SUB-OPTIMUM RECEIVER FILTERS

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ABSTRACT

This paper presents a method for analyzing the performance of a digital receiver when using standard analog filters in place of the ideal matched filter. Expressions are developed for the probability of error and performance loss of the sub-optimum receiver as functions of the minimum eye value and noise bandwidth of the sub-optimum receiver filter. A method is developed for choosing the best sub-optimum filter in the sense of minimizing the probability of error. The best sub-optimum Bessel filter of order less than or equal to 6 is specified in terms of 3-dB bandwidth and filter order for a system with a rectangular transmit pulse. This method is applicable to other transmit pulse shapes and can be applied to channels with limited bandwidth. The optimum 3-dB bandwidth obtained here can be scaled relative to the symbol rate to correspond to any practical system.

I. INTRODUCTION

The optimum receiver filter for a given channel is one matched to the signaling waveform. However, for various reasons an exact matched filter may not be available

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for a given application. This paper analyzes the performance, in terms of probability of error, of a system with a sub-optimum receiver filter and presents a method for gauging the performance loss for any sub-optimum filter. The method includes sampling the frequency response of the sub-optimum filter, filtering a modulated pseudo random bit sequence, finding the minimum eye opening at the optimum sampling point, and calculating the loss associated with the filter.

Closed form expressions for analysis of standard analog filters (Bessel, Gaussian, Butterworth, and Chebyshev) in terms of the Nyquist I criterion, sampling clock extraction, and sensitivity to sampling clock jitter, are developed in [1]. The Nyquist I criterion is a measure of the ISI at the sampling instant. The paper, [1], finds that for the minimization of ISI, linear phase filters perform best with the Bessel filter giving the least amount of ISI. However, [1] does not consider choosing the best filter order and 3-dB bandwidth which minimizes the probability of error. There is the Saltzberg bound however, which relates MSE (mean square error) and probability of error [5]. In the work presented here consideration is given to choosing the best sub-optimum filter in the sense of minimizing the probability of error.

This paper is organized as follows. Section II develops expressions for the probability of error of the sub-optimum filter and its loss relative to an ideal system. The expressions are functions of the minimum eye opening at the sampling point and the noise bandwidth of the receiver filter. Section III illustrates how to sample the frequency response of the receiver filter and find the minimum eye opening at the output of the filter corresponding to the optimum sampling point. Section IV presents a technique for choosing the best sub-optimum filter from among various choices and section V presents the conclusions.

II. COMMUNICATION SYSTEM PERFORMANCE

The communication system model assumed is illustrated in Figure 1. The input signal x(t) is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a(k) \cdot u(t-kT),$$

where a(k) ∈ {-1,1}, and the transmit pulse is given by $u(t) = \begin{cases} 1, & 0 < t \le T \\ 0, & \text{otherwise} \end{cases}$

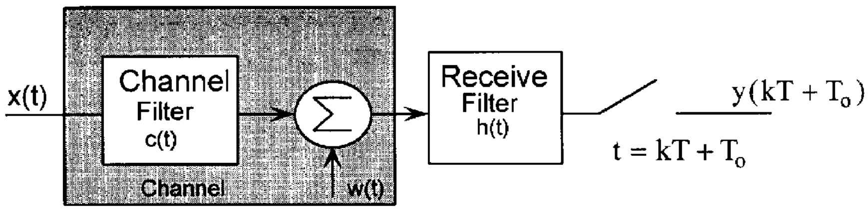

Figure 1. Communication System Model

The channel consists of a linear filter, c(t), and the addition of white Gaussian noise, w(t), with two sided power spectral density $N_o/2$ W/Hz. The offset, T_o, is the signal delay associated with the channel and receiver filter, without loss of generality it is assumed to be zero. If h(t) is matched to c(t)*u(t), the convolution of the channel impulse response with the transmit pulse, and the combined signaling waveform, h(t)*c(t)*u(t), is ISI free then the well known probability of error is given by

$$P(e) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) \quad (2\text{-}1)$$

where $Q(\beta) = \frac{1}{\sqrt{2\pi}} \int_{\beta}^{\infty} \exp\left(\frac{-\alpha^2}{2}\right) d\alpha,$

the received signal energy in a symbol is $E_b = \int_0^T u^2(T)dt = T$. If h(t)*c(t)*u(t) does not satisfy the Nyquist I criterion, then the combined signaling waveform will not be ISI free. In this case, the ideal receiver is given by Maximum Likelihood Sequence Estimation [6]. If the impulse response h(t)*u(t)*c(t) is zero outside of the interval

