SPECTRAL SHAPING WITHOUT SUBCARRIERS

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ABSTRACT

For proper operation of the phase lock loop which tracks a carrier it is important to minimize the spectral energy at frequencies near the carrier. A traditional method is to modulate the data onto a subcarrier in such a way that there is little energy near D.C. The resulting signal then is used to modulate the carrier. The problem with such a scheme is that the total bandwidth is much larger than necessary to transmit the data.

This paper proposes and analyzes a simpler scheme which increases the data bandwidth by a very small fraction, yet reduces the energy near D.C. to nearly zero.

INTRODUCTION

We will do our analysis at baseband and begin with a statistic which will allow us to estimate the energy of a process between the frequencies -B and +B.

For a stationary process X(t) with spectral density $S_X(f)$, define a new process by

$$Y_B(t) = \frac{1}{T}\int_0^T X(t-\tau)\,d\tau$$

where T = l/2B. Then the spectral density of Y is

$$S_Y(f) = S_X(f)\left(\frac{\sin \pi fT}{\pi fT}\right)^2 \qquad (1)$$

and the power in Y is

$$E\{Y^2(t)\} = \int_{-\infty}^{\infty} S_Y(f)\,df = \int_{-\infty}^{\infty} S_X(f)\left[\frac{\sin(\pi fT)}{\pi fT}\right]^2 df \qquad (2)$$

Now

$$\left[\frac{\sin(\pi f T)}{\pi f T}\right]^2 \geq \left(\frac{2}{\pi}\right)^2 \qquad \text{for} \qquad |f| \leq \frac{1}{2T} = B$$

and ≥ 0 otherwise

So equation (2) implies

$$E\{Y^2(t)\} \geq \left(\frac{2}{\pi}\right)^2 \int_{-B}^{B} S_X(f)\,df$$

or

$$\int_{-B}^{B} S_X(f)\,df \leq \left(\frac{\pi}{2}\right)^2 E\{Y^2(t)\} \tag{4}$$

Thus the second moment of Y(t) gives an estimate of the amount of energy in X between frequencies -B and +B.

For an application of this statistic, consider the process X(t) which is +1 or -1 on each interval $[nT_o,(n+1)T_o)$. Assume the values on different intervals are independent and have probability ½. Then

$$E\{Y^2(t)\} = E\left\{\frac{1}{T}\int_0^T X(t)\,dt\right\}$$

$$= E\left\{\frac{T_o}{T}\sum_{n=1}^{T/T_o} X[(n-1)T_o]\right\}$$

$$= \left(\frac{T_o}{T}\right)^2 \frac{T}{T_o} = \frac{T_o}{T} = 2BT_o \tag{5}$$

and the bound is

$$\int_{-B}^{B} S_X(f)\,df \leq \left(\frac{\pi}{2}\right)^2 2BT_o \tag{6}$$

Of course,

$$S_X(f) = \left(\frac{\sin(\pi T_o f)}{\pi f}\right)^2$$

and for small B the energy between -B and +B is $2BT_o$. The factor $\left(\frac{\pi}{2}\right)^2$ indicates the looseness of the bound.

The signal design problem is to encode the data into a signal X(t) such that $E\{Y^2(t)\}$ is small.

PROPOSED SOLUTION

The proposed solution is to expand the data stream by inserting a redundant bit every L-th bit, the value of the bit being chosen to bring the total number of +1's and -1's into balance.

More precisely:

Let X_n be a sequence of ±1's, defined below.

Define $X(t) = X_n$ for $t \in [(n-1)T_o, nT_o]$.

Define $C_n = \sum_{m=1}^{n} X_n$. (7)

Let L be an even integer.

Then X_n is defined as follows: When n is not a multiple of L, X_n is a data bit (±1). When n is a multiple of L then

$$X_n = -\text{sgn}\,[C_{n-1}]$$

(since L is even n-1 is odd. Then from its definition C_{n-1} must be odd and cannot be zero).

