ON OVERLAPPED FAST FOURIER TRANSFORMS

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ABSTRACT

Many signal processing applications require the averaging of transforms taken over partitioned sets of data. We show that the required overlap for the partitions is window dependent and that is varies from 50% to 75% depending upon the sidelobe levels of the window.

INTRODUCTION

When processing long time sequence with the FFT, a partition length N is selected to establish the required spectral resolution of the analysis. A window is applied to the partition to control spectral leakage (those effects due to processing finite blocks of data). The reduction in effective time duration due to the window is compensated for the overlapping successive partitions. The sequence of overlapped transforms are then averaged either coherently to realize additional processing gain, or incoherently to realize reduced variance. We now demonstrate that the partitions should overlap between 50% to 75% depending upon the sidelobe levels of the given window.

COHERENT PROCESSING

Coherent post processing of FFT's is employed in generating estimates of the Delay and Doppler Corrected Coherence Function, in cascade transforms, in (sweep frequency, impulse, or broadband noise) transfer function generation, and in cross power spectrum estimation. In all cases, the transforms are windowed to obtain leakage free estimates of the spectral components needed for the post processing (Fig. 1a). The successive sequences processed by the FFT must be overlapped to avoid discarding the data that occurs near the trailing boundary of the previous partition interval (Fig. 1b). We now address the question, by how much should the intervals overlap?

We demonstrate the existence of a minimum required overlap by recognizing that any particular FFT bin is simply the output of a non-recursive filter as shown in Figure 2a. The Fourier transform of the impulse response is of course the frequency response of the filter (Fig. 2b). To control the high sidelobe levels of the frequency response, the impulse response is windowed. (In fact, in the FFT, the data is windowed, but in a block process, the effects are identical.) The windowed impulse response exhibits a frequency response with reduced sidelobe levels and with an increased width mainlobe (1). Since data passed through this filter has experienced a significant reduction in bandwidth, we are permitted to reduce the output sample rate without a loss of spectral information. We can reduce the output sample rate till the replicated spectrum aliases onto itself as shown in Figure 3a.

Good windows, i.e., windows with sidelobes more than 70 dB down (the noise floor of a 12 bit converter), exhibit mainlobes with single sided bandwidths of approximately four FFT bins (Fig. 3b) (1). To keep an FFT bit alias free the output sample rate must be sufficiently high to prevent the replicated mainlobe from overlapping into the bin width. This requires an output sample rate of $4(f_s/N)$, or a sample period of NT/4. But NT is the sequence duration for the filter (or FFT). Hence, to satisfy the Nyquist sampling criterion, the FFT must be sampled four times per period. This is equivalent to performing the FFT with 75% overlap or with four-to-one redundancy. Of course if the windows being used exhibit higher sidelobes, (which is permitted if there are fewer bits in the converter), the mainlobes will be narrower, and the replicates are allowed to be closer. Table 1 shows aliasing levels and required overlap intervals for some classic windows (1).

WINDON	ALIASING LEVEL	OVERLAP
Rectangle	-13 dB	4/8
Triangle	-27 dB	4/8
Hann	-32 dB	5/8
Hamming	-43 dB	5/8
Gaussian	-70 dB	7/8
Dolph-Tchebyshev	-70 dB	5/8
Kaiser Bessel	-70 dB	6/8
Blackman-harris	-70 dB	6/8

TABLE 1. Overlap to nearest eight to maintain an aliasing level less than the highest sidelobe level in a bandwidth equal to one FFT bin,

Figure 4 shows a Coherency doppler-time ambiguity surface computed for'the four conditions listed below.

4a.	No Window	No Overlap
4b.	Window	No Overlap
4c.	No Window	75% Overlap
4d.	Window	75% Overlap

Note how important the overlap is to reducing the many artifacts on the ambiguity surface.

Figure 5 shows a linear sweep transfer function computed by averaging eight successive sequence by the three conditions listed below.

