c_1 > x\} \sim kx^{-\alpha} \tag{1}$$

The exponent α is the noisy channel error exponent (Ref. 1), and k is a small constant, found by Heller (Ref. 2) to be 1.9.

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The computation distribution is somewhat changed when an entire code frame is considered. The number of computations needed by the decoder to penetrate from a depth of N-1 to a depth of N is certainly not independent of the number of computations needed to penetrate from depth N to depth N+l. However, the number of computations needed to penetrate from depth N-1 to depth N is independent of the number of computations needed to penetrate from depth N+j-1 to depth N+j, if $|j|$ is large enough. The magnitude $|j|$ which is large enough to establish independence is believed to be a function of the signal-to-noise ratio (SNR). The Pareto distribution has the property that for moderately large N, the probability of a single long computation of length 2N is much greater than the probability of two smaller computations, each of length approximately N. As a result, whenever the number of computations needed to decode a code frame is large, its distribution is dominated by single long computations, representing decoder penetration from $M-\ell$ to M, for some M,ℓ. Where the number of computations is small, however, the distribution function represents the sum of many small computations.

III. Carrier Loop Effects. The receivers of the JPL Deep Space Network use a narrowband phase-locked loop, tracking the carrier component of the signal received from the spacecraft, to provide a coherent reference for demodulation of the telemetry sidebands on that signal. The bandwidth of the phase-locked loop is generally wide enough to track out received doppler, yet narrow with respect to the telemetry data rate, so that the phase of the reference signal is essentially constant while several tens of bits are being received. If a phase error ϕ exists between the received carrier and the local carrier reference, the amplitude of the signal entering the decoder is degraded by a factor $\cos\phi$.

The probability distribution of the phase error ϕ in a phase-locked loop has been derived elsewhere (Ref. 3) to be

$$P(\phi) = \frac{\exp(\rho_L \cos \phi)}{2\pi I_0(\rho_L)} \qquad (2)$$

where $I_0(\ )$ is the zeroth-order modified Bessel function and P_L is $2P_c/N_oW_L$.

Lindsey (Ref. 4) has used this phase error distribution to derive performance curves for the biorthogonal block code which account for noise in the reference signal under the (reasonable) assumption that the phase error ϕ is constant while a code block is being received. This assumption is valid when the bandwidth of the phase-locked loop is narrow with respect to the rate at which code blocks are received. The theoretical bit-error probability curves, which are functions of bit SNR, can thus be considered functions of the phase error ϕ that existed while each block was being received, and the bit SNR that would exist if the carrier reference were perfect. Averaging over the probability distribution of phase error ϕ results in performance curves which show the expected bit-error probability of the coded system, and account correctly for the losses due to a noisy carrier reference.

For sequential decoding, if the phase error ϕ is essentially constant over a frame of data, then it is clear that we can average the erasure probability curves conditioned on bit SNR (and ϕ) over the distribution of phase error, and derive a valid estimate of decoding performance with a noisy reference. This condition, however, requires that the phase-locked loop be extremely narrow with respect to data rate, an unrealistic assumption at medium and low data rates.

Let us consider the characteristics of the distribution of the number of computations per frame in the region where the number of computations is large. As noted before, the computations on any block in this region are dominated by single large computations that result from the decoder extending its penetration from depth $M-\ell$ to depth M, for some M and some ℓ much less than the frame length. If the phase error ϕ is essentially constant for these ℓ or more bits, then the distribution of computations can be considered as being conditioned on ϕ for large numbers of computations per frame. The validity of this high-rate approach appears to extend at the lowest to 10^3 bits/sec.

At extremely low data rates, the sequential decoding noisy reference performance is again relatively well behaved. At low data rates, the time-varying carrier phase error varies rapidly enough that its effect is almost completely averaged out within one symbol time; there is no correlation between carrier reference noise in adjacent symbols, and the resultant channel model is white and Gaussian, with a somewhat degraded signal power.

