A GEOMETRIC MOMENT BOUNDING ALGORITHM

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ABSTRACT

There are many important problems in the field of communications theory whose solution is the expectation of a function of a random variable. Examples include linear interference problems such as intersymbol interference and co-channel interference. In these cases, it is often not computationally feasible to evaluate the expectation exactly.

This paper presents an algorithm that will compute tight upper and lower bounds to generalized moments of a broad class of random variables. The procedure is based on an isomorphism theorem from Game Theory. The technique is easily understood while yielding excellent results for this class of communication problems.

INTRODUCTION

Problem Description

There are many important problems in the field of communications theory whose solution is the expectation of a function of a random variable. Perhaps the classic example of such a problem is that of computing the probability of bit error for a binary signal being transmitted on a channel with linear intersymbol interference (1) - (15). A block diagram for this example is given in Figure 1.

The binary source selects a value for a_i with equal probability each T seconds. These source symbols are encoded into waveforms suitable for transmission across the channel. The time function x(t) represents a string of these channel waveforms. The channel is assumed to act upon the waveform string x(t) as a linear filter. Thus, the waveform associated with a particular source symbol will typically be distorted in shape and spread in time by the action of the channel. Let the distorted waveform string at the channel output be represented by y(t). This signal is assumed to be further distorted by the addition of a white Gaussian noise process that is denoted by n(t). The waveform string finally presented to the receiver is r(t) where

$$r(t) = y(t) + n(t) \tag{1}$$

This signal is detected and sampled. The sampled output at time zero can be represented by

$$r_o = a_o h_o + \sum_{i=-M}^{M}{}' a_i h_i + n_o \tag{2}$$

where r_i is the detected and sampled output at time i, ... a_{-M} ... $a_{-1}a_oa_1$ ... a_M ... is the binary input signal string, $\{h_i\}$ is the sampled impulse response of the channel, and n_o is the Gaussian noise sample at time zero. The primed summation in Eq. (2) is a standard symbolism for a summation that is missing its central term. It is further assumed that the channel impulse response has only 2M + 1 significant terms.

The probability of bit error at time zero can be shown (11) to be given by the expressions

$$P_e = E_U\left[Q\left(\frac{h_o + u}{\sigma}\right)\right] \tag{3a}$$

$$= E_U\left\{\left[Q\left(\frac{h_o + |u|}{\sigma}\right) + Q\left(\frac{h_o - |u|}{\sigma}\right)\right]\right\}/2 \tag{3b}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-y^2/2)\,dy, \tag{4}$$

$$u = \sum_{i=-M}^{M}{}' a_i h_i, \tag{5}$$

σ is the standard deviation of the Gaussian noise, and U represents the space of all strings of 2M binary syrnbols. The expressions (3a) and (3b) are clearly equal mathematically, but it has been shown (11) that one form or the other can have analytical advantages when evaluating the probability.

Unfortunately, the expressions in Eqs. (3a) and (3b) may be difficult or computationally impractical to evaluate exactly. For example, if the intersymbol interference extends for forty samples preceeding and trailing the actual signal sample time (M = 40), the exact evaluation of the probability of error would involve the summation of $2^{80} \approx 10^{24}$ terms of the form of Eq. (4). Thus, even if Eq. (4) could be solved in 1 nsec of computer time, exact computation of P_e would require 3 X 10^5 centuries. Although channels having an impulse response that is significant over 80 bit times may be rare, it is clear that the computation involved in calculating Eq. (3a) or (3b) can still be large even for fairly modest impulse responses.

Expressions similar to (3a) can be derived for the probability of bit error on any additive Gaussian noise channel with linear interference. Examples would include spread spectrum multiple access channels and channels with co-channel interference, (16) - (19). Thus, the evaluation methods that will be discussed are more generally applicable than those for only intersymbol interference problems. They will apply to Gaussian channels with other kinds of linear interference as well.

