QUANTIZATION FOR SIGNAL DETECTION AND REPRESENTATION*

Saleem A. Kassam
Moore School of Electrical Engineering
University of Pennsylvania
Philadelphia, Pennsylvania

ABSTRACT

For digital representation of analog data the minimum mean-squared-error criterion is commonly used as a criterion for the basis of optimum quantizer design. In this paper we show that in some situations measures other than the minimum mean-squared-error may be more appropriate. For the signal representation problem, it is shown that the mean-absolute-error criterion has theoretical justification, as again for some signal detection problems it is shown that the mean-squared-error criterion is not the most appropriate criterion.

INTRODUCTION

Because of the widespread use of digital signal processing methods, the conversion of analog data into digital form is a necessary step in many signal processing systems. A commonly encountered need is to obtain a good representation of analog data with a finite number of bits. We will call this the problem of quantizing data for representation. In other applications it is not the goodness of the representation or "fit" obtained which is of prime importance but rather the extent to which some particular feature of the analog data is preserved in its quantized version. For example, if analog data is to be used to detect the presence or absence of a signal in noise, then the quantization should be performed to maintain as much of the separation of the characteristics of the data under the two hypotheses (signal present or noise only present).

Most previous considerations of quantization have been based on the criterion of minimizing the mean-squared-error (MSE) between the analog and quantized data. The use of this criterion cannot be theoretically justified in many instances. In this paper we will show that for both signal representation and signal detection applications, other criteria may be more appropriate and justifiable as a basis for optimum quantizer design. It should be noted that this idea is analogous to one developed in a recent paper[1] where the effects

* This work is supported by the National Science Foundation under Grant ENG76-10444

of sampling a continuous waveform for a signal detection application is analyzed directly using detection criteria, rather than the criterion of mean-squared reconstruction error.

Some of the results discussed in this paper are based on recent published work by the author [2,3].

SIGNAL REPRESENTATION - THE MEAN-ABSOLUTE-ERROR CRITERION

We will assume that the analog source input S to the quantizer has an even density function f and distribution function F. Thus we will consider symmetric quantizers q, which are described by the positive input transition values $0 < x_1 < x_2 < \ldots < x_{M-1}$ and levels $y_1\ y_2, \ldots y_M$ for 2M-level quantization. We have $q(s) = y_i$ for $s \in (x_{i-1},\ x_i)$, where we also define $x_0 \triangleq 0,\ x_M \triangleq \infty$.

Let ℓ be an absolutely continuous even function which is increasing on $[0,\infty)$, with $\ell(0) \geq 0$. A *distortion* measure D_ℓ may be defined by

$$D_\ell = E\{\ell[S - q(S)]\} \tag{1}$$

(where the integral is assumed to exist), and the quantizer minimizing D_ℓ may be derived easily [4]. The result is an optimum quantizer parameter set defined by the equations

$$x_i = \frac{y_i + y_{i+1}}{2}, \qquad i = 1,2,\ldots,M-1, \tag{2}$$

$$\int_{x_{i-1}}^{x_i} \ell'(s-y_i)\,dF(s) = 0, \quad i = 1,2,\ldots M \tag{3}$$

where the prime denotes the first derivative of ℓ.

Now consider the distribution function F_q of the output q(S) of the quantizer; with g a weight function with the same properties as ℓ, we may define a *distance* Δ_g between F and F_q by

$$\Delta_g = \int_{-\infty}^{\infty} g[F(s) - F_q(s)]ds \tag{4}$$

It is reasonable to look for a quantizer minimizing Δ_g for a given g; this is because q(S) is completely dependent on S. We proceed to do this by first expressing Δ as

$$\Delta_g = 2\sum_{i=0}^{M}\int_{y_i}^{y_{i+1}} g[F(s) - F(x_i)]ds \tag{5}$$

where $y_0 \stackrel{\Delta}{=} 0$ and $y_{M+1} \stackrel{\Delta}{=} \infty$. Thus Δ_g is a weighted integral of distance between F and F_q. To minimize Δ_g, we set partial derivatives equal to zero and find the following necessary conditions:

$$\int_{y_i}^{y_{i+1}} g'[F(s) - F(x_i)]ds = 0, \quad i = 1,2,...M-1, \tag{6}$$

$$F(y_i) = \frac{F(x_i)+F(x_{i-1})}{2}, \qquad i = 1,2,...M \tag{7}$$

(we assume f(s) > 0 when 0 <F(s) < 1).