[0, MT], then since the output of the receiver filter is sampled at 1/T symbols/sec., the output of the receiver filter can be represented by a discrete time model with finite memory M. Denote the noiseless receiver output for each of the 2^M possible combinations of M-bits as y_m, where y_m represents the receiver output corresponding to the m-th input combination. The probability that a receiver output, y_m, is in error is given by the probability that the Gaussian noise at the output of the receiver causes the receiver output to cross a decision boundary. It follows that the probability of error, given that the M-bit sequence corresponding to m is transmitted, is [2]

$$P(e\,|m) = Q\left(\frac{|y_m|}{\sigma}\right) \qquad (2\text{-}2)$$

where $|y_m|$ represents the distance to the decision boundary and σ^2 is the variance of the noise at the output of the receiver. This is filtered white Gaussian noise, so that σ^2 can be expressed as [3]

$$\sigma^2 = \int_{-\infty}^{\infty} S_{ww}(f)\cdot|H(f)|^2 df = N_o \cdot NBW_1 \qquad (2\text{-}3)$$

where $S_{ww}(f)$ is the spectrum of the noise w(t), H(f) is the frequency response of the filter, and NBW_1 is the one-sided noise bandwidth, $\int_0^{\infty}|H(f)|^2 df$. The probability of error for each m can be averaged over all possible 2^M M-bit combinations to obtain the probability of error for the sub-optimum receiver

$$P(e) = \sum_{\forall m} P(m)\cdot Q\left(\frac{|y_m|}{\sigma}\right) = \frac{1}{2^M}\sum_{\forall m} Q\left(\frac{|y_m|}{\sigma}\right) \qquad (2\text{-}4)$$

where $P(m) = 1/2^M$ represents the probability of the M-bit combination which results in the receiver output y_m. Denote y_{mwc} as the receiver output corresponding to the minimum eye opening (worst case ISI). Then $P(e\,|y_m) \le P(e\,|\,y_{mwc})$ for all m, so that

$$P(e) \le Q\left(\frac{|y_{mwc}|}{\sigma}\right) \qquad (2\text{-}5)$$

Because errors due to the worst case ISI dominate, this bound becomes a reasonable approximation to the probability of error for high signal-to-noise-ratios, so that

$$P(e) \approx Q\left(\frac{|y_{mwc}|}{\sigma}\right), \text{ (high signal-to-noise-ratios)} \tag{2-6}$$

We can define the signal to noise ratio for the sub-optimum receiver as

$$SNR_s = \frac{y^2{}_{mwc}}{\sigma^2} \tag{2-7}$$

and the signal to noise ratio for the ideal receiver filter with probability of error given by (2-1) is

$$SNR_o = \frac{2 \cdot E_b}{N_o} \tag{2-8}$$

The loss, L, of the filter relative to the ideal filter can be defined in terms of the ratio of (2-7) and (2-8), using (2-3) and $E_b = T$ we obtain

$$L = 10 \cdot Log_{10}\left(\frac{y^2{}_{mwc}}{2 \cdot T \cdot NBW_I}\right) \tag{2-9}$$

This expression gives a loss of 0 dB when the ideal matched filter is used as the receiver filter.

III. ESTIMATING SUB-OPTIMUM FILTER PERFORMANCE

Estimating the sub-optimum filter performance requires determining the worst case ISI sustained with the receiver filter of interest and from this calculating the loss. In calculating the loss, one must obtain the noise bandwidth of the receiver filter which may be found in tables or by evaluating the associated integral either analytically or numerically. Estimating the ISI may be done by numerically evaluating the expressions given in [1], however this would require having an analytical expression for y(t), the output of the receiver filter. Although this may be a useful approach for the case of transmitting a square pulse and an all pass channel, in general it may not be tractable to obtain the channel output analytically. Consequently, the method used here is to filter a waveform containing the worst case ISI bit sequence, finding the minimum eye opening, and calculating the loss. Methods for performing this analysis are described in [4] and will be presently outlined.