To obtain numerical results for the high rate extreme model, an experimental computation distribution family is used as a basis. The author has used the distribution determined by Dolainsky (Ref. 5) for the Helios frame of 1152 bits. This data, shown in Figure 1, was approximated by functions of bit SNR(R), and average number of computations per bit (N). The chosen approximating functions are of the form

$$\Pr\{c_L > N^*L\} = \exp\left\{ \sum_{\substack{n=-1,1 \\ r=0,2}} A_{n,r} R^r (\ln N)^n \right\} \qquad (3)$$

The coefficients $\{A_n,r\}$ were determined by a two-dimensional, least-squares polynomial fit, and appear in Table 1. The frame length is L.

The solid lines of Fig. 1 show this approximation. Having thus been defined as functions of bit SNR, it is a trivial task to express these distributions as functions of total bit SNR and carrier phase error ϕ, and to numerically integrate them over the distribution of ϕ (Eq. 2) for various values of the carrier tracking loop SNR. As noted above, this approach is valid only at high data rates.

IV. Medium Rate Model. Sequential decoding data rates between 10 and 10^3 bits/s must be categorized as medium data rates from the performance modeling standpoint. Their performance lies somewhere between the performance predicted by the high- and low-rate models. There are two primary difficulties associated with establishing an accurate performance model for these medium data rates. The first is that we do not really know over what interval of data record the sequential decoding search is defined. It could be argued that the computation distributions for most long searches are defined by a noisy “barrier” of perhaps 3 to 10 bits in length. However, the backward search depth in sequential decoding is on the order of a constraint-length, or two, and the noise at each symbol encountered within a search must necessarily affect the number of computations needed in that search. Finally, we note that individual searches can interact up to the limit of the frame length, where they are forcibly terminated. None of these correctly represents the effective memory duration of the decoder, yet all are partially correct. The second problem is that the carrier reference errors interact with the record length that defines the searches. For example, the data rate/loop bandwidth ratio δ may be such that the carrier phase reference is essentially constant over the 1 to 3 bit times that define most short searches; yet when a large carrier reference phase error occurs a long search results, with a number of computations, which is dependent not only upon the section of data over which the carrier reference is poor, but upon a long preceding section of data within which the carrier reference varies significantly. From these considerations, it is not expected that the noisy reference performance of sequential decoding can be accurately modeled with any simple technique.

Fairly tractable techniques exist for calculating the performance of uncoded telemetry at medium data rates (Ref. 6). They exist because the error probability in uncoded telemetry depends uniformly upon the signal, noise, and carrier reference statistics over a predetermined interval of the data signal, and not at all outside that interval. The approach which has been followed in modeling the sequential decoding performance at medium data rates has been to use an uncoded medium rate technique to extend the validity of the high-rate model to lower data rates.

Extension of the high-rate model into the medium-rate region involves a number of assumptions and approximations: (1) the decoding computation distribution depends predominantly upon isolated long searches that are defined in structure over some fixed length segment of the data record, called T_m; (2) the computation distribution for long searches depends uniformly upon the carrier reference noise throughout the T_m interval; (3) the correlation between T_m intervals within a frame is independent of their position within that frame. The analysis technique implied by these assumptions is as follows: the channel signal-strength statistics are computed for the signal average over the T_m interval using the techniques for uncoded telemetry. The sequential decoding performance is computed conditioned upon the channel signal strength, and then averaged over the distribution of the channel signal strength. Specifically we compute

$$P_{deletion} = \int_0^2 \Pr\left\{ \text{deletion} \middle| \text{No. of computations, SNR} = \frac{E_b}{N_0}(1-x/2)^2 \right\} h(x)dx + \int_2^\infty h(x)dx \qquad (4)$$

where

$$\left.\begin{aligned} h(x) &= \frac{\alpha}{\pi} e^{2\sqrt{ab}} (\rho_L' x)^{-\frac{1}{2}} \exp\{-a\rho_L' x - b/(\rho_L' x)\} \rho_L' \\ \rho_L' &= \text{effective loop signal-to-noise ratio (SNR)} \\ a &= \frac{B(\delta)}{4} \left(1+\sqrt{1+4/B(\delta)}\right) \\ b &= a-1+1/4a \\ \delta &= 1/W_L T_m \\ B(\delta) &= 1/\left[\delta - (\delta^2/4)(1-e^{-4/\delta})\right] \end{aligned}\right\} \qquad (5)$$

This approximate distribution has been developed for analysis of mediumrate uncoded communications (Ref. 6). The effective loop signal-to-noise ratio ρ_L', is determined parametrically by $\rho_L = \rho_L' \exp\{1/(2\rho_L')\}$, where ρ_L is the true carrier loop SNR in the operating bandwidth, as computed by Lindsey (Ref. 7), and includes the effects of bandwidth expansion of the limiter-phase-locked loop. Results computed with this distribution merge smoothly with high-rate, and low-rate extremes as a function of δ, the normalized data rate.