Isomorphism Theorem

One approach to problems in communications and information theory that appear's difficult or impossible to solve in their exact form is to find easily computed bounds to the exact solution. A technique that has been proven to be useful in providing bounds to problems of this kind is the Geometric Moment Bounding Technique (11) - (19). This technique is based on an Isomorphism Theorem from Game Theory (20), (21). In this approach, the moment of the function of the random variable that is of interest is bounded in terms of moments of other functions of the same random variable. These other functions are called auxiliary functions. The auxiliary functions are chosen such that their moments are relatively easy to evaluate. This approach has the unique advantage in that both upper and lower bounds can be found with the same computational technique. In addition, the moment bounds that are derived using one or two auxiliary functions yield a relatively simple geometrical understanding of the bounding process.

The Isomorphism Theorem can be stated as follows:

Let u be a random variable with probability distribution $G_U(u)$ defined over a finite closed interval I = [a, b]. Let $k_1(u), k_2(u), \ldots, k_n(u)$ be n continuous functions defined on I. Let m_i, $i = 1, \ldots, n$, denote the n generalized moments of the random variable u induced by the functions $\{k_i(u)\}$.

$$m_i = \int_I k_i(u)\, dG_U(u) = E_U[k_i(u)], \quad i = 1, \ldots, n \tag{6}$$

Denote the moment space M as

$$M = \left\{ \underline{m} = (m_1, m_2, \ldots, m_n) \in E^n \right\} \tag{7}$$

where $G_U(u)$ ranges over the set of all probability distribution functions defined on I. M is a closed, bounded, and convex set.

Let C denote the generalized curve $\underline{r} = (r_1, r_2, ..., r_n)$ traced out in E^n by $r_i = k_i(u)$ for $u \in I$. Let H be the convex hull of C. Then M = H.

The application of the Isomorphism Theorem to bounding problems can be seen from the following two-dimensional (n = 2) example. Given a function $k_1(u)$ of the random variable u, the value of whose moment, $m_1 = E_U[k_1(u)]$, is desired, select a second function, $k_2(u)$, whose moment, $m_2 = E_U[k_2(u)]$, can be easily evaluated. By identifying the functions $k_1(u)$ and $k_2(u)$ with the two orthogonal axes of a two-dimensional coordinate system, as in Figure 2, a curve C can be traced out as u varies through its finite range of values. The convex hull, H, of the curve C can be found, as in Figure 3. According to the Isomorphism Theorem the set of all moment pairs

$$M = \{(m_1, m_2) | m_1 = E_U[k_1(u)], \ m_2 = E_U[k_2(u)]\} \tag{8}$$

where the distribution of u varies over all possible distributions defined on the range of u, is identical to the convex hull H. Thus, from Figure 4, upper and lower bounds to the exact value of m_1 for any particular distribution $G_U(u)$ occur at the points where the line $k_2(u) = m_2$ intersects the surface of the convex hull. In Figure 4, the values of the bounds are denoted P_{e_U} and P_{e_L}, respectively.

Most of the applications of the moment bounding technique to problems in communications theory have used two-dimensional moment bounds (a single auxiliary function) (8) - (13), (16) - (19). This is because two-dimensional bounds are quite intuitive and inherently tractible (they can always be found graphically). Higher dimensional moment bounds are desirable because they are typically much tighter than their two-dimensional counterparts. Unfortunately, the mathematical description of the surface of a convex hull of greater than two dimensions is often very difficult to derive. The remainder of this paper describes a computational algorithm that can be used to evaluate geometric moment bounds of any given dimension.

ALGORITHM

Basic Concepts

The algorithm that will be outlined below provides a unified approach to the solution of a broad class of bounding problems. A major advantage of this approach over other means of evaluating geometric moment bounds is that the bounds can be computed without a detailed preliminary investigation of the geometry of the convex hull. For purposes of illustration, a four-dimensional algorithm will be outlined. The procedures for extending the technique to any number of dimensions will be apparent.

