Kisin-Ren classification of ϖ-divisible $\mathcal{O}$-modules via the Dieudonné Crystal

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Signed: Alex Henniges

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Abstract

Let k be a perfect field of characteristic $p > 2$ and K a totally ramified extension of $K_0 = \mathrm{Frac}\, W(k)$ with uniformizer π. Let $F \subseteq K$ be a subfield with uniformizer ϖ, ring of integers $\mathcal{O}$, and residue field $k_F \subseteq k$ with $|k_F| = q$. Let $W_F = \mathcal{O} \otimes_{W(k_F)} W(k)$ and consider the ring $\mathfrak{S} = W_F[\![u]\!]$ with an endomorphism φ that lifts the q-power Frobenius of k on W_F and satisfies $\varphi(u) \equiv u^q \bmod \varpi$ and $\varphi(u) \equiv 0 \bmod u$. In this dissertation, we use $\mathcal{O}$-divided powers to define the analogue of Breuil-Kisin modules over the rings $\mathfrak{S}$ and S, where S is an $\mathcal{O}$-divided power envelope of the surjection $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$ sending u to π. We prove that these two module categories are equivalent, generalizing the case when $F = \mathbb{Q}_p$ and $\varpi - p$. As an application of our theory, we generalize the results of Kisin [17] and Cais-Lau [8] to relate the Faltings Dieudonné crystal of a ϖ-divisible $\mathcal{O}$-module, which gives a Breuil module over S in our sense, to the modules of Kisin-Ren, providing a geometric interpretation to the latter.

1. Introduction

An important class of objects in both algebraic geometry and number theory are abelian varieties. For example, elliptic curves are one-dimensional abelian varieties and were key in Wiles' proof of Fermat's Last Theorem, and elliptic curves are also important in cryptography.

One way to study an abelian variety A is through its p-divisible group $A[p^\infty]$. A theorem of Serre-Tate says, essentially, that the deformation theory of an abelian scheme is the same as the deformation theory of its p-divisible group. The power of Serre-Tate is that it states that to understand an abelian variety, it is enough to understand its p-divisible group, a much simpler object.

This motivates a desire to classify p-divisible groups. In characteristic p, one way to do this is via the Dieudonné module. Let k be a finite field of characteristic p and let $W := W(k)$. There is an equivalence between the category of p-divisible groups over k and a certain category of $W(k)$-modules with additional semilinear algebra structure (so-called Dieudonné modules). In this setting, when the p-divisible group comes from an abelian variety over k, a theorem of Mazur-Messing relates this classification to the crystalline cohomology of A over W.

Let K be a finite, totally ramified extension of Frac $W(k)$ and let $\mathcal{O}_K$ be its ring of integers. Let $\mathfrak{S} = W[\![u]\!]$ be the ring of power series in one variable over W. A similar classification for p-divisible groups over $\mathcal{O}_K$ by a certain category of $\mathfrak{S}$-modules with additional semilinear algebra structure was conjectured by Breuil and shown by Kisin in [17], and moreover Kisin relates this equivalence to crystalline cohomology via the Dieudonné crystal. We will describe this classification in more detail below, but we point out that one difficulty in this setting opposed to the setting of p-divisible groups over k is that the kernel of the map $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K/(p)$ does not have divided powers and so the classification via the Dieudonné crystal must go through a divided power envelope of $\mathfrak{S}$.

Often, for example in the study of Shimura varieties, one wants to classify abelian varieties with extra structure. This structure may come in the form of an action of $\mathcal{O}$, where $\mathcal{O}$ is the ring of integers for some, possibly ramified, field extension $F/\mathbb{Q}_p$ with uniformizer ϖ. In this case, the associated p-divisible group also has an action of $\mathcal{O}$, and we call such objects *ϖ-divisible $\mathcal{O}$-modules.* The results of Kisin-Ren [19] provide a classification of such ϖ-divisible $\mathcal{O}$-modules over $\mathcal{O}_K$ by a category of modules with semilinear data by using Galois representations via the Tate module. Our goal in this dissertation is to relate this construction of Kisin-Ren to crystalline cohomology via a Dieudonné functor given by Faltings [12].

Let us first discuss the classical setting in more detail. As above, let K be a local field of mixed characteristic p, where p is an odd prime. Let π be a uniformizer of K and k its residue field. The Witt vector ring $W := W(k)$ has a canonical lift of the p-power Frobenius on k. Define $\mathfrak{S} := W[\![u]\!]$ to be the power series ring over W with indeterminate u. On $\mathfrak{S}$ we can define an endomorphism φ extending the Frobenius on W as $\varphi(u) = u^p$. Denote by $E(u) \in \mathfrak{S}$ the minimal polynomial for π over W.

Let $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-semilinear and additive map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the linearization

$$\mathrm{id} \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} := \mathfrak{S} \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M} \to \mathfrak{M}$$

is killed by some power of $E(u)$. Kisin [17] showed that the category of crystalline representations of G_K admits a fully faithful embedding into the isogeny category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi} \otimes \mathbb{Q}_p$ of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$. Kisin also described the essential image and used this to give a classification of $\mathbb{Z}_p$-lattices in crystalline G_K-representations.

When G is a p-divisible group over $\mathcal{O}_K$, the Tate module T_pG is a lattice in a crystalline representation with Hodge-Tate weights equal to 0 or 1, and Kisin shows in [17] Theorem 2.2.7 that there is an object $\mathfrak{M}(G)$ of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ corresponding to T_pG. In fact, $\mathfrak{M}(G) \in \mathrm{BT}_{\mathfrak{S}}^{\varphi}$, the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ consisting of objects such that the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)$. Moreover, the functor $G \rightsquigarrow \mathfrak{M}(G)$ induces an equivalence between the category of p-divisible groups over $\mathcal{O}_K$ and $\mathrm{BT}^{\varphi}_{\mathfrak{S}}$, shown also in Theorem 2.2.7 of [17].

Let S be the p-adic completion of the divided power envelope of $\mathfrak{S}$ with respect to the ideal $E(u)\mathfrak{S}$, and let $\mathrm{Fil}^i S$ be the usual divided power filtration on S. The ring S is equipped with a unique continuous extension of φ. Denote by $\mathrm{BT}^{\varphi}_{S}$ the category of finite free S-modules with a submodule $\mathrm{Fil}^1 \mathcal{M}$ containing $\mathrm{Fil}^1 S\cdot\mathcal{M}$ and a φ-semilinear map $\varphi_{\mathcal{M},1} : \mathrm{Fil}^1 \mathcal{M} \to \mathcal{M}$ whose image spans $\mathcal{M}$ over S. Using the Dieudonné crystal (see [17] Appendix A) of a p-divisible group G, one can functorially associate to G an object $\mathcal{M}(G) := \mathbb{D}(G)(S)$ in $\mathrm{BT}^{\varphi}_{S}$. Furthermore, this association $G \to \mathcal{M}(G)$ induces an equivalence between the category of p-divisible groups over $\mathcal{O}_K$ and $\mathrm{BT}^{\varphi}_{S}$ as demonstrated in [17] Proposition A.6.

There is a functor $\mathrm{BT}^{\varphi}_{\mathfrak{S}} \to \mathrm{BT}^{\varphi}_{S}$ given in part by sending the $\mathfrak{S}$-module $\mathfrak{M}$ to the S-module $S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, where the left tensor is considered an $\mathfrak{S}$-module by a twist with φ. Kisin [17] showed that for a p-divisible group G, the modules $\mathfrak{M}(G)$ and $\mathcal{M}(G)$ are compatible in the sense that

$$S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M}(G) \cong \mathcal{M}(G)$$

functorially in G. This provides a geometric link between Kisin's classification of p-divisible groups, also called Barsotti-Tate groups, by their Galois representations with crystalline cohomology. Notice that this also gives an equivalence between $\mathrm{BT}^{\varphi}_{\mathfrak{S}}$ and $\mathrm{BT}^{\varphi}_{S}$, albeit indirectly.

On the other hand, Kisin-Ren [19] generalize the theory of Wach modules developed by Wach [26], Colmez [10], and Berger-Breuil [1] in the following way. If $F \subseteq K$ is a subfield with uniformizer ϖ and residue field k_F with $q = |k_F|$, construct the ring $\mathfrak{S}_F = W_F[\![u]\!]$ where $W_F = W(k) \otimes_{W(k_F)} \mathcal{O}_F$. On W_F we now consider the canonical lift of the q-power Frobenius on k. Define a Frobenius lift φ on all of $\mathfrak{S}_F$ as follows: Let $\mathcal{G}$ be a Lubin-Tate group over $\mathcal{O}_F$ corresponding to ϖ. By fixing a local coordinate X on $\mathcal{G}$, the formal Hopf algebra $\mathcal{O}_{\mathcal{G}}$ may be identified with $\mathcal{O}_F[\![X]\!]$. That is, for $a \in \mathcal{O}_F$, there is an associated power series $[a](X)$ with coefficients in $\mathcal{O}_F$ and $[a](X) \equiv aX \bmod X^2$. Moreover, $[\varpi](X) \equiv X^q \bmod \varpi$. Then $\varphi(u)$ is defined to be $[\varpi](u)$. Define K_n to be the extension of K generated by the $[\varpi^n]$-torsion points of $\mathcal{G}$ and define $K_\infty := \bigcup K_n$. If $\Gamma := \mathrm{Gal}(K_\infty/K)$, we get a character $\chi : \Gamma \to \mathcal{O}_F^\times$ and an action of Γ on $\mathfrak{S}$ given by $\gamma(u) = [\chi(\gamma)](u)$.

Let $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\Gamma}$ denote the category of finite free $\mathfrak{S}_F$ modules equipped with a φ-semilinear map and commuting semilinear action of Γ such that the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by a power of $E(u)$ and the action of Γ is trivial on $\mathfrak{M}/u\mathfrak{M}$. Kisin-Ren [19] give a functor

$$T : \mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\Gamma} \to \mathrm{Rep}_{\mathcal{O}_F}(G_K)$$

and they show that T induces an equivalence between the category of $\mathcal{O}_F$-lattices in F-*crystalline* G_K-representations and the full subcategory of $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\Gamma}$ consisting of so-called F-*analytic* objects. We will denote by T^* the contravariant version of this functor and by $\mathfrak{M}_{\mathrm{KR}}$ the inverse on F-crystalline representations. This recovers the theory of Wach modules in the case where $F = \mathbb{Q}_p$ and $\varpi = p$, which corresponds to the Lubin-Tate group $\mathcal{G} = \hat{\mathbb{G}}_m$ and the cyclotomic extension K_∞/K. If G is a ϖ-divisible $\mathcal{O}_F$-module such that $T_\varpi G$ is an $\mathcal{O}_F$-lattice in an F-crystalline G_K-representation, this result of Kisin-Ren associates to the dual $(T_\varpi G)^\vee$ an object

$$\mathfrak{M}_{\mathrm{KR}}(G) := \mathfrak{M}_{\mathrm{KR}}((T_\varpi G)^\vee)$$

in $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\Gamma}$ of E-height 1. A natural question then is if one can relate the crystalline cohomology of G to $\mathfrak{M}_{\mathrm{KR}}(G)$. Cais-Lau [8] gives this relationship when $F/\mathbb{Q}_p$ is unramified and $\varpi = p$.

When F is ramified over $\mathbb{Q}_p$, however, the analogue of the ring S over $\mathfrak{S}$ cannot be defined using classical divided powers. A reason for this is that the ideal (ϖ) of $\mathcal{O}_F$ will fail to have divided powers if the ramification degree of $F/\mathbb{Q}_p$ is greater than $p-1$. A definition of $\mathcal{O}$-divided powers was first given by Gross and Hopkins in 1994 [15] and refined by Faltings in [12]. In short, an ideal of an $\mathcal{O}_F$-algebra has $\mathcal{O}_F$-divided powers if there is a map $\gamma : I \to I$ which behaves like $\gamma(x) = x^q/\varpi$, and then define γ_n to be the n-fold iteration of γ.

In this dissertation, we give a construction for the ring S_F, now by taking the $\mathcal{O}_F$-divided power envelope of $\mathfrak{S}_F$ with respect to the ideal $E(u)\mathfrak{S}_F$. We then develop the analogue of Breuil-Kisin modules ([4],[6]) in this setting. More precisely, we define the categories $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,r}$ and $\mathrm{Mod}_{S_F}^{\varphi,r}$ for $r < q-1$ and show directly that the two categories are equivalent in Theorem 4.3.7. This entails adapting the methods of Breuil ([4], [5], [6]), Kisin [18], and Caruso-Liu [9] and represents the technical heart of this thesis.

We then apply Theorem 4.3.7 to generalize some of the results of Cais-Lau [8] to the more general setting of Kisin-Ren. We use the analogue of the Dieudonné crystal, as given by Faltings [12], to functorially associate to a ϖ-divisible $\mathcal{O}_F$-module G an object $\mathcal{M}(G)$ of $\mathrm{BT}_{S_F}^{\varphi,\Gamma}$. The main result of this thesis is the following:

Theorem 1.1. *For a ϖ-divisible $\mathcal{O}_F$-module G with $T_\varpi G \in \mathrm{Rep}_{\mathcal{O}_F}^{F-\mathrm{cris}}(G_K)$, there is an isomorphism of objects in $BT_{S_F}^{\varphi,\Gamma}$:*

$$S_F \otimes_{\varphi,\mathfrak{S}_F} \mathfrak{M}_{KR}(G) \cong \mathcal{M}(G).$$

Let $\mathfrak{M}(G)$ be the descent of $\mathcal{M}(G)$ to $\mathrm{BT}_{\mathfrak{S}_F}^{\varphi,\Gamma}$ which follows from Theorem 4.3.7. To prove Theorem 1.1, we establish that the following diagram commutes in Proposition 5.3.1:

$$\begin{array}{ccc} \mathrm{BT}_{\mathfrak{S}_F}^{\varphi,\Gamma} & \xrightarrow{T^*} & \mathrm{Rep}_{\mathcal{O}_F}(G_K) \\ \downarrow & \overset{T^*_{\mathrm{cris}}}{\nearrow} & \uparrow T_\varpi \\ \mathrm{BT}_{S_F}^{\varphi,\Gamma} & \xleftarrow{\mathcal{M}} & \varpi\text{-div}/\mathcal{O}_K \end{array}$$

One deduces from the theory of Faltings [12] that for G a ϖ-divisible $\mathcal{O}_F$-module, one has $T^*_{\mathrm{cris}}(\mathcal{M}(G)) \cong T_\varpi G$ and we show in Proposition 5.3.1 that for $\mathfrak{M} \in \mathrm{BT}_{\mathfrak{S}_F}^{\varphi,\Gamma}$, one has

$$T^*(\mathfrak{M}) \cong T^*_{\mathrm{cris}}(S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}).$$

Then using the above diagram and Theorem 1.6 of [19] as in the theory of Fontaine [14], one shows that the two (φ, Γ)-modules given by

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}_F} \mathfrak{M}_{\mathrm{KR}}(G) \quad \text{and} \quad \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}_F} \mathfrak{M}(G)$$

are isomorphic, where $\mathcal{O}_{\mathcal{E}}$ is the ϖ-adic completion of $\mathfrak{S}_F[1/u]$. After demonstrating that the functor $\mathrm{Mod}^{\varphi,\Gamma}_{\mathfrak{S}_F} \to \mathrm{Mod}^{\varphi,\Gamma}_{\mathcal{O}_{\mathcal{E}}}$ is fully-faithful, we can then conclude that $\mathfrak{M}(G) \cong \mathfrak{M}_{\mathrm{KR}}(G)$ and so Theorem 1.1 follows.

In Chapter 2, we give background information on Lubin-Tate groups. A familiar reader can safely skip or skim over this material. In Chapter 3, we give the definition of $\mathcal{O}$-divided powers and a description of how to take the $\mathcal{O}$-divided power envelope for $\mathcal{O}$-algebras without ϖ-torsion. The literature on $\mathcal{O}$-divided powers is still scarce and we choose to give an alternative (but equivalent to [12]) development of $\mathcal{O}$-divided powers. It is the author's opinion that this alternative notation more closely matches that of the classical divided powers and provides some advantages.

We define the rings $\mathfrak{S}_F$ and S_F in Chapter 4 as well as their respective φ-module categories. We then consider the categories in the case where the modules are killed by ϖ and show that they are equivalent by demonstrating that the functor $\mathfrak{M} \to S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is both fully faithful and essentially surjective. Many of the arguments as given by Breuil in [4],[5], and [6] hold up in this more general setting. We then lift the results to modules free over $\mathfrak{S}$ and S. Lemma 4.3.3 and Proposition 4.3.4 give the descent of an S-module to an $\mathfrak{S}$-module and for this the methods of Caruso-Liu [9] extend nicely. Proposition 4.3.6 completes the equivalence by showing that $\mathfrak{M} \to S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is fully faithful. We use here the generalization of the arguments as established by Kisin in [18]. Throughout this chapter, we give careful details of the arguments.

Chapter 5 then provides the link between the results of Kisin-Ren and the construction of Faltings. For any $\mathfrak{M} \in \mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$, we show that there is a G_K-equivariant isomorphism

$$T^*(\mathfrak{M}) \to T^*_{\mathrm{cris}}(S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}).$$

The arguments used in the proof follow those of Kisin in [17] and we rely on results from [19] and [12]. The remaining necessary piece is to show that the functor given by $\mathfrak{M} \to \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is fully faithful. The arguments of Kisin in [17] and Cais and Lau in [8] largely hold, but a generalized result given in [10] is needed, which we give in Lemma 5.3.3. Combined, this shows that when $\mathfrak{M}_{\mathrm{KR}}(G)$ exists for a ϖ-divisible $\mathcal{O}_F$-module G, we have

$$S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{\mathrm{KR}}(G) \cong \mathcal{M}(G).$$

This gives a geometric description to $\mathfrak{M}_{\mathrm{KR}}(G)$ via the Dieudonné crystal.

2. Lubin-Tate Groups

We borrow from the notes of Milne [23] I.2 throughout this chapter.

