

# **THE RECURSIVE ALGORITHMS FOR GDOP AND POSITIONING SOLUTION IN GPS**

**Chang Qing Liu Zhongkan Zhang Qishan**

## **ABSTRACT**

This paper proves theoretically that GDOP decreases as the number of satellites is increased. This paper proposes two recursive algorithms for calculating the GDOP and positioning solution. These algorithms not only can recursively calculate the GDOP and positioning solution, but also is very flexible in obtaining the best four-satellite positioning solution, the best five-satellite positioning solution and the all visible satellite positioning solution according to given requirements. In the need of the two algorithms, this paper extends the definition of the GDOP to the case in which the number of visible satellites is less than 4.

## **KEY WORDS**

GPS, GDOP, Positioning solution, Algorithm, Generalized inverse.

## **INTRODUCTION**

The existing method for Global Positioning System (GPS) positioning is to linearize the GPS positioning model and then solve the linear positioning model iteratively. This method is, in fact, the Gauss-Newton method [5]. The main work in each iteration is to solve the normal equations of the linear positioning model, i.e., to obtain the least squares solution of the linear positioning model. Since the normal equations form a linear system of equations, there are many methods that can be used to solve it [6]. The method to obtain the least squares solution of the linear positioning model is not unique. In addition to solving normal equations, there are still other methods [7]. Here we present two recursive algorithms. The two algorithms not only can recursively calculate Geometric Dilution Of Precision (GDOP) and the least squares solution of the linear positioning model, but also is very flexible in obtaining the best four-satellite positioning solution, the best five-satellite positioning solution and the all visible satellite positioning solution according to given requirements.

## THE EXTENSION OF THE DEFINITION OF GDOP

The positioning model of GPS is[1][4]

$$G_n r = A_n S_n - \mathbf{r}_n \underline{\Delta} \mathbf{r}_n^* \quad (1)$$

where  $G_n = [e_1, e_2, \dots, e_n]^T$ ,  $e_i = [l_i, m_i, n_i, 1]^T$ ,  $l_i, m_i, n_i$  are the direction cosine from the receiver to the  $i$ th satellite;  $r = [x, y, z, -l_u]^T$ ,  $x, y, z$  are the coordinates of the receiver in Earth Centered, Earth Fixed (ECEF) coordinate system,  $l_u$  is the error in the receiver clock times the speed of light;

$$A_n = \begin{bmatrix} e_1^T & 0 & \dots & 0 & 0 \\ 0 & e_2^T & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & e_n^T \end{bmatrix}$$

$S_n = [s_1^T, s_2^T, \dots, s_n^T]^T$ ,  $s_i = [x_i, y_i, z_i, l_i^s]^T$ ,  $x_i, y_i, z_i$  are the coordinates of the  $i$ th satellite in ECEF coordinate system,  $l_i^s$  is the error in the  $i$ th satellite clock times the speed of light;  $n$  is the number of visible satellites,  $n \geq 4$ ;  $\mathbf{r}_n = [\mathbf{r}_{1,n}, \mathbf{r}_{2,n}, \dots, \mathbf{r}_{n,n}]^T$ ,  $\mathbf{r}_{i,n}$  is the measured pseudorange from the receiver to the  $i$ th satellite. The least squares solution of equation (1) is[1][4]

$$\hat{\mathbf{r}}_n = (G_n^T G_n)^{-1} G_n^T \mathbf{r}_n^* \quad (2)$$

If it is assumed that  $E(\Delta \mathbf{r}_n^*) = 0$  and  $\text{cov}(\Delta \mathbf{r}_n^*) = \mathbf{S}_0^2 I$ , where  $\Delta \mathbf{r}_n^*$  denotes the error vector of  $\mathbf{r}_n^*$  and  $I$  is  $n \times n$  identity matrix, then GDOP is defined as

$$GDOP = [\text{Trace}(G_n^T G_n)^{-1}]^{\frac{1}{2}} \quad (3)$$

In fact, the least squares solution of equation (1) and the least squares solution of equation (1) with the least norm can be unitedly expressed as

$$\hat{\mathbf{r}}_n = G_n^+ \mathbf{r}_n^* \quad (4)$$

where  $G_n^+$  is Moore-Penrose generalized inverse of  $G_n$ . When  $n \geq 4$ , equation (4) is the least squares solution of equation (1). When  $n < 4$ , equation (4) is the least squares solution of equation (1) with the least norm. So when  $n \geq 4$ , equation (4) is equal to equation (2) and the GDOP defined by (3) can be written as

$$GDOP = [\text{Trace}(G_n^+ G_n^+)^{\frac{1}{2}}] = \{\text{Trace}[G_n^+ (G_n^+)^T]\}^{\frac{1}{2}} \quad (5)$$