Consider a continuous function $k_4(u)$ of a random variable u with distribution $G_U(u)$ defined on a finite closed interval I = [a, b]. Bounds on the generalized moment of $k_4(u)$,

$$m_4 = \int_I k_4(u)\, dG_U(u) \tag{9}$$

are desired.

Select three auxiliary functions (for a fourth-dimensional bound), $k_1(u)$, $k_2(u)$, $k_3(u)$, whose generalized moments m_1, m_2, and m_3, are easily evaluated. These four functions can be associated with the orthogonal coordinate axes of a standard four-dimensional Euclidean space, E^4. Label these axes w, x, y, and z. Consider the curve C defined by

$$C = \{(w, x, y, z) \mid w = k_1(u),\ x = k_2(u),\ y = k_3(u),\ z = k_4(u);\ u \in I\} \tag{10}$$

Let H denote the convex hull of C. According to the Isomorphism Theorem, bounds to the true value of m_4 will be given by the points of intersection of the surface of H with the line ℓ given by

$$\ell = \{(w, x, y, z) \mid w = m_1,\ x = m_2,\ y = m_3,\ z = z\} \tag{11}$$

where m_1, m_2, and m_3 are the generalized moments of $k_1(u)$, $k_2(u)$, and $k_3(u)$, respectively. The line ℓ can be seen to be parallel to the z -axis and will intersect the surface of H in two places, corresponding to the upper and lower bounds.

The key to the algorithmic solution method is that the curve C is approximated in a computer by an array of N sample points taken from the curve. This is equivalent to replacing the curve C with a piecewise linear approximation $\hat{C}$. Denote the convex hull of $\hat{C}$ by $\hat{H}$. The convex hull $\hat{H}$ will be strictly interior to the original hull H, but since the fit can be made arbitrarily tight by increasing the number of points, N, this fact is unimportant in practice (23).

The big advantage of this approach to the bounding problem is that regardless of the shape of the original hull $\hat{H}$, the new hull $\hat{H}$ will be a polytope. Thus, while a mathematical description of the surface of H may be intractable, the surface of $\hat{H}$ can be described in terms of sections of hyperplanes. The solution of the bounding problem then becomes a matter of determining which two of the hyperplanar sections on the surface of $\hat{H}$ are pierce by the line ℓ , Eq. (11).

Algorithm Outline

In this section, a particular algorithm is outlined that will solve the four-dimensional geometric moment bounding problem. This algorithm has the advantages of being conceptually simple and easily extensible to problems of higher dimension. In this outline, only the computation of the lower bound will be considered. The alterations required to compute the upper bound will be clear.

Consider a set of sample points of a continuous curve C in E^4. Denote this set by

$$\{\underline{v}_i\} = \{(w_i,\ x_i,\ y_i,\ z_i)\},\ i = 1,\ N \qquad (12)$$

The objective is to compute the four-dimensional geometric moment lower bound for the moment

$$m_4 = E_U[z(u)] = E_U[k_4(u)] \qquad (13)$$

based on this set of sample points.

<u>Step 1</u> – Find a point $\underline{v}_1$ from the set $\{\underline{v}\}$ that is on the surface of the convex hull $\hat{H}$. In particular, select as $\underline{v}_1$ the point with the minimum value of z -coordinate.

<u>Step 2</u> – Consider the projection of the set $\{\underline{v}\}$ from the space E^4 into the E^2 subspace formed by the w and z coordinates axes. Denote-this projected set by

$$\{\underline{v}^{(2)}\} = \{(w,\ z) \mid (w,\ x,\ y,\ z) \in \{\underline{v}\}\} \qquad (14)$$

and denote the two-dimensional convex hull of $\{\underline{v}^{(2)}\}$ by $\hat{H}^{(2)}$. Similarly, let $\ell(2)$ denote the projection of the line $\ell 1$, Eq. 11, into this subspace.