5a.	No Window	No Overlap
5b.	Window	No Overlap
5c.	Window	75% Overlap

Again note how important the overlap has been to the stability of the transfer function estimate.

INCOHERENT PROCESSING

Incoherent post processing is used to generate stable estimates of auto power spectrums. Transforms are windowed to realize spectral smoothing as well as to avoid spectral aliasing. Successive magnitude squared transforms are averaged to reduce the variance of the estimates. For an unlimited run of data, transforms over K independent (non overlapped) intervals will reduce the variance of a spectral estimate by the factor K, For a limited amount of data, it is prudent to overlap the successive intervals and process more than K intervals over the data set. Alternatively, we can overlap and acquire the K transforms in a shorter interval. We recognize of course that the overlapped transforms are correlated and the reduction in variance obtained by averaging correlated data is not proportional to the number of averages. In fact Welsh (2) gives an expression for the reduction in variance to be obtained from averaging overlapped spectral estimates, see Eq. (1).

$$\frac{1}{K_{EFF}} = \frac{1}{K'}\left[1 + \sum_{k=1}^{K'} \left(1 - \frac{k}{K'}\right) c^2(k\, s)\right] \qquad (1)$$

where	K'	is number of transforms averaged
	s	is fractional shift of intervals
	c(s)	is correlation coefficient

Equation 2 is used to evaluate the actual number of overlapped transforms (K') in the interval covered by K adjacent non-overlapped transforms.

$$\lceil K' (1 - OL) \rceil > K \geq \lfloor K' (1 - OL) \rfloor \qquad (2)$$

where $\lceil \ \rceil$ implies quantizing up (the smallest integer not less than)

$\lfloor \ \rfloor$ implies quantizing down (the largest integer not greater than)

Table 2 lists the K' corresponding to K equal thirty two for overlaps equal to increments of one eighth.

K'	32	36	42	51	64	85	128	256
OL	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8

TABLE 2. Number of overlapped intervals for the given amounts of overlap in a non-overlapped interval of 32 units.

Figure 6 demonstrates the behavior of Eqs. 1 and 2 for the windows listed in Table 1 by presenting K_{EFF} (the variance reducing measure) and K' (the work count measure) as functions of fractions overlap. We can conclude from this figure that we realize variance reductions for overlapped processing of up to 6/8 overlap. After 6/8 overlap we are merely consuming processor time. Figure 7 is a graph of the ratio K_{EFF}/K' as a function of fractional overlap. We can think of this ratio as the efficiency of overlapped transforms. (We don’t mind increasing the work by a factor or two if we reduce the variance by a factor of two; we do object, however, to increasing the work by a factor of two and realizing a variance reduction of only 1.2.) Note that processing with more than 6/8 overlap is less than 50% efficient relative to that which can be obtained without overlap.

Now our final observation relative to Figure 9. Near the position where the individual curves level off, the ratios of K_{EFF} for the different windows to the K_{EFF} for the rectangle window, is very nearly the Equivalent Noise Bandwidth of that window (1) . Thus when we apply the window to the data we incurr a penalty in terms of larger variance, but by proper overlapping of the windows we remove that penalty. Table 4 is a listing of windows with the K_{EFF} ratios and the ENBW for that window

WINDOW	K_{EFF} (Window)/K_{EFF}(Rect)	ENBW
Rectangle	1.0	1.0
Triangle	1.33 at 4/8 OL	1.33
Hann	1.48 at 5/8 OL	1.50
Hamming	1.35 at 5/8 OL	1.36
Gaussian	1.73 at 6/8 OL	1.80
Dolph-Tchebyshev	1.59 at 5/8 OL	1.62
Kaiser-Bessel	1.90 at 6/8 OL	1.90
Blackman-harris	1.73 at 6/8 OL	1.79

TABLE 3 Comparison of K_{EFF} ratios to Equivalent Noise Bandwidth of classic windows.