2.1. Formal group laws

Let $\mathcal{O}$ be the ring of integers for a finite extension of $\mathbb{Q}_p$, let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$, let q be the size of the residue field $\mathcal{O}/\mathfrak{m}$, and fix ϖ a uniformizer of $\mathcal{O}$. Denote the ring of power series in the variables $X_1, ..., X_n$ and with coefficients in $\mathcal{O}$ by $\mathcal{O}[\![X_1, ..., X_n]\!]$. We write an $f \in \mathcal{O}[\![X_1, ..., X_n]\!]$ as

$$f = \sum_{i_1,...,i_n \geq 0} a_{i_1,...,i_n} X_1^{i_1} \cdots X_n^{i_n}$$

where $a_{i_1,...,i_n} \in \mathcal{O}$ and we say that the term $a_{i_1,...,i_n} X_1^{i_1} \cdots X_n^{i_n}$ has degree equal to $i_1 + \cdots + i_n$. For simplicity, we use the notation $\overset{d}{\cdots}$ to represent an arbitrary power series whose nonzero terms all have degree at least d. We will make use of this notation as a means to describe just the first few terms of a power series.

We recall some properties of $\mathcal{O}[\![X_1, ..., X_n]\!]$. If we have $f \in \mathcal{O}[\![X_1, ..., X_n]\!]$ and if for $1 \leq j \leq n$, we have $g_j \in \mathcal{O}[\![Y_1, ..., Y_m]\!]$ such that the constant terms, or degree 0 terms, of g_j are all 0, then $f(g_1, ..., g_n)$ makes sense in $\mathcal{O}[\![Y_1, ..., Y_m]\!]$. Moreover, if $f \in X\mathcal{O}[\![X]\!]$ and the coefficient of the degree 1 term of f is a unit in $\mathcal{O}$, then there is a unique $g \in X\mathcal{O}[\![X]\!]$ such that $f \circ g = X$. For $f = \sum_{i_1,...,i_n \geq 0} a_{i_1,...,i_n} X_1^{i_1} \cdots X_n^{i_n}$ and for $c_j \in \mathfrak{m}$ with $1 \leq j \leq n$, define

$$f(c_1, ..., c_n) \coloneqq \sum_{i_1,...,i_n \geq 0} a_{i_1,...,i_n} c_1^{i_1} \cdots c_n^{i_n},$$

which converges in $\mathcal{O}$.

Definition 2.1.1. A *commutative formal group law* is a power series $F \in \mathcal{O}[\![X, Y]\!]$ with the following properties:

1. $F(X,Y) = X + Y + .^2.$

2. $F(X, F(Y,Z)) = F(F(X,Y), Z)$

3. There exists a unique $i_F(X) \in \mathcal{O}[\![X]\!]$ such that $F(X, i_F(X)) = 0$.

4. $F(X,Y) = F(Y,X)$.

Example 2.1.2. If $F(X,Y) = X + Y + XY = (1+X)(1+Y) - 1$, then F is easily seen to be a formal group law with $i_F(X) = (1+X)^{-1} - 1$.

A formal group law can endow certain sets with a group operation, as the next two examples show.

Example 2.1.3. If F is a formal group law, then $T\mathcal{O}[\![T]\!]$ becomes a commutative group under the operation

$$f +_F g = F(f(T), g(T)).$$

The first property ensures that $F(f(T), g(T)) \in T\mathcal{O}[\![T]\!]$. The second property is precisely the associativity of $+_F$, and the fourth property gives the commutativity of $+_F$. The identity is given by 0. To see this, first consider $h(X) = F(X, 0) \in X\mathcal{O}[\![X]\!]$. Then there exists $h^{-1}(X) \in X\mathcal{O}[\![X]\!]$ such that $h^{-1} \circ h = X$. Also, the second property of F says that

$$\begin{aligned} h(X) &= F(X, 0) \\ &= F(X, F(0,0)) \\ &= F(F(X,0), 0) \\ &= h \circ h. \end{aligned}$$

Composing this equation with h^{-1} then gives that $h(X) = X$. So $F(X,0) = X$, and hence $f +_F 0 = f$. Finally, the inverse of f under $+_F$ is $i_F \circ f$ by the third property of F.

Example 2.1.4. Suppose that F is a formal group law over $\mathcal{O}$. Then for $x, y \in \mathfrak{m}$, we get $F(x, y) \in \mathfrak{m}$. If we denote $F(x, y)$ by $x +_F y$, then the set $\mathfrak{m}$ together with the operation $+_F$ forms a commutative group.

When $F(X, Y) = X + Y + XY$, the group $(\mathfrak{m}, +_F)$ is isomorphic to the group $(1 + \mathfrak{m})$ under multiplication.

Definition 2.1.5. Let $F(X, Y)$ and $G(X, Y)$ be formal group laws. A *homomorphism* $h : F \to G$ is a power series $h \in T\mathcal{O}[\![T]\!]$ such that

$$h(F(X, Y)) = G(h(X), h(Y)).$$

If there is a homomorphism $h^{-1} : G \to F$ such that $h \circ h^{-1} = h^{-1} \circ h = X$, then F and G are said to be *isomorphic*. A homomorphism $h : F \to F$ is called an *endomorphism* of F.

Remark 2.1.6. Let $\mathrm{Hom}(F, G)$ denote the set of homomorphisms $F \to G$. Using Example 2.1.3, it is not difficult to show that for $f, g \in \mathrm{Hom}(F, G)$, the power series $f +_G g$ remains a homomorphism, and so $\mathrm{Hom}(F, G)$ becomes an abelian group under this addition. Likewise, define $\mathrm{End}(F) \coloneqq \mathrm{Hom}(F, F)$, and then this set becomes a ring under addition by $+_F$, and multiplication given by composition. Since $f \circ (g + h)$ does not generally equal $f \circ g + f \circ h$ for the standard addition of power series, we emphasize the distributivity of composition with $+_F$. To see the distributivity, let $f, g, h \in \mathrm{End}(F)$. Then,

$$\begin{aligned} f \circ (g +_F h) &= f(F(g(T), h(T))) \\ &= F(f \circ g(T), f \circ h(T)) \\ &= f \circ g +_F f \circ h. \end{aligned}$$

Example 2.1.7. If $F(X, Y) = (1 + X)(1 + Y) - 1$, then the power series given by $f(T) = (1 + T)^n - 1$ is an endomorphism of F since

$$f(F(X, Y)) = (1 + (1 + X)(1 + Y) - 1)^n - 1 = (1 + X)^n(1 + Y)^n - 1$$

and

$$F(f(X), f(Y)) = (1 + (1 + X)^n - 1)(1 + (1 + Y)^n - 1) - 1.$$

2.2. Lubin-Tate formal group laws

Before we define the Lubin-Tate formal group laws, recall that ϖ is a uniformizer for $\mathcal{O}$ and q is the size of the residue field.

Definition 2.2.1. Denote by $\mathcal{F}_\varpi$ the set of $f(X) \in \mathcal{O}[\![X]\!]$ such that

1. $f(X) = \varpi X + \overset{2}{\dots}$.

2. $f(X) \equiv X^q \bmod \varpi$.

Example 2.2.2. If $K = \mathbb{Q}_p$ and $\varpi = p$, then by binomial expansion,

$$f(x) = (1 + x)^p - 1 \in \mathcal{F}_p.$$

A key lemma is the following.

Lemma 2.2.3. *Let $f, g \in \mathcal{F}_\varpi$, and let $\eta_1(X_1, ..., X_n)$ be a linear form with coefficients in $\mathcal{O}$. Then there exists a unique $\eta \in \mathcal{O}[\![X_1, ..., X_n]\!]$ such that*

$$\eta(X_1, ..., X_n) = \eta_1 + \overset{2}{\dots}, \qquad \textit{and}$$
$$f(\eta(X_1, ..., X_n)) = \eta(g(X_1), ..., g(X_n)).$$

Proof. We build η inductively. That is, we show inductively that for a positive integer r, there is a unique polynomial $\eta_r(X_1, ..., X_n)$ of degree r such that $\eta_r = \eta_1 + \overset{2}{\dots}$ and

$$f(\eta_r(X_1, ..., X_n)) = \eta_r(g(X_1, ..., X_n)) + \overset{r+1}{\dots}.$$

For $r = 1$, the only possible choice given the first condition is η_1 itself. To see that η_1 satisfies the second condition, note that $f(\eta_1(X_1, ..., X_n)) = \varpi\eta_1 + \overset{2}{\dots}$ whereas

$$\begin{aligned}\eta_1(g(X_1), ..., g(X_n)) &= \eta_1(\varpi X_1, ..., \varpi X_n) + \overset{2}{\dots} \\ &= \varpi\eta_1(X_1, ..., X_n) + \overset{2}{\dots}\end{aligned}$$

as η_1 is a linear form.

Now suppose that for some $r \geq 1$, we have a unique η_r. We show that there exists a unique homogeneous polynomial P of degree $r+1$ such that $\eta_{r+1} = \eta_r + P$ satisfies the conditions. If there were an η'_{r+1} that also satisfied the conditions, then write $\eta'_{r+1} = \eta'_r + P'$ where η'_r has degree r and P is homogeneous of degree $r+1$. Since η'_r would thus satisfy the conditions for r, we would have $\eta_r = \eta'_r$ by uniqueness of η_r and hence $P = P'$ by uniqueness of P, and so η_{r+1} is unique. To see that such a P exists and is unique, note that on the one hand we would have

$$f(\eta_{r+1}(X_1, ..., X_n)) = f(\eta_r(X_1, ..., X_n)) + \varpi P(X_1, ..., X_n) + \overset{r+2}{\cdots} .$$

On the other hand,

$$\eta_{r+1}(g(X_1), ..., g(X_n)) = \eta_r(g(X_1), ..., g(X_n)) + P(\varpi X_1, ..., \varpi X_n) + \overset{r+2}{\cdots},$$

and we know $P(\varpi X_1, ..., \varpi X_n) = \varpi^{r+1} P(X_1, ..., X_n)$. Therefore, we define P uniquely as

$$\frac{f \circ \eta_r - \eta_r \circ g}{\varpi^{r+1} - \varpi} = P(X_1, ..., X_n) + \overset{r+2}{\cdots} .$$

The right-hand side has coefficients in $\mathcal{O}$ since, modulo ϖ,

$$f \circ \eta_r - \eta_r \circ g \equiv (\eta_r(X_1, ..., X_n))^q - \eta_r(X_1^q, ..., X_n^q) \equiv 0,$$

where the latter congruence follows since $(x+y)^q = x^q + y^q$ for $\mathcal{O}/\mathfrak{m}$-algebras and $a^q = a$ for $a \in \mathcal{O}/\mathfrak{m}$. So $f \circ \eta_r - \eta_r \circ g$ is divisible by ϖ and $\varpi^r - 1$ is a unit in $\mathcal{O}$. Lastly, since all of the nonzero terms of $f \circ \eta_r - \eta_r \circ g$ have degree at least $r+1$, we get that P is homogeneous of degree $r+1$.

To complete the proof, define η to be the unique power series such that $\eta = \eta_r + \overset{r+1}{\cdots}$ for every $r \geq 1$ as $\eta_{r+1} = \eta_r + \overset{r+1}{\cdots}$ for every r. Then for any $r \geq 1$,

$$\begin{aligned} f \circ \eta &= f \circ \eta_r + \overset{r+1}{\cdots} \\ &= \eta_r \circ g + \overset{r+1}{\cdots} \\ &= \eta \circ g + \overset{r+1}{\cdots} . \end{aligned}$$

Since this is true for any r, it must be that $f \circ \eta = \eta \circ g$. □

Proposition 2.2.4. *For every $f \in \mathcal{F}_\varpi$, there exists a unique formal group law F_f for which f is an endomorphism, i.e.*

$$f(F_f(X,Y)) = F_f(f(X), f(Y)).$$

Proof. By Lemma 2.2.3, let $F_f(X,Y)$ be the unique power series with coefficients in $\mathcal{O}$ such that

$$\begin{aligned} F_f(X,Y) &= X + Y + \overset{2}{\ldots}, \qquad \text{and} \\ f(F_f(X,Y)) &= F_f(f(X), f(Y)). \end{aligned}$$

We still have to check that F_f satisfies the formal group laws. For commutativity, let $G(X,Y) = F_f(Y,X)$. Then G also satisfies the conditions

$$\begin{aligned} G(X,Y) &= X + Y + \overset{2}{\ldots}, \qquad \text{and} \\ f(G(X,Y)) &= f(F_f(Y,X)) \\ &= F_f(f(Y), f(X)) \\ &= G(f(X), f(Y)). \end{aligned}$$

But then by the uniqueness of F_f, we get $F_f(X,Y) = F_f(Y,X)$. Now let $i_F(X)$ be the unique power series such that

$$\begin{aligned} i_F(X) &= -X + \overset{2}{\ldots}, \qquad \text{and} \\ f(i_F(X)) &= i_F(f(X)). \end{aligned}$$

Then it is easy to see that both $F_f(X, i_F(X))$ and 0 are power series that commute with f and whose linear and constant terms are 0. So by the uniqueness guaranteed from Lemma 2.2.3, $F_f(X, i_F(X)) = 0$. For associativity, let

$$G_1(X,Y,Z) = F_f(X, F_f(Y,Z)) \quad \text{and} \quad G_2(X,Y,Z) = F_f(F_f(X,Y), Z).$$

Then both G_1 and G_2 satisfy

$$G(X,Y,Z) = X + Y + Z + \overset{?}{\ldots}, \qquad \text{and}$$
$$f(G(X,Y,Z)) = G(f(X), f(Y), f(Z)).$$

Therefore, by Lemma 2.2.3, $G_1 = G_2$. □

The formal group laws F_f are called the *Lubin-Tate formal group laws.* We will show that if $f, g \in \mathcal{F}_\varpi$, then F_f and F_g are isomorphic, and hence, up to a change of variables, the Lubin-Tate formal group laws depend only on ϖ.

Definition 2.2.5. For $f, g \in \mathcal{F}_\varpi$ and $a \in \mathcal{O}$, let $[a]_{g,f}(X)$ be the unique power series such that

$$[a]_{g,f}(X) = aX + \overset{?}{\ldots}, \qquad \text{and}$$
$$g([a]_{g,f}(X)) = [a]_{g,f}(f(X)).$$

Proposition 2.2.6. *Let $f, g, h \in \mathcal{F}_\varpi$ and $a, b \in \mathcal{O}$.*

1. *The power series $[a]_{g,f}$ is a homomorphism $F_f \to F_g$.*
2. $[a+b]_{g,f} = [a]_{g,f} +_{F_g} [b]_{g,f}$.
3. $[ab]_{h,f} = [a]_{h,g} \circ [b]_{g,f}$.

Proof. Each of the proofs make use of the uniqueness given by Lemma 2.2.3 to show that two power series are equal. We will just give a brief sketch for each.

1. One can show that both $[a]_{g,f}(F_f(X,Y))$ and $F_g([a]_{g,f}(X), [a]_{g,f}(Y))$ have the form $H = aX + aY + \overset{?}{\ldots}$ and that $H \circ f = g \circ H$ in both cases.
2. Let $H(X) = F_g([a]_{g,f}(X), [b]_{g,f}(X))$, which is by definition $[a]_{g,f} +_{F_g} [b]_{g,f}$. Then it is not difficult to see that $H(X) = (a+b)X + \overset{?}{\ldots}$, and using that g is an endomorphism of F_g, we get $g \circ H = H \circ f$. Since this precisely satisfies the conditions of the definition of $[a+b]_{g,f}$, we have $[a+b]_{g,f} = [a]_{g,f} +_{F_g} [b]_{g,f}$.

3. Certainly, $[a]_{h,g} \circ [b]_{g,f} = (ab)X + \overset{?}{\dots}$, and $h \circ ([a]_{h,g} \circ [b]_{g,f}) = ([a]_{h,g} \circ [b]_{g,f}) \circ f$, and this satisfies the conditions of the definition of $[ab]_{h,f}$.

$\square$

Corollary 2.2.7. *For $f, g \in \mathcal{F}_\varpi$, the formal groups laws F_f and F_g are isomorphic.*

Proof. If u is any unit in $\mathcal{O}$, then by Proposition 2.2.6, $[u]_{g,f} \circ [u^{-1}]_{f,g} = [1]_{g,g}$ and Lemma 2.2.3 assures us that $[1]_{g,g}(X) = X$. Likewise,

$$[u^{-1}]_{f,g} \circ [u^{-1}]_{g,f}(X) = [1]_{f,f}(X) = X.$$

$\square$

When $f = g$, write $[a]_f := [a]_{f,f}$. Then by Proposition 2.2.6 we clearly have the following.

Corollary 2.2.8. *For $f \in \mathcal{F}_\varpi$, the map $[\cdot]_f : \mathcal{O} \to End(F_f)$ given by $a \mapsto [a]_f$ is a ring homomorphism.*

Remark 2.2.9.

- Since $f \in \mathcal{F}_\varpi$ satisfies the conditions that $f(X) = \varpi X + \overset{?}{\dots}$ and commutes with itself, it follows that $[\varpi]_f = f$.
- By Proposition 2.2.6, we have that $[a]_f = [1]_{f,g} \circ [a]_g \circ [1]_{g,f}$ and the power series $[1]_{f,g}$ and $[1]_{g,f}$ are units in $\mathcal{O}[\![X]\!]$, so this gives a conversion from $[a]_f$ to $[a]_g$.
- When $K = \mathbb{Q}_p$, and $\varpi = p$ and $f(x) = (1+x)^p - 1$, then the above remark gives the cyclotomic extensions $\mathbb{Q}_p(\mu_{p^n})$.

3. $\mathcal{O}$-divided Powers

3.1. Definition

Let F be a finite extension of $\mathbb{Q}_p$, let $\mathcal{O} = \mathcal{O}_F$ be its ring of integers, k_F its residue field with $|k_F| = q$, and ϖ a uniformizer. In this chapter, we define a generalization of divided powers to allow for ramification of F over $\mathbb{Q}_p$. There are two goals: first, we need a definition such that the ideal (ϖ) of $\mathcal{O}$ has divided powers. Second, we want to consider the q-power Frobenius map on k_F as opposed to the p-power Frobenius map. A key property we need is that, if x is an element of an ideal with divided powers, then x^q is congruent to 0 modulo ϖ.

We will use the description of $\mathcal{O}$-divided powers as given by Faltings in [12] and we introduce some of our own notation. To explore the combinatorial properties of $\mathcal{O}$-divided powers, we first define them on F. Then we define more generally $\mathcal{O}$-divided powers on an $\mathcal{O}$-algebra and lastly give a description of how to obtain the $\mathcal{O}$-divided power envelope with respect to some ideal.