The surface of $\hat{H}^{(2)}$ is a set of chords joining points of $\{\underline{v}^{(2)}\}$. The point $\underline{v}_1$ from Step 1 is involved in the definition of two of these surface chords (22). Label these chords $ch_2 = (\underline{v}_1, \underline{v}_2)$ and $ch_3 = (\underline{v}_1, \underline{v}_3)$. If either ch_2 or ch_3 intersect $\ell^{(2)}$, label that chord ch_1. If they both intersect $\ell^{(2)}$ observe which of $\{\underline{v}_2, \underline{v}_3\}$ has the smaller z -coordinate value and define ch_1 accordingly. If neither chord intersects $\ell^{(2)}$, observe which of $\{\underline{v}_2, \underline{v}_3\}$ is closer to $\ell^{(2)}$ in terms of Euclidean distance and define ch_1 accordingly. In all cases, relabel the points (if necessary) such that $ch_1 = (\underline{v}_1, \underline{v}_2)$.

<u>Step 3</u> – In a similar fashion, define the projected set

$$\{\underline{v}^{(3)}\} = \{(w,\ x,\ z) \mid (w,\ x,\ y,\ z) \in \{\underline{v}\}\} \qquad (15)$$

the line $\ell^{(3)}$, and the convex hull $\hat{H}^{(3)}$. for the projection from E^4 onto the E^3 subspace of the w, x, and z coordinates. The surface of $\hat{H}^{(3)}$ is a set of planar sections. Each planar section is bounded by surface chords. The chord ch_1 is involved in the boundary of two of these planar sections (22). Label these planar sections $ps_2 = (ch_1, \underline{v}_4) = (\underline{v}_1, \underline{v}_2, \underline{v}_4)$ and $ps_3 = (ch_1,$

$\underline{v}_5$) = ($\underline{v}_1$, $\underline{v}_2$, $\underline{v}_5$). If either ps_2 or ps_3 is pierce by $\ell^{(3)}$ label that planar section ps_1. If both are pierced by $\ell^{(3)}$, observe which of {$\underline{v}_4$, $\underline{v}_5$} has the smaller z-coordinate value and define ps_1 accordingly. If neither planar section is pierced by $\ell^{(3)}$, observe which of {$\underline{v}_4$, $\underline{v}_5$} is closer to $\ell^{(3)}$ in terms of Euclidean distance and define ps_1 accordingly. In any case, relabel the points such that ps_1 = (ch_1, $\underline{v}_3$) = ($\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$)-

Step 4 – The surface of $\hat{H}$ is a set of hyperplanar sections. Each hyperplanar section is bounded by surface planar sections. The planar section ps_1 is involved in the boundary of two of these hyperplanar sections (22). Label these two sections hp_2 = (ps_1, $\underline{v}_6$) = ($\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$, $\underline{v}_6$,) and hp_3 = (ps_1, $\underline{v}_7$) = ($\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$, $\underline{v}_7$). If either hp_2 or hp_3 is pierced by ℓ, label that hyperplanar section hp_1. If both are pierced by ℓ, observe which of {$\underline{v}_6$, $\underline{v}_7$} has the smaller z-coordinate value and define hp_1 accordingly. If neither hyper planar section is pierced by ℓ, observe which of {$\underline{v}_6$, $\underline{v}_7$} is closer to ℓ in terms of Euclidean distance and define hp_1 accordingly. In any case, relabel the points such that hp_1 = (ps_1, $\underline{v}_4$) = ($\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$, $\underline{v}_4$,).

Step 5 – If hp_1 is pierced by ℓ, the lower bound has been found. The value of the lower bound will be the z -coordinate value of the point of intersection between the line ℓ and hp_1. If hp_1 is not pierced by ℓ, proceed to Step 6.

Step 6 – The hyperplanar section hp_1 is bounded by $\binom{4}{3} = 4$ planar sections, one of which is ps_1. Label the other three ps_2, ps_3, and ps_4. Determine which member of the set {ps_2, ps_3, ps_4} is closest to ℓ in terms of Euclidean distance. Relabel the system such that this planar section is denoted ps_1 = ($\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$).

