

# CASCADED COHERENT TRACKING SYSTEMS WITH TIME-VARYING CHANNELS\*

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**Summary.** An approach is given to the analysis of two-way coherent tracking systems in which the transmitted signals have passed through linear time-varying channels. The specific channel considered is the lognormal fading channel, although the results and techniques can be extended to other channels such as the Rice and Rayleigh channels. The performance of the system is characterized by the steady state probability density function of the reduced phase error process in the second tracking loop. Particular numerical examples and system performance curves are given to illustrate the theory for channel models and two-way systems of practical interest.

**I. Introduction.** Two-way coherent tracking systems have been used for many years for the purpose of spacecraft tracking and navigation. Two-way Doppler measurements of range rate have been used in the Goddard Range and Range Rate<sup>[1]</sup> System for tracking scientific satellites, and the Unified S-band System<sup>[2, 3]</sup> for tracking deep space vehicles and the Apollo spacecraft. In both of these systems a signal is transmitted from the ground station to a spacecraft and coherently transponded on another frequency to the ground. In most cases, telemetry is added to the downlink signal, which is coherently demodulated at the ground station. Thus, two of the functions performed by the ground station receiver are Doppler measurement and coherent demodulation. It is this latter function that is analyzed in this paper, although the extension of this work to Doppler extraction is straight forward<sup>[4, 5, 6]</sup>.

While the analysis of two-way systems is fairly complete for the case of the usual additive Gaussian noise channel, another class of channels has arisen in deep-space communications, namely, time-varying channels<sup>[7, 8]</sup>. Communications through turbulent planetary atmospheres and the solar corona, for example, deviate from the usual additive noise channel in that random amplitude and phase perturbations are imparted on the signal itself. This situation has arisen in many planetary missions in the usual two-way tracking

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mode and could also occur in the case of a planetary orbiter serving as a coherent relay for a landing probe. While the one-way case was treated in [9], the purpose of this paper is to model the cascaded phase-locked system in the presence of a time-varying channel and to present results that can be used to assess the performance degradation of a coherent communication system.

For simplicity in both analysis and presentation, the system considered in this paper consists of two first-order phase-locked loops (PLL) in cascade, The performance measure used to show degradation in the two-way system due to the fading channels is the steady state probability density function (pdf) of the system phase error process  $\phi_2(t)$  reduced modulo  $2\pi$ . From these pdf's the variance of the phase error has been computed and is presented for a number of cases of interest. While the lognormal channel, which arises in the communication through planetary atmospheres, has been used as the channel model, the results can be extended to the other channels, such as the Rice and Rayleigh channel models<sup>[9]</sup>.

**II System Model.** The model of the cascaded coherent communication system is shown in Fig. 1. The first transmitter sends a carrier signal of the form

$$s_1(t) = \sqrt{2} A_1 \sin \omega_1 t \quad (1)$$

where  $A_1^2$  is the signal power and  $\omega_1$  is the radian frequency, This signal passes through a time varying channel that produces random amplitude and phase fluctuations on this signal. Including the effects of added Gaussian noise, the received signal at the transponder is thus of the form

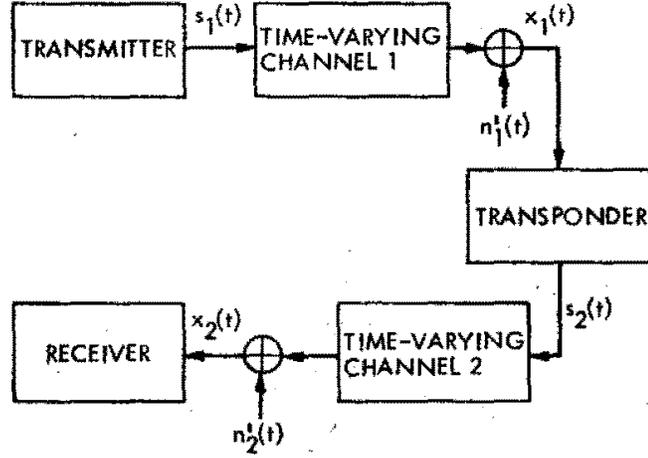
$$x_1(t) = \sqrt{2} a_1(t) \sin [\omega_1 t + \theta_1(t) + d_1(t) + \theta_{01}] + n_1'(t) \quad (2)$$

