Summary. - Multiple-amplitude and phase-shift-keyed (MAPSK) signal set selection is influenced by factors such as average and/or peak signal-to-noise ratio for a given error probability, dynamic range of signal amplitudes, simplicity of generation and detection, and number of bit errors per adjacent symbol error. This paper compares two possible quadrature-amplitude-shift-keyed (QASK) signal sets for the case where the number of bits per symbol is odd (for the even-bit case, the square array is the only viable QASK choice). The symmetric QASK version outperforms the rectangular QASK set at a very modest implementation penalty. This permits symmetric QASK to be considered in future odd-bit system studies.

Introduction. - Quadrature Amplitude Shift Keying (QASK) is a term used to characterize signal sets with essentially rectangular signal space arrays. The maximum likelihood decision regions in the signal space are essentially all rectangular. Actually, these regions are all square or rectangular if, and only if, the two quadrature signal components are independent, as in the various square and rectangular signal arrays of Fig. 1. Although the square arrays of Fig. 1 are the best QASK arrays for data symbols consisting of even numbers of bits, the rectangular arrays for the odd-bit cases are not. These rectangular arrays can be improved upon (reduced packing coefficients, peak coefficient, and dynamic range) by a more symmetric array structure (see, for example, Fig. 2), but at a penalty in increased complexity of implementation. Further, since the quadrature components are not independent, the maximum-likelihood regions are not all rectangular, as shown in Fig. 3a for a particular 5-bit array. However, suboptimal, but rectangular, decision regions such as that of Fig. 3b are possible which yield insignificantly degraded performance at typical error levels.

For the case of interest, the data source is assumed to be a random binary stream, hence block buffering and two-dimensional Gray coding are required. Figure 4 shows the basic system with rectangular QASK shown in Fig. 4a and symmetric QASK in Fig. 4b. Successive blocks of k bits \( k = 2n + 1, \ k \geq 5 \) are accumulated in a buffer in the transmitter; then input to two Gray-coded D/A converters. Rectangular arrays do not require any processing between buffers and converters since the \( 2n + 1 \) bits are simply split into \( n + 1 \) and \( n \) bits (Fig. 4a). Symmetric arrays require processing to add one bit and manipulate the \( 2n + 2 \) bits before splitting into two groups of \( n + 1 \) bits each (Fig. 4b). The two analog outputs are multiplied by cosine and sine carriers, and thus are called, respectively, the in-phase and quadrature signals. The receiver requires

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Dr. Smith is with the Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California.
a pair of quadrature references, matched filters, A/D converters, buffer, and, in the case of a symmetric array, further processing during transfer to recover the original data stream (Fig. 4).

The three implementation penalties associated with a symmetric pattern are: (1) an impure Gray code penalty (defined later), (2) the extra one bit in the quadrature channel converters, and (3) the digital processing necessary in both transmitter and receiver to create the symmetric structure. The penalties must be weighed, however, against the three performance parameter improvements: (1) a reduced packing coefficient (a measure of average symbol signal-to-noise ratio required), (2) a smaller peak coefficient, and (3) a lower signal set dynamic range.

This paper examines the improvement in performance and the corresponding implementation penalty for a particular class of symmetric QASK signal sets with 5, 7, 9 ... bits per symbol (the 3-bit set is not considered). The average symbol signal-to-noise ratio is reduced at least 1 dB by using a symmetric QASK signal rather than the rectangular one, and the peak power and dynamic range are reduced 1.5 to 2 dB. The added complexity (ignoring the added bit requirement in the quadrature converters) is a negligible amount of digital processing, independent of the number of bits per symbol.