Definition 3.1.1. Define the $\mathcal{O}$-divided power on F, denoted $\gamma : F \to F$, by

$$\gamma(x) = \frac{x^q}{\varpi},$$

and for $n \geq 1$ denote by γ_n the n-th iterate of γ:

$$\gamma_n(x) = \frac{x^{q^n}}{\varpi^{1+q+\cdots+q^{n-1}}}.$$

In the usual definition of divided powers, in [2] for example, $\gamma_n(x) = \frac{x^n}{n!}$, which is not quite the analog of the $\mathcal{O}$-divided power γ_n. We note that an important aspect of divided powers is the growth of the p-adic valuation of the denominator. It is easy to check that, if n is written p-adically as $n = a_0 + a_1 p + \cdots + a_k p^k$ with $0 \leq a_i < p$, then the p-adic valuation of $n!$ is

$$v_p(n!) = a_1 + a_2(1+p) + \cdots + a_k(1+p+\cdots+p^{k-1}).$$

This motivates a different notation for $\mathcal{O}$-divided powers. For a nonnegative integer m, we can uniquely write $m = a_0 + a_1 q + \cdots + a_n q^n$ with $0 \le a_i < q$. Then define $m_{(q)}$ to be the number

$$m_{(q)} = a_1 + a_2(1+q) + \cdots + a_n(1+q+\cdots+q^{n-1}). \tag{3.1.1}$$

It can sometimes be helpful to view $m_{(q)}$ equivalently as

$$\begin{aligned} m_{(q)} &= \frac{1}{q-1}[a_1(q-1) + a_2(q^2-1) + \cdots + a_n(q^n-1)], \\ m_{(q)} &= \frac{1}{q-1}[m - (a_0 + a_1 + \cdots + a_n)]. \end{aligned} \tag{3.1.2}$$

Definition 3.1.2. Define the $\mathcal{O}$-divided power $\delta_m : F \to F$ by

$$\begin{aligned} \delta_m(x) &= x^{a_0}(\gamma_1(x))^{a_1} \cdots (\gamma_n(x))^{a_n} \\ &= \frac{x^m}{\varpi^{m_{(q)}}}. \end{aligned} \tag{3.1.3}$$

Then δ_m behaves like a divided power in the sense that the ϖ-adic valuation of the denominator is $m_{(q)}$, whereas the p-adic valuation of $m!$ is precisely $m_{(p)}$. Also, when $F = \mathbb{Q}_p$ and $\mathcal{O} = \mathbb{Z}_p$, the two definitions differ by an element of $\mathcal{O}^\times$. Furthermore, δ_m has the following properties:

Proposition 3.1.3. *Given $x, y, a \in F$,*

1. $\delta_m(ax) = a^m \delta_m(x)$
2. $\delta_m(x+y) = \sum_{k+l=m} a_{k,l}\delta_k(x)\delta_l(y)$, *where*

$$a_{k,l} = \binom{k+l}{k} \cdot \varpi^{(k_{(q)}+l_{(q)})-(k+l)_{(q)}}$$

lies in $\mathcal{O}$ and does not depend on x or y.

3. *$\delta_m(x)\delta_k(x) = b_{m,k}\delta_{m+k}(x)$, where*

$$b_{m,k} = \varpi^{(m+k)_{(q)} - (m_{(q)} + k_{(q)})}$$

lies in $\mathcal{O}$ and does not depend on x.

4. *$\delta_m(\delta_k(x)) = c_{m,k}\delta_{mk}(x)$, where*

$$c_{m,k} = \varpi^{(mk)_{(q)} - (mk_{(q)} + m_{(q)})}$$

lies in $\mathcal{O}$ and does not depend on x.

Proof. The proof uses standard combinatorial arguments, where the p-adic valuation of $m!$ is replaced with $m_{(q)}$.

1. The first property is clear.

2. We just need to show that

$$\binom{k+l}{k} \cdot \varpi^{(k_{(q)} + l_{(q)}) - (k+l)_{(q)}}$$

is in $\mathcal{O}$. Note that the p-adic valuation of $\binom{k+l}{k}$ is precisely $(k+l)_{(p)} - (k_{(p)} + l_{(p)})$, so it suffices to show that

$$(k+l)_{(q)} - (k_{(q)} + l_{(q)}) \leq (k+l)_{(p)} - (k_{(p)} + l_{(p)}).$$

Write $k = a_0 + a_1 q + \cdots + a_n q^n$ and $l = b_0 + b_1 q + \cdots + b_n q^n$ (where a_n or b_n could be zero). For each $i = 0, ..., n$, there is a d_i equal to 0 or 1 such that if we write

$$k + l = c_0 + c_1 q + \cdots + c_n q^n + c_{n+1} q^{n+1},$$

we have

$$\begin{aligned} a_0 + b_0 &= c_0 + d_0 q \\ a_1 + b_1 + d_0 &= c_1 + d_1 q \\ &\vdots \\ a_n + b_n + d_{n-1} &= c_n + d_n q \\ d_n &= c_{n+1}. \end{aligned}$$

Hence, d_i counts if there is *carry-over* from the i-th position to the $(i+1)$-st in the q-ary form for the sum. We can then compute

$$\begin{aligned} (k+l)_{(q)} - (k_{(q)} + l_{(q)}) &= c_1 + \cdots + c_n(1 + \cdots + q^{n-1}) + c_{n+1}(1 + \cdots + q^n) \\ &\quad - [(a_1 + b_1) + \cdots + (a_n + b_n)(1 + \cdots + q^{n-1})] \\ &= (d_0 - d_1 q) + \cdots + (d_{n-1} - d_n q)(1 + \cdots + q^{n-1}) \\ &\quad + c_{n+1}(1 + \cdots + q^n) \\ &= d_0 + d_1 + \cdots + d_n. \end{aligned}$$

This is precisely the number of carry-over operations needed in the q-ary form. The same result holds true for $(k+l)_{(p)} - (k_{(p)} + l_{(p)})$ regarding the p-ary form. But if there is carry-over in the i-th position, so $a_i + b_i + d_{i-1} \geq q$ and $q = p^f$, then writing $a_i = a'_0 + a'_1 p + \cdots + a'_{f-1} p^{f-1}$ and $b_i = b'_0 + b'_1 p + \cdots + b'_{f-1} p^{f-1}$, there must be p-ary carry-over in at least the final position when adding the terms $a_i + b_i + d_{i-1}$. In other words, the number of carry-overs in the q-ary form is at most the number of carry-overs in the p-ary form. This shows that $a_{k,l} \in \mathcal{O}$.

3. For the third property, clearly $b_{m,k} = \varpi^{(m+k)_{(q)} - (m_{(q)} + k_{(q)})}$ and we've seen that $(m+k)_{(q)} - (m_{(q)} + k_{(q)})$ is the number of carry-overs from the sum, which is at least 0. So $b_{m,k} \in \mathcal{O}$. Note in particular that if there are no carry-overs, for example if $m = a_n q^n$ and $k = b_l q^l$ with $n \neq l$, then $\delta_m(x)\delta_k(x) = \delta_{m+k}(x)$.

4. We will first show that $\delta_m(\delta_k(x)) = c_{m,k}\delta_{mk}(x)$ for some $c_{m,k} \in \mathcal{O}$. Then, by comparing the definitions of $\delta_m(\delta_k(x))$ and $\delta_{mk}(x)$, it will be clear that $c_{m,k}$ must be given by

$$c_{m,k} = \varpi^{(mk)_{(q)} - (mk_{(q)} + m_{(q)})}.$$

We first consider the case that $m = a_n q^n$ and $k = b_l q^l$. We also assume that a_n, b_l are nonzero. Write $a_n b_l = c_0 + c_1 q$ with $0 \leq c_0, c_1 \leq q-1$. We can do this since $a_n b_l < q^2$. If $c_1 = 0$, then $mk = (a_n b_l) q^{n+l}$ and

$$\delta_{mk}(x) = \frac{x^{mk}}{\varpi^{a_n b_l (1+q+\cdots+q^{n+l-1})}}.$$

If $c_1 > 0$, then $mk = c_0 q^{n+l} + c_1 q^{n+l+1}$ and

$$\begin{aligned}\delta_{mk}(x) &= \frac{x^{mk}}{\varpi^{c_0(1+\cdots+q^{n+l-1}) + c_1(1+\cdots+q^{n+l})}} \\ &= \frac{x^{mk}}{\varpi^{a_n b_l (1+\cdots+q^{n+l-1}) + c_1}}.\end{aligned}$$

Therefore, in either case we can write $\delta_{mk}(x) = \frac{x^{mk}}{\varpi^{a_n b_l (1+\cdots+q^{n+l-1})+c_1}}$. On the other hand,

$$\begin{aligned}\delta_m(\delta_k(x)) &= \frac{x^{mk}}{\varpi^{m(b_l(1+\cdots+q^{l-1}))}\varpi^{a_n(1+\cdots+q^{n-1})}} \\ &= \frac{x^{mk}}{\varpi^{a_n b_l q^n (1+\cdots+q^{l-1}) + a_n(1+\cdots+q^{n-1})}} \\ &= \varpi^{a_n(b_l-1)(1+\cdots+q^{n-1})+c_1}\delta_{mk}(x),\end{aligned}$$

and clearly $c_{m,k} = \varpi^{a_n(b_l-1)(1+\cdots+q^{n-1})+c_1}$ is in $\mathcal{O}$. Now suppose $m = a_n q^n$ and $k = b_0 + \cdots + b_l q^l$. By the proof of the third property, since there are no carry-overs in any of the sums, we know we can write

$$\delta_{b_0+b_1q+\cdots+b_lq^l}(x) = \delta_{b_0}(x)\delta_{b_1q}(x)\cdots\delta_{b_lq^l}(x).$$

Therefore, using the first and third properties and our already proven case, as well as the identity $y^m = \varpi^{m_{(q)}} \delta_m(y)$, we have

$$\begin{aligned}
\delta_m(\delta_k(x)) &= (\delta_{b_0}(x) \cdots \delta_{b_{l-1}q^{l-1}}(x))^m \delta_m(\delta_{b_l q^l}(x)) \\
&= (\varpi^{m_{(q)}})^l \delta_m(\delta_{b_0}(x)) \cdots \delta_m(\delta_{b_l q^l}(x)) \\
&= (\varpi^{m_{(q)}})^l \left(\prod_{i=0}^{l} c_{m, b_i q^i} \right) \delta_{m b_0}(x) \cdot \delta_{m b_1 q}(x) \cdots \delta_{m b_l q^l}(x) \\
&= c_{m,k} \delta_{mk}(x),
\end{aligned}$$

for some $c_{m,k} \in \mathcal{O}$. Finally, if $m = a_0 + \cdots + a_n q^n$, then we have

$$\begin{aligned}
\delta_m(\delta_k(x)) &= \delta_{a_0}(\delta_k(x)) \cdots \delta_{a_n q^n}(\delta_k(x)) \\
&= c_{a_0,k} \delta_{a_0 k}(x) \cdots c_{a_n q^n, k} \delta_{a_n q^n k}(x) \\
&= c_{m,k} \delta_{mk}(x),
\end{aligned}$$

for some $c_{m,k} \in \mathcal{O}$. Using property (1) to compute $\delta_m(\delta_k(x))$, it is clear that we must have $c_{m,k} = \varpi^{(mk)_{(q)} - (m k_{(q)} + m_{(q)})}$.

□

We now define $\mathcal{O}$-divided powers more generally, in the language of Berthelot-Ogus [2]. Respecting the tradition, we write P.D. for "*puissances divisée*", that is, divided powers.

Definition 3.1.4. An *$\mathcal{O}$-P.D.-algebra* is a triple (A, I, δ) where A is an $\mathcal{O}$-algebra with an ideal I and maps $\delta = \{\delta_m\}_{m \geq 0}$ where $\delta_0 : I \to A$ and $\delta_m : I \to I$ for every integer $m \geq 1$ such that

1. $\delta_0(x) = 1, \delta_1(x) = x$, and $\varpi^{m_{(q)}} \delta_m(x) = x^m$ for every $x \in I$.
2. For $a \in A$ and $x \in I$, $\delta_m(ax) = a^m \delta_m(x)$.
3. For $x, y \in I$, $\delta_m(x + y) = \sum_{k+l=m} a_{k,l} \delta_k(x) \delta_l(y)$.
4. For $x \in I$, $\delta_m(x) \delta_k(x) = b_{m,k} \delta_{m+k}(x)$.

5. For $x \in I$, $\delta_m(\delta_k(x)) = c_{m,k}\delta_{mk}(x)$.

The numbers $a_{k,l}, b_{m,k}$, and $c_{m,k}$ are the same as those in Proposition 3.1.3.

Definition 3.1.5. If (A, I, δ) is an $\mathcal{O}$-P.D.-algebra, we say (I, δ) is the *$\mathcal{O}$-P.D. ideal*. An ideal $J \subseteq I$ is a *sub-$\mathcal{O}$-P.D. ideal* if $\delta_m(x) \in J$ for any $x \in J$ and all $m \geq 1$. If (B, J, δ') is another $\mathcal{O}$-P.D.-algebra, we say that an $\mathcal{O}$-algebra morphism $f : A \to B$ is an *$\mathcal{O}$-P.D. morphism* if $f(I) \subseteq J$ and $\delta'_m(f(x)) = f(\delta_m(x))$ for any m and all $x \in I$.

Remark 3.1.6.

- If A is an F-algebra and I an ideal of A, then there is a unique set of maps $\delta = \{\delta_m : I \to A\}$ making (A, I, δ_m) an $\mathcal{O}$-P.D.algebra, and these maps are defined as $\delta_m(x) = x^m/\varpi^{m_{(q)}}$.

- Note that since $\varpi^m/\varpi^{m_{(q)}} \in \mathcal{O}$, if (A, I, δ) is an $\mathcal{O}$-P.D.-algebra, then δ can be extended to $I + \varpi A$. That is, the maps δ_m on I can be extended to maps on $I + \varpi A$ with image in $I + \varpi A$ for $m \geq 1$ and this makes $(A, I + \varpi A, \delta)$ an $\mathcal{O}$-P.D.-algebra for which (I, δ) is a sub-$\mathcal{O}$-P.D.-ideal.

Lemma 3.1.7. *If (A, I, δ) is an $\mathcal{O}$-P.D. algebra and $J \subseteq A$ is an ideal, then there exists a $\overline{\delta}$ on $\overline{I} = I(A/J)$ such that $(A, I, \delta) \to (A/J, \overline{I}, \overline{\delta})$ is an $\mathcal{O}$-P.D. morphism if and only if $I \cap J \subseteq I$ is a sub-$\mathcal{O}$-P.D. ideal.*

Proof. If such a $\overline{\delta}$ exists, and if $x \in I \cap J$, and if $m \geq 1$, then certainly $\delta_m(x) \in I$ and since $0 = \overline{\delta}_m(\overline{x}) = \overline{\delta_m(x)}$, we know $\delta_m(x)$ is also in J. So $I \cap J$ is a sub $\mathcal{O}$-P.D.-ideal. For the converse, define $\overline{\delta}_m(\overline{x}) = \overline{\delta_m(x)}$ where x is any preimage of $\overline{x}$. Then this will be an $\mathcal{O}$-divided power if it is well defined. But if $y \in I \cap J$, then

$$\begin{aligned}\delta_m(x + y) &= \sum_{k+l=m} a_{k,l}\delta_k(x)\delta_l(y) \\ &= \delta_m(x) + \sum_{l=1}^{m} a_{m-l,l}\delta_{m-l}(x)\delta_l(y),\end{aligned}$$

and the latter sum is in $I \cap J$ since $\delta_l(y) \in I \cap J$ for all $l \geq 1$. □

3.2. $\mathcal{O}$-divided power envelope

We now want a way to construct an $\mathcal{O}$-P.D.-algebra from an $\mathcal{O}$-algebra A and an ideal I of A. That is, given an ideal of A that does not have $\mathcal{O}$-divided powers, construct a ring that contains A and an ideal which contains I on which we can define $\mathcal{O}$-divided powers. For regular divided powers, Berthelot and Ogus [2] give a universal construction, called the divided power envelope of A with respect to the ideal I. This universal construction is involved and given in the appendix of their paper. However, for our purposes we only give a construction of the $\mathcal{O}$-divided power envelope for relatively nice A (as defined below), in which we give an ad hoc construction.

Definition 3.2.1. Suppose that A is an $\mathcal{O}$-algebra with $A \to A \otimes_{\mathcal{O}} F$ an injection, and suppose I is an ideal of A. Then on all of the F-algebra $A \otimes_{\mathcal{O}} F$ there is a unique $\mathcal{O}$-divided power δ as in Remark 3.1.6. Define the *$\mathcal{O}$-divided power envelope* of A with respect to the ideal I as the triple (B, I^+, δ), where B is the A-subalgebra of $A \otimes_{\mathcal{O}} F$ given by adjoining to A the elements $\delta_m(x \otimes 1)$ for all $m \geq 1$ and $x \in I$, and where I^+ is the ideal of B generated by all such $\delta_m(x)$ for $m \geq 1$.

Proposition 3.2.2. *The triple (B, I^+, δ) is an $\mathcal{O}$-P.D.-algebra.*

Proof. We just need to check that $\delta_m(y) = y^m/\varpi^{m_{(q)}}$ is in I^+ for any $y \in I^+$ and $m \geq 1$ since δ_m satisfies all of the other properties of $\mathcal{O}$-divided powers by Proposition 3.1.3. Using the first and second properties of Proposition 3.1.3, it suffices to show that $\delta_m(y) \in I^+$ for any y of the form $y = \delta_k(x)$ where $k \geq 1$ and $x \in I$. But by the fourth property,

$$\delta_m(\delta_k(x)) = c_{m,k}\delta_{mk}(x)$$

and this is in I^+ by the definition of I^+. □

Note that if I is principally generated by some $x \in I$, then B is generated as an *A-module* by $\delta_m(x)$ for $m \geq 1$ since $\delta_m(x)\delta_n(x) = \varpi^{(m+n)_{(q)}-(m_{(q)}+n_{(q)})}\delta_{m+n}(x)$.

On B we can define a filtration:

Definition 3.2.3. For $r \geq 1$, let $\operatorname{Fil}^r B$ be the ideal of B generated by $\delta_i(x)$ for all $i \geq r$ and $x \in I$. So in particular, $\operatorname{Fil}^1 B = I^+$.

Using property 4 of δ_m in Definition 3.1.4, it is easy to see that this gives a filtration of B in the sense that $\operatorname{Fil}^i B \cdot \operatorname{Fil}^j B \subseteq \operatorname{Fil}^{i+j} B$. Note that for $i < q$ and $x \in I$, we have $\delta_i(x) = x^i \in A$ and therefore we can write $B = A + \operatorname{Fil}^q B$. If I is principally generated by some x, then $\operatorname{Fil}^r B$ is generated as an A-submodule by $\delta_i(x)$ for $i \geq r$ and so any element of $\operatorname{Fil}^r B$ can be written as

$$\sum_{j \geq r} a_j \delta_j(x) \tag{3.2.1}$$

with $a_j \in A$, and $a_j = 0$ for j larger than some n.