It is instructive to compare the entries of Table 1 and 3. We find that the amount of overlap to minimize aliasing also minimizes variance.

CONCLUSIONS

We have demonstrated the need to perform overlap processing of FFT's when they are followed by either coherent or incoherent averaging. In coherent processing we were led to use overlap to control spectral aliasing, and in incoherent processing, we were led to use overlap to reduce the variance of our spectral estimates. From both viewpoints, we have been led to the same required amount of fractional overlap for the set of windows we have examined.

REFERENCES

(1) f.j. harris, "On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform," Proceedings of the IEEE, Vol. 66, No. 1, pp. 51-83, January 1978.

(2) P.D. Welch, "The use of fast Fourier transform for the estimation of power spectra: A method based on time averaging over short, modified periodograms," IEEE Trans. Audio Electracoust., Vol. AU-15, pp. 70-73, June 1967.

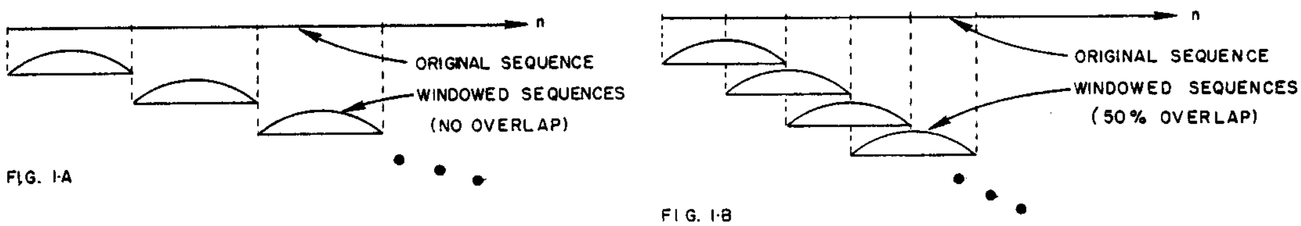

FIGURE 1. PARTITION OF SEQUENCES FOR NONOVERLAPPED AND OVERLAPPED PROCESSING

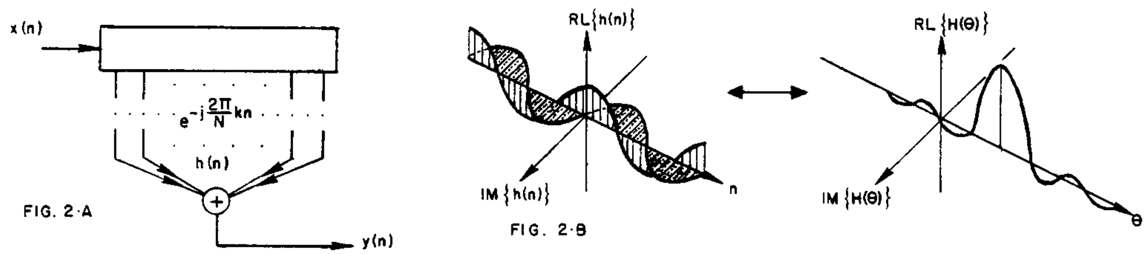

FIGURE 2. NONRECURSIVE FILTER, IMPULSE RESPONSE, AND FREQUENCY RESPONSE

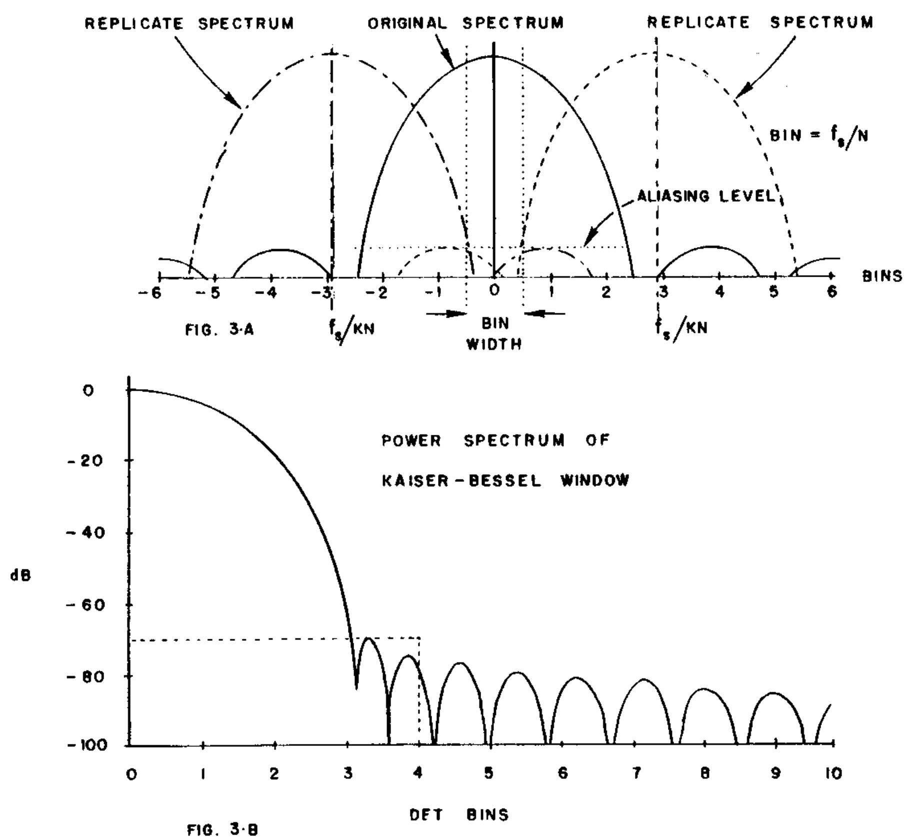

FIGURE 3. ALIASING ASSOCIATED WITH DESAMPLING A NONRECURSIVE FILTER AND FREQUENCY RESPONSE OF WINDOW.