where the signal amplitude,  $a_1(t)$ , and the phase process,  $\theta_1(t)$ , are random processes introduced by the time-varying channel. The term  $\theta_{01}$  is an arbitrary constant phase angle, while  $d_1(t)$  represents any Doppler effects due to the relative motions of the transmitter and transponder. For simplicity,  $d_1(t)$  is assumed to be zero in the remainder of this paper. The additive noise,  $n_1'(t)$  is assumed to be the white Gaussian noise process of two-sided spectral density  $N_{01}/2$ .

The output of the voltage-controlled oscillator (VCO) of the transponder PLL is

$$r_1(t) = \sqrt{2} K_{v_1} \cos [\omega_1 t + \hat{\theta}_1(t)] \quad (3)$$

where  $\hat{\theta}_1(t)$  is the phase estimate of the received signal by the transponder. Thus, the phase error process of this loop can be defined as



**Fig. 1 - Cascaded Coherent Communication System Model**

$$\varphi_1(t) \triangleq \theta_1(t) + \theta_{01} - \hat{\theta}_1(t) \quad (4)$$

The stochastic differential equation of loop operation is derived in Ref. [10] and is

$$\dot{\varphi}_1(t) = \dot{\theta}_1(t) - K_1 F_1(p) \left[ a_1(t) \sin \varphi_1(t) + n_1(t) \right] \quad (5)$$

where  $K_1$  is the usual open loop gain,  $F_1(p)$  represents the loop filter transfer function, and  $n_1(t)$  can be shown to be <sup>[10, 11]</sup> approximately white Gaussian noise of two-sided spectral density  $N_{01}/2$ .

The transmitted signal from the transponder derives its frequency reference from the VCO and is of the form

$$s_2(t) = \sqrt{2} A_2 \sin \left[ \omega_2 t + G_f \theta_1(t) \right] \quad (6)$$

where  $G_f \triangleq \omega_2 / \omega_1$

This signal passes through the time-varying channel and is received in the form

$$x_2(t) = \sqrt{2} a_2(t) \sin \left[ \omega_2 t + \theta_2(t) + d_2(t) + \theta_{02} + G_f \hat{\theta}_1(t) \right] + n_2'(t) \quad (7)$$

The terms in Eq. (7) are analogous to those in Eq. (2) with  $n_2'(t)$  having a spectral density of  $N_{02}/2$ . Again as in Eq. (2) this paper will treat the case of  $d_2 = 0$ .

The VCO output in the second receiver is of the form

$$r_2(t) = \sqrt{2} K_{v2} \cos \left[ \omega_2 t + \hat{\theta}_2(t) \right] \quad (8)$$

where  $\theta_2(t)$  is the phase estimate of the received signal by the second receiver. The resulting phase error process is defined as

$$\varphi_2(t) \triangleq \theta_2(t) + \theta_{02} + G_f \hat{\theta}_1(t) - \hat{\theta}_2(t) \quad (9)$$

Thus the equation of loop operation for the second receiver is

$$\dot{\varphi}_2(t) = \dot{\theta}_2(t) + G_f \dot{\hat{\theta}}_1(t) - K_2 F_2(p) [a_2(t) \sin \varphi_2(t) + n_2(t)] \quad (10)$$

where  $K_2$  and  $F_2(p)$  are the open loop gain and loop filter transfer function respectively and  $n_2(t)$  can be assumed to be white Gaussian noise of spectral density  $N_{02}/2$ <sup>[10, 11]</sup>

For the sake of simplicity, this paper treats only the first-order loop case, that is  $F_1(p) = F_2(p) = 1$ . However, the results given here can be extended to the second- and higher-order loops.