When  $n < 4$ , if we define the error covariance matrix of equation (4) as

$$\text{cov}(\Delta \hat{\mathbf{r}}_n) = E[(\hat{\mathbf{r}}_n - G_n^+ G_n^+ \mathbf{r}_n^*)(\hat{\mathbf{r}}_n - G_n^+ G_n^+ \mathbf{r}_n^*)^T] \quad (6)$$

then substituting equation (4) into equation (6) yields

$$\text{cov}(\Delta \hat{\mathbf{r}}_n) = \mathbf{S}_0^2 G_n^+ (G_n^+)^T = \mathbf{S}_0^2 (G_n^T G_n)^+ \quad (7)$$

From equation (7), we know that when  $n < 4$ , the error of  $\hat{\mathbf{r}}_n$  in the sense of (6) is

$\mathbf{S}_0 [\text{Trace}(G_n^T G_n)^+]^{\frac{1}{2}}$ . Thus for an arbitrary natural number  $n$ , we can define the GDOP of  $\hat{\mathbf{r}}_n = G_n^+ \mathbf{r}_n^*$  as

$$GDOP_n = [\text{Trace}(G_n^T G_n)^+]^{\frac{1}{2}} = \{\text{Trace}[G_n^+ (G_n^+)^T]\}^{\frac{1}{2}} \quad (8)$$

## THE RECURSIVE ALGORITHM FOR GDOP AND POSITIONING SOLUTION

For an arbitrary natural number  $n$ , the matrix  $G_{n+1}$  can be written as  $G_{n+1} = [G_n^T \quad e_{n+1}]^T$ .

From [2], we know that

$$G_{n+1}^+ = [G_n^+ - b_{n+1}d_{n+1}^T \quad b_{n+1}] \quad (9)$$

where  $d_{n+1}^T = e_{n+1}^T G_n^+$  (10)

$$c_{n+1}^T = e_{n+1}^T - d_{n+1}^T G_n^+ \quad (11)$$

$$b_{n+1} = \begin{cases} (c_{n+1}^T)^+ & , \quad c_{n+1} \neq 0 \\ (1 + d_{n+1}^T d_{n+1})^{-1} G_n^+ d_{n+1} & , \quad c_{n+1} = 0 \end{cases} \quad (12)$$

If we express  $G_n^+ d_{n+1}$  as  $p_{n+1}$ , i.e.,

$$p_{n+1} = G_n^+ d_{n+1} \quad (14)$$

then

$$\text{Trace}(G_n^+ d_{n+1} b_{n+1}^T) = p_{n+1}^T b_{n+1} \quad (15)$$

Calculating  $GDOP_{n+1}$  using equations (8),(9), (15) yields

$$GDOP_{n+1} = [(GDOP_n)^2 - 2p_{n+1}^T b_{n+1} + (1 + d_{n+1}^T d_{n+1}) b_{n+1}^T b_{n+1}]^{\frac{1}{2}} \quad (16)$$

From Lemma 3.6.10 of [2], it follows that for the  $c_{n+1}$  in equation (11),  $c_{n+1} = 0$  if and only if  $\text{rank}(G_{n+1}) = \text{rank}(G_n)$ . From this conclusion, it is known that when  $n < 4$ ,  $c_{n+1} \neq 0$ ; when  $n \geq 4$ ,  $c_{n+1} = 0$ . So  $p_{n+1} = (1 + d_{n+1}^T d_{n+1}) b_{n+1}$  when  $n \geq 4$ . Substituting this  $p_{n+1}$  into equation (16) yields

$$GDOP_{n+1} = [(GDOP_n)^2 - (1 + d_{n+1}^T d_{n+1}) b_{n+1}^T b_{n+1}]^{\frac{1}{2}} \quad (n \geq 4) \quad (17)$$

From equation (17), we know that

$$GDOP_{n+1} < GDOP_n \quad (n \geq 4) \quad (18)$$

Equation (18) is the relationship between GDOP and the number of satellites. It indicates that the GDOP decreases as the number of satellites increases. From equations (11),(12),(14), we know that  $p_{n+1}^T b_{n+1} = 0$  when  $n < 4$ . So

$$GDOP_{n+1} = [(GDOP_n)^2 + (1 + d_{n+1}^T d_{n+1}) b_{n+1}^T b_{n+1}]^{\frac{1}{2}} \quad (n < 4) \quad (19)$$