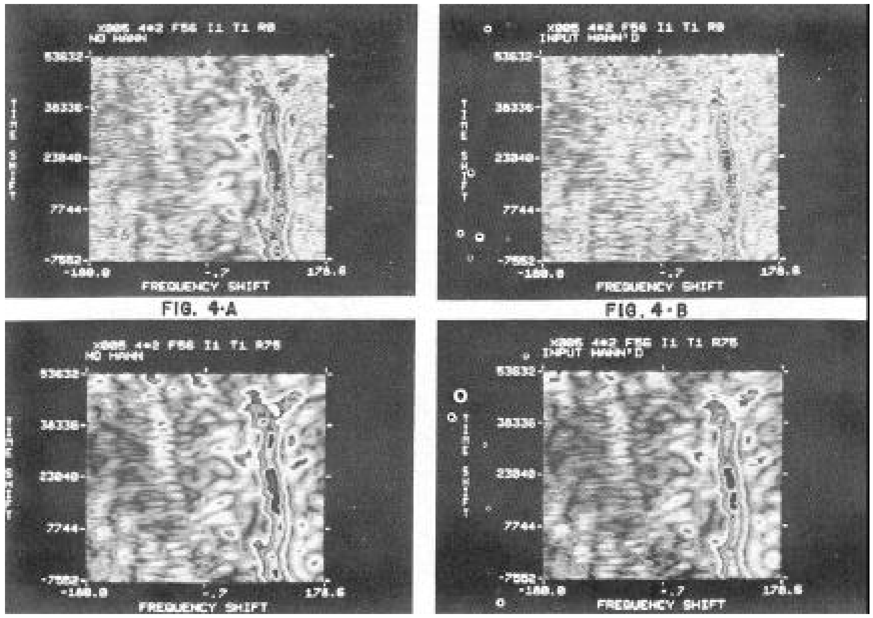

FIGURE 4. NORMALIZED (TIME-dOPPLER) AMBIGUITY SURFACE WITH VARIOUS COMBINATIONS OF OVERLAP AND WINDOWING

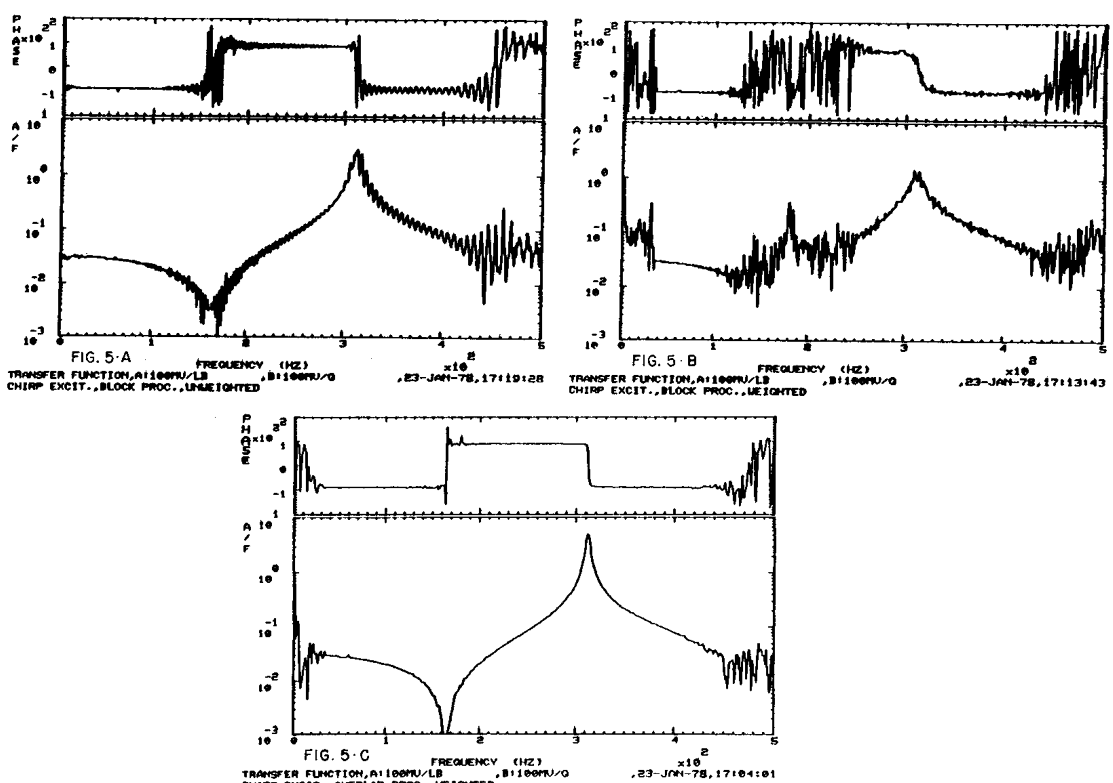

FIGURE 5. SWEEP FREQUENCY TRANSFER FUNCTION WITH VARIOUS COMBINATIONS OF OVERLAP AND WINDOWING