Step 7 – This new planar section ps_1 is involved in the boundary of two hyperplanar sections on the surface of $\hat{H}$, one of which is hp_1. Let the second one be denoted hp_2 = ($\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$, $\underline{v}_8$). Discard hp_1 and replace it in memory with hp_2. Rename this new hyperplanar section hp_1 and return to Step 5.

This algorithm can be thought of as a search routine that "slides" across the surface of $\hat{H}$. The "slide" is from one hyperplanar section to an adjacent one. This is accomplished by replacing one point in the set of four points that defines the current section hp_1 in a recursive manner. Clearly, an n-dimensional search could be implemented by systematically replacing one point at a time in a set of n points that defines an n-dimensional hyperplanar section on the surface of an n-dimensional convex hull.

As an aid in the visualization of the basic ideas behind the algorithm, consider the two-dimensional example illustrated in Figure 5. It is clear that all higher dimensional algorithms could be viewed as extensions of this basic situation.

In Step 1 the algorithm will find the point $\underline{v}_1$, as shown in the figure. There will be two chords on the surfaces of $\hat{H}$ that involve $\underline{v}_1$. These are the chords ($\underline{v}_2$, $\underline{v}_1$) and ($\underline{v}_3$, $\underline{v}_1$). Since $\underline{v}_2$ is closer to the line ℓ than $\underline{v}_3$, the chord ($\underline{v}_2$, $\underline{v}_1$) is the chord ch_1 found in §tep 2.

Because this is only a two-dimensional illustration, Step 3 and Step 4 do not apply. The algorithm determines that the chord ch_1 does not intersect ℓ, and, therefore, does not contain the solution, at Step 5. The point $\underline{v}_2$ is also involved in the definition of two chords on the surface of $\hat{H}$. One of these is ch_1. The second chord will be found in Steps 6 and 7 and checked to determine if it contains the solution. When it is found that the chord does not the algorithm will continue. In this manner, the algorithm "slides" across the surface of $\hat{H}$ in the direction of the solid arrows on Figure 5 until the solution is found.

The algorithm can be seen to be convergent in a finite number of steps (23). This is because the algorithm always proceeds toward the line ℓ and, therefore, it cannot cycle back on itself. In addition, since {$\underline{v}$} contains a finite number of points, only a finite number of steps are required to find the bounds even under worst case conditions. Reasonably efficient routines for implementing the individual steps in this algorithm are found in References 23-28.

NUMERICAL RESULTS AND CONCLUSIONS

Results

The algorithm outlined in the preceding section has been coded in the FORTRAN IV programming language. Two sample cases have been run. Both cases are intersymbol interference problems of the type described in the first section. The auxiliary functions used in obtaining the bounds were the second, fourth, and sixth powers of the amplitude of the interference. The parametric equations for the curve C are

$$x = u^2 \tag{16}$$

$$y = u^4 \tag{17}$$

$$W = u^6 \tag{18}$$

$$z = \frac{1}{2}\left[Q\left(\frac{h_o + u}{\sigma}\right) + Q\left(\frac{h_o - u}{\sigma}\right)\right] \tag{19}$$

where u is the value of the amplitude of the intersymbol interference, h_o is the amplitude of the desired signal, σ is the standard deviation of the additive white Gaussian noise, and Q(·) is the usual complementary error function given by

$$Q(y) = \int_y^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \tag{20}$$

The limits on the value of u are given by

$$0 \leq u \leq D \leq h_o \tag{21}$$

where D represents the maximum possible amplitude of the intersymbol interference.

The first example is that of a channel with a Chebyshev filter impulse response. This is a channel impulse response that is often used in comparing intersymbol interference bounding techniques (1) - (11). For this channel D = 0. 28. The results presented in Table I were obtained by approximating the curve C, given parametrically by Eqs. (16)-(19) by an array of 50 points. These points were equally spaced in terms of the parameter u. Fifty points were found to be sufficient to ensure the accuracy of the. results (23).