Performance of MAPSK Sets. - The "Gilbert approximation" to the probability of symbol error $P_E$ for any multiple-amplitude and phase-shift-key (MAPSK) array is [1, 2]

$$P_E \approx N Q(\Delta)$$

where $Q$ is the Gaussian tail function [3, equation 26.2.3]; $\Delta$, the normalized "Gilbert distance" of the array, is the least distance (in signal-to-noise ratio) from a signal point to a decision boundary; and $N$, the "Gilbert number" of the array, is the average (over the array) number of distinct decision boundaries at an exact distance $\Delta$ from a signal point. $N$ may be thought of as the average number of "nearest neighbors" in the array, since a decision boundary at the minimum distance $\Delta$ implies another signal point equidistant beyond it. The "packing coefficient" $C_p$ is defined by

$$C_p \Delta (\Delta)$$

$$\sum_{i=1}^{M} \frac{1}{M} \sum_{i=1}^{M} \frac{d_i^2}{d_i^2}$$

where $M$ is the number of signal points in the array, and $d_i$ is the distance from the origin to the $i$th signal point (normalized by the Gilbert distance). The average symbol signal-to-noise ratio $R_d$ is then given by [2]

$$R_d = \frac{C_p \Delta^2}{2}$$

To a good approximation at low symbol error probability $P_E$, equations (1) and (3) can be combined to yield [2]

$$R_d = \frac{C_p R_{do} \ln N}{1 - (\ln N)/R_{do}}$$

where

$$R_{do} = \left[ Q^{-1}(P_E) \right]^2 / 2$$
Thus, since \( C_p \) and \( N \) are fixed numbers for any specified array, \( PE \) can be described as a function of \( R_d \) using equations (1) and (3), or \( R_d \) as a function of \( PE \), using equations (4) and (5). Therefore, \( C_p \) and \( N \) of an array suffice to characterize the average symbol performance of the array. Further, in comparing two arrays with essentially equal Gilbert numbers, the ratio of required average symbol signal-to-noise ratios equals the ratio of packing coefficients. (Later, inclusion of the Gray code penalty \( G_p \) will permit comparison of arrays at equal average bit error probabilities).

Similarly, the peak symbol signal-to-noise ratio \( R_p \) is related to the "peak coefficient" \( C_{pk} \), the maximum of the \( d_i \)'s in (2), by a similar relation [4]

\[
R_p = C_{pk} \Delta^2 / 2
\]

and in comparing two arrays, the ratio of peak coefficients is a measure of the ratio of peak powers required by the arrays. Finally, the signal set dynamic range \( D \) is defined as the ratio of the highest and lowest signal energy levels in an array.

Thus, in comparing any two arrays, the packing coefficients, Gilbert numbers, peak coefficients, and dynamic ranges characterize differences required in average, maximum, and minimum power (or signal-to-noise ratios).

Rectangular QASK. - It is geometrically obvious (and easily proved mathematically) that the rectangular array of \( 2^n \) by \( 2^{n+1} \) signal points has the smallest packing coefficient of all rectangular arrays of \( 2^k \) points, where \( k = 2n + 1 \). We define \( L_1 = 2^{n-1}, L_2 = 2^n \), and

\[
S(\ell) \triangleq \sum_{i=1}^{L_1} (2i-1)^2 = \ell (4\ell^2 - 1)/3
\]

Since there are \( L_1L_2 \) signal points in each quadrant, the packing coefficient \( C_p \) can be written as

\[
C_p = \frac{1}{L_1L_2} \sum_{i=1}^{L_1} \sum_{j=1}^{L_2} [(2i-1)^2 + (2j-1)^2]
\]

\[
= \frac{1}{L_1L_2} \left[ L_2S(L_1) + L_1S(L_2) \right]
\]

\[
= \frac{2}{3} (40 \ 2^{2n-4} - 1)
\]

(8)

For reasons which will be obvious later, we define

\[
\gamma \triangleq 2^{n-2} = L_1/2 = L_2/4
\]

(9)

and note that \( M = 32\gamma^2 \). Thus, we can view the rectangular signal set as an array of 32 blocks of \( \gamma^2 \) points each, arranged in 4 rows and 8 columns.
Using (8) and (9) the packing coefficient becomes

$$C_p = \frac{2}{3}(40\gamma^2 - 1)$$

(10)