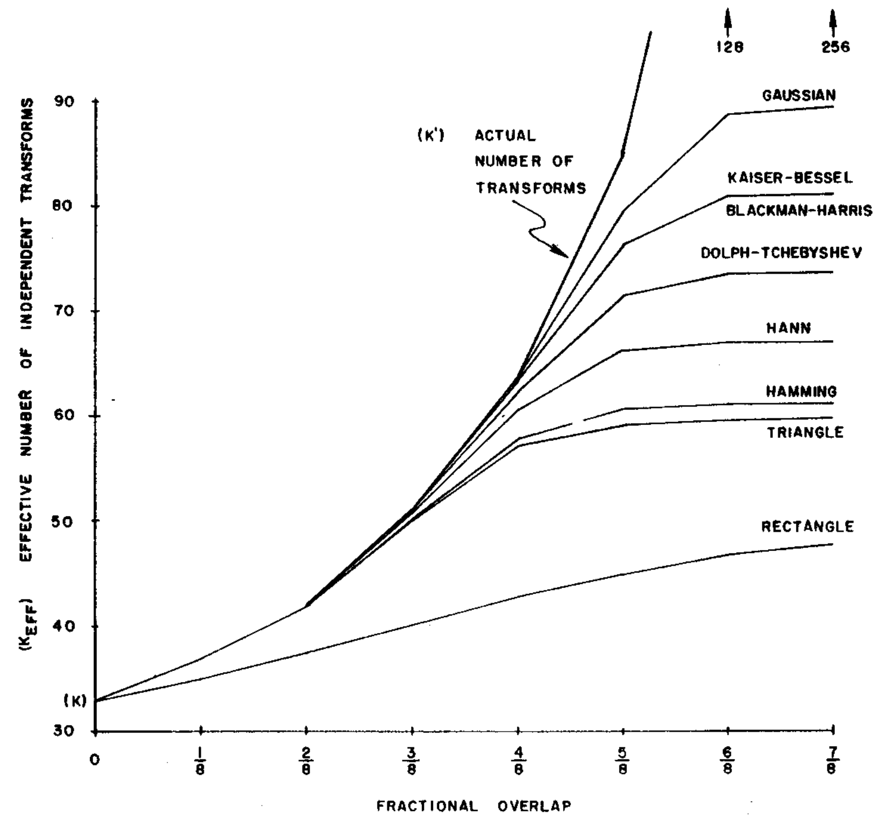

FIGURE 6. EQUIVALENT NUMBER OF INDEPENDENT TRANSFORMS AS A FUNCTION OF FRACTIONAL OVERLAP FOR VARIOUS CLASSIC WINDOWS

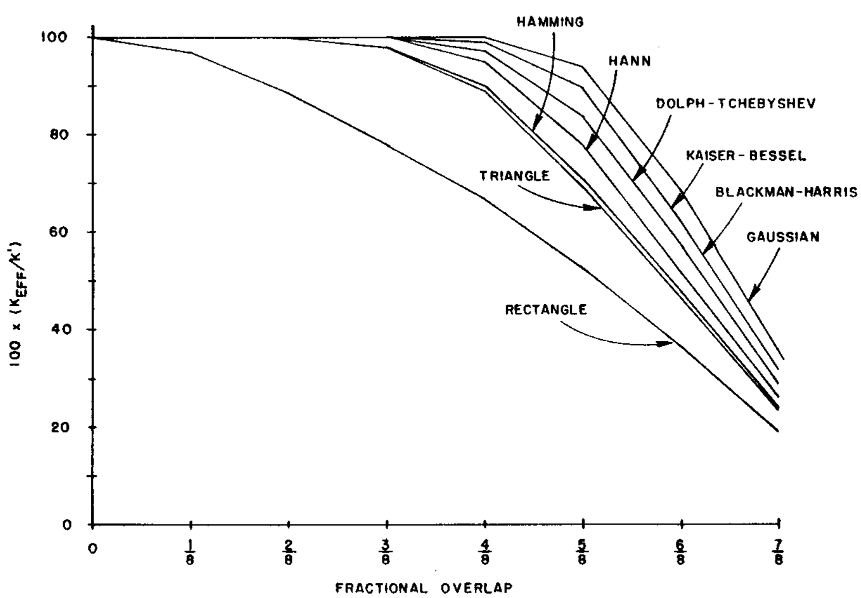

FIGURE 7. RATIO (K_{EFF}/K') AS A FUNCTION OF FRACTIONAL OVERLAP FOR VARIOUS CLASSIC WINDOWS