The system model can now be completed if the form of the time-varying channels is specified. The form of the channel is assumed to be that of the lognormal channel as presented in<sup>[9]</sup>. While this model seems restrictive, the effect of using this model in the PLL analysis produces results that can be easily generalized to other channel models, in particular, the Rayleigh and Rician channel models<sup>[9]</sup>. Specifically, then, the channel assumptions are that  $a_1(t)$ ,  $a_2(t)$ ,  $\theta_1(t)$ , and  $\theta_2(t)$  are independent random processes and furthermore that  $\theta_1$  and  $\theta_2$  are Gaussian random processes of zero mean. The probability density function of  $a_1$  and  $a_2$  can be considered arbitrary in the following analysis although the lognormal density is used in specific examples. That is,

$$a_i(t) = A_i e^{\chi(t)}, \quad i = 1, 2 \quad (11)$$

where  $\chi(t)$  is a Gaussian random process of mean  $m_\chi$  and variance  $\sigma_\chi^2$ <sup>[8]</sup>. Thus,

$$p\left(\frac{a_i}{A_i}\right) = \frac{A_i}{a_i} \frac{1}{\sqrt{2\pi\sigma_\chi^2}} \exp\left\{-\frac{[\ln(a_i/A_i) - m_\chi]^2}{2\sigma_\chi^2}\right\}, \quad i = 1, 2 \quad (12)$$

Using a conservation of power argument that assumes  $E(a_i^2) = A_i^2$  it can be easily shown that  $m_\chi = -\sigma_\chi^2$ . This assumption is used in all further analysis in this paper.

It is also necessary for part of the analysis to specify spectral densities for  $\theta_1$  and  $\theta_2$ . While the spectral densities can be assumed to have arbitrary rational spectra<sup>[12]</sup>, the analysis in this paper assumes a first-order Butterworth spectrum for both  $\theta_1$  and  $\theta_2$ . That is, the two-sided spectral density of  $\theta_i$  can be written as

$$S_{\theta_i}(\omega) = \frac{\sigma_{\theta_i}^2 / W_{\theta_i}}{1 + \left(\frac{\omega}{2W_{\theta_i}}\right)^2}; \quad i = 1, 2 \quad (13)$$

where

$$\sigma_{\theta_i}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\theta_i}(\omega) d\omega \quad (14)$$

and

$$W_{\theta_i} = \frac{1}{2\pi} \frac{\int_{-\infty}^{\infty} S_{\theta_i}(\omega) d\omega}{S_{\theta_i}(0)} \quad (15)$$

The two processes  $\theta_1$  and  $\theta_2$  can be characterized by the following state equations:

$$\dot{\theta}_i = -2W_{\theta_i} \theta_i + 2\sigma_{\theta_i} \sqrt{W_{\theta_i}} u_i; \quad i = 1, 2 \quad (16)$$

where the time variable has been omitted and  $u_1$  and  $u_2$  are independent white Gaussian noises having unity two-sided spectral densities.

Thus, combining Eqs. (5), (10), and (16), the complete set of equations describing the operation of the cascaded, first-order loops can be written as

$$\left. \begin{aligned} \dot{\phi}_1 &= -2W_{\theta_1} \theta_1 + 2\sigma_{\theta_1} \sqrt{W_{\theta_1}} u_1 - K_1 [a_1 \sin \phi_1 + n_1] \\ \dot{\phi}_2 &= -2W_{\theta_2} \theta_2 + 2\sigma_{\theta_2} \sqrt{W_{\theta_2}} u_2 + G_f K_1 [a_1 \sin \phi_1 + n_1] \\ &\quad - K_2 [a_2 \sin \phi_2 + n_2] \\ \dot{\theta}_1 &= -2W_{\theta_1} \theta_1 + 2\sigma_{\theta_1} \sqrt{W_{\theta_1}} u_1 \\ \dot{\theta}_2 &= -2W_{\theta_2} \theta_2 + 2\sigma_{\theta_2} \sqrt{W_{\theta_2}} u_2 \end{aligned} \right\} \quad (17)$$

where the time dependence has been omitted.