Since the array has four "corner" points, each with two "nearest" neighbors, $4L_1 + 4L_2 - 8$ "side" points each with three nearest neighbors, and $(2L_1-2)(2L_2-2)$ "interior" points each with four nearest neighbors, the Gilbert number can be shown to be

$$N = 4(1 - 3/16\gamma)$$

(11)

Also, the peak coefficient $C_{pk}$ is

$$C_{pk} = (2L_1 - 1)^2 + (2L_2 - 1)^2 = 80\gamma^2 - 24\gamma + 2$$

(12)

and the dynamic range $D$, the ratio of the highest to the lowest signal level, is

$$D = 40\gamma^2 - 12\gamma + 1$$

(13)

Symmetric QASK. - Rectangular QASK was characterized in the last section by an $8 \times 4$ array of blocks of $\gamma^2$ points each. The symmetric QASK set described here is characterized by a $6 \times 6$ array of these blocks, with the four corner blocks deleted. (See Fig. 5b.)*

The packing coefficient $C_p$ is

$$C_p = \frac{1}{L_1 L_2} \left[ \sum_{i=1}^{L_1} \sum_{j=1}^{L_2} \left[ (2i - 1)^2 + (2j - 1)^2 \right] 
+ \sum_{i=1}^{L_1} \sum_{j=L_1+1}^{L_2} \left[ (2i - 1)^2 + (2j - 1)^2 \right] \right]$$

$$= \frac{2}{L_1 L_2} \left[ L_1 S(L) + (L - L_1) S(L_1) \right] = \frac{2}{3} (31\gamma^2 - 1)$$

(14)

The array has eight blocks of $\gamma^2$ signal points in each quadrant. One block has one "corner" point, $2(\gamma - 1)$ "side" points, and $(\gamma - 1)^2$ "interior" points. Three blocks have $\gamma$ "side" points and $\gamma(\gamma - 1)$ "interior" points. The remaining four blocks each have $\gamma^2$ "interior" points. Thus, the Gilbert number can be shown to be

$$N = 4(1 - 5/32\gamma)$$

(15)

*The symmetric 7-bit QASK signal described here differs from Compopiano and Glazer's original 7-bit signal set [5]. Both have the same performance, but the original requires more processing.
The peak coefficient $C_{pk}$ is

$$C_{pk} = (2L - 1)^2 + (2L_1 - 1)^2 = 52\gamma^2 - 20\gamma + 2$$

(16)

and the dynamic range is

$$D = 26\gamma^2 - 10\gamma + 1$$

(17)

Equations (10) through (17) are shown in Table 1 for comparison. Note that the differences in Gilbert numbers of the two kinds of arrays are negligible, and that the packing coefficient suffices for comparison of average symbol error probability performance. It can be shown that no more than 0.025-dB error results from this approximation. Table 2 gives the actual performance parameter values of the two QASK kinds of arrays for several values of $k$. Note that the difference in packing coefficients is exactly $6\gamma^2$, while the ratio decreases asymptotically (for large $k$) to 1.29 or about 1.1 dB. The ratio of peak coefficients and dynamic range decreases asymptotically to 1.54 or about 1.9 dB.

Symmetric QASK Implementation Penalties. A bit-symbol correspondence that yields exactly one-bit differences in bit representations of adjacent symbols is called a "pure" Gray code. All one-dimensional Gray codes, and two-dimensional Gray codes of orthogonal components are pure. All other Gray codes are "impure," and suffer a "Gray code penalty," $G_p$, the average numbers of bit differences per adjacent decision.