Remark 3.2.4. Note that we would get the same ring B if we took the $\mathcal{O}$-divided power envelope of A with respect to $I + \varpi A$, but we would get a different filtration.

4. Breuil-Kisin modules over $\mathfrak{S}$ and S

4.1. Definitions

Let k be a finite field of characteristic p, an odd prime. Let $K_0 = \operatorname{Frac} W(k)$, and let K be a totally ramified finite extension of K_0 with uniformizer π. Furthermore, let $F/\mathbb{Q}_p$ be a finite extension contained in K. Denote its ring of integers by $\mathcal{O}_F$, the residue field by k_F and a uniformizer by ϖ. Also write $q = |k_F|$. Denote by φ_q the q-power Frobenius map $\varphi_q(x) = x^q$ on k_F. Write $\mathcal{O}_{F_0} = W(k_F)$ and $F_0 = \operatorname{Frac} W(k_F)$. For an $\mathcal{O}_{F_0}$-algebra A, we will write $A_F = A \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$. In particular, $W(k)_F$ is a discrete valuation ring with uniformizer ϖ and residue field k. Its fraction field is the compositum of F with K_0.

There is a unique lift $\varphi_q : W(k)_F \to W(k)_F$ of the q-power Frobenius on k, which is trivial on $\mathcal{O}_F$. Define $\mathfrak{S} = W(k)_F[\![u]\!]$ to be the ring of power series in u over $W(k)_F$. Let $\varphi : \mathfrak{S} \to \mathfrak{S}$ be a continuous (for the (u, ϖ)-adic topology) endomorphism of $W(k)_F$ that is φ_q on $W(k)_F$ and such that

$$\varphi(u) \equiv u^q \bmod \varpi \quad \text{and} \quad \varphi(u) \equiv 0 \bmod u.$$

Let $E(u)$ be the monic, minimal polynomial of π over $\mathcal{O}_F$. Then $E(u)$ is an Eisenstein polynomial in u, so we can write $E(u) = u^e + \varpi g(u)$ for some integer e with $g(u)$ a unit in $\mathfrak{S}$.

Using 3.2.1, let S_0 be the $\mathcal{O}_F$-divided power envelope of $\mathfrak{S}$ with respect to the ideal $E(u)\mathfrak{S}$. Define S to be the ϖ-adic completion of S_0 and give S the (ϖ)-adic topology. Denote by $\operatorname{Fil}^r S_0$ the ideal generated by $\delta_m(x)$ for $m \geq r$ and $x \in E(u)\mathfrak{S}$ and by $\operatorname{Fil}^r S$ the closure of $\operatorname{Fil}^r S_0$ in S. Note that $\operatorname{Fil}^r S$ is topologically generated by the $\mathcal{O}_F$-divided powers $\delta_m(E(u))$ for $m \geq r$. We will now give some useful properties of S.

Proposition 4.1.1.

1. *There is a unique, continuous extension of* φ *on* $\mathfrak{S}$ *to* $\varphi : S \to S$*.*

2. $\varphi(E(u)) \in \varpi S$

3. $\varphi(E(u))/\varpi \in S^{\times}$*.*

4. $\varphi(x) \in \varpi^r S$ *for* $x \in \operatorname{Fil}^r S$ *and* $r \leq q-1$*.*

Proof. Recall from Remark 3.2.4 that we get the same rings S_0 and S if we consider the $\mathcal{O}_F$-divided power envelope with respect to $(E(u), \varpi)\mathfrak{S}$, and so we can write, for example, $\delta_m(\varpi)$ or $\delta_m(u^e)$, and these make sense in S_0 and S.

We first extend φ to S_0, which requires defining φ on the $\mathcal{O}_F$-divided powers. Using that $\varphi(\varpi) = \varpi$, we know that if φ is an endomorphism of S_0 compatible with φ on $\mathfrak{S}$, then it must be that $\varpi^{m_{(q)}}\varphi(\delta_m(x)) = \varphi(x^m)$ for any $x \in E(u)\mathfrak{S}$, and so it follows that if an extension of φ exists, it must be unique on S_0. We therefore define

$$\varphi(\delta_m(x)) = \delta_m(\varphi(x)),$$

but it is not clear *a priori* that $\delta_m(\varphi(x)) \in S_0$. It suffices to show that $\delta_m(\varphi(E(u)))$ is in S_0, and writing $E(u) = u^e + \varpi g(u)$, we get

$$\varphi(E(u)) = \varphi(u)^e + \varpi(\varphi(g(u))) = u^{eq} + \varpi(f(u) + \varphi(g(u))), \tag{4.1.1}$$

where $f(u) \in u\mathfrak{S}$. Then $\varphi(E(u)) \in (E(u), \varpi)\mathfrak{S}$, and so $\delta_m(\varphi(E(u))) \in S_0$. Now, because $\varphi(\varpi) = \varpi$ and S is the ϖ-adic completion of S_0, we have that φ extends uniquely and continuously to S.

Consider (4.1.1) further. Since $u^{eq} = \varpi\delta_q(u^e)$, it is clear that $\varphi(E(u)) \in \varpi S$, and that

$$\frac{\varphi(E(u))}{\varpi} = \frac{u^{eq}}{\varpi} + (f(u) + \varphi(g(u))).$$

Now, $(\frac{u^{eq}}{\varpi})^q = \varpi\delta_{q^2}(u^e) \in \varpi S$, and thus, as $n \to \infty$,

$$(\frac{u^{eq}}{\varpi})^n \to 0 \quad \varpi\text{-adically}.$$

Since $f(u) \in u\mathfrak{S}$ and $g(u) \in \mathfrak{S}^\times$, both $\varphi(g(u))$ and hence $f(u) + \varphi(g(u))$ are units in $\mathfrak{S}$ and so it follows that $\varphi(E(u))/\varpi$ is a unit in S as well.

Since $\varphi(E(u)) \in \varpi S$, we know $\varphi(E(u)^i)$ is divisible by ϖ^i in S. Now for $m \geq q$, by (3.1.2), writing $m = a_0 + a_1 q + \cdots + a_n q^n$, we have

$$\begin{aligned} m - m_{(q)} &\geq (m - \frac{1}{q-1} m) + (a_0 + ... + a_n) \\ &\geq (q-2) + 1 = q - 1. \end{aligned}$$

Moreover, if $m < q$, then $m - m_{(q)} = m$. This means that for $r \leq q-1$ and $i \geq r$, we know $i - i_{(q)} \geq r$ and so $\varphi(\delta_i(E(u)))$ is divisible by ϖ^r in S. This shows that $\varphi(x) \in \varpi^r S$ for $x \in \operatorname{Fil}^r S$ and $r \leq q-1$. □

Definition 4.1.2. For $r \leq q-1$, define the φ-semilinear map $\varphi_r : \operatorname{Fil}^r S \to S$ by $\varphi_r = \varphi/\varpi^r$. Denote by c the unit $\varphi_1(E(u))$.

Using Proposition 4.1.1 (4), we can see that for $r < q-1$, we have

$$\varphi_r(\operatorname{Fil}^{r+1} S) \subseteq \varpi S. \tag{4.1.2}$$

This is, however, not true when $r = q-1$, since $\varphi_{q-1}(\delta_q(E(u))) = \frac{\varphi(E(u)^q)}{\varpi^q}$ is not divisible by ϖ in S. This indicates that there is some difference between $r = q-1$ and $r < q-1$. In fact our major results will be restricted to the case that $r < q-1$. While many of the arguments to come work equally well in the case $r = q-1$, there are some instances in which it does not. For example, this fact (4.1.2) is needed in the statement of (4.2.1) in Section 4.2 and this in turn is necessary in Definition 4.2.9.

Since $\delta_m(E(u)) \in \mathfrak{S}$ for $m < q$, by (3.2.1), any element of S can be written as a sum of an element of $\mathfrak{S}$ and an element of $\operatorname{Fil}^q S$. This leads to the following result.

Proposition 4.1.3. *For $r \leq q$ we have that*

$$\mathfrak{S} \cap \operatorname{Fil}^r S = E(u)^r \mathfrak{S} \quad \textit{and} \quad S/\operatorname{Fil}^r S \cong \mathfrak{S}/E(u)^r \mathfrak{S}.$$

Proof. We will prove the claim that $S/\operatorname{Fil}^r S \cong \mathfrak{S}/E(u)^r\mathfrak{S}$ by induction on r. If $r = 1$, then both $\mathfrak{S}/E(u)\mathfrak{S}$ and $S/\operatorname{Fil}^1 S$ are isomorphic to $\mathcal{O}_K$ as the maps

$$\mathfrak{S} \to \mathcal{O}_K \quad \text{and } S \to \mathcal{O}_K$$

given by $u \mapsto \pi$ and $\delta_m(E(u)) \mapsto 0$ for $m \geq 1$ are surjective with kernels $E(u)\mathfrak{S}$ and $\operatorname{Fil}^1 S$, respectively. Now suppose the claim is true for $r < q$, and note that this implies that $\mathfrak{S} \cap \operatorname{Fil}^r S = E(u)^r\mathfrak{S}$. Then consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & E(u)^r\mathfrak{S}/E(u)^{r+1}\mathfrak{S} & \longrightarrow & \mathfrak{S}/E(u)^{r+1}\mathfrak{S} & \longrightarrow & \mathfrak{S}/E(u)^r\mathfrak{S} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \operatorname{Fil}^r S/\operatorname{Fil}^{r+1} S & \longrightarrow & S/\operatorname{Fil}^{r+1} S & \longrightarrow & S/\operatorname{Fil}^r S & \longrightarrow 0
\end{array}$$

Because $\operatorname{Fil}^1 S \cdot \operatorname{Fil}^r S \subseteq \operatorname{Fil}^{r+1} S$, the left vertical map is a map of $\mathcal{O}_K$-modules. Moreover, since $S = \mathfrak{S} + \operatorname{Fil}^q S$ and $\operatorname{Fil}^q S \subseteq \operatorname{Fil}^{r+1} S$ as $r + 1 \leq q$, then for $i = r$ or $r + 1$ we have that $\operatorname{Fil}^i S = (\mathfrak{S} \cap \operatorname{Fil}^i S) + \operatorname{Fil}^q S$ and so by the second isomorphism theorem for modules,

$$\operatorname{Fil}^i S/\operatorname{Fil}^q S \cong (\mathfrak{S} \cap \operatorname{Fil}^i S)/(\mathfrak{S} \cap \operatorname{Fil}^q S).$$

Then by the third isomorphism theorem for modules we have that

$$\begin{aligned}
\operatorname{Fil}^r S/\operatorname{Fil}^{r+1} S &\cong (\operatorname{Fil}^r S/\operatorname{Fil}^q S)/(\operatorname{Fil}^{r+1} S/\operatorname{Fil}^q S) \\
&\cong [(\mathfrak{S} \cap \operatorname{Fil}^r S)/(\mathfrak{S} \cap \operatorname{Fil}^q S)]/[(\mathfrak{S} \cap \operatorname{Fil}^{r+1} S)/(\mathfrak{S} \cap \operatorname{Fil}^q S)] \\
&\cong (\mathfrak{S} \cap \operatorname{Fil}^r S)/(\mathfrak{S} \cap \operatorname{Fil}^{r+1} S).
\end{aligned}$$

Finally, by the induction hypothesis, we have

$$\begin{aligned}
\operatorname{Fil}^r S/\operatorname{Fil}^{r+1} S &\cong (\mathfrak{S} \cap \operatorname{Fil}^r S)/(\mathfrak{S} \cap \operatorname{Fil}^{r+1} S) \\
&= E(u)^r\mathfrak{S}/(\mathfrak{S} \cap \operatorname{Fil}^{r+1} S).
\end{aligned}$$

Since $E(u)^{r+1}\mathfrak{S} \subseteq \mathfrak{S} \cap \operatorname{Fil}^{r+1} S$, the map $E(u)^r\mathfrak{S}/E(u)^{r+1}\mathfrak{S} \to \operatorname{Fil}^r S/\operatorname{Fil}^{r+1} S$ is a surjective map of rank one $\mathcal{O}_K$-modules, and so is an isomorphism (see Lemma

4.1.15). The right vertical map is also an isomorphism by induction, and so we get an isomorphism $\mathfrak{S}/E(u)^{r+1}\mathfrak{S} \cong S/\operatorname{Fil}^{r+1} S$.

□

Corollary 4.1.4. *The* $\mathfrak{S}$*-module* $S/\operatorname{Fil}^r S$ *has no* ϖ*-torsion for* $r \leq q$*.*

Proof. By Proposition 4.1.3, we have $S/\operatorname{Fil}^r S \cong \mathfrak{S}/E(u)^r\mathfrak{S}$, so suppose that for some $y \in \mathfrak{S}$, we get $\varpi y = E(u)^r g$ for $y, g \in \mathfrak{S}$. Since $E(u)$ is irreducible in $\mathfrak{S}$ and ϖ is a prime element, we know ϖ divides g and so $y \in E(u)^r\mathfrak{S}$. So $S/\operatorname{Fil}^r S$ has no ϖ-torsion. □

On $\mathfrak{S}$, we define a category of modules, as well as some subcategories, whose structure matches that of Breuil modules in the classical setting (see, for example, [6] or [9]).

Definition 4.1.5.

- Denote by $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where $\mathfrak{M}$ is an $\mathfrak{S}$-module equipped with a φ-semilinear endomorphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ where $E(u)^r\mathfrak{M}$ is contained in the submodule of $\mathfrak{M}$ generated by the image of $\varphi_{\mathfrak{M}}$. Morphisms in $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ consist of $\mathfrak{S}$-module homomorphisms $f : \mathfrak{M} \to \mathfrak{N}$ such that $f \circ \varphi_{\mathfrak{M}} = \varphi_{\mathfrak{N}} \circ f$. Furthermore a sequence in $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ is said to be *exact* if the underlying sequence of $\mathfrak{S}$-modules is exact.

- Denote by $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ consisting of $\mathfrak{S}$-modules $\mathfrak{M}$ that are free of finite rank over $\mathfrak{S}$.

- Denote by $\operatorname{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ consisting of $\mathfrak{S}$-modules $\mathfrak{M}$ that are killed by ϖ and free of finite rank as $\mathfrak{S}_1 := \mathfrak{S}/\varpi\mathfrak{S}$-modules.

- Denote by $\operatorname{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ consisting of $\mathfrak{S}$-modules $\mathfrak{M}$ such that

$$\mathfrak{M} \cong \bigoplus_{i \in I} \mathfrak{S}/\varpi^{n_i}\mathfrak{S},$$

where I is finite and the n_i are positive integers. Note that $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ is the full subcategory of $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$ of objects killed by ϖ. This is commonly referred to as the subcategory of modules with invariant factors, or *"Facteurs Invariants"* in French.

- Denote by $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ the full subcategory of $'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ stable under extensions in $'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and generated by $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$. That is, $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ contains $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and if $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ and $\mathfrak{M}$ is an object of $'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ such that

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$$

is an exact sequence in $'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, then $\mathfrak{M}$ is also an object of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.

Remark 4.1.6. There is another way to state the main condition of $'\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ by considering the $\mathfrak{S}$-linearization of the Frobenius map $\varphi_{\mathfrak{M}}$. Let

$$\varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M},$$

where the left tensor is $\mathfrak{S}$ as an $\mathfrak{S}$-module with a twist of Frobenius φ, and consider the $\mathfrak{S}$-linear map $\mathrm{id} \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$. Then the condition that $E(u)^r\mathfrak{M}$ is contained in the $\mathfrak{S}$-span of the image of $\varphi_{\mathfrak{M}}$ is equivalent to the condition that $E(u)^r\mathfrak{M}$ is contained in the image of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$. Stated another way, since $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ is linear, the cokernel of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ is killed by $E(u)^r$.

Definition 4.1.7. We say that $\mathfrak{M}$ has *E-height h* if h is the smallest integer such that $E(u)^h$ kills the cokernel of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$. When the r is omitted, $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ consists of all finite, free φ-modules over $\mathfrak{S}$ of finite E-height.

Proposition 4.1.8. *If* $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ *is an object of* $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ *or* $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$*, then the pair* $(\mathfrak{M}/\varpi^n\mathfrak{M}, \overline{\varphi}_{\mathfrak{M}})$ *is an object of* $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$*, where* $\overline{\varphi}_{\mathfrak{M}}$ *is the reduction of* $\varphi_{\mathfrak{M}}$ *modulo* ϖ^n*.*

Proof. Clearly in either case $\mathfrak{M}/\varpi^n\mathfrak{M}$ is isomorphic as an $\mathfrak{S}$-module to a finite direct sum of modules of the form $\mathfrak{S}/\varpi^{n_i}\mathfrak{S}$ with n_i at most n. Moreover, the map $\overline{\varphi}_{\mathfrak{M}}$ is well-defined since $\varphi_{\mathfrak{M}}$ is φ-semilinear and $\varphi(\varpi) = \varpi$. It is also easy to see that since any element of $E(u)^r\mathfrak{M}$ is in the $\mathfrak{S}$-span of the image of $\varphi_{\mathfrak{M}}$, then reduction modulo ϖ^n gives that any element of $E(u)^r\mathfrak{M}/\varpi^n\mathfrak{M}$ is in the $\mathfrak{S}$-span of the image of $\overline{\varphi}_{\mathfrak{M}}$. $\square$

Corollary 4.1.9. *The category* $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$ *is a full subcategory of* $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$*.*

Proof. This can be seen by an induction on the smallest power of ϖ that kills an object $\mathfrak{M} \in \text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$. If $\mathfrak{M}$ is killed by ϖ, then $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ which is contained in $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. Suppose that for any object $\mathfrak{M} \in \text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$ that is killed by ϖ^n, we have that $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. Now suppose that $\mathfrak{M}$ is an object of $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$ that is killed by ϖ^{n+1}. Note that $\varpi\mathfrak{M}$ is a subobject of $\mathfrak{M}$ in $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$ (this is easy to check) and $\varpi\mathfrak{M}$ is killed by ϖ^n and so is an object of $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. Moreover, using Proposition 4.1.8 and the fact that the objects killed by ϖ in $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$ are precisely those in $\text{Mod}_{\mathfrak{S}_1}^{\varphi,r}$, we have that $\mathfrak{M}/\varpi\mathfrak{M}$ is an object of $\text{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and hence an object of $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. We therefore have an exact sequence in $'\text{Mod}_{\mathfrak{S}}^{\varphi,r}$

$$0 \to \varpi\mathfrak{M} \to \mathfrak{M} \to \mathfrak{M}/\varpi\mathfrak{M} \to 0,$$

and so by the definition of $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$, we have shown that $\mathfrak{M}$ is an object of $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. $\square$

The following induction argument will be used frequently when proving properties of the category $\text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$, and we give the argument here so that we may refer to it later.