We begin the analysis with a receiver filter with a known frequency response H(f). We may have an analytical expression for this filter or we may measure the frequency characteristics on a spectrum analyzer. The memory of our communication system is known to be less than M symbols. M may be determined by injecting an impulse into the system and measuring the length of the resulting response. First, we modulate a sequence of bits containing all M-bit combinations. One method of obtaining this bit sequence is with an M-bit pseudo-random bit generator. Next, we sample this sequence with a sample rate, f_s, at least twice the Nyquist limit. In order to align the frequency representation of the input sequence and the sampled frequency response of the receiver filter, let $f_s = K/T$ so that the sample rate is K times the symbol rate. This gives us an input sequence, x[n], with $N = 2^M$ points.

Next, we sample the frequency response, H(f), of the filter starting at f=0, at frequency steps of f_s/N up to $f_s/2$. Now, mirror the sampled frequency response about $f_s/2$ such that $H\left(\frac{N}{2}+i\right) = H^*\left(\frac{N}{2}-i\right)$ for i=1,...N/2-1. This gives the sampled frequency response, H(n), with N points and |H(n)| has even symmetry about $f_s/2$ and arg(H(n)) has odd symmetry. As an example, Figure 2 illustrates the sampled frequency response of a 4th order Bessel filter, for which $H(s) = \frac{1}{s^4+10s^3+45s^2+105s+105}$, ,where $s=j2\pi f$. The frequency response from zero to 4 Hz is plotted, the 3-dB cutoff frequency has been scaled to 0.5 Hz, f_s=32, and N=8192 (M=8, K=32, $N = 2^M \cdot K$).

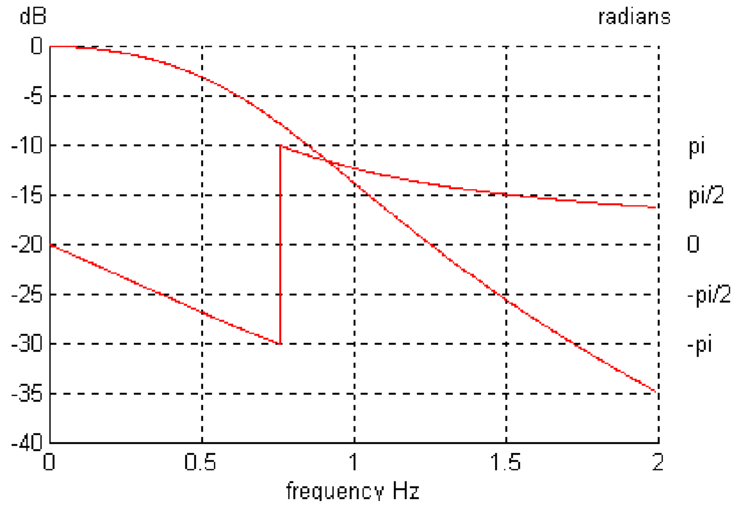

Figure 2. Sampled Frequency Response of 4-th Order Bessel Filter (f 3-dB = 0.5 Hz)

The filtered output of the receiver filter, in the frequency domain, is given by Y(k) = X(k) · H(k), and y(n) is the inverse discrete time Fourier transform of Y(k). We obtain the frequency representation, X(k), of x[n] by taking the discrete time Fourier transform of x[n]. We may now obtain the minimum eye opening at the optimum sampling time for y(n). This may be read off an eye diagram, as illustrated in Figure 3 for a signal with a rectangular transmit pulse, T=1s, f_s=32, and filtered with the filter of Figure 2. The worst case eye opening at the optimum sampling point is determined to be y_{mwc} = 0.89331. The single sided noise bandwidth, NBW_1, for the filter of Figure 2 is evaluated to be 0.520, so that the loss may be computed, using (2-9), to be 1.14 dB.