Table 1. Performance Equations of Symmetric and Rectangular QASK

<table>
<thead>
<tr>
<th>Rectangular</th>
<th>Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_p = 2(4\gamma^2 - 1)/3$</td>
<td>$C_p = 2(31\gamma^2 - 1)/3$</td>
</tr>
<tr>
<td>$N = 4(1 - 3/16\gamma)$</td>
<td>$N = 4(1 - 5/32\gamma)$</td>
</tr>
<tr>
<td>$C_{pk} = 80\gamma^2 - 24\gamma + 2$</td>
<td>$C_{pk} = 52\gamma^2 - 20\gamma + 2$</td>
</tr>
<tr>
<td>$D = 40\gamma^2 - 12\gamma + 1$</td>
<td>$D = 26\gamma^2 - 10\gamma + 1$</td>
</tr>
</tbody>
</table>

Table 2. Performance Parameter Values of Symmetric and Rectangular QASK

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\gamma$</th>
<th>$C_p$ (dB)</th>
<th>$N$</th>
<th>$C_{pk}$ (dB)</th>
<th>$D$ (dB)</th>
<th>$C_p$ (dB)</th>
<th>$N$</th>
<th>$C_{pk}$ (db)</th>
<th>$D$ (db)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>14.15</td>
<td>3.25</td>
<td>17.63</td>
<td>14.63</td>
<td>13.01</td>
<td>3.38</td>
<td>15.31</td>
<td>12.31</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>26.30</td>
<td>3.81</td>
<td>30.74</td>
<td>27.74</td>
<td>25.19</td>
<td>3.84</td>
<td>28.77</td>
<td>25.77</td>
</tr>
</tbody>
</table>
Thus, on the average, a single symbol threshold error will cause \( G_p \) bits to be in error. The "peak Gray penalty," \( G_{pk} \), is defined as the maximum (over the array) number of bit errors per single symbol threshold error. An acceptable Gray code has a \( G_p \) near unity and a \( G_{pk} \) of two. Although \( G_p \) could be incorporated into equations (1-5) relating error probability and signal-to-noise ratio, it suffices here to note that a small value of \( G_p \) has minimal impact on the equations and can be neglected. That is, it suffices to establish that \( G_p \) is near unity, since the bit error probability \( P_B \) is approximately

\[
P_B \approx \frac{G_n}{\log_2 M} Q(\Delta)
\]

(18)

Two-dimensional Gray coding is achieved by processing received bits and splitting the results, which are then input to two conventional Gray coded D/A converters (Fig. 4b). The Gray codes used are reflected binary [6] and conform to bit assignments typical of commercial Gray-coded D/A and A/D converters (see Table 3). In rectangular QASK, the \( n+1 \) bits of the in-phase and \( n \) bits of the quadrature inputs are independently Gray-coded as shown, for example, in Fig. 6a for 5-bit QASK. There the first three bits encode the in-phase signals; the second two bits encode the quadrature signal. The Gray coding is perfect! The problem is how to generate Gray-coded signals for symmetric QASK. A solution is shown in Fig. 6b for the 5-bit QASK. There the eight "end" words have been shifted to new positions to form the symmetric array. For convenience, the original 5-bit words of the rectangular array of Fig. 6a have been written as a pair of 3-bit words by adding a zero bit, \( X_1X_2X_3/Y_1Y_20 \). Below this pair of 3-bit words is written the pair of 3-bit words to be produced by the processing in order to generate the correct analog signals. The function of the two digital processing units placed between buffer pairs in Fig. 4b is to perform the changes in bits shown. Note that if \( X_2X_3\neq1 \), no "end" word has occurred and no processing need occurs. Note further that if \( X_2X_3Y_2=1 \), then \( X_2, X_3, \) and \( Y_3 \) must be complemented; while if \( X_2X_3Y_2=1 \), then \( X_3, Y_2, \) and \( Y_3 \) must be complemented. Clearly the logic necessary to implement the decisions for this 5-bit symmetric QASK array is very minor.

The most important point of this paper is that the logic just described for generating the array for 5-bit symmetric QASK is sufficient for any odd-bit symmetric QASK array. In general, an extra zero bit is added to the \( (2n+1) \)-bit input word to yield \( X_1, X_2, \ldots, X_n, X_{n+1}/Y_1, Y_2, \ldots, Y_nY_{n+1} \). The logic is (as before): if \( X_nX_{n+1}Y_{n+1}=1 \), then \( X_n, X_{n+1}, \) and \( Y_{n+1} \) must be complemented; while if \( X_nX_{n+1}Y_{n+1}=1 \), then \( X_{n+1}, Y_n, \) and \( Y_{n+1} \) must be complemented.