Finally it should be noted that the time varying channels may or may not be one and the same. An example in which they are, the same is the two-way coherent communication link between a ground station and a spacecraft<sup>[4, 5, 6]</sup>. In this case, the statistical properties of  $a_1$  and  $a_2$  and those of  $\theta_1$  and  $\theta_2$  are identical. However, in the case of a coherent relay, the two transmission paths could be quite different so that the amplitude and phase fluctuations would have different characteristics.

**III. System Performance Analysis.** The function of interest in this paper is the steady state probability density function (pdf) of the phase error process  $\phi_2(t)$  reduced modulo  $2\pi$  and defined as  $\phi_2(t) \triangleq \{\varphi_2(t) + \pi\}(\text{mod } 2\pi) - \pi$ . While a general solution to this problem appears formidable, solutions to three cases of particular interest are presented. The cases depend on the bandwidths and variances of the fading components relative to the PLL bandwidths and signal amplitudes respectively.

**A. Channels with Slow Amplitude and Phase Fluctuations.** When the fading components  $a_1(t)$ ,  $a_2(t)$ ,  $\theta_1(t)$ , and  $\theta_2(t)$  change slowly with time, the phase angle of the received signal can be tracked by the loop provided the loop bandwidth is large compared to the spectral bandwidth of the fading components. Thus, in Eqs. (5) and (10),  $\theta_1$  and  $\theta_2$  can be assumed to be zero and the equations of loop operation can be solved for the conditional density,  $p(\phi_2|a_1, a_2)$ , as in the usual two-way case<sup>[5]</sup>. This also means that  $a_1(t)$  and  $a_2(t)$  are approximately constant over a relatively long period of loop operation. Hence,

$$p\{\phi_2(t)|a_1(\xi), a_2(\xi), t_0 \leq \xi \leq t\} \approx p\{\phi_2(t)|a_1(t), a_2(t)\} \quad (18)$$

Then the amplitudes  $a_1$  and  $a_2$  can be integrated out to obtain the pdf

$$p(\phi_2) = \int_{a_1} \int_{a_2} p(\phi_2|a_1, a_2) p(a_1, a_2) da_1 da_2 \quad (19)$$

where  $p(a_1, a_2)$  is the joint pdf of the amplitudes, and

$$p(a_1, a_2) = p(a_1) p(a_2) \quad (20)$$

where  $p(a_i)$  is given in Eq. (12).

For cascaded first-order PLL's in slow-fading channels, the equations of operation (17) become

$$\begin{aligned} \dot{\phi}_1 &= -K_1 a_1 \sin \phi_1 - K_1 n_1 \\ \dot{\phi}_2 &= G_f K_1 a_1 \sin \phi_1 + G_f K_1 n_1 - K_2 a_2 \sin \phi_2 - K_2 n_2 \end{aligned} \quad (21)$$

By assuming  $a_1$  and  $a_2$  known, these equations are exactly those for cascaded PLL's in additive white Gaussian channels <sup>[5]</sup>. Hence the pdf  $p(\phi_2|a_1, a_2)$  can be found by using generalized Fokker-Planck techniques as in Ref. [5] and is

$$p(\phi_2|a_1, a_2) = \frac{\exp(\alpha_2 \cos \phi_2)}{2\pi I_0(\alpha_2)} \quad (22)$$