IV. FINDING THE BEST SUB-OPTIMUM FILTER

We may use the technique developed above to find the best sub-optimum filter. For example, suppose we want to know the best classical analog filter to use as a receiver filter. This includes Bessel, Gaussian, Butterworth, and Chebyshev. We know from

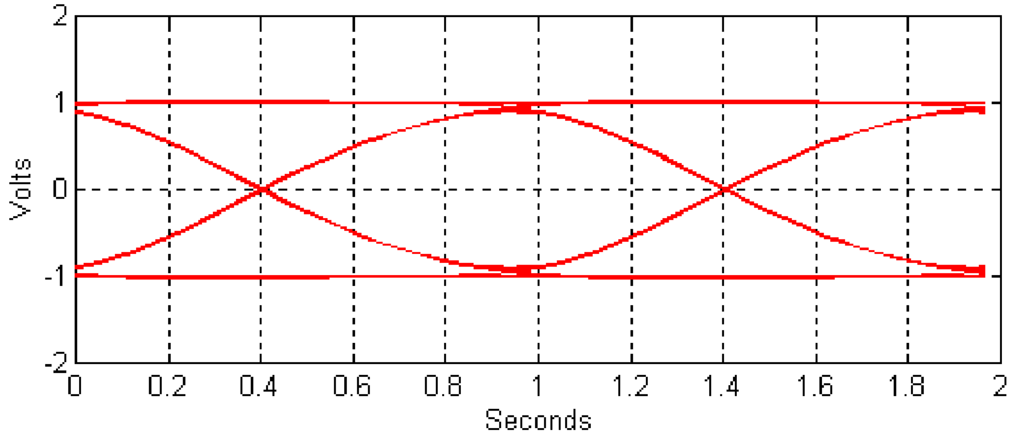

Figure 3. Eye Diagram

[1] that from these choices linear phase filters give the least amount of ISI, and in fact the Bessel filter is the best choice. Figure 4 illustrates the method for finding the optimum parameters. The loss for Bessel filters of order 3 and 4 versus 3-dB cutoff frequency is plotted. The symbol rate, Rs = 1/T, is 1 symbol/sec. and the 3-dB cutoff frequency is swept from 0.267 Rs to 1.33 Rs. The 4-th order Bessel filter gives the least amount of loss at 0.546 Rs and L=1.07 dB. For 3-dB cutoff frequencies above 0.546 Rs the loss is noise dominated since the noise bandwidth of the receiver is large in this case, however the ISI is small. For 3-dB cutoff frequencies below 0.546 Rs the

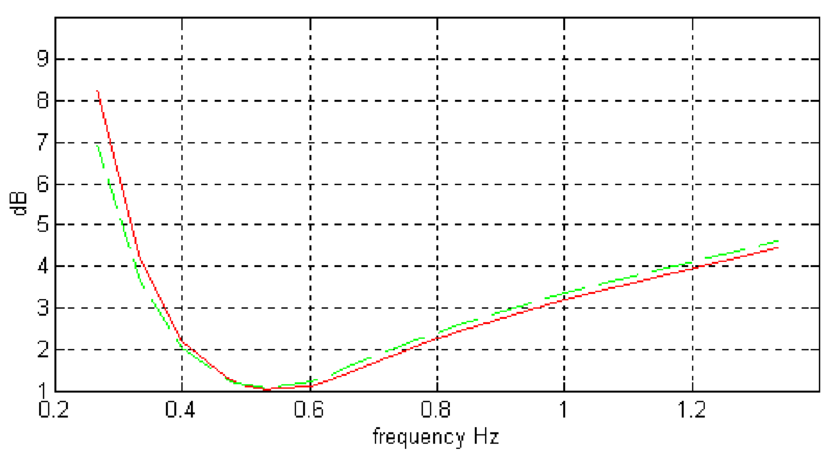

Figure 4. Bessel Filter Loss vs. 3-db cutoff frequency for filter orders 3 and 4.

loss is ISI dominated. The same analysis was performed for filter orders of 1 to 6 with The 4-th order Bessel having the minimum loss. These results can be scaled to any symbol rate, and the same method can be applied to other transmit pulse shapes, channel responses, filter types, and combinations thereof.

V. CONCLUSIONS

Although for the ideal situation, the optimum receiver for a communication system is known to be a matched filter, it may not be available. Under such circumstances, it may be necessary to utilize a sub-optimum receiver filter. Expressions were developed for calculating the probability of error and performance loss of the sub-optimum receiver as a function of the minimum eye value and noise bandwidth of the receiver filter. A method was presented for finding the best sub-optimum filter in the sense of minimizing the probability of error. The best Bessel filter of order less than or equal to 6 was found for a rectangular pulse shape and an all pass channel in terms of 3-dB bandwidth. This method is applicable to other transmit pulse shapes channel responses, filter types, and combinations thereof.

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