Two interesting observations can be made: one is that Eq. (7), like Eq. (2), is independent of the weight function defining the error, and secondly, the two pairs of equations (2),(3) and (6),(7) may be considered as being duals of each other. A necessary condition for minimum Δ_g is given by the set of equations (7), and a necessary condition for minimum D_ℓ is given by the set of equations (2). These two sets of equations are independent of g and ℓ, respectively, and thus it would seem reasonable to use Eqs. (2) and (7), which give a total of 2M-1 equations, for the parameters of a quantizer for S.

The signifcant property of the resulting quantizer is that it is the minimum MAE quantizer, that is, Eq. (3) reduces to Eq. (7) when ℓ is the absolute value function. This is easily seen by substituting sgn (s-y.) for $\ell'(s-y_i)$ in Eq. (3). Furthermore, we find that with g the absolute value functions Eq. (6) reduces to Eq. (2). Thus the quantizer with parameters satisfying Eqs. (2) and (7) is the minimum MAE quantizer and also the minimum integratedabsolute-distance (IAD) quantizer, i.e., minimizing Δ_g where g is the absolute value function.

There does exist a strong connection between the distortion measure D_ℓ and the distance measure Δ_g. To see this, we rewrite D_ℓ by changing variables in Eq. (1) to get

$$D_\ell = 2\sum_{i=0}^{M-1}\int_{x_i}^{x_{i+1}} \ell(s-y_{i+1})\, dF(s)$$

$$= 2\sum_{i=0}^{M-1}\int_{F(x_i)}^{F(x_{i+1})} \ell[F^{-1}(s)-y_{i+1}]\, ds, \tag{8}$$

assuming that F^{-1} is well-defined. We see that Eq. .(8) also defines a weighted integral over area between F and F_q, the weights being applied to the *horizontal* distances between F and F_q. It is now clear why we get the same quantizer under the minimum MAE and minimum IAD criteria. This also explains why for a uniform density f, the uniform quantizer minimizes both D_ℓ and ℓ_g for any ℓ and g.

It is thus clear that the distortion and distance criteria are quite closely related, both being interpreted as expressing the deviation of the distribution function of the quantizer output from the input distribution function. We have also demonstrated that the absolute-value weight function is the only one for which the same quantizer is obtained under either criterion, for arbitrary input signal probability distribution function.

The quantizer equations (2) and (7) also imply that simple adaptive schemes based on estimating the distribution functions F can be implemented [2].

QUANTIZATION FOR SIGNAL DETECTION

(a) One-Input, Known Signal

Let $\{\chi_i\}_{i=1}^{n}$ be a sequence of n independent samples, the samples being described by

$$\chi_i = \theta s_i + N_i, \qquad 1 \leq i \leq n,\ \theta \geq 0. \tag{9}$$

Here the sequence $\{s_i\}_{i=1}^{n}$ is a known signal sequence, and the sequence $\{N_i\}_{i=1}^{n}$ is composed of independent random variables, representing noise, with common density and distribution functions f and F respectively. We assume that f is symmetric about the origin and is absolutely continuous. Given a positive integer k, our problem is to specify the k-level quantizer such that with Y_i defined as

$$Y_i = q(\chi_i), \tag{10}$$

the statistic

$$Q = \sum_{i=1}^{n} Y_i s_i \tag{11}$$

is the optimum statistic based on k-level quantized data for deciding between the null hypothesis H_o:$\theta = 0$ versus the Alternative H_1: $\theta > o$. We also confine attention to the case of quantization with an even number of levels k, that is to the case where k = 2m for some positive integer m. The optimum quantizer with odd number of levels k can be derived in an exactly similar manner.