Table I. Bounds for Chebyshev Channel

SNR (dB)	Upper Bound B_U	Lower Bound B_L	Difference $B_U - B_L$
0	1.5931×10^{-1}	1.5931×10^{-1}	5.4907×10^{-12}
4	5.7725×10^{-2}	$5.772\ 5 \times 10^{-2}$	3.3281×10^{-10}
8	6.7577×10^{-3}	6.7577×10^{-3}	1.7327×10^{-9}
12	6.6205×10^{-5}	6.6198×10^{-5}	7.0937×10^{-9}
16	5.5281×10^{-9}	3.8831×10^{-9}	1.6451×10^{-9}
20	3.8729×10^{-16}	1.3277×10^{-18}	3.8596×10^{-16}

Chebyshev Channel:

$$h_i = 0.4023 \cos(2.839\,|i| - 0.7553)\exp(-0.4587\,|i|$$

$$+ 0.7162 \cos(1.176\,|i| - 0.1602)\exp(-1.107\,|i|$$

Table I contains the upper and lower bounds that were computed using the algorithm described in the second section. Also tabulated is the difference between the bounds. The difference serves as a measure of tightness of the four-dimensional bounding technique. The bounds are seen to be extremely tight at low values of signal-to-noise ratio (SNR) and useful everywhere. These bounds are about two orders of magnitude tighter than corresponding three-dimensional bounds (23).

The second example is that of a channel with a modified version of a Chebyshev impulse response. In this example, the maximum distortion, D, is taken to be three times that of the standard Chebyshev channel. The channel impulse response and other appropriate parameters are scaled accordingly.

The numerical results are presented in Table II. They can be seen to remain quite useful for this example of more severe intersymbol interference. These four-dimensional bounds were found to represent about an order of magnitude improvement over corresponding three-dimensional bounds.

Table II. Modified Chebyshev Channel

SNR (dB)	Upper Bound B_U	Lower Bound B_L	Difference $B_U - B_L$
0	1.6445×10^{-1}	1.6445×10^{-1}	3.5336×10^{-8}
4	6.7405×10^{-2}	6.7403×10^{-2}	1.7210×10^{-6}
8	1.4003×10^{-2}	1.4001×10^{-2}	2.5350×10^{-6}
12	1.2972×10^{-3}	1.2113×10^{-3}	8.5834×10^{-5}
16	1.7687×10^{-4}	3.0458×10^{-5}	1.4642×10^{-4}
20	6.7317×10^{-5}	6.2781×10^{-9}	6.7310×10^{-5}

Conclusions

The results of this section demonstrate the usefulness of the algorithm of the second section. This algorithm systematically stepped across the surface of the convex hull until the appropriate surface feature was found. This was accomplished by considering a sequence of hyperplanar surface features in terms of the sets of four points that define them. The algorithm proceeded by identifying and eliminating the point out of the four-point set that was "furthest" from the desired direction of travel. The eliminated point was then replaced by an appropriate point that was in the direction of convergence. This new four point set defined another hyperplanar surface feature that was "closer" to the solution of the bounding problem than was the previous section. This procedure was repeated until convergence to the solution of the bounding problem was achieved.

The extension to bounding problems of dimensions greater than four seems clear. For a five-dimensional problem, the hyperplanes would be defined by sets of five points. In a manner similar to the four-dimensional algorithm, one of these points that is "furthest" from the desired direction of travel could be identified and eliminated. This point could be replaced with an appropriate point that is "closer" to the solution of the bounding problem. This would define a new hyperplanar surface section that is closer to the solution of the problem. Clearly, if this procedure is repeated a sufficient number of times, the algorithm will converge to the desired solution. It is also clear that the concept of stepping across the surface of a convex hull by modifying the set of points one point at a time is a concept that will work independently of the total number of points in the set. That is, the procedure of stepping across the surface by systematically going from one surface feature to an adjacent one is a procedure that will work independently of the dimensionality of the problem.

An important point to investigate is the relationship between problem dimension and algorithm run time. This is a difficult problem, but some insight can be gained from the form of the four -dimensional algorithm.