Calculating  $\hat{r}_{n+1}$  using equation (9) and  $\mathbf{r}_{n+1}^* = [(\mathbf{r}_n^*)^T \quad \mathbf{r}_{n+1,n+1}^*]^T$  yields

$$\hat{r}_{n+1} = G_{n+1}^+ \mathbf{r}_{n+1}^* = \hat{r}_n + (\mathbf{r}_{n+1,n+1}^* - e_{n+1}^T \hat{r}_n) b_{n+1} \quad (20)$$

Equation (20) is the recursive relationship for the positioning solution.

We can obtain a recursive algorithm for the positioning solution and GDOP by using the recursive relationship given in equations (17),(19),(20). Let the number of visible satellites be  $M$  at a given time,  $M > 4$ . The algorithm begins with the satellite with the largest elevation angle. The second step turns to the satellite with the smallest elevation angle. If we let  $E_i$ ,  $A_i$  represent the elevation angle and azimuth angle of the satellite in the  $i$ th step, then the satellites in the third step and fourth step are those whose azimuth angles are in the

vicinity of  $A_2 + 120^\circ$  and  $A_2 + 240^\circ$  respectively and whose elevation angles are closest to  $E_2$  (if there are still other satellites in the vicinity of  $A_2 + 120^\circ$  and  $A_2 + 240^\circ$ ). From the fifth step to the last step, the satellite in the  $i$ th step ( $i = 5, 6, \dots, M$ ) is the one with largest elevation angle among all surplus visible satellites. Let the direction cosine from the receiver to the satellite with the largest elevation angle be  $l_1, m_1, n_1$ , then we have

$$G_1 = [l_1, m_1, n_1, 1], G_1^+ = \frac{1}{2} G_1^T, GDOP_1 = \frac{\sqrt{2}}{2}, \hat{r}_1 = \left(\frac{\mathbf{r}_{1,1}^*}{2}\right) G_1^T \quad (21)$$

Our algorithm will begin with equation (21). The algorithm will calculate  $GDOP_i$  and  $\hat{r}_i$  from  $GDOP_{i-1}$  and  $\hat{r}_{i-1}$  in  $i$ th step ( $i = 2, 3, \dots, M$ ). Let  $n+1 = i$ , then  $n = i - 1$ . So  $b_{n+1}$  and  $GDOP_{n+1}$  should be calculated according to equations (11), (12), (19) from the second step to the fourth step and be calculated according to equations (13), (17) from the fifth step to last step.

To sum up, Algorithm 1 is obtained: Calculate equation (21) and let  $n = 1$  in equations (10), (11), (12), (19), (20), (9). From the second step to the fourth step, calculate equations (10), (11), (12), (19), (20), (9) successively after the satellite in the  $i$ th step ( $i = 2, 3, 4$ ) has been chosen according to the method given above. From the fifth step to the last step, calculate equations (10), (13), (17), (20), (9) successively after the satellite in the  $i$ th step ( $i = 5, 6, \dots, M$ ) has been chosen according to the method given above.

If we let Algorithm 1 stop at the fourth step, then the best four-satellite positioning solution  $\hat{r}_4$  and the smallest  $GDOP_4$  can be obtained [3]; if not, Algorithm 1 will continue and an all visible satellite positioning solution  $\hat{r}_M$  and  $GDOP_M$  can be obtained at last.

If we use Algorithm 1 to calculate  $GDOP_4$  and  $\hat{r}_4$ , then a fewer calculations are needed. First, Algorithm 1 does not involve the multiplication and inverse of matrix. It only involves the multiplication of a matrix and a vector. Second,  $d_2$  is a number, so the first two steps of Algorithm 1 do not involve the multiplication of a matrix and a vector, only the third and the fourth step involve the multiplication of a matrix and a vector.

From equation (17), after  $\hat{r}_4$  and  $GDOP_4$  have been obtained by using Algorithm 1, if we choose the fifth satellite which enables  $\mathbf{b} = (1 + d_5^T d_5) b_5^T b_5 = (1 + d_5^T d_5)^{-1} p_5^T p_5$  to be maximized, then the smallest  $GDOP_5$  can be obtained. The following calculated results will provide a simple method for the choice of the fifth satellite.