The derivation of a bound on the power between -B and +B is given below, resulting in equation (15). For non-redundant data (flat random data) the amount of power is $2T_oB$, so the factor $\left[T_oB\,\frac{3\pi^2}{8}\,L^2\right]$ indicates what the gain has been when a redundancy of 1/L has been inserted. In particular, when T_o = 1/30 MHz and B = 1 KHz, if the value of L is 30 then the factor is 1/8 or a gain of 9 dB. If L = 10 then the gain is 18. 5 dB.

ANALYSIS

It is clear from the definitions that, for large n

$$Y(nT_o) = [C_n - C_{n-(T/T_o)}] \frac{T_o}{T}$$

Therefore the second moments of $\{C_n\}$ must be studied. We will assume n so large that the stationary distributions have been obtained so that

$$E\{C_n^2\} = E\{C^2_{n-(T/T_o)}\}$$

In the case that the data bits are independent it can be shown that

$$E\{C_n\ C_{n-(T/T_o)}\} \geq 0$$

From this we have

$$E\{Y^2(nT_o)\} \leq \frac{2T_o^2}{T^2} E\{C_n^2\} \tag{8}$$

To analyze C_n, let

$$Z_k = \sum_{n=kL+1}^{kL+L-1} X_n$$

That is Z_k is the sum of L-1 consecutive data bits. For most of the analysis we will assume only that the odd moments of Z_k are 0 but for the best result we must also assume that the X_n contributing to Z_k are mutually independent.

From the definition of X_n we have

$$X_{(k+1)L} = -\text{sgn}\,[C_{kL} + Z_k]$$

and (9)

$$C_{(k+1)L} = C_{kL} + Z_k - \text{sgn}\,[C_{kL} + Z_k]$$

Multiplying through by sgn $[C_{kL} + Z_k]$ gives

$$C_{(k+1)}L \cdot \text{sgn}\,[C_{kL} + Z_k] = |C_{kL} + Z_k| -1 \tag{10}$$

Since subtracting 1 from a positive odd integer cannot change the sign, the left side of Equation (10) must be non-negative and we have

$$|C_{(k+1)L}| + 1 = |C_{kL} + Z_K|$$

Next define

$$\mu_k = E\{Z^k\}$$

and

$$m_k = E\{|C_{kL}|^k\}$$

Then from equation (10) and the assumption that $\mu_k = 0$ for odd k gives

$$m_2 + 2m_1 + 1 = m_2 + \mu_2$$

$$m_4 + 4m_3 + 6m_2 + 4m_1 + 1 = m_4 + 6m_2\mu_2 + \mu_4 \qquad (12)$$

From these equations and the Swartz inequality $m_1\, m_3 \geq m_2^2$ the following inequality can be derived

$$\left[m_2 - \frac{3}{8}(\mu_2 - 1)^2\right]^2 \leq (\mu_2 - 1)\left[\frac{\mu_4 - 4}{8} - \frac{\mu_2 - 1}{4} + \frac{9}{64}(\mu_2 - 1)^3\right]$$

or

$$m_2 \leq \frac{3}{8}(\mu_2 - 1)^2 + \frac{1}{8}\sqrt{(\mu_2 - 1)[8\mu_4 - 16 - 16\mu_2 + 9(\mu_2 - 1)^3]} \qquad (13)$$

When the data bits are independent $\mu_2 = L-2$ and equation (13) implies

$$m_2 \leq \frac{3}{4}L^2 \qquad (14)$$

This combined with equations (4) and (8) give

$$\int_{-B}^{B} S_X(f)\,df \leq \left(\frac{\pi}{2}\right)^2 (2T_oB)^2 \cdot 2 \cdot \frac{3}{4}L^2$$

$$= (2T_oB)^2\left[\frac{3\pi^2}{8}L^2\right]$$

or (15)

$$\int_{-B}^{B} S_X(f)\,df \leq 2T_oB\left[T_oB \cdot \frac{3\pi^2}{8}L^2\right]$$