Proposition 4.1.10. *Suppose that* $P(\mathfrak{M})$ *is a statement that is true or false for each* $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$*. Suppose that*

1. $P(\mathfrak{M})$ *is true for every* $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_1}^{\varphi,r}$*.*

2. If there is an exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$$

in $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ *and* $P(\mathfrak{M}')$ *and* $P(\mathfrak{M}'')$ *are true, then* $P(\mathfrak{M})$ *is also true.*

Then $P(\mathfrak{M})$ *is true for all* $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.

Proof. By the definition of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$, we can define the *extension length* of an object $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ as follows. We say that $\mathfrak{M}$ has extension length 0 if $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and extension length $n \geq 1$ if n is the smallest integer such that there exist $\mathfrak{M}'$ and $\mathfrak{M}''$ in $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ of extension lengths at most $n-1$ and there is an exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0.$$

For any $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ there exists such an n by the very definition of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.

We can then prove $P(\mathfrak{M})$ by a strong induction on the extension length of $\mathfrak{M}$, in which case the two assumptions on P are the base case and induction step, respectively. □

Lemma 4.1.11. *The objects killed by* ϖ *in* $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ *are all objects of* $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$. *Moreover, every object of* $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ *is a finite type* $\mathfrak{S}$*-module killed by some power of* ϖ.

Proof. If $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ is killed by ϖ, then we have a short exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0,$$

where $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. Since $\mathfrak{M}$ is killed by ϖ, then so are $\mathfrak{M}'$ and $\mathfrak{M}''$, and so by Proposition 4.1.10, it suffices to assume that $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects of $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$. This means that $\mathfrak{M}'$ and $\mathfrak{M}''$ are free, finite type modules over $\mathfrak{S}_1 = \mathfrak{S}/\varpi\mathfrak{S} = k[\![u]\!]$. Then we know that $\mathfrak{M}$ is a finite type module over the principal ideal domain $\mathfrak{S}_1$. But $\mathfrak{M}$ has no torsion over $\mathfrak{S}_1$ since $\mathfrak{M}'$ and $\mathfrak{M}''$ are free. So $\mathfrak{M}$ is free of finite type over $\mathfrak{S}_1$. So $\mathfrak{M}$ is an object of $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$.

To show that any object $\mathfrak{M}$ of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ is a finite type $\mathfrak{S}$-module and is killed by some power of ϖ, we again use Proposition 4.1.10. If $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$, then clearly $\mathfrak{M}$ is finite type and killed by ϖ. Now suppose for some $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ we have a short exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0,$$

where $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ that are finite type $\mathfrak{S}$-modules killed by some powers of ϖ. Then from the exact sequence it is clear that $\mathfrak{M}$ is a finite type $\mathfrak{S}$-module and killed by a power of ϖ no greater than the sum of the powers that kill $\mathfrak{M}'$ and $\mathfrak{M}''$. This completes the proof. □

Lemma 4.1.12. *The map* $\mathrm{id} \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ *is injective for objects* $\mathfrak{M}$ *of* $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ *or* $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$.

Proof. We follow Lemma 1.1.9 of [18]. First suppose that $\mathfrak{M}$ is an object of $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$, and free as an $\mathfrak{S}_1$-module of rank m. Fix a basis $\{e_1, ..., e_m\}$ of $\mathfrak{M}$ and consider the basis $\{1 \otimes e_1, ..., 1 \otimes e_m\}$ of $\varphi^*\mathfrak{M}$. Since $u^{er}\mathfrak{M}$ is contained in the image of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$, the matrix of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ with respect to these bases must have a determinant that divides $(u^{er})^m$. But u^{erm} is not a zero divisor of $\mathfrak{S}_1$. So $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ is injective.

Now, suppose that for $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ we have an exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0,$$

where $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ such that $\mathrm{id} \otimes \varphi_{\mathfrak{M}'}$ and $\mathrm{id} \otimes \varphi_{\mathfrak{M}''}$ are injective. Then we have the following commutative diagram of exact sequences since $\varphi : \mathfrak{S} \to \mathfrak{S}$ is a flat map:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varphi^*\mathfrak{M}' & \longrightarrow & \varphi^*\mathfrak{M} & \longrightarrow & \varphi^*\mathfrak{M}'' & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes \varphi_{\mathfrak{M}'}} & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes \varphi_{\mathfrak{M}}} & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes \varphi_{\mathfrak{M}''}} & & \\ 0 & \longrightarrow & \mathfrak{M}' & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{M}'' & \longrightarrow & 0 \end{array}$$

Then a simple diagram chase shows that $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ is injective. So by Proposition 4.1.10, $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ is injective for objects of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.

Now suppose that $\mathfrak{M}$ is an object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$. Since $\varphi(\varpi) = \varpi$, the reduction of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ modulo any power of ϖ, say ϖ^n, is the linearization of $\overline{\varphi}_{\mathfrak{M}}$ on $\mathfrak{M}/\varpi^n\mathfrak{M}$. By Proposition 4.1.8 and Corollary 4.1.9, $\mathfrak{M}/\varpi^n\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ and so $\mathrm{id} \otimes \overline{\varphi}_{\mathfrak{M}}$ is injective. Hence, any element of the kernel of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ must be in $\varpi^n \varphi^* \mathfrak{M}$ for any n, and so $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ is injective. $\square$

Lemma 4.1.13. *If $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ and $\mathfrak{N}$ is an $\mathfrak{S}$-module with no ϖ-torsion, then $\mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) = 0$.*

Proof. First, if $\mathfrak{M}$ is a free $\mathfrak{S}_1$-module, say

$$\mathfrak{M} \cong \bigoplus_{i=1}^{d} \mathfrak{S}/\varpi\mathfrak{S},$$

then we have

$$\begin{aligned} \mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) &= \bigoplus_{i=1}^{d} \mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{S}/\varpi\mathfrak{S}, \mathfrak{N}) \\ &= \bigoplus_{i=1}^{d} \mathfrak{N}[\varpi] = 0. \end{aligned}$$

Now suppose that for some $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ there is an exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0,$$

in $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ such that

$$\mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{N}) = \mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}'', \mathfrak{N}) = 0.$$

Then tensoring this short exact sequence with $\mathfrak{N}$ yields a long exact sequence of $\mathfrak{S}$-modules which is in part

$$\mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{N}) \to \mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}'', \mathfrak{N}).$$

But then clearly $\mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) = 0$. So by Proposition 4.1.10, we get that

$$\mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) = 0$$

for all objects of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. $\square$

The next two lemmas are some commutative algebra facts that we will need.

Lemma 4.1.14. *Let R and T be rings and $R \to T$ a ring homomorphism. Suppose that M is a free R-module. If A_1 and A_2 are R-submodules of T, then as submodules of $T \otimes_R M$ we have an equality*

$$(A_1 \underset{R}{\otimes} M) \cap (A_2 \underset{R}{\otimes} M) = (A_1 \cap A_2) \underset{R}{\otimes} M$$

where the intersection on the left-hand side occurs in $T \otimes_R M$ and the intersection on the right-hand side occurs in T.

Proof. Because M is a free R-module it is flat as an R-module. Then $A_1 \otimes_R M$ and $A_2 \otimes_R M$ are naturally submodules of $T \otimes_R M$. Moreover, by tensoring the following short exact sequence of R-modules with M

$$0 \to A_i \to T \to T/A_i \to 0$$

we get the short exact sequence

$$0 \to A_i \underset{R}{\otimes} M \to T \underset{R}{\otimes} M \to (T/A_i) \underset{R}{\otimes} M \to 0.$$

This shows that $(T \otimes_R M)/(A_i \otimes_R M) \cong (T/A_i) \otimes_R M$. Now consider the left exact sequence

$$0 \to A_1 \cap A_2 \to T \to T/A_1 \oplus T/A_2,$$

where the last map is $s \mapsto (s \bmod A_1, s \bmod A_2)$. Then tensoring with M gives a left exact sequence

$$0 \to (A_1 \cap A_2) \underset{R}{\otimes} M \to T \underset{R}{\otimes} M \to ((T/A_1) \underset{R}{\otimes} M) \oplus ((T/A_2) \underset{R}{\otimes} M).$$

But we then know that this final R-module is isomorphic to

$$(T \otimes_R M)/(A_1 \otimes_R M) \oplus (T \otimes_R M)/(A_2 \otimes_R M)$$

and thus

$$(A_1 \cap A_2) \underset{R}{\otimes} M = (A_1 \underset{R}{\otimes} M) \cap (A_2 \underset{R}{\otimes} M).$$

□

Lemma 4.1.15. *Let R be a commutative ring with unity. If M and N are free R-modules of the same finite rank and $f : M \to N$ is a surjective homomorphism, then f is an isomorphism.*

Proof. This statement is given more generally, without the condition that M and N are free, as the main theorem of [24]. □

We now define a similar collection of categories of S-modules:

Definition 4.1.16. For $r < q - 1$:

- Denote by $'\operatorname{Mod}_S^{\varphi,r}$ the category of triples $(\mathcal{M}, \operatorname{Fil}^r \mathcal{M}, \varphi_{\mathcal{M},r})$ where $\mathcal{M}$ is an S-module with a submodule $\operatorname{Fil}^r \mathcal{M}$ such that $\operatorname{Fil}^r S \cdot \mathcal{M} \subseteq \operatorname{Fil}^r \mathcal{M}$ and $\varphi_{\mathcal{M},r}$ is a φ-semilinear map $\varphi_{\mathcal{M},r} : \operatorname{Fil}^r \mathcal{M} \to \mathcal{M}$ such that, for $s \in \operatorname{Fil}^r S$ and $x \in \mathcal{M}$,

$$\varphi_{\mathcal{M},r}(sx) = c^{-r}\varphi_r(s)\varphi_{\mathcal{M},r}(E(u)^r x), \tag{4.1.3}$$

  where c is the unit defined in 4.1.2. Morphisms in $'\operatorname{Mod}_S^{\varphi,r}$ consist of S-module homomorphisms $f : \mathcal{M} \to \mathcal{N}$ such that $f(\operatorname{Fil}^r \mathcal{M}) \subseteq \operatorname{Fil}^r \mathcal{N}$ and $f \circ \varphi_{\mathcal{M},r} = \varphi_{\mathcal{N},r} \circ f$. Furthermore, a sequence in $'\operatorname{Mod}_S^{\varphi,r}$ is said to be *exact* if it is an exact sequence of S-modules and induces an exact sequence on the submodules Fil^r.

- Denote by $\operatorname{Mod}_S^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_S^{\varphi,r}$ consisting of objects $\mathcal{M}$ with the following additional properties:

  - $\mathcal{M}$ is free of finite rank.
  - $\mathcal{M}/\operatorname{Fil}^r \mathcal{M}$ has no ϖ-torsion.
  - The image of $\varphi_{\mathcal{M},r}$ generates $\mathcal{M}$ as an S-module.

- Denote by $\operatorname{Mod}_{S_1}^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_S^{\varphi,r}$ consisting of objects $\mathcal{M}$ with the following additional properties:

  - $\mathcal{M}$ is free of finite rank over $S_1 := S/\varpi S$.

  - The image of $\varphi_{\mathcal{M},r}$ generates $\mathcal{M}$ as an S_1-module.

- Denote by Mod-FI$_S^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_S^{\varphi,r}$ consisting of S-modules $\mathcal{M}$ such that

  - $\mathcal{M} \cong \bigoplus_{i\in I} S/\varpi^{n_i}S$ as S-modules, where I is finite and the n_i are positive integers.

  - The image of $\varphi_{\mathcal{M},r}$ generates $\mathcal{M}$ as an S-module.

  Note that $\operatorname{Mod}_{S_1}^{\varphi,r}$ is the full subcategory of Mod-FI$_S^{\varphi,r}$ of objects killed by ϖ.

- Denote by $\operatorname{Mod}_{S_\infty}^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_S^{\varphi,r}$ stable under extensions in $'\operatorname{Mod}_S^{\varphi,r}$ and generated by $\operatorname{Mod}_{S_1}^{\varphi,r}$.

Remark 4.1.17. When $\mathcal{M} \in \operatorname{Mod}_S^{\varphi,r}$, then the condition of (4.1.3) is guaranteed. If $s \in \operatorname{Fil}^r S$ and $x \in \mathcal{M}$, then both sx and $E(u)^r x$ are in $\operatorname{Fil}^r \mathcal{M}$, and we can write

$$\begin{aligned}
\varpi^r \varphi_{\mathcal{M},r}(sx) &= c^{-r}\varpi^r \varphi_r(E(u)^r)\varphi_{\mathcal{M},r}(sx)\\
&= c^{-r}\varphi(E(u)^r)\varphi_{\mathcal{M},r}(sx)\\
&= c^{-r}\varphi_{\mathcal{M},r}(sE(u)^r x)\\
&= c^{-r}\varphi(s)\varphi_{\mathcal{M},r}(E(u)^r x)\\
&= \varpi^r(c^{-r}\varphi_r(s)\varphi_{\mathcal{M},r}(E(u)^r x)).
\end{aligned}$$

Then since $\mathcal{M}$ is a free S-module, we can say that

$$\varphi_{\mathcal{M},r}(sx) = c^{-r}\varphi_r(s)\varphi_{\mathcal{M},r}(E(u)^r x)$$

Lemma 4.1.18. *For $(\mathcal{M}, \operatorname{Fil}^r \mathcal{M}, \varphi_{\mathcal{M},r}) \in \operatorname{Mod}_S^{\varphi,r}$ and any integer $n \geq 0$,*

$$\varpi^n \mathcal{M} \cap \operatorname{Fil}^r \mathcal{M} = \varpi^n \operatorname{Fil}^r \mathcal{M}.$$

Proof. One can easily check that the proof follows from the fact that $\mathcal{M}/\operatorname{Fil}^r \mathcal{M}$ is assumed to be ϖ-torsion-free. $\square$

Lemma 4.1.18 shows that the image of $\operatorname{Fil}^r \mathcal{M}$ in $\mathcal{M} \to \mathcal{M}/\varpi^n\mathcal{M}$ factors through $\varpi^n \operatorname{Fil}^r \mathcal{M}$ and the $S/\varpi^n S$-module

$$\overline{\operatorname{Fil}^r}\mathcal{M} \coloneqq \operatorname{Fil}^r \mathcal{M}/\varpi^n \operatorname{Fil}^r \mathcal{M}$$

is exactly this image, so is actually a submodule of $\mathcal{M}/\varpi^n\mathcal{M}$. We can thus define the map

$$\overline{\varphi}_{\mathcal{M},r} : \overline{\operatorname{Fil}^r}\mathcal{M} \to \mathcal{M}/\varpi^n\mathcal{M}$$

as the reduction of $\varphi_{\mathcal{M},r}$ modulo ϖ^n and this will be well-defined since

$$\varphi_{\mathcal{M},r}(\varpi^n x) = \varpi^n \varphi_{\mathcal{M},r}(x)$$

for $x \in \operatorname{Fil}^r \mathcal{M}$. It is then easy to check that this gives an object in $\text{Mod-FI}_S^{\varphi,r}$:

Proposition 4.1.19. *If* $(\mathcal{M}, \operatorname{Fil}^r \mathcal{M}, \varphi_{\mathcal{M},r}) \in \operatorname{Mod}_S^{\varphi,r}$, *then*

$$(\mathcal{M}/\varpi^n\mathcal{M}, \overline{\operatorname{Fil}^r}\mathcal{M}, \overline{\varphi}_{\mathcal{M},r}) \in \text{Mod-FI}_S^{\varphi,r}.$$

We will want to show that the results of 4.1.18 and 4.1.19 also hold if we begin with an object of $\text{Mod-FI}_S^{\varphi,r}$, so that we can also give an analogue of Corollary 4.1.9. However, we no longer have the condition that $\mathcal{M}/\operatorname{Fil}^r \mathcal{M}$ is ϖ-torsion free. To circumvent this difficulty, we will have to delay the proof until 4.2.20.

There is a natural way to obtain a module in $'\operatorname{Mod}_S^{\varphi,r}$ given one in $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ or $\operatorname{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.

Definition 4.1.20. For an object $\mathfrak{M}$ of $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ or $\operatorname{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ and $r < q-1$, define the functor Θ_r by

$$\Theta_r(\mathfrak{M}) = S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M},$$

where the left tensor S is an $\mathfrak{S}$-module by the composite $\mathfrak{S} \hookrightarrow S \xrightarrow{\varphi} S$. On morphisms $f : \mathfrak{M} \to \mathfrak{N}$, we define $\Theta_r(f) = \operatorname{id} \otimes f$. Note that the map

$$\operatorname{id} \otimes \varphi_{\mathfrak{M}} : S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M} \to S \underset{\mathfrak{S}}{\otimes} \mathfrak{M}$$

is S-linear, where the second tensor product is simply over the natural inclusion $\mathfrak{S} \hookrightarrow S$. Define an S-submodule of $\mathcal{M} := \Theta_r(\mathfrak{M})$ by

$$\operatorname{Fil}^r \mathcal{M} := \{x \in \mathcal{M} \mid (\operatorname{id} \otimes \varphi_{\mathfrak{M}})(x) \in \operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \subseteq S \otimes_{\mathfrak{S}} \mathfrak{M}\}.$$

Define the map $\varphi_{\mathcal{M},r} : \operatorname{Fil}^r \mathcal{M} \to \mathcal{M}$ as the composite

$$\operatorname{Fil}^r \mathcal{M} \xrightarrow{\operatorname{id} \otimes \varphi_{\mathfrak{M}}} \operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_r \otimes \operatorname{id}} \mathcal{M}.$$

Proposition 4.1.21. *The functor Θ_r is well-defined, and $\Theta_r(\mathfrak{M}) \in \operatorname{Mod}_S^{\varphi,r}$ when $\mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$, and $\Theta_r(\mathfrak{M}) \in \text{Mod-FI}_S^{\varphi,r}$ when $\mathfrak{M} \in \text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$.*

Proof. We first need to check that, given $\mathfrak{M}$ in $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ or $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$, the S-module $\mathcal{M} := \Theta_r(\mathfrak{M})$ is actually in $\operatorname{Mod}_S^{\varphi,r}$ or $\text{Mod-FI}_S^{\varphi,r}$, respectively. Note that $S \otimes_{\varphi,\mathfrak{S}} \mathfrak{S} \cong S$ as an S-module since $\varphi : S \to S$ is injective. Therefore, if $\mathfrak{M}$ is free and finitely generated as an $\mathfrak{S}$-module, then $\mathcal{M}$ is free and finitely generated as an S-module. Likewise, if

$$\mathfrak{M} \cong \bigoplus_{i \in I} \mathfrak{S}/\varpi^{n_i}\mathfrak{S},$$

then

$$\mathcal{M} \cong \bigoplus_{i \in I} S/\varpi^{n_i} S.$$

Also, $\operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}$ is a submodule of $S \otimes_{\mathfrak{S}} \mathfrak{M}$ and this can be seen by tensoring the following exact sequence of $\mathfrak{S}$-modules with $\mathfrak{M}$:

$$0 \to \operatorname{Fil}^r S \to S \to S/\operatorname{Fil}^r S \to 0.$$

Indeed, if $\mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$, then $\mathfrak{M}$ is a free (and hence flat) $\mathfrak{S}$-module. If $\mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$, then since $S/\operatorname{Fil}^r S$ has no ϖ-torsion by Corollary 4.1.4, we can apply Lemma 4.1.13 and conclude that $\operatorname{Tor}^1_{\mathfrak{S}}(S/\operatorname{Fil}^r S, \mathfrak{M}) = 0$. So in either case, $\operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}$ is a submodule of $S \otimes_{\mathfrak{S}} \mathfrak{M}$ and $\operatorname{Fil}^r \mathcal{M}$ is well-defined.