where

$$\alpha_2 = \frac{2}{K_{22}} \left\{ K_2 a_2 - G_f K_1 a_1 \frac{1}{\sigma_{\sin \phi_2}^2} R_{\sin \phi_1 \sin \phi_2}(0) \right\} \quad (23)$$

$$K_{22} = \frac{K_2^2 N_{02}}{2} + \frac{G_f^2 K_1^2 N_{01}}{2} \quad (24)$$

The quantity  $\sigma_{\sin \phi_2}^2$  is the variance of  $\sin \phi_2$  and  $R_{\sin \phi_1 \sin \phi_2}(0)$  is the correlation function of  $\sin \phi_1$  and  $\sin \phi_2$  evaluated at the origin. These quantities, of course, in themselves are not available without knowing the joint pdf of  $\phi_1$  and  $\phi_2$ . However, they can be approximated by using the linear approximations  $\sin \phi_1 \approx \phi_1$  and  $\sin \phi_2 \approx \phi_2$ , which results from a strong signal assumption. The results are

$$\sigma_{\sin \phi_2}^2 \approx G_f \frac{1}{\rho_1 \left( 1 + \frac{W_{L2}}{W_{L1}} \frac{a_2 A_1}{A_2 a_1} \right)} + \frac{A_2}{a_2} \frac{1}{\rho_2} \quad (25)$$

and

$$R_{\sin \phi_1 \sin \phi_2}(0) \approx -G_f \frac{1}{\rho_1 \left( 1 + \frac{W_{L2}}{W_{L1}} \frac{a_2 A_1}{A_2 a_1} \right)} \quad (26)$$

where

$$\rho_i = \frac{2A_i^2}{N_{0i} W_{Li}}, \quad i = 1, 2 \quad (27)$$

$$W_{Li} = \frac{A_i K_i}{2}, \quad i = 1, 2 \quad (28)$$

Using Eqs. (12) and (22) in Eq. (19) with the appropriate substitutions of Eqs. (23) through (28),  $p(\phi_2)$  can be evaluated. Numerical results in terms of the variance of  $\phi_2$  as a function of  $\rho_1$  and  $\rho_2$  are presented in Figs. 2 and 3. The variances of  $\chi$  are chosen from Woo's Venus atmosphere study<sup>[8]</sup>, while  $G_f$  of 240/221 is the usual S-band transponder ratio for deep-space probes.

**B. Channels with Weak Amplitude Fluctuations.** In many cases the amplitude fluctuations of the signal are small after passing through the time-varying channel. The system performance degradation in this case is due almost entirely to the phase fluctuations in the signal. The problem of system analysis is thus reduced to the usual cascaded PLL case with the addition of a phase noise term in both the uplink and downlink signals. The equations of loop operation are thus given by Eq. (17) with the assumption that  $a_1(t) = A_1$  and  $A_2(t) = A_2$ .

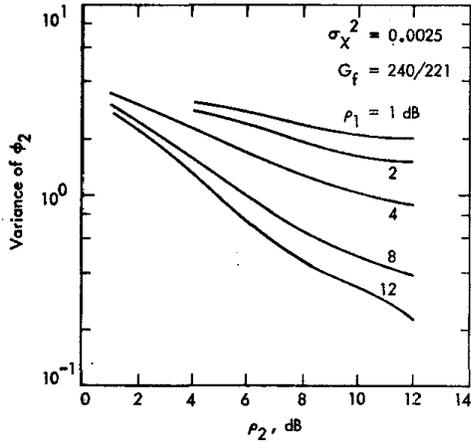


Fig. 2 - Slowly Fading Amplitude and Phase Fluctuations ( $\sigma_{\chi}^2 = 0.0025$ )