Table 3. Reflected Binary Gray Code

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Gray Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>001</td>
</tr>
<tr>
<td>5</td>
<td>011</td>
</tr>
<tr>
<td>3</td>
<td>010</td>
</tr>
<tr>
<td>1</td>
<td>000</td>
</tr>
<tr>
<td>-1</td>
<td>100</td>
</tr>
<tr>
<td>-3</td>
<td>110</td>
</tr>
<tr>
<td>-5</td>
<td>111</td>
</tr>
<tr>
<td>-7</td>
<td>101</td>
</tr>
</tbody>
</table>
The impact of this logic on the signal set is shown in Fig. 7. With no logic, the eight blocks $B_{11}$ through $B_{14}$ and $B_{21}$ through $B_{24}$ comprise the signal set in the first quadrant. The effect of the logic is to convert blocks $B_{14}$ and $B_{24}$ into blocks $B_{14}$ and $B_{24}$, respectively (and correspondingly in other quadrants). It can be shown that blocks $B_{14}$ and $B_{24}$ differ only in one bit positions, while edge elements in blocks $B_{14}$ and $B_{24}$ differ in exactly two bit positions. Thus "end" blocks are effectively converted to new better-placed blocks by the logic.

The logic required to effect this shifting is trivial. AND and OR gates examine the $n$th, $(n+1)$th, and $k$th bits to see if the signal point lies in blocks $B_{14}$ or $B_{24}$ (or corresponding blocks in other quadrants). If $B_{14}$, then the $n$th, $(n+1)$th, and $(k+1)$th bits are changed; if $B_{24}$, then the $(n+1)$th, $k$th, and $(k+1)$th bits are changed. Similarly in the receiver, if the $(k+1)$th bit is a 1, the $(n+1)$th bit is changed, as is either the $n$th or $k$th, depending on whether the signal block is $B_{14}$ or $B_{24}$ (which depends on the $n$th bit value).

The only double-bit changes occurring at symbol threshold boundaries are the $8\gamma$ signal point pairs where the shifted blocks interface. For $k=5$ (and $\gamma=1$), there are 4 points with a Gray code penalty of $3/2$ (i.e., an average of $3/2$ bit errors per symbol error), 4 at $4/3$, 8 at $5/4$, and the rest at 1, so the average Gray code penalty $G_p = 7/6$. For $k > 5$ (and $\gamma > 1$), there are 4 points at $4/3$, $16\gamma-4$ points at $5/4$, and the rest at 1, so $G_p = 1 + 1/6\gamma + 1/96\gamma^2$. In general, then, the Gray code penalty is no more than $7/6$, a perfectly tolerable level.

Conclusion. - Comparison of odd-bit symmetric and rectangular QASK reveals that symmetric QASK requires about 1 dB less average signal-to-noise ratio, about 1.5 to 2 dB less peak power and dynamic range, has an average Gray code penalty of less than $7/6$ with a maximum Gray code penalty of 2, and requires a trivial amount of digital processing, independent of the array size. Clearly the performance improvements more than warrant the minor implementation penalties.

References

Fig. 1 - Rectangular and Square QASK Arrays

Fig. 2 - Odd-Bit Symmetric QASK Arrays
Fig. 3a - Maximum Likelihood Decision Regions for 5-Bit Symmetric QASK

Fig. 3b - Rectangular Decision Regions for 5-Bit Symmetric QASK

Fig. 4a - Transmitter/Receiver for Rectangular QASK

Fig. 4b - Transmitter/Receiver for Symmetric QASK
Fig. 5a - $\gamma^2$ Block Structure of Rectangular QASK

Fig. 5b - $\gamma^2$ Block Structure of Symmetric QASK

Fig. 6a - Pure Gray Code for 5-Bit Rectangular QASK
Fig. 6b - Gray Code for 5-Bit Symmetric QASK

Fig. 7 - Block Notation for Odd-Bit QASK