The rest of the proof is identical whether $\mathfrak{M}$ is in $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ or $\text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$. It is clear that for $x \in \operatorname{Fil}^r S \cdot \mathcal{M}$ we have $\operatorname{id} \otimes \varphi_{\mathfrak{M}}(x) \in \operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}$, so $\operatorname{Fil}^r S \cdot \mathcal{M}$ is contained in $\mathrm{Fil}^r \mathcal{M}$. To see that $\varphi_{\mathcal{M},r}$ is φ-semilinear, suppose that $x \in \mathrm{Fil}^r \mathcal{M}$ with $(\mathrm{id} \otimes \varphi_{\mathfrak{M}})(x) = \sum_{i=1}^{n} t_i \otimes m_i$ in $\mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}$. Then because $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ is S-linear and φ_r on S is φ-semilinear, for $s \in S$,

$$\begin{aligned} \varphi_{\mathcal{M},r}(sx) &= \sum_{i=1}^{n} \varphi_r(st_i) \otimes m_i \\ &= \varphi(s) \sum_{i=1}^{n} \varphi_r(t_i) \otimes m_i \\ &= \varphi(s)\varphi_{\mathcal{M},r}(x). \end{aligned}$$

We will now show that $\varphi_{\mathcal{M},r}(\mathrm{Fil}^r \mathcal{M})$ generates $\mathcal{M}$. Given $m \in \mathfrak{M}$, we know that $E(u)^r m = \sum_{i=1}^{n} s_i \varphi_{\mathfrak{M}}(m_i)$ for some $s_i \in \mathfrak{S}$ and $m_i \in \mathfrak{M}$. Consider

$$x = \sum_{i=1}^{n} s_i \otimes m_i \in \mathcal{M}.$$

Then

$$\begin{aligned} (\mathrm{id} \otimes \varphi_{\mathfrak{M}})(x) &= \sum_{i=1}^{n} s_i \otimes \varphi_{\mathfrak{M}}(m_i) \\ &= \sum_{i=1}^{n} 1 \otimes s_i \varphi_{\mathfrak{M}}(m_i) \qquad (4.1.4) \\ &= 1 \otimes E(u)^r m \\ &= E(u)^r \otimes m, \end{aligned}$$

which is in $\mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}$ and so $x \in \mathrm{Fil}^r \mathcal{M}$. Therefore,

$$\varphi_{\mathcal{M},r}(x) = \varphi_r(E(u)^r) \otimes m = c^r \otimes m. \qquad (4.1.5)$$

But c is a unit and so it follows that the image of $\varphi_{\mathcal{M},r}$ generates $\mathcal{M}$ as an S-module. To check (4.1.3), suppose $t \in \mathrm{Fil}^r S$ and suppose $x \in \mathcal{M}$ is such that

$$\mathrm{id} \otimes \varphi_{\mathfrak{M}}(x) = \sum s_i \otimes m_i$$

for $s_i \in S$ and $m_i \in \mathfrak{M}$. Then

$$\begin{aligned} \varphi_{\mathcal{M},r}(tx) &= (\varphi_r \otimes \mathrm{id})(\sum (t \cdot s_i) \otimes m_i) \\ &= \varphi_r(t) \sum \varphi(s_i) \otimes m_i. \end{aligned}$$

Therefore, applying this calculation with $t = E(u)^r$, for any $s \in \operatorname{Fil}^r S$ we have

$$\begin{aligned} c^{-r}\varphi_r(s)\varphi_{\mathcal{M},r}(E(u)^r x) &= c^{-r}\varphi_r(s)\left(\varphi_r(E(u)^r)\sum \varphi(s_i)\otimes m_i\right) \\ &= \varphi_r(s)\sum \varphi(s_i)\otimes m_i \\ &= \varphi_{\mathcal{M},r}(sx). \end{aligned}$$

This shows that $\mathcal{M} \in \operatorname{Mod}_S^{\varphi,r}$ whenever $\mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ and $\mathcal{M} \in \text{Mod-FI}_S^{\varphi,r}$ whenever $\mathfrak{M} \in \text{Mod-FI}_{\mathfrak{S}}^{\varphi,r}$.

Finally, it remains to check that Θ_r is compatible with morphisms. Let $f : \mathfrak{M} \to \mathfrak{N}$ be a morphism in $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ so that $\Theta_r(f) = \operatorname{id}\otimes f$. We need to show that $h = \Theta_r(f)$ is a morphism in $'\operatorname{Mod}_S^{\varphi,r}$. We first need to check that $h(\operatorname{Fil}^r\mathcal{M}) \subseteq \operatorname{Fil}^r\mathcal{N}$. Let $x = \sum_{i=1}^n s_i \otimes m_i \in \operatorname{Fil}^r\mathcal{M}$ with $(\operatorname{id}\otimes\varphi_{\mathfrak{M}})(x) = \sum_{j=1}^l t_j \otimes n_j \in \operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}$. Then since $\varphi_{\mathfrak{N}} \circ f = f \circ \varphi_{\mathfrak{M}}$,

$$\begin{aligned} (\operatorname{id}\otimes\varphi_{\mathfrak{N}})(h(x)) &= (\operatorname{id}\otimes\varphi_{\mathfrak{N}})\left(\sum_{i=1}^n s_i \otimes f(m_i)\right) \\ &= \sum_{i=1}^n s_i \otimes \varphi_{\mathfrak{N}}(f(m_i)) \\ &= \sum_{i=1}^n s_i \otimes f(\varphi_{\mathfrak{M}}(m_i)) \\ &= h(\sum_{i=1}^n s_i \otimes \varphi_{\mathfrak{M}}(m_i)) \\ &= h(\sum_{j=1}^l t_j \otimes n_j) \\ &= \sum_{j=1}^l t_j \otimes f(n_j), \end{aligned}$$

which is in $\operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{N}$. It is then easy to see that $\Theta_r(f)$ commutes with $\varphi_{\mathcal{M},r}$ and $\varphi_{\mathcal{N},r}$. □

Lemma 4.1.22. *Let $\mathfrak{M}$ be an object of $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ and let $\mathcal{M} = \Theta_r(\mathfrak{M})$. Then for any n,*

$$\Theta_r(\mathfrak{M}/\varpi^n\mathfrak{M}) \cong \mathcal{M}/\varpi^n\mathcal{M}$$

as objects of $\text{Mod-FI}_S^{\varphi,r}$.

Proof. Write $\mathcal{M}' := \Theta_r(\mathfrak{M}/\varpi^n\mathfrak{M})$. We will show that we have the following short exact sequence in $'\operatorname{Mod}_S^{\varphi,r}$:

$$0 \to \varpi^n\mathcal{M} \to \mathcal{M} \to \mathcal{M}' \to 0. \tag{4.1.6}$$

First, we have a short exact sequence in $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ given by

$$0 \to \varpi^n\mathfrak{M} \to \mathfrak{M} \to \mathfrak{M}/\varpi^n\mathfrak{M} \to 0.$$

Consider S as an $\mathfrak{S}$-module via φ. Then $\operatorname{Tor}^1_{\mathfrak{S}}(S, \mathfrak{M}/\varpi^n\mathfrak{M}) = 0$ by Lemma 4.1.13 since S has no ϖ-torsion. Then applying $S \underset{\varphi,\mathfrak{S}}{\otimes} (-)$, we get the short exact sequence of S-modules:

$$0 \to S \underset{\varphi,\mathfrak{S}}{\otimes} (\varpi^n\mathfrak{M}) \to S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M} \to S \underset{\varphi,\mathfrak{S}}{\otimes} (\mathfrak{M}/\varpi^n\mathfrak{M}) \to 0.$$

Since $\varphi(\varpi) = \varpi$, we know that $S \otimes_{\varphi,\mathfrak{S}} (\varpi^n\mathfrak{M}) = \varpi^n(S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$, and note that as an S-module, $\Theta_r(\mathfrak{M}/\varpi^n\mathfrak{M}) = S \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}/\varpi^n\mathfrak{M})$. Therefore we get the exact sequence (4.1.6) as S-modules.

We now wish to show that we get an induced map $\operatorname{Fil}^r \mathcal{M} \to \operatorname{Fil}^r \mathcal{M}'$ and that this is surjective. Suppose that $x \in \operatorname{Fil}^r \mathcal{M}$, which by the definition of $\Theta_r(\mathfrak{M})$ means that we can write $\operatorname{id} \otimes \varphi_{\mathfrak{M}}(x) = \sum t_i \otimes m_i$ for some $t_i \in \operatorname{Fil}^r S$. Since $\mathfrak{M} \to \mathfrak{M}/\varpi^n\mathfrak{M}$ is a morphism in $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ by Proposition 4.1.8, we know that the following diagram commutes

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{M}' \\ \downarrow {\scriptstyle \operatorname{id}\otimes\varphi_{\mathfrak{M}}} & & \downarrow {\scriptstyle \operatorname{id}\otimes\varphi_{\mathfrak{M}/\varpi^n\mathfrak{M}}} \\ S \underset{\mathfrak{S}}{\otimes} \mathfrak{M} & \longrightarrow & S \underset{\mathfrak{S}}{\otimes} (\mathfrak{M}/\varpi^n\mathfrak{M}) \end{array} \tag{4.1.7}$$

and therefore the image of x in $\mathcal{M}'$ is in $\operatorname{Fil}^r \mathcal{M}'$. On the other hand, suppose that $y \in \operatorname{Fil}^r \mathcal{M}'$, so we can write $\operatorname{id} \otimes \varphi_{\mathfrak{M}/\varpi^n\mathfrak{M}}(y) = \sum t_i \otimes \overline{m}_i$ for some $t_i \in \operatorname{Fil}^r S$ and

$\overline{m}_i \in \mathfrak{M}/\varpi^n\mathfrak{M}$. If x is a lift of y in $\mathcal{M}$, and m_i a lift of $\overline{m}_i$ in $\mathfrak{M}$, then again by the commutativity of (4.1.7), we have that $\mathrm{id}\otimes\varphi_{\mathfrak{M}}(x) = \sum t_i \otimes (m_i + \varpi m_i')$ for some $m_i' \in \mathfrak{M}$, and this clearly shows that $x \in \mathrm{Fil}^r\,\mathcal{M}$. By composing the columns of (4.1.7) with $\varphi_r \otimes \mathrm{id}$, we see that the map $\mathrm{Fil}^r\,\mathcal{M} \to \mathrm{Fil}^r\,\mathcal{M}'$ commutes with $\varphi_{\mathcal{M},r}$ and $\varphi_{\mathcal{M}',r}$. Finally, the kernel of $\mathrm{Fil}^r\,\mathcal{M} \to \mathrm{Fil}^r\,\mathcal{M}'$ is $\varpi^n\mathcal{M} \cap \mathrm{Fil}^r\,\mathcal{M}$ from above, but this is $\varpi^n\,\mathrm{Fil}^r\,\mathcal{M} = \mathrm{Fil}^r(\varpi^n\mathcal{M})$ by Lemma 4.1.18, and so we also have an exact sequence of S-modules given by

$$0 \to \mathrm{Fil}^r(\varpi^n\mathcal{M}) \to \mathrm{Fil}^r\,\mathcal{M} \to \mathrm{Fil}^r\,\mathcal{M}' \to 0.$$

Hence the sequence (4.1.6) is exact in $'\,\mathrm{Mod}_S^{\varphi,r}$ and this shows that $\Theta_r(\mathfrak{M}/\varpi^n\mathfrak{M})$ and $\mathcal{M}/\varpi^n\mathcal{M}$ are isomorphic as objects of $\text{Mod-FI}_S^{\varphi,r}$. □

Proposition 4.1.23. *The functor Θ_r is exact on $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. In particular, $\Theta_r(\mathfrak{M}) \in \mathrm{Mod}_{S_\infty}^{\varphi,r}$ when $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.*

Proof. The proof follows [18] 1.1.11. We will first show that Θ_r is exact on $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ or $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ simultaneously. Suppose that

$$0 \to \mathfrak{M}' \to \mathfrak{M} \xrightarrow{f} \mathfrak{M}'' \to 0$$

is a short exact sequence in either $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ or $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. Consider S as an $\mathfrak{S}$-module via φ. If $\mathfrak{M}''$ is a free $\mathfrak{S}$-module, then $\mathrm{Tor}^1_{\mathfrak{S}}(S, \mathfrak{M}'') = 0$. If $\mathfrak{M}'' \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$, then $\mathrm{Tor}^1_{\mathfrak{S}}(S, \mathfrak{M}'') = 0$ by Lemma 4.1.13 since S has no ϖ-torsion. In either case, we get an exact sequence of S-modules by applying $\Theta_r = S \underset{\varphi,\mathfrak{S}}{\otimes} (-)$:

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0.$$

We also need an exact sequence of $\mathrm{Fil}^r(-)$ as S-modules. To see this, view the submodule $\mathrm{Fil}^r\,\mathcal{M}$ (see Definition 4.1.20) as

$$\mathrm{Fil}^r\,\mathcal{M} = \ker\left(S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M} \xrightarrow{\mathrm{id}\otimes\varphi_{\mathfrak{M}}} S \underset{\mathfrak{S}}{\otimes} \mathfrak{M} \to (S \underset{\mathfrak{S}}{\otimes} \mathfrak{M})/(\mathrm{Fil}^r\, S \underset{\mathfrak{S}}{\otimes} \mathfrak{M})\right). \tag{4.1.8}$$

Now, $S/\operatorname{Fil}^r S$ has no ϖ-torsion by Corollary 4.1.4, so in either case we have that $\operatorname{Tor}^1_{\mathfrak{S}}(S/\operatorname{Fil}^r S, \mathfrak{M})$ is 0 using the same argument as above, and therefore tensoring the following short exact sequence with $\mathfrak{M}$

$$0 \longrightarrow \operatorname{Fil}^r S \longrightarrow S \longrightarrow S/\operatorname{Fil}^r S \longrightarrow 0$$

yields the short exact sequence

$$0 \longrightarrow \operatorname{Fil}^r S \underset{\mathfrak{S}}{\otimes} \mathfrak{M} \longrightarrow S \underset{\mathfrak{S}}{\otimes} \mathfrak{M} \longrightarrow (S/\operatorname{Fil}^r S) \underset{\mathfrak{S}}{\otimes} \mathfrak{M} \longrightarrow 0\,.$$

This shows that $(S \otimes_{\mathfrak{S}} \mathfrak{M})/(\operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}) \cong (S/\operatorname{Fil}^r S) \otimes_{\mathfrak{S}} \mathfrak{M}$. We can therefore rewrite (4.1.8) as

$$\operatorname{Fil}^r \mathcal{M} = \ker\left(\rho \otimes \varphi_{\mathfrak{M}} : S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M} \to (S/\operatorname{Fil}^r S) \underset{\mathfrak{S}}{\otimes} \mathfrak{M}\right)$$

where ρ is the projection $S \twoheadrightarrow S/\operatorname{Fil}^r S$. Moreover, we have the following diagram of exact sequences of S-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M}' & \longrightarrow & S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M} & \longrightarrow & S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M}'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho\otimes\varphi_{\mathfrak{M}'}} & & \downarrow{\scriptstyle \rho\otimes\varphi_{\mathfrak{M}}} & & \downarrow{\scriptstyle \rho\otimes\varphi_{\mathfrak{M}''}} & & \\
0 & \longrightarrow & (S/\operatorname{Fil}^r S) \underset{\mathfrak{S}}{\otimes} \mathfrak{M}' & \longrightarrow & (S/\operatorname{Fil}^r S) \underset{\mathfrak{S}}{\otimes} \mathfrak{M} & \longrightarrow & (S/\operatorname{Fil}^r S) \underset{\mathfrak{S}}{\otimes} \mathfrak{M}'' & \longrightarrow & 0
\end{array}$$

where the bottom row is exact using $\operatorname{Tor}^1_{\mathfrak{S}}(S/\operatorname{Fil}^r S, \mathfrak{M}'') = 0$. Since both of the maps $\mathfrak{M}' \to \mathfrak{M}$ and $\mathfrak{M} \to \mathfrak{M}''$ commute with the respective Frobenius maps, it follows that the above diagram is commutative. Hence, by the Snake Lemma, we have the exact sequence

$$0 \to \operatorname{Fil}^r \mathcal{M}' \to \operatorname{Fil}^r \mathcal{M} \to \operatorname{Fil}^r \mathcal{M}''$$

To see that the final map is actually surjective, let $y \in \operatorname{Fil}^r \mathcal{M}''$. Using the fact that $S = \mathfrak{S} + \operatorname{Fil}^q S$, we can write $y = \overline{y} + z$ where $\overline{y} \in \mathfrak{S} \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M}'' = \varphi^* \mathfrak{M}''$ and $z \in \operatorname{Fil}^q S \cdot \mathcal{M}''$. But $\mathcal{M} \to \mathcal{M}''$ is surjective, so $\operatorname{Fil}^q S \cdot \mathcal{M}$ maps onto $\operatorname{Fil}^q S \cdot \mathcal{M}''$, and since $r < q$, we know $\operatorname{Fil}^q S \cdot \mathcal{M} \subseteq \operatorname{Fil}^r \mathcal{M}$. We can conclude that there exists some element of $\operatorname{Fil}^r \mathcal{M}$ that maps to z. Now, $(\operatorname{id} \otimes \varphi_{\mathfrak{M}''})(\overline{y}) \in \mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M}'' \cong \mathfrak{M}''$, but also by (4.1.8) the image of $(\operatorname{id} \otimes \varphi_{\mathfrak{M}''})(\overline{y})$ is 0 in $(S/\operatorname{Fil}^r S) \otimes_{\mathfrak{S}} \mathfrak{M}''$ since $\overline{y} = y - z \in \operatorname{Fil}^r \mathcal{M}''$. Using Proposition 4.1.3, $(S/\operatorname{Fil}^r S) \otimes_{\mathfrak{S}} \mathfrak{M}'' \cong \mathfrak{M}''/E(u)^r \mathfrak{M}''$. Therefore, we can write $(\operatorname{id} \otimes \varphi_{\mathfrak{M}''})(\overline{y}) = E(u)^r m''$ for some $m'' \in \mathfrak{M}''$. Since $f : \mathfrak{M} \to \mathfrak{M}''$ is surjective, there exists $m \in \mathfrak{M}$ with $f(m) = m''$ and because $\mathfrak{M}$ has E-height (4.1.7) at most r, there must be an $x \in \varphi^*\mathfrak{M}$ such that $(\operatorname{id} \otimes \varphi_{\mathfrak{M}})(x) = E(u)^r m$. Since $f \circ \varphi_{\mathfrak{M}} = \varphi_{\mathfrak{M}''} \circ f$, we see that $(\operatorname{id} \otimes \varphi_{\mathfrak{M}''})((\operatorname{id} \otimes f)(x)) = (\operatorname{id} \otimes \varphi_{\mathfrak{M}''})(\overline{y})$. But $\operatorname{id} \otimes \varphi_{\mathfrak{M}''}$ is injective by Lemma 4.1.12, so $(\operatorname{id} \otimes f)(x) = \overline{y}$ and $x \in \operatorname{Fil}^r \mathcal{M}$ by the definition of $\operatorname{Fil}^r \mathcal{M}$ since $(\operatorname{id} \otimes \varphi_{\mathfrak{M}})(x) \in \operatorname{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M}$. This shows that $\operatorname{id} \otimes f$ maps $\operatorname{Fil}^r \mathcal{M}$ onto $\operatorname{Fil}^r \mathcal{M}''$ and this completes the proof that Θ_r is exact on $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$ and on $\operatorname{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.