## THE ANALYSIS OF CALCULATED RESULTS

Algorithm 1 is used to calculate  $GDOP$  for the following three cases. Case 1: The number of visible satellites is 4.  $E_1 = 90^\circ, A_1 = 0^\circ, A_2 = 0^\circ, A_3 = 120^\circ, A_4 = 240^\circ, E_2 = E_3 = E_4 = E$ . Case 2: The number of visible satellites is 5.  $E_1 = 90^\circ, A_1 = 0^\circ, A_2 = 0^\circ, A_3 = 90^\circ, A_4 = 180^\circ, A_5 = 270^\circ, E_2 = E_3 = E_4 = E_5 = E$ . Case 3: The number of visible satellites is 6.  $E_1 = 90^\circ, A_1 = 0^\circ, A_2 = 0^\circ, A_3 = 72^\circ, A_4 = 144^\circ, A_5 = 216^\circ, A_6 = 288^\circ, E_2 = E_3 = E_4 = E_5 = E_6 = E$ . The calculated results are shown in Table 1. The results in Table 1 are the same as those calculated according to equation (3). Three  $GDOP$ s decrease successively for the same elevation angle  $E$  as shown in Table 1. When  $E = -20^\circ, GDOP_i (i = 4, 5, 6)$  is minimized. Let  $E = 30^\circ$  in Case 1, then calculating  $p_5^T p_5$  and  $(1 + d_5^T d_5)^{-1}$  using those angles in Case 1 yields

$$p_5^T p_5 = \left(\frac{16}{3} \sin E_5 - \frac{10}{3}\right)^2 + \left(\frac{8}{9} \cos E_5\right)^2 + \left(-\frac{10}{3} \sin E_5 + \frac{7}{3}\right)^2 \quad (22)$$

$$(1 + d_5^T d_5) = 1 + \left(\frac{16}{3} \sin E_5 - \frac{10}{3}\right) \sin E_5 + \frac{8}{9} (\cos E_5)^2 + \left(-\frac{10}{3} \sin E_5 + \frac{7}{3}\right) \quad (23)$$

From equations (22),(23), it is known that  $\mathbf{b}$  is irrelevant to the azimuth angle  $A_5$ , and only relevant to the elevation angle  $E_5$ . In fact, because the  $GDOP$  is irrelevant to the choice of coordinate system,  $\mathbf{b}$  is irrelevant to the choice of coordinate system. So  $\mathbf{b}$  does not rely on the azimuth angle  $A_5$ , and only relies on the elevation angle  $E_5$ . When the elevation angles and azimuth angles of the four-satellite positioning constellation satisfy that  $70^\circ < E_1 < 90^\circ, 10^\circ < E_i < 50^\circ (i = 2, 3, 4), |A_i - a_i| < 20^\circ (i = 2, 3, 4)$ , where  $a_i$  is at an angle of  $120^\circ$  to  $a_j (i, j = 2, 3, 4, i \neq j)$ ,  $\mathbf{b}$  decreases and then increases as  $E_5$  changes from  $0^\circ$  to  $90^\circ$  (See Figure 1). The elevation angles and azimuth angles of the curves in Fig.1 are  $E_1 = 90^\circ - 4j, A_1 = 0^\circ - 3j, E_2 = 30^\circ + 4j, A_2 = 0^\circ - 4j, E_3 = 30^\circ - 4j, A_3 = 120^\circ + 4j, E_4 = 30^\circ + 2j, A_4 = 240^\circ - 4j, j$  from  $0^\circ$  to  $4^\circ$ . To decrease  $GDOP_4$  effectively, the fifth satellite should be the one with an elevation angle either larger than  $70^\circ$  or smaller than  $25^\circ$  as shown in Fig. 1. In actual application, since the pseudorange measurement error increases as the decrease of the elevation angle of the satellite, the satellite whose elevation angle is in the vicinity of  $25^\circ$  is not chosen as the fifth satellite unless the satellite whose elevation angle is larger than  $70^\circ$  does not exist.