The performance computed by this technique is significantly dependent upon the value of T_m used. The true value is, as noted above, unknown. It is clear that the effective value for T_m depends upon the number of computations in each search. If the number of computations-per-bit is very small, then all decoding decisions are made on the basis of very short pieces of the received data, and $T_m = 1$ is appropriate. On the other hand, when the number of computations-per-bit is large, at least some of the decoding decisions must be made over long segments of the received data, and T_m may be much greater than one. If we let N be the average number of computations per bit in the frame, and approximate

$$T_m \simeq 2(1-\log_2(1+N/2)/N)\cdot T_b \tag{6}$$

we compute medium rate computation distribution curves which agree, on the average, satisfactorily with experimental ones. This agreement is only approximate, since it represents a compromise between differing modes of behavior at different data rates. The true behavior of the sequential decoding algorithm depends in a complex way upon the data-rate/phase-process-bandwidth which cannot be modeled exactly by separating them as is done here. It would, of course, be much more correct to compute the numerical sequential decoding model using the true joint distribution of T_m,W_L, and N, but such would require much more computing time than is used by the current model, and the needed statistics are not currently available.

V. Modeled Decoding Performance. The modeled decoding performance estimate is perhaps best displayed graphically. Figures 2a and 2b show the deletion probability as a function of the total-power-to-noise density ratio (P_T/N_0) for the Helios modulation indices (MI) and data rates. A 12-Hz carrier-tracking loop is assumed in the DSN receiver. These Mod. indices are a compromise optimum set for the overall range of data rates, and were chosen based largely upon the sequential decoding model described herein.

Figure 3 shows the modeled total-power-to-noise density ratio required to achieve a 10^{-4} deletion probability for the Helios rates and modulation indices. The results of a series of system tests is also shown. Viewed conservatively, the modeled performance is an adequate, although far from exact, representation of total system performance of sequential decoding.

VI. Commentary and Future Work. At this point, modeling of the sequential decoding noisy reference performance by the techniques described here appears to be at, or near, a dead-end. The results are close, and perhaps usable for system design, but they are not exact. There is no obvious physically-justifiable change to the modeling technique or parameters that can be applied with assurance of improving the result, or of representing more exactly the physical process of sequential decoding.

The choice of T_m is perhaps the weakest link within the model. T_m has been used by Stolle (Ref. 8) as a free parameter to manipulate a model similar to the one presented here into agreement with experimental data. The approach is successful, and clearly a good one for improving a model with experimental data. However, the effective T_m/T_b ratio determined this way is largest at 128 bits/s, and the physical interpretation of that fact is not clear.

To step from the simple but serviceable model presented here to an exact model for sequential decoding seems extremely difficult; for medium data rates, this step will only be achieved through an in-depth understanding of the complex interaction between the sequential decoding algorithm and the noisy carrier reference phase process.

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Table 1. $A_{n,r}$ for Helios frame

r \ n	-1	0	1
0	2.397	8.824	-0.9887
1	-0.5331	-6.788	1.569
2	0.02303	0.8848	-0.8543