□

4.2. Modules in characteristic p

Our goal is to show that Θ_r induces an equivalence of categories $\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r} \to \operatorname{Mod}_{S}^{\varphi,r}$, and we will do this by demonstrating that Θ_r is both fully faithful and essentially surjective. We will first show that we have an equivalence on modules killed by ϖ. The advantage in this setting is that the ring $\mathfrak{S}/\varpi\mathfrak{S} \cong k[\![u]\!]$ is a principal ideal domain and that we are also able to pass from the ring $S/\varpi S$ to $S/(\varpi, \operatorname{Fil}^q S) \cong k[u]/u^{eq}$, which is a much simpler ring than S. The general structure of this section is modeled after Breuil in [4],[5], and [6].

Definition 4.2.1. By Proposition 4.1.19, the ring $S_1 \coloneqq S/\varpi S$ is naturally an object of $\operatorname{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ for any $r < q - 1$, where $\operatorname{Fil}^r S_1 = \operatorname{Fil}^r S/\varpi \operatorname{Fil}^r S$ is the image of $\operatorname{Fil}^r S$ in $S/\varpi S$ and $\varphi_r : \operatorname{Fil}^r S_1 \to S_1$ is given by reduction of φ_r on $\operatorname{Fil}^r S$ modulo ϖ. We further give S_1 a filtration for $r \geq q - 1$ by defining

$$\operatorname{Fil}^r S_1 \coloneqq \operatorname{Im}(\operatorname{Fil}^r S \hookrightarrow S \twoheadrightarrow S_1).$$

We also denote again by $\varphi : S_1 \to S_1$ the reduction of the Frobenius φ on S modulo ϖ. Moreover, by Lemma 3.1.7, $(S_1, \operatorname{Fil}^1 S_1, \overline{\delta})$ is an $\mathcal{O}_F$-P.D.-algebra where the divided power maps $\{\overline{\delta}_m\}$ are defined by $\overline{\delta}_m(\overline{x}) := \overline{\delta_m(x)}$ for $x \in \operatorname{Fil}^1 S$ and $m \geq 0$.

Remark 4.2.2.

1. The Frobenius φ on S_1 is well-defined since $\varphi(\varpi) = \varpi$.

2. Note that the reduction of $E(u)$ modulo ϖ is u^e. We can also give an explicit description of the ideal $\operatorname{Fil}^r S_1$. The ideals $\operatorname{Fil}^r S$ are generated over S by $\delta_m(E(u))$ for $m \geq r$, and by the definition of $\overline{\delta}_m$, we have $\overline{\delta_m(E(u))} = \overline{\delta}_m(u^e)$, and so the ideals $\operatorname{Fil}^r S_1$ are generated by $\overline{\delta}_m(u^e)$ for $m \geq r$.

3. Note that for $r < q - 1$, as in (4.1.2), we know that $\varphi_r(\operatorname{Fil}^{r+1} S) \subseteq \varpi S$, and so $\varphi_r(\operatorname{Fil}^{r+1} S_1) = 0$.

4. The image of u^{eq} is 0 in S_1 since $u^{eq} \in \varpi S$ as seen in the proof of Proposition 4.1.1, whereas the image of an element such as u^{eq}/ϖ is not 0 in S_1. Using Proposition 4.1.3, we know that $S/(\varpi, \operatorname{Fil}^q S) \cong \mathfrak{S}/(\varpi, E(u)^q) \cong k[u]/u^{eq}$, so we have
$$k[u]/u^{eq} \to S/\varpi S \to S/(\varpi, \operatorname{Fil}^q S) \cong k[u]/u^{eq},$$
and this map is the identity map. Thus, we can in fact view $k[u]/u^{eq}$ as living naturally inside of S_1. Then the complicated portion of the ring S_1 comes from the ideal $\operatorname{Fil}^q S_1$ in the sense that any element of S_1 can be written as the sum of an element of $k[u]/u^{eq}$ and an element of $\operatorname{Fil}^q S_1$.

The category of Breuil modules over S_1, denoted $\operatorname{Mod}_{S_1}^{\varphi,r}$ was given in Definition 4.1.16. Note that the condition (4.1.3) on $\varphi_{\mathcal{M},r}$ can be written as

$$\varphi_{\mathcal{M},r}(sx) = \overline{c}^{-r} \varphi_r(s) \varphi_{\mathcal{M},r}(u^{er} x).$$

Moreover, since φ_r is zero on $\operatorname{Fil}^{r+1} S_1$, this equation implies that

$$\varphi_{\mathcal{M},r}\big|_{(\operatorname{Fil}^{r+1} S)\cdot\mathcal{M}} = 0. \tag{4.2.1}$$

The final remark of 4.2.2 is motivation for the definition of the following ring.

Definition 4.2.3. Let $\widetilde{S}_1 := S_1/\operatorname{Fil}^q S_1$, which can be naturally identified with $k[u]/u^{eq}$ via $k[u]/u^{eq} \hookrightarrow S_1 \twoheadrightarrow S_1/\operatorname{Fil}^q S_1$. For $r < q$, define $\operatorname{Fil}^r \widetilde{S}_1$ to be the image of $\operatorname{Fil}^r S_1$ in $\widetilde{S}_1$ and define the Frobenius φ on $\widetilde{S}_1$ by the reduction of φ on S_1. Lastly, for $0 < r < q-1$, define the φ-semilinear maps $\varphi_r : \operatorname{Fil}^r \widetilde{S}_1 \to \widetilde{S}_1$ by $\varphi_r(u^{er}) = \widetilde{c}^r$ where $\widetilde{c}$ is the image of $\overline{c}$ in $\widetilde{S}_1$.

The ideal $\operatorname{Fil}^r \widetilde{S}_1$ is equal to $u^{er}\widetilde{S}_1$ since $\overline{\delta}_m(u^e) = u^{em} \in u^{er}S_1$ for $r \leq m < q$. Therefore, φ_r is well-defined. Because φ is 0 on $\operatorname{Fil}^q S_1$, we know φ on $\widetilde{S}_1$ is well-defined. Moreover, if we identify $\widetilde{S}_1$ as $k[u]/u^{eq}$, then $\varphi(u) = u^q$. Thus, the kernel of φ on $\widetilde{S}_1$ is generated by u^e and we have an injection $k[u]/u^e \overset{\varphi}{\hookrightarrow} \widetilde{S}_1$

We define a category of $\widetilde{S}_1$-modules in a similar way as $\operatorname{Mod}_{S_1}^{\varphi,r}$ except that the condition (4.1.3) is omitted, as it is automatic (see remark below).

Definition 4.2.4.

- Denote by $'\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ the category of triples $(\widetilde{\mathcal{M}}, \operatorname{Fil}^r \widetilde{\mathcal{M}}, \varphi_{\widetilde{\mathcal{M}},r})$, where $\widetilde{\mathcal{M}}$ is an $\widetilde{S}_1$-module with a submodule $\operatorname{Fil}^r \widetilde{\mathcal{M}}$ such that $\operatorname{Fil}^r \widetilde{S}_1 \cdot \widetilde{\mathcal{M}} \subseteq \operatorname{Fil}^r \widetilde{\mathcal{M}}$ and $\varphi_{\widetilde{\mathcal{M}},r}$ is a φ-semilinear map $\varphi_{\widetilde{\mathcal{M}},r} : \operatorname{Fil}^r \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$. Morphisms in $'\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ consist of $\widetilde{S}_1$-module homomorphisms $f : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ such that $f(\operatorname{Fil}^r \widetilde{\mathcal{M}}) \subseteq \operatorname{Fil}^r \widetilde{\mathcal{N}}$ and $f \circ \varphi_{\widetilde{\mathcal{M}},r} = \varphi_{\widetilde{\mathcal{N}},r} \circ f$.

- Denote by $\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ the full subcategory of $'\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ consisting of objects with the following additional properties:

  - $\widetilde{\mathcal{M}}$ is free of finite rank over $\widetilde{S}_1$.

  - The image of $\varphi_{\widetilde{\mathcal{M}},r}$ generates $\widetilde{\mathcal{M}}$ as an $\widetilde{S}_1$-module.

Remark 4.2.5. Note that if $\widetilde{\mathcal{M}}$ is in $'\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ and $x \in \widetilde{\mathcal{M}}$ and $s \in \operatorname{Fil}^r \widetilde{S}_1$, then writing $s = u^{er}t$ for $t \in \widetilde{S}_1$, we get

$$\begin{aligned}\varphi_{\widetilde{\mathcal{M}},r}(sx) &= \varphi(t)\varphi_{\widetilde{\mathcal{M}},r}(u^{er}x) \\ &= \tilde{c}^{-r}\varphi(t)\varphi_r(u^{er})\varphi_{\widetilde{\mathcal{M}},r}(u^{er}x) \\ &= \tilde{c}^{-r}\varphi_r(s)\varphi_{\widetilde{\mathcal{M}},r}(u^{er}x), \end{aligned} \tag{4.2.2}$$

and so the analogue of (4.1.3) is always satisfied for objects of $'\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$.

Our first course of action is to show that the categories $\operatorname{Mod}_{S_1}^{\varphi,r}$ and $\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ are equivalent.

Lemma 4.2.6. *Fix a positive integer n. Let $\mathcal{M}$ be a finite, free $k[u]/u^n$-module, and let $\mathcal{N}$ be a submodule of $\mathcal{M}$. Then for any positive integer $m < n$*

1. *If $u^m \cdot x = 0$ for some $x \in \mathcal{M}$, then $x \in u^{n-m}\mathcal{M}$.*
2. *$\mathcal{N}/u^m\mathcal{N}$ is a k-vector space with dimension*

$$\dim_k(\mathcal{N}/u^m\mathcal{N}) \leq \dim_k(\mathcal{M}/u^m\mathcal{M})$$

and the dimensions are equal if and only if $u^{n-m}\mathcal{M} \subseteq \mathcal{N}$.

Proof.

1. Let $\{e_1, ..., e_d\}$ be a basis for $\mathcal{M}$ and suppose $u^m \cdot x = 0$ with $x = \sum f_i(u)e_i$ where $f_i(u) \in k[u]/u^n$. Then $\sum(u^m f_i(u))e_i = 0$, so $u^m f_i(u) = 0$. If this implies that $f_i(u) \in u^{n-m}k[u]/u^n$, then $x \in u^{n-m}\mathcal{M}$. We have thus reduced the claim to $k[u]/u^n$ itself. Suppose $u^m f = 0$ in $k[u]/u^n$ and let $\hat{f} \in k[u]$ be a lift of f. Then there is some $\hat{g} \in k[u]$ with $u^m\hat{f} = u^n\hat{g}$. But $k[u]$ is a domain and so it must be that $\hat{f} = u^{n-m}\hat{g}$, which shows that $f \in u^{n-m}k[u]/u^n$.

2. The proof is similar to that of Breuil [4] 2.2.1.1. Consider the following commutative diagram of exact sequences of k-vector spaces:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & u^m\mathcal{N} & \rightarrow & u^m\mathcal{M} & \rightarrow & u^m\mathcal{M}/u^m\mathcal{N} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{N} & \rightarrow & \mathcal{M} & \rightarrow & \mathcal{M}/\mathcal{N} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{N}/u^m\mathcal{N} & \rightarrow & \mathcal{M}/u^m\mathcal{M} & \rightarrow & (\mathcal{M}/u^m\mathcal{M})/(\mathcal{N}/u^m\mathcal{N}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Thus we have that

$$\dim_k(\mathcal{M}/u^m\mathcal{M}) = \dim_k(\mathcal{N}/u^m\mathcal{N}) + \dim_k[(\mathcal{M}/u^m\mathcal{M})/(\mathcal{N}/u^m\mathcal{N})].$$

This gives the desired inequality since $\dim_k[(\mathcal{M}/u^m\mathcal{M})/(\mathcal{N}/u^m\mathcal{N})] \geq 0$. Moreover, by dimension counting along the third column, we see that $\dim_k(\mathcal{M}/u^m\mathcal{M})$ and $\dim_k(\mathcal{N}/u^m\mathcal{N})$ are equal if and only if the k-linear surjective map

$$\mathcal{M}/\mathcal{N} \overset{\cdot u^m}{\to} u^m\mathcal{M}/u^m\mathcal{N}$$

is an isomorphism. This map is an isomorphism if and only if it is injective, which is equivalent to the condition that $\{x \in \mathcal{M} \text{ s.t. } u^m x \in u^m\mathcal{N}\} = \mathcal{N}$. We will show $\{x \in \mathcal{M} \text{ s.t. } u^m x \in u^m\mathcal{N}\} = \mathcal{N}$ if and only if $u^{n-m}\mathcal{M} \subseteq \mathcal{N}$. First suppose that $\{x \in \mathcal{M} \text{ s.t. } u^m x \in u^m\mathcal{N}\} = \mathcal{N}$ is true and let $y \in \mathcal{M}$. Then $u^m(u^{n-m}y) = 0 \in u^m\mathcal{N}$ implies that $u^{n-m}y \in \mathcal{N}$. On the other hand, suppose that $u^{n-m}\mathcal{M} \subseteq \mathcal{N}$ and consider $x \in \mathcal{M}$ such that $u^m x \in u^m\mathcal{N}$. Let $y \in \mathcal{N}$ be such that $u^m x = u^m y$. Then $u^m(x-y) = 0$, and 4.2.6 (1) implies that $x - y \in u^{n-m}\mathcal{M} \subseteq \mathcal{N}$. Since $y \in \mathcal{N}$, this shows that $x \in \mathcal{N}$. Hence, $\dim_k(\mathcal{N}/u^m\mathcal{N}) = \dim_k(\mathcal{M}/u^m\mathcal{M})$ if and only if $u^{n-m}\mathcal{M} \subseteq \mathcal{N}$.

□

Lemma 4.2.7. *For* $\widetilde{\mathcal{M}} \in \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$*, if* $\widetilde{\mathcal{M}}$ *is free of rank* d *over* $\widetilde{S}_1 = k[u]/u^{eq}$*, then for* $1 \leq m \leq e$*,* $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^m \mathrm{Fil}^r \widetilde{\mathcal{M}}$ *is free of rank* d *as a* $k[u]/u^m$*-module.*

Proof. Because $r \leq q-1$, we get that $u^{eq-1}\widetilde{\mathcal{M}} \subseteq \mathrm{Fil}^r \widetilde{\mathcal{M}}$ and so by Lemma 4.2.6 (2) with $\mathcal{M} := \widetilde{\mathcal{M}}$ and $\mathcal{N} := \mathrm{Fil}^r \widetilde{\mathcal{M}}$ and $m = 1$ and $n = eq$, we have

$$\dim_k(\mathrm{Fil}^r \widetilde{\mathcal{M}}/u \,\mathrm{Fil}^r \widetilde{\mathcal{M}}) = \dim_k(\widetilde{\mathcal{M}}/u\widetilde{\mathcal{M}}).$$

By Nakayama's lemma, $\dim_k(\widetilde{\mathcal{M}}/u\widetilde{\mathcal{M}}) = \mathrm{rk}_{k[u]/u^{eq}}(\widetilde{\mathcal{M}}) = d$, and so the claim is true when $m = 1$. We will show it is true for all $1 \leq m \leq e$ inductively, so suppose $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^m \mathrm{Fil}^r \widetilde{\mathcal{M}}$ is free of rank d as a $k[u]/u^m$-module for some $m < e$. Let $\{b_1, ..., b_d\}$ in $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ be such that the images form a basis for $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^m \mathrm{Fil}^r \widetilde{\mathcal{M}}$. Then by Nakayama's Lemma, the images of $\{b_1, ..., b_d\}$ generate $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^{m+1} \mathrm{Fil}^r \widetilde{\mathcal{M}}$ as a $k[u]/u^{m+1}$-module.