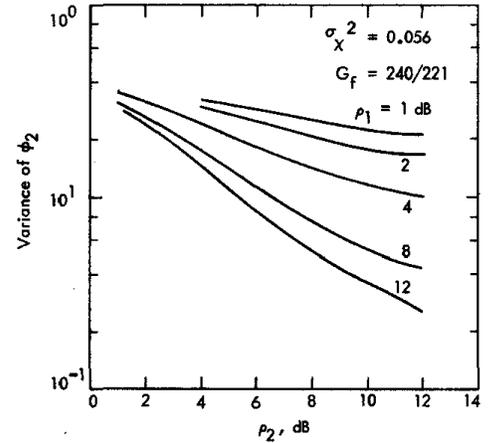


Fig. 3 - Slowly Fading Amplitude and Phase Fluctuations ( $\sigma_{\chi}^2 = 0.056$ )

The solution of  $p(\phi_2)$  from Eq. (17) proceeds along the lines of the generalized Fokker-Planck (F-P) approach as used in Refs. [5], [9], [12], and [13] and used in the previous section. In this case the F-P intensity coefficients become

$$K_{12}(\phi_2) = -2W_{\theta_2} E(\theta_2|\phi_2) + G_f K_1 A_1 E(\sin \phi_1|\phi_2) - K_2 A_2 \sin \phi_2 \quad (29)$$

$$K_{22} = 4\sigma_{\theta_2}^2 W_{\theta_2} + \frac{G_f^2 K_1^2 N_{01}}{2} + \frac{K_2^2 N_{02}}{2} \quad (30)$$

The solution for  $p(\phi_2)$  now depends on approximating the conditional expectations in Eq. (29). The techniques for using linear analysis to approximate these expectations is outlined in Refs. [12] and [13] and are not presented here because the resulting four-dimensional equations can best be solved using computer manipulation of the linear matrices. A simplified approach is presented here, however, which is known to give quite accurate results when applied to other related PLL analysis[12, 14]. First of all, it has been shown in Refs. [10], [12], and [13] that  $E(\theta_2|\phi_2)$  and  $E(\sin \phi_1|\phi_2)$  can be assumed to be directly proportional to  $\sin \phi_2$ . Thus, the intensity coefficient,  $K_{12}(\phi_2)$  is of the form  $\sin \phi_2$  times some constant. Therefore, from this form of  $K_{12}$  and  $K_{22}$  given in Eqs (29) and (30) the form of  $p(\phi_2)$  is thus the usual Tikhonov density <sup>[10,11]</sup>, i.e.,

$$p(\phi_2) = \frac{\exp(\alpha \cos \phi_2)}{2\pi I_0(\alpha)} \quad (31)$$

where  $\alpha$  is a parameter that must be approximated. The approximation technique used here <sup>[14]</sup> is to let  $\alpha$  equal the reciprocal of the linear variance,  $\sigma_{\phi_2}^2$ , obtained by linearizing Eqs. (17), i. e., letting  $\sin(\phi_i) = \phi_i$ .

By taking the Laplace transforms of the linearized form of Eq. (17) the Laplace transform of  $\phi_2$  can be obtained. From this the spectral density of  $\phi_2$  can be solved for and hence  $\sigma_{\phi_2}^2$ . From this approach, the result is

$$\begin{aligned} \sigma_{\phi_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_A(\omega)|^2 S_{u_2}(\omega) d\omega \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_B(\omega)|^2 S_{u_1}(\omega) d\omega \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_C(\omega)|^2 S_{n_1}(\omega) d\omega \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_D(\omega)|^2 S_{n_2}(\omega) d\omega \end{aligned} \quad (32)$$

where

$$\begin{aligned}
H_A(s) &= \frac{2\sigma_{\theta_2} \sqrt{W_{\theta_2}} s}{s^2 + s(A_2 K_2 + 2W_{\theta_2}) + 2W_{\theta_2} A_2 K_2} \\
H_B(s) &= \frac{2\sigma_{\theta_1} \sqrt{W_{\theta_1}} G_f A_1 K_1 s}{s^3 + s^2(A_1 K_1 + A_2 K_2 + 2W_{\theta_1}) + s(A_1 K_1 A_2 K_2 + 2W_{\theta_1} A_1 K_1 + 2W_{\theta_1} A_2 K_2) + 2W_{\theta_1} A_1 K_1 A_2 K_2} \\
H_C(s) &= \frac{G_f K_1 s}{s^2 + s(A_1 K_1 + A_2 K_2) + A_1 K_1 A_2 K_2} \\
H_D(s) &= \frac{K_2}{s + A_2 K_2}
\end{aligned} \tag{33}$$