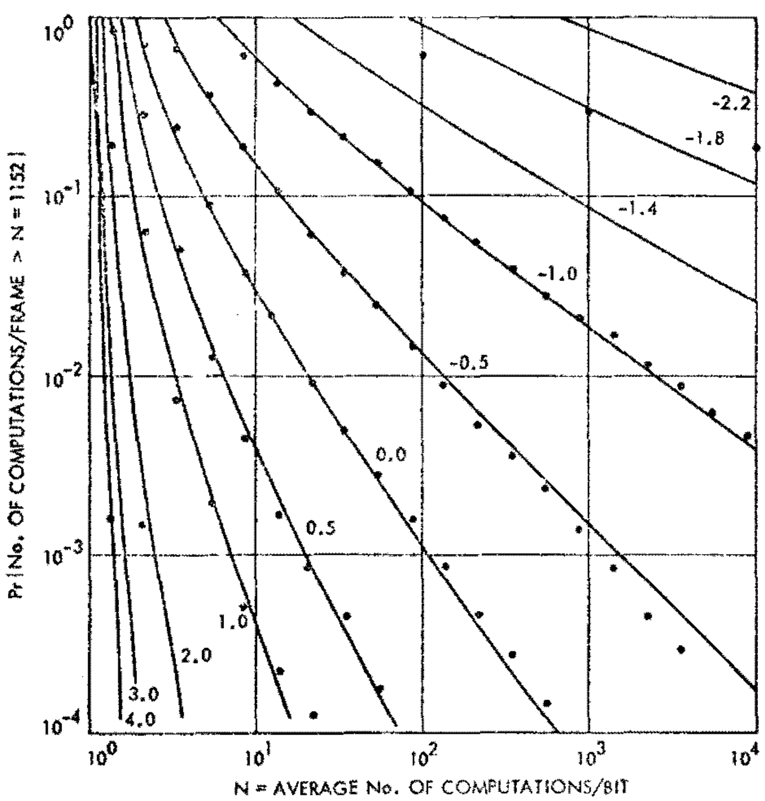

Fig. 1 . Distribution Of computations for sequential decoding of Helios frame

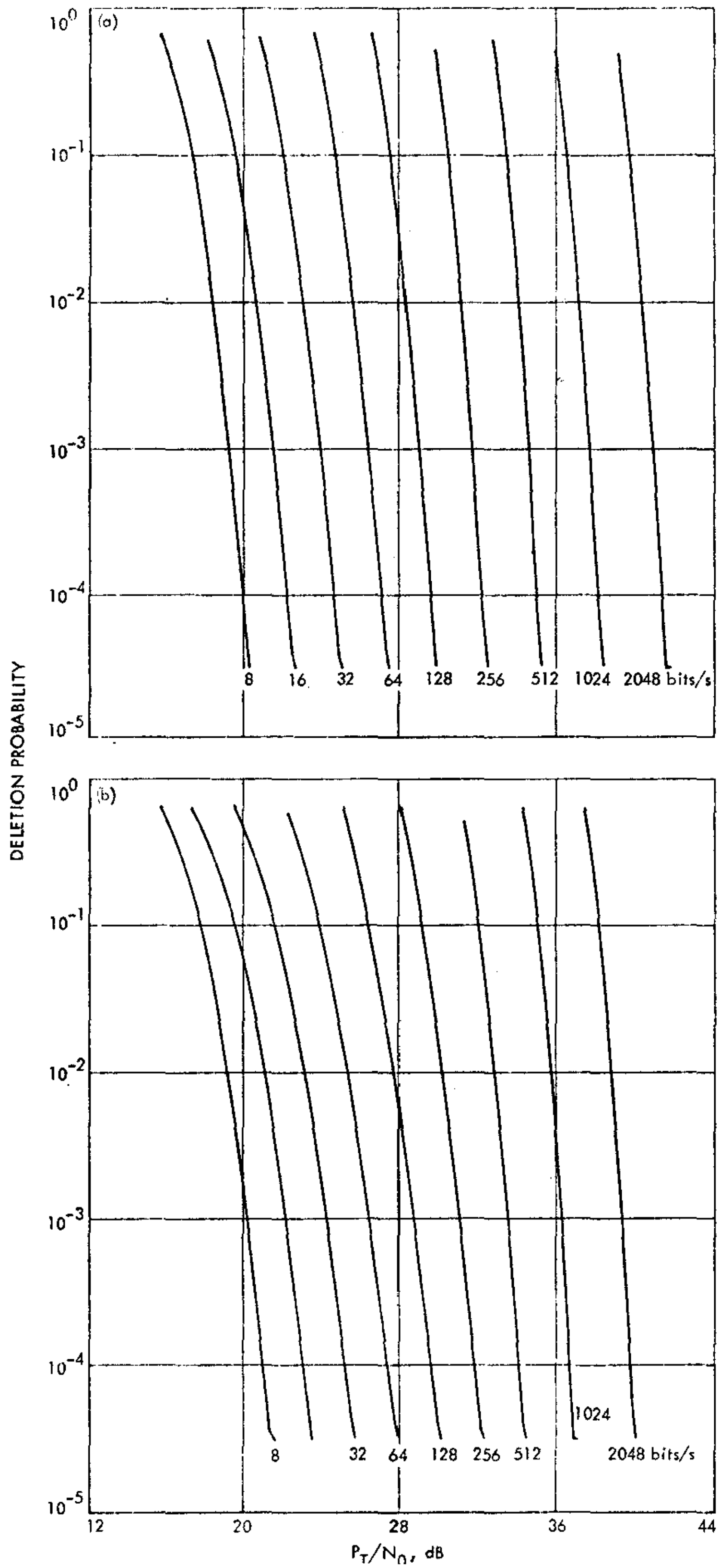

Fig. 2. Modeled deletion probability for