The *efficacy* F of a threshold test for H_o versus H_1 based on a test statistic Q is defined as [5]

$$EF = \lim_{n \to \infty} \frac{1}{n} \frac{\left[\frac{d}{d\theta} E_\theta\{Q\} \Big|_{\theta=o} \right]^2}{Var_o \{Q\}} \tag{12}$$

It is an asymptotic measure of the effectiveness of the test in detecting small signals, and may be described as an incremental-signal-to-noise ratio for the test statistic. Using efficacy as a measure of performance, we then look for the optimum (2m)- level quantizer maximizing efficacy.

Based on this criterion of quantizer performance, after some algebraic manipulations, we can derive the optimum quantizer parameter equations [3]:

$$y_i = \frac{f(x_{i-1}) - f(x_i)}{F(x_i) - F(x_{i-1})}, \quad i = 1,2,\ldots,m, \tag{13}$$

$$\frac{y_{i+1} + y_i}{2} = -\frac{f'(x_i)}{f(x_i)}, \quad i = 1,2,\ldots,m-1 \tag{14}$$

This same set of quantizer equations can also be obtained by considering the optimum weighting of data which has been partitioned by the set of break points $\{x_i\}$. There is yet another interesting interpretation of the optimum quantizer equations (13) and (14) .

According to the theory of locally optimum tests, the locally optimum test statistic for our detection problem is given by

$$Q_{\ell o} = \sum_{i=1}^{n} h(X_i) s_i \tag{15}$$

where

$$h(x) = \frac{-\frac{d}{dx} f(x)}{f(x)} \tag{16}$$

f being the density function of the noise. A comparison of Eqs.(11) and (15) leads one to expect a direct relationship between the optimum quantizer q and the locally optimum nonlinearity h. Indeed, consider the mean-squared-error ϵ between q (X_i) and $h(X_i)$:

$$\varepsilon = E\left\{\left[q(\chi_i) + \frac{f'(\chi_i)}{f(\chi_i)}\right]^2\right\} \tag{17}$$

where f'(x) = [df(x)/dx]. This can be written as

$$\varepsilon = E\left\{[q(\chi_i)]^2 + 2\,q(\chi_i)\,\frac{f'(\chi_i)}{f(\chi_i)} + \left[\frac{f'(\chi_i)}{f(\chi_i)}\right]^2\right\} \tag{18}$$

so that ϵ is finite if the quantity

$$I_f = E\left\{\left[\frac{f'(\chi_i)}{f(\chi_i)}\right]^2\right\} = \int_{-\infty}^{\infty} \frac{[f'(x)]^2}{f(x)}\,dx \tag{19}$$

is finite. If defined by Eq. (19) is known as the Fisher Information function of the density f(·) when it is finite, as we henceforth assume it to be. Writing the mean-squared-error ϵ more explicitly as a function of the quantizer parameters, and setting the partial derivatives of ϵ with respect to the x_i and y_i equal to zero we get again the sets of equations (13) and (14) .

Thus we find that the minimum distortion quantizer which minimizes the mean-squared error between X_1 and its quantized version coincides with the optimum quantizer based on our detection criteria only for Gaussian noise. For other noise densities better detection results can be obtained it the quantization is a good approximation to h(x) rather than to x itself.

For example, when noise has the double-exponential density function, -f'(x)/f(x) = sgn (x), the sign function. In this case only a two-level quantizer (hard-limiter) need be used, and any higher level quantizer of data is unnecessary.

(b) <u>Multi-Input, Random Signals</u>

The detection of a weak signal common to two or more input channels against a background of additive noise is a requirement in many applications, such as in underwater sound systems and geophysical signal processing. For two-input systems such problems with sampled observations may be formulated as statistical hypotheses testing problems. We can define a null hypothesis Ho and an alternative H_0 such that

$$\left.\begin{aligned} &\text{under } H_0: \quad X_1^i = N_1^i, \quad X_2^i = N_2^i \\ &\text{under } H_1: \quad X_1^i = N_1^i + \Theta s^i, \quad X_2^i = N_2^i + \Theta s^i \end{aligned}\right\} \quad i=1,2,\ldots,n, \tag{20}$$

where $\underline{X}_1 = (X_1^1, X_1^2, \ldots, X_1^n)$ and $\underline{X}_2 = (X_2^1, X_2^2, \ldots, X_2^n)$ are the observation vectors and $\underline{s} = (s_1,\ldots,s_n)$ represents the vector of common random signal components. We assume that the components of the additive noise vectors $\underline{N}_1 = (N_1^1, N_1^2, \ldots, N_1^n)$ and $\underline{N}_2 = (N_2^1, N_2^2, \ldots, N_2^n)$ are independent and identically distributed and their common density function f is symmetric about the origin. The components of the vector $\underline{s}$ are assumed to be independent and to have zero mean and unit variance. The parameter θ^2 is then the signal strength parameter.