After solving the necessary integrations and rearranging terms, Eq. (32) can be written as

$$\begin{aligned}
\sigma_{\phi_2}^2 &= \frac{1}{\rho_2} + \frac{G_f^2}{\rho_1} \left( \frac{1}{1 + \frac{W_{L2}}{W_{L1}}} \right) + \frac{\sigma_{\theta_2}^2}{1 + \frac{W_{L2}}{W_{\theta_2}}} \\
&\quad + \frac{G_f^2 \sigma_{\theta_1}^2}{\left(1 + \frac{W_{L2}}{W_{L1}}\right)^2 + \left(1 + \frac{W_{L2}}{W_{L1}}\right) \left(\frac{W_{L2}}{W_{\theta_1}} + \frac{W_{\theta_1}}{W_{L1}}\right)}
\end{aligned} \tag{34}$$

where  $\rho_i$  and  $W_{Li}$  are defined in Eqs. (27) and (28).

The resulting  $p(\phi_2)$  is thus given by Eq. (31) with  $\alpha = 1/\sigma_{\phi_2}^2$  as given in Eq. (34). It is interesting to note the significance of the four terms in Eq. (34). The first term results from the usual one-way PLL linear analysis [0, 11]. The first and second term together represent the results of the two-way cascaded PLL linear analysis<sup>[5]</sup>. The first and third terms together represent the one-way linear analysis including the fading<sup>[9]</sup> while the fourth term represents the contribution to the variance of  $\phi_2$  due to the phase perturbations on the uplink.

Numerical results of Eq. (31) are presented in Figs. 4, 5, and 6. The variance of  $\phi_2$  was obtained by numerically integrating Eq. (31), using Eq. (34) as described above. It is not Eq. (34) directly. The values of  $\sigma_{\theta_i}^2$  and  $W_{\theta_i}$  are again typical of Woo's Venus study<sup>[8]</sup> while  $W_{L1}/W_{L2}$  is typical for a deep-space probe and a Deep Space Network (DSN) ground station.

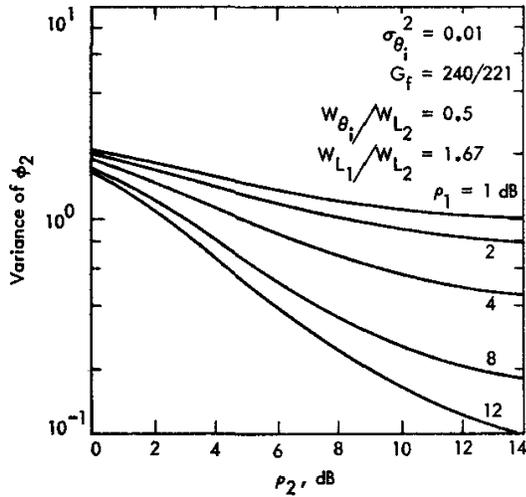


Fig. 4 -Weak Amplitude  
Fluctuations  
(  $\sigma_{\theta_i}^2 = 0.01$  )