Helios; (a) MI = 42 deg, (b) MI = 55 deg

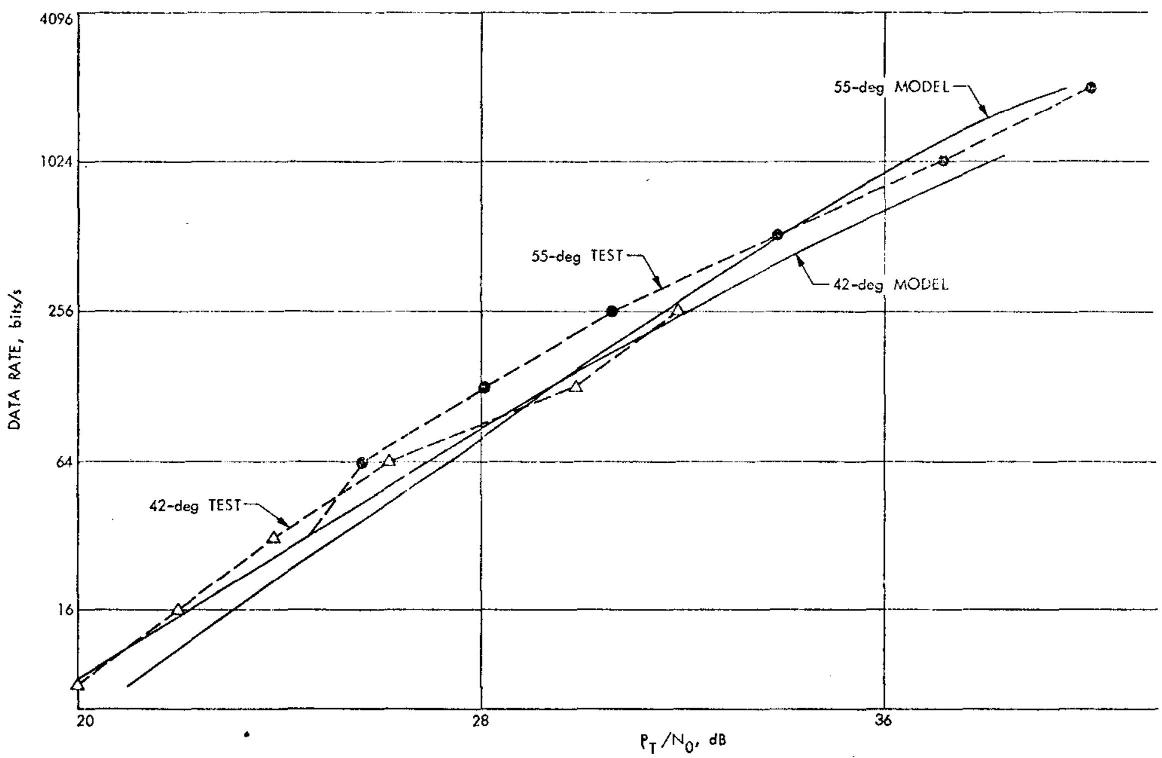

Fig. 3. P_T/N_o thresholds for Helios for 10^{-4} deletion probability; comparison of model and extrapolated experiment