In order to determine whether the desired solution has been found, the algorithm must compute the point of intersection of a hyperplane and the line ℓ. In E^4, this amounts to inverting a 4 x 4 matrix. In an extension to a K-dimensional problem, it would mean inverting a K x K matrix. The naive method of inverting a K x K matrix (straightforward use of the method of cofactors) would require more than 2(K!) multiplications. While more sophisticated numerical methods would undoubtedly require fewer multiplications, it appears that run time can be expected to increase very quickly as the dimensionality of the problem increases. Fortunately, the numerical results presented here appear to indicate the extensions to very high dimensionality will rarely be necessary. This is because the bounding results appear to tighten very quickly with increasing problem dimension.

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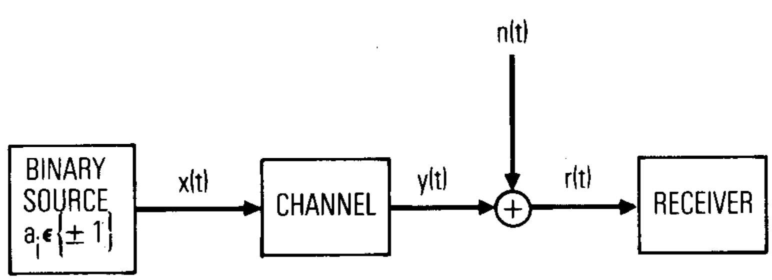

Figure 1. Binary Communication Link

Figure 2. Curve C

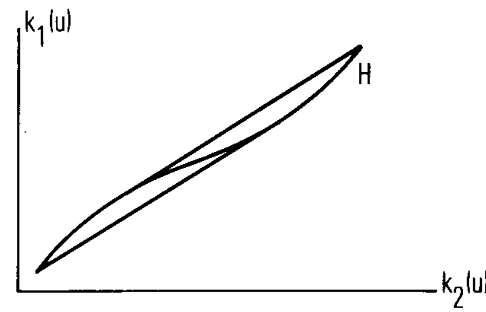

Figure 3. Convex Hull H Generated by C

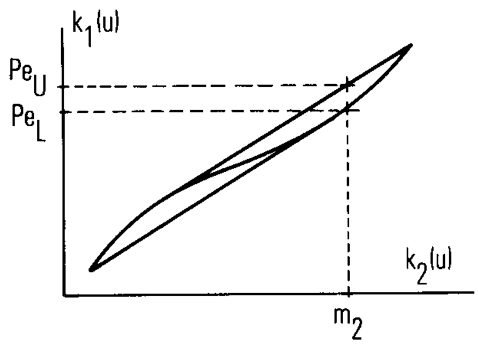

Figure 4. Computation of Bounds

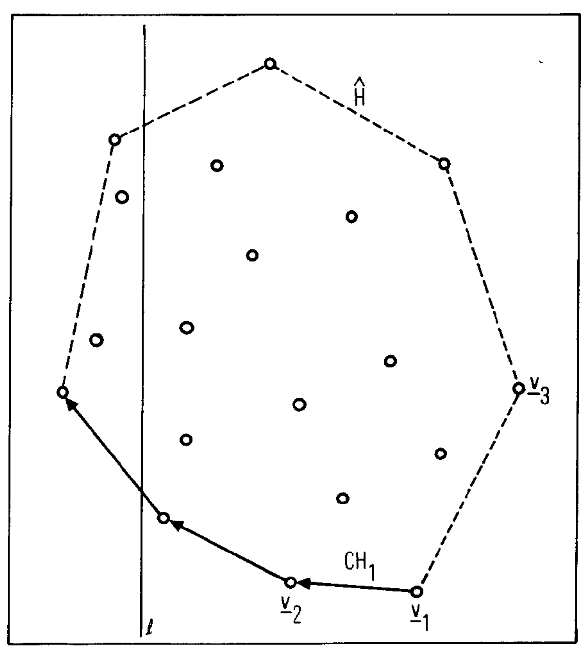

Figure 5. Bounding Technique Illustration