The locally optimum correlator detector maximizing detector efficacy, based on analog data, uses the test statistic

$$Q_{\ell o} = \sum_{i=1}^{n} h(x_1^i)\, h(x_2^i). \tag{21}$$

For quantization of the two-demensional data, the analog function $h(x_i^i)\ h(x_2^i) = h(x_1^i,\ x_2^i)$ is replaced by a partitioning function defined by

$$h(x_1^i, x_2^i) = \sum_{j=1}^{m} w_j\, g_{ij} \tag{22}$$

where

$$g_{ij} = \begin{cases} 1 & (x_1^i, x_2^i) \in P_j \\ -1, & (x_1^i, x_2^i) \in N_j \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

the P_j's , N_j's being 2m symmetric regions partitioning the two-dimensional observation space. The problem then is to find the optimum weights w_j and optimum partitions.

Using the criterion of detector efficacy or requiring the detector to be a locally-optimum detector, and after some algebraic manipulations, one ends up with the result that the optimum partitioning of the two-dimensional space should follow level curves of the function $h(x_1^i)h(x_2^i)$. Once again, this partitioning turns out to correspond directly to the locally-optimum function h(x) = -f'(x)/f(x). The optimum weight w_j can then easily be shown to satisfy the equations

$$c_j = \frac{w_j + w_{j+1}}{2}, \quad j = 1,2,\ldots,m-1, \tag{24}$$

where the optimum region P_j is defined by

$$P_j = \{ (x_1, x_2): \quad c_j \leq h(x_1)h(x_2) < c_{j-1} \}, \; j=1,2,\ldots,m \tag{25}$$

It can also be demonstrated that equations (24) and (25) define the partitioning and weighting which gives the minimum mean-squared-error fit to $h(x_1)h(x_2)$ of the general partitioning function $h(x_1,x_2)$of equation(22). Thus we see that in this case too it is a modified MSE criterion which is more appropriate for optimum quantizer design. Only in the Gaussian case is this the same as the minimum MSE quantizer for the product of twodimensional data.

CONCLUSION

The minimum MSE criterion is not alwaysthe most appropriate criterion for the design of optimum quantizers. The MAE criterion has been shown to have interesting properties which may justify its use as a basis for optimum quantization. The minimum MAE quantizer parameters can be obtained as the solutions of fairly simple equations, which also imply the possibility of simple adaptive structures for optimum quantization in unknown environments.

It has also been shown that for detection based on quantized data the minimum MSE quantizer does not give the locally optimum detection scheme, or maximum detection efficacy, except in the case of Gaussian noise. For these detection criteria the optimum quantizer is the “best” approximation to the locally optimum detection nonlinearity.

REFERENCES

1. J.G. Shin and S.A. Kassam, “Sampling rate and the J-divergence in detection based on continuous-time observations.” Proc. Conf. Inform. Sciences and Systems, Johns Hopkins Univ., pp. 19-24, April 1978.

2. S.A. Kassam, “Quantization based on the mean-absolute-error criterion,” IEEE Trans. Commun., vol. COM-26, pp. 267-270, February 1978.

3. S.A. Kassam, “Optimum quantization for signal detection,” IEEE Trans. Commun., vol. COM-25, pp. 479-488, May 1977.

4. J. Max, “Quantizing for minimum distortion,” IRE Trans. Inform Theory, vol. IT-6, pp. 7-12, March 1960.

5. J. Capon, “On the asymptotic efficiency of locally optimum detectors,” IEEE Trans. Infom. Theory, vol. IT-7, pp. 67-71, April 1961.