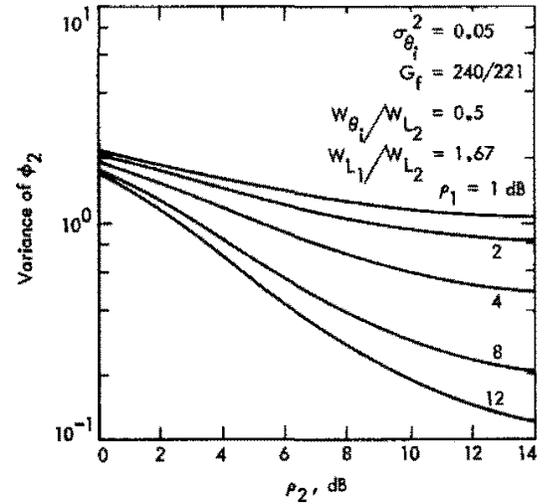


Fig. 5 -Weak Amplitude  
Fluctuations  
(  $\sigma_{\theta_i}^2 = 0.05$  )

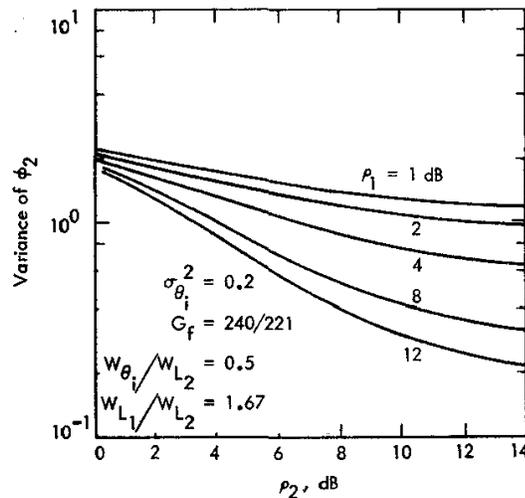


Fig. 6 - Weak Amplitude  
Fluctuations  
(  $\sigma_{\theta_i}^2 = 0.2$  )

**C. Channels with Slowly Fading Amplitudes.** The results of the previous section can be expanded to include the case when the amplitude fluctuations are not weak but are slowly fading with respect to the loop bandwidths. This case can be easily handled by combining the results of the last section with the techniques of the section on slow fading. That is, in this case

$$\begin{aligned}
 p(\phi_2) &= \int_0^\infty \int_0^\infty p(\phi_2 | a_1, a_2) p(a_1) \\
 &\quad \times p(a_2) da_1 da_2
 \end{aligned} \tag{35}$$

where  $p(\phi_2 | a_1, a_2)$  is the result of the last section as given by Eqs. (31) and (34) with the exception that all the  $\rho_i$  and  $W_{Li}$  are functions of  $a_1$  and  $a_2$  instead of  $A_1$  and  $A_2$ . Equation (35) can thus be integrated as in the section on channels with slow amplitude and phase fluctuations.

The difference between Eq. (35) and the previous slow fading results is that the previous results assumed that the phase process was also slowly fading and was thus tracked by the loop. The above analysis includes the effects on the loop when the phase process bandwidth is not small compared to the loop bandwidth. The results of Eq. (35) are thus good for any values of phase variance or bandwidth and for any amplitude variance. The only restriction is on the amplitude bandwidth.

**IV. Conclusions.** The results of the analysis in this paper have shown that the variance of  $\phi_2$  is increased by large fading bandwidths (of  $a_1$ ,  $a_2$ ,  $\theta_1$ , and  $\theta_2$ ) relative to the loop bandwidths and by large variances of these same fading components. It is clear from the results that there exist optimum settings of the loop gains,  $K_1$  and  $K_2$ , to minimize the resulting variance of  $\phi_2$ . This optimization depends on the setting of the loop bandwidths with respect to each other and with respect to the corresponding bandwidths of the fading components.

A number of selected cases have been presented. This work is by no means complete but extensions of this work can be made in the following areas: the simple Butterworth spectrum can be extended to any arbitrary rational spectrum; other channel models, such as Rice and Rayleigh, can be analyzed with the techniques presented; and, finally, the analysis can be extended to higher order loops, the effects of bandpass limiters, and Doppler effects.

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