Table 1 The calculated results of GDOP

$E$ (deg.)	$GDOP_4$	$GDOP_5$	$GDOP_6$	$E$ (deg.)	$GDOP_4$	$GDOP_5$	$GDOP_6$
-35-3	1.67123	1.49296	1.37495	5	1.83106	1.67835	1.57966
-30	1.62161	1.45297	1.34164	10	1.9646	1.80884	1.70858
-25	1.59218	1.43068	1.32437	15	2.14124	1.98097	1.87826
-20-2	1.58124	1.42488	1.32222	20	2.37273	2.20621	2.09998
-15-1	1.58843	1.4355	1.33537	25	2.67553	2.50054	2.38941
-10-1	1.61451	1.46354	1.36497	30	3.07318	2.88675	2.76887
-5-5	1.66136	1.51103	1.41318	35	3.60028	3.39834	3.27119
0	1.73205	1.58114	1.48324				

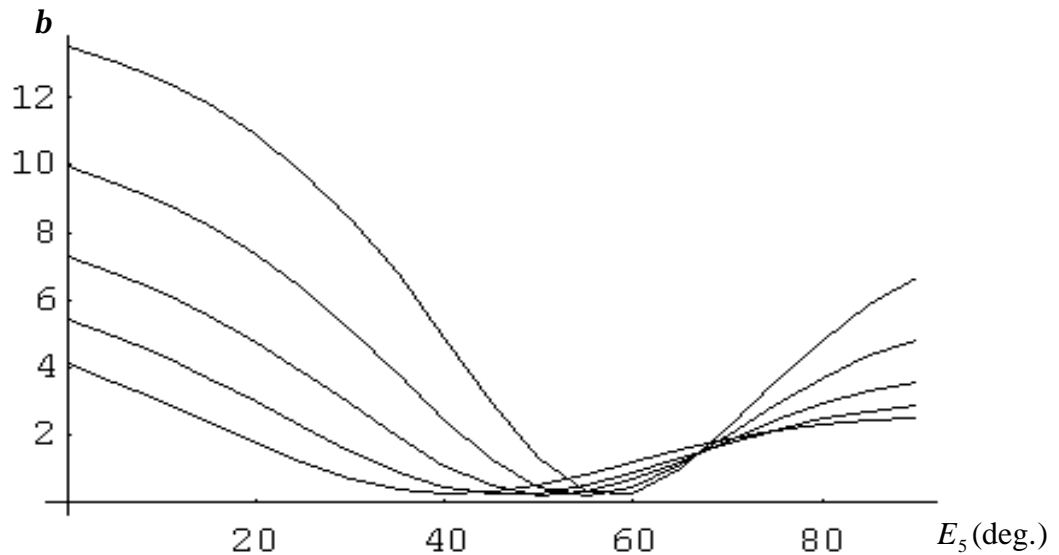


Fig.1 The relationship between  $\beta$  and  $E_5$

To sum up, Algorithm 2 is obtained: After  $\hat{r}_4$  and  $GDOP_4$  have been obtained by using Algorithm 1, choose the fifth satellite according to the method given above and then calculate equations (10),(13),(17),(20) successively to yield the  $\hat{r}_5$  and  $GDOP_5$ .

## CONCLUSION

This paper presents two complete recursive algorithms for GPS positioning .The algorithms presented not only can recursively calculate  $GDOP$  and positioning solutions , but also are very flexible to use. If Algorithm 1 is stopped at the fourth step ,then the best four-satellite positioning solution  $\hat{r}_4$  and  $GDOP_4$  can be obtained; if not, Algorithm 1 will continue and an all visible satellite positioning solution  $\hat{r}_M$  and  $GDOP_M$  can be obtained at last.If we combine Algorithm 1 with Algorithm 2,i.e.,let Algorithm 1 stop at the fourth step and then turn to Algorithm 2, the best five-satellite positioning solution  $\hat{r}_5$  and  $GDOP_5$  can be obtained.

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### Adress and phone number:

#### **Chang Qing and Zhang Qishan**

Dept. of Electronic Engineering, BUAA  
37 Xue Yuan Rd.  
Beijing, China 100083  
86-010-62017251-7644

#### **Liu Zhongkan**

Dept. of Applied Mathematics, BUAA  
37 Xue Yuan Rd.  
Beijing, China 100083  
86-010-62026677-5205

### Author Biographies

**Chang Qing** received the B. Sc. in mathematics department from Beijing Normal University in 1986 and M. Sc. in applied mathematics department from Beijing University of Aeronautics and Astronautics in 1994 respectively. He is currently working towards the Ph. D degree in electronic engineering department at Beijing University of Aeronautics and Astronautics, Beijing, China.

**Liu Zhongkan** is a professor in applied mathematics department at Beijing University of Aeronautics and Astronautics, Beijing, China.

**Zhang Qishan** is a professor in electronic engineering department at Beijing University of Aeronautics and Astronautics, Beijing, China.