Suppose that there is a relation on the b_i of the form

$$\sum p_i(u) b_i \in u^{m+1} \mathrm{Fil}^r \widetilde{\mathcal{M}} \subseteq u^m \mathrm{Fil}^r \widetilde{\mathcal{M}}$$

for some $p_i(u) \in k[u]/u^{eq}$. Since the images of the b_i freely generate $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^m \mathrm{Fil}^r \widetilde{\mathcal{M}}$, it follows that $p_i(u) = u^m p_i'(u)$ for some $p_i'(u) \in k[u]/u^{eq}$. Let $\alpha \in \mathrm{Fil}^r \widetilde{\mathcal{M}}$ be such that $\sum p_i(u) b_i = u^{m+1}\alpha$. Then $u^m \cdot (\sum p_i'(u) b_i - u\alpha) = 0$ and so by Lemma 4.2.6 (1), there exists some $\beta \in \widetilde{\mathcal{M}}$ such that $\sum p_i'(u) b_i - ua = u^{eq-m}\beta$. We can rewrite this as

$$\sum p_i'(u) b_i = u^{e(q-1-r)+(e-m)}(u^{er}\beta) + u\alpha.$$

Moreover, $u^{er}\beta$ and α are both in $\mathrm{Fil}^r \widetilde{\mathcal{M}}$, and since $r \leq q-1$ and $m < e$, the sum $\sum p_i'(u) b_i$ must be in $u \,\mathrm{Fil}^r \widetilde{\mathcal{M}}$. But the images of b_i freely generate $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u \,\mathrm{Fil}^r \widetilde{\mathcal{M}}$ and so $p_i'(u) \in uk[u]/u^{eq}$ and hence $p_i(u) \in u^{m+1}k[u]/u^{eq}$ which shows that the images of $\{b_1, ..., b_d\}$ form a basis for $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^{m+1} \mathrm{Fil}^r \widetilde{\mathcal{M}}$. □

Lemma 4.2.8. *Suppose that* $\widetilde{\mathcal{M}}$ *is an object of* $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$. *Then the map*

$$\mathrm{id}\otimes\varphi_{\widetilde{\mathcal{M}},r} : \widetilde{S}_1 \underset{\varphi,k[u]/u^e}{\otimes} \mathrm{Fil}^r\,\widetilde{\mathcal{M}}/u^e\,\mathrm{Fil}^r\,\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}} \tag{4.2.3}$$

is an isomorphism of $\widetilde{S}_1$*-modules. Moreover,* $\varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r\,\widetilde{\mathcal{M}})$ *is a* $k[u^q]/u^{eq}$*-module and the natural multiplication map*

$$\widetilde{S}_1 \underset{k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r\,\widetilde{\mathcal{M}}) \longrightarrow \widetilde{\mathcal{M}} \tag{4.2.4}$$

is an isomorphism of $\widetilde{S}_1$*-modules.*

Proof. Since $\varphi(u^e)=0$ in $\widetilde{S}_1$, the map

$$\varphi_{\widetilde{\mathcal{M}},r} : \mathrm{Fil}^r\,\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$$

factors through $\mathrm{Fil}^r\,\widetilde{\mathcal{M}}/u^e\,\mathrm{Fil}^r\,\widetilde{\mathcal{M}}$. Then, the condition that $\varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r\,\widetilde{\mathcal{M}})$ generates $\widetilde{\mathcal{M}}$ over $\widetilde{S}_1$ shows that (4.2.3) and (4.2.4) are both surjections.

By Lemma 4.2.7, the $\widetilde{S}_1$-module

$$\widetilde{S}_1 \underset{\varphi,k[u]/u^e}{\otimes} \mathrm{Fil}^r\,\widetilde{\mathcal{M}}/u^e\,\mathrm{Fil}^r\,\widetilde{\mathcal{M}}$$

is free of the same rank over $\widetilde{S}_1 = k[u]/u^{eq}$ as $\widetilde{\mathcal{M}}$, so the map of (4.2.3) must be an isomorphism of $\widetilde{S}_1$-modules by Lemma 4.1.15. Then (4.2.4) follows immediately from decomposing (4.2.3) as surjective $\widetilde{S}_1$-linear maps

$$\widetilde{S}_1 \underset{\varphi,k[u]/u^e}{\otimes} \mathrm{Fil}^r\,\widetilde{\mathcal{M}}/u^e\,\mathrm{Fil}^r\,\widetilde{\mathcal{M}} \overset{\mathrm{id}\otimes\varphi_{\widetilde{\mathcal{M}},r}}{\longrightarrow} \widetilde{S}_1 \underset{k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r\,\widetilde{\mathcal{M}}) \longrightarrow \widetilde{\mathcal{M}}$$

□

Definition 4.2.9. Define the functor $T_r : \mathrm{Mod}_{S_1}^{\varphi,r} \to \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ by

$$\widetilde{\mathcal{M}} := T_r(\mathcal{M}) = \widetilde{S}_1 \underset{\sigma,S_1}{\otimes} \mathcal{M},$$

where σ is the natural surjection of S_1 onto $\widetilde{S}_1$ and let $\iota : \widetilde{S}_1 \hookrightarrow S_1$ be the natural inclusion of $k[u]/u^{eq}$ in S_1, which is a section of σ. Denote by $\sigma_{\mathcal{M}}$ the surjection

$$\sigma_{\mathcal{M}} : \mathcal{M} \to T_r(\mathcal{M}), \quad x \mapsto 1\otimes x.$$

Then, set

$$\operatorname{Fil}^r T_r(\mathcal{M}) \coloneqq \sigma_{\mathcal{M}}(\operatorname{Fil}^r \mathcal{M}) \tag{4.2.5}$$

and for $y \in \operatorname{Fil}^r T_r(\mathcal{M})$, define

$$\varphi_{\widetilde{\mathcal{M}},r}(y) \coloneqq \sigma_{\mathcal{M}}(\varphi_{\mathcal{M},r}(x)) \tag{4.2.6}$$

where x is any lift of y in $\operatorname{Fil}^r \mathcal{M}$. Define T_r on morphisms in $\operatorname{Mod}_{S_1}^{\varphi,r}$ by $T_r(f) = \operatorname{id} \otimes f$.

Note that σ commutes with φ on S_1 and $\widetilde{S}_1$. Moreover, since ι is the natural inclusion, ι also commutes with φ on S_1 and $\widetilde{S}_1$.

Proposition 4.2.10. *The functor* $T_r : \operatorname{Mod}_{S_1}^{\varphi,r} \to \operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ *is well-defined.*

Proof. First, $\mathcal{M}$ is free of finite rank over S_1, so it follows that $T_r(\mathcal{M})$ is free of finite rank as an $\widetilde{S}_1$-module (of rank equal to the S_1-rank of $\mathcal{M}$). Also, by Definition 4.2.3, $\sigma(\operatorname{Fil}^r S_1) = \operatorname{Fil}^r \widetilde{S}_1$, and so it follows that $(\operatorname{Fil}^r \widetilde{S}_1) \cdot T_r(\mathcal{M}) \subseteq \operatorname{Fil}^r T_r(\mathcal{M})$. The map $\varphi_{\widetilde{\mathcal{M}},r}$ is well-defined because the choice of lift does not matter. To see this, note that if $\sigma_{\mathcal{M}}(x) = \sigma_{\mathcal{M}}(x')$, then this implies that $x - x' \in \operatorname{Fil}^q S_1 \cdot \mathcal{M}$ on which $\varphi_{\mathcal{M},r}$ is 0, by (4.2.1).

Since $\sigma \circ \varphi = \varphi \circ \sigma$, the map $\varphi_{\widetilde{\mathcal{M}},r}$ is φ-linear. Moreover, by definition of $\varphi_{\widetilde{\mathcal{M}},r}$,

$$\varphi_{\widetilde{\mathcal{M}},r}(\operatorname{Fil}^r T_r(\mathcal{M})) = \operatorname{id} \otimes \varphi_{\mathcal{M},r}(\operatorname{Fil}^r \mathcal{M})$$

and then the fact that $\varphi_{\mathcal{M},r}(\operatorname{Fil}^r \mathcal{M})$ generates $\mathcal{M}$ as an S_1-module and $\sigma : S_1 \to \widetilde{S}_1$ is a surjection shows that $T_r(\mathcal{M})$ is generated over $\widetilde{S}_1$ by the image of $\varphi_{\widetilde{\mathcal{M}},r}$.

Finally, if $f : \mathcal{M} \to \mathcal{N}$ is a morphism in $\operatorname{Mod}_{S_1}^{\varphi,r}$, then $T_r(f) = \operatorname{id} \otimes f$ is a morphism in $\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$. To check that $T_r(f)$ respects the Fil^r's and φ_r's, note that $(\operatorname{id} \otimes f)(\sigma_{\mathcal{M}}(x)) = \sigma_{\mathcal{M}}(f(x))$ for any $x \in \mathcal{M}$. Then these compatibilities will follow from the fact the f respects the Fil^r's and φ_r's in $\operatorname{Mod}_{S_1}^{\varphi,r}$. ☐

We also define a functor from $\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$ to $\operatorname{Mod}_{S_1}^{\varphi,r}$.

Definition 4.2.11. Define the functor $T_r^{-1} : \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r} \to \mathrm{Mod}_{S_1}^{\varphi,r}$ by

$$\begin{aligned} T_r^{-1}(\widetilde{\mathcal{M}}) &:= S_1 \underset{\iota,\widetilde{S}_1}{\otimes} \left(\widetilde{S}_1 \underset{k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}}) \right) \\ &= S_1 \underset{\iota,k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}}). \end{aligned}$$

We use the the second equality for simplicity when giving a description of elements in $\mathcal{M} := T_r^{-1}(\widetilde{\mathcal{M}})$. Let $\widetilde{\sigma}_{\mathcal{M}}$ be the composite

$$T_r^{-1}(\widetilde{\mathcal{M}}) \xrightarrow{\sigma\otimes\mathrm{id}} \widetilde{S}_1 \underset{k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}}) \longrightarrow \widetilde{\mathcal{M}}$$

and define

$$\mathrm{Fil}^r\, T_r^{-1}(\widetilde{\mathcal{M}}) := \widetilde{\sigma}_{\mathcal{M}}^{-1}(\mathrm{Fil}^r \widetilde{\mathcal{M}}). \tag{4.2.7}$$

For $x \in \mathrm{Fil}^r\, T^{-1}(\widetilde{\mathcal{M}})$, define

$$\varphi_{\mathcal{M},r}(x) := 1 \otimes \varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(x)). \tag{4.2.8}$$

Define T_r^{-1} on morphisms in $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ by $T_r^{-1}(\widetilde{f}) := \mathrm{id} \otimes \widetilde{f}$

Proposition 4.2.12. *The functor T_r^{-1} is well-defined.*

Proof. By (4.2.4), $\widetilde{S}_1 \otimes_{k[u^q]/u^{eq}} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}})$ is a free finite rank $\widetilde{S}_1$-module, so $T_r^{-1}(\widetilde{\mathcal{M}})$ is free of finite rank as an S_1-module (of rank equal to the $\widetilde{S}_1$-rank of $\widetilde{\mathcal{M}}$). To see that $\varphi_{\mathcal{M},r}$ is φ-semilinear, let $x \in \mathrm{Fil}^r\, T_r^{-1}(\widetilde{\mathcal{M}})$ and $s \in S_1$. Then

$$\begin{aligned} \varphi_{\mathcal{M},r}(sx) &= 1 \otimes \varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(sx)) \\ &= 1 \otimes \varphi(\sigma(s))\varphi_{\mathcal{M},r}(\widetilde{\sigma}_{\mathcal{M}}(x)) \\ &= \iota \circ \varphi(\sigma(s)) \otimes \varphi_{\mathcal{M},r}(\widetilde{\sigma}_{\mathcal{M}}(x)) \\ &= \varphi(\iota \circ \sigma(s)) \otimes \varphi_{\mathcal{M},r}(\widetilde{\sigma}_{\mathcal{M}}(x)) \\ &= \varphi(s) \otimes \varphi_{\mathcal{M},r}(\widetilde{\sigma}_{\mathcal{M}}(x)) \\ &= \varphi(s)\varphi_{\mathcal{M},r}(x), \end{aligned}$$

where in the fifth equality we use the fact that $\iota \circ \sigma(s)$ and s differ by an element of $\operatorname{Fil}^q S_1$, on which φ is 0.

We now check the condition of (4.1.3). By using (4.2.2), we get that for $s \in \operatorname{Fil}^r S_1$ and $x \in T_r^{-1}(\widetilde{\mathcal{M}})$,

$$\begin{aligned} \varphi_{\mathcal{M},r}(sx) &= 1 \otimes \varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(sx)) \\ &= 1 \otimes \varphi_{\widetilde{\mathcal{M}},r}(\sigma(s)\widetilde{\sigma}_{\mathcal{M}}(x)) \\ &= 1 \otimes \tilde{c}^{-r}\varphi_r(\sigma(s))\varphi_{\widetilde{\mathcal{M}},r}(u^{er}\widetilde{\sigma}_{\mathcal{M}}(x)) \\ &= \iota(\tilde{c}^{-r}\varphi_r(\sigma(s))) \otimes \varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(u^{er}x)) \\ &= \iota(\tilde{c}^{-r}\varphi_r(\sigma(s)))\varphi_{\mathcal{M},r}(u^{er}x). \end{aligned}$$

Now, we can write $s = u^{er}t + y$ where $t \in k[u]/u^{eq}$ and $y \in \operatorname{Fil}^q S_1$ (see Remark 4.2.2 (4)) and notice that $\varphi_r(s) = \varphi_r(u^{er}t)$ and $\sigma(s) = u^{er}t$. Therefore,

$$\begin{aligned} \iota(\tilde{c}^{-r}\varphi_r(\sigma(s))) &= \iota(\tilde{c}^{-r}\varphi_r(u^{er})\varphi(t)) \\ &= \iota(\varphi(t)) \\ &= \varphi(t) \\ &= \overline{c}^{-r}\varphi_r(u^{er}t) \\ &= \overline{c}^{-r}\varphi_r(s). \end{aligned}$$

This shows that $\varphi_{\mathcal{M},r}$ satisfies (4.1.3). It remains to check that the image of $\varphi_{\mathcal{M},r}$ generates $T_r^{-1}(\widetilde{\mathcal{M}})$ over S_1, but this follows easily using the very definition of $T_r^{-1}(\widetilde{\mathcal{M}})$ and from the fact that the image of $\varphi_{\widetilde{\mathcal{M}},r}$ generates $\widetilde{\mathcal{M}}$ over $\widetilde{S}_1$.

Finally, for a morphism $\widetilde{f}$ in $\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$, let $f = \operatorname{id} \otimes \widetilde{f}$. To show that this is a morphism in $\operatorname{Mod}_{S_1}^{\varphi,r}$, we first show that the following diagram commutes

$$\begin{array}{ccc} T_r^{-1}(\widetilde{\mathcal{M}}) & \xrightarrow{f} & T_r^{-1}(\widetilde{N}) \\ {\scriptstyle \widetilde{\sigma}_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \widetilde{\sigma}_{\mathcal{N}}} \\ \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{f}} & \widetilde{N} \end{array}$$

To see this, suppose $x \in T_r^{-1}(\widetilde{\mathcal{M}})$ can be written as $x = \sum s_i \otimes y_i$ for $s_i \in S_1$ and $y_i \in \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}})$. Then $\widetilde{f} \circ \varphi_{\widetilde{\mathcal{M}},r} = \varphi_{\widetilde{N},r} \circ \widetilde{f}$ and so $\widetilde{f}(y_i) \in \varphi_{\widetilde{N},r}(\mathrm{Fil}^r \widetilde{N})$. Therefore, we get on the one hand that

$$\begin{aligned}\widetilde{\sigma}_{\mathcal{N}}(f(x)) &= \widetilde{\sigma}_{\mathcal{N}}(\sum s_i \otimes \widetilde{f}(y_i)) \\ &= \sum \sigma(s_i)\widetilde{f}(y_i).\end{aligned}$$

On the other hand, $\widetilde{f}$ is $\widetilde{S}_1$-linear, and so

$$\begin{aligned}\widetilde{f}(\widetilde{\sigma}_{\mathcal{M}}(x)) &= \widetilde{f}(\sum \sigma(s_i)y_i) \\ &= \sum \sigma(s_i)\widetilde{f}(y_i).\end{aligned}$$

Now, by the definition of $\mathrm{Fil}^r T^{-1}(\widetilde{\mathcal{M}})$ and $\mathrm{Fil}^r T^{-1}(\widetilde{N})$ (4.2.7), it is easy to check that f respects the filtrations, and by the definition of $\varphi_{\mathcal{M},r}$ and $\varphi_{\mathcal{N},r}$ (4.2.8), it is likewise easy to check that $f \circ \varphi_{\mathcal{M},r} = \varphi_{\mathcal{N},r} \circ f$. □

Theorem 4.2.13. *The functor* $T_r : \mathrm{Mod}_{S_1}^{\varphi,r} \to \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ *induces an equivalence of categories and* T_r^{-1} *is a quasi-inverse for* T_r.

Proof. We follow the proof of [4] 2.2.2.1. We just need to show that T_r^{-1} is a quasi-inverse of T_r.

Suppose $\widetilde{\mathcal{M}} \in \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$. Then by Lemma 4.2.8,

$$\begin{aligned}T_r(T_r^{-1}(\widetilde{\mathcal{M}})) &= \widetilde{S}_1 \underset{\sigma,S_1}{\otimes} (S_1 \underset{k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}})) \\ &= \widetilde{S}_1 \underset{k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}}) \\ &\cong \widetilde{\mathcal{M}},\end{aligned}$$

where the isomorphism is an isomorphism of $\widetilde{S}_1$-modules. Let I denote this isomorphism. We need to show that I respects Fil^r and commutes with the φ_r. Notice that the composite

$$T_r^{-1}(\widetilde{\mathcal{M}}) \xrightarrow{\sigma_{\mathcal{M}}} T_r(T_r^{-1}(\mathcal{M})) = \widetilde{S}_1 \underset{\sigma,S_1}{\otimes} (S_1 \underset{k[u^q]/u^{eq}}{\otimes} \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}})) \xrightarrow{I} \widetilde{\mathcal{M}}$$

is precisely the map $\widetilde{\sigma}_{\mathcal{M}}$. Since $\operatorname{Fil}^r T_r(T_r^{-1}(\widetilde{\mathcal{M}})) = \sigma_{\mathcal{M}}(\widetilde{\sigma}_{\mathcal{M}}^{-1}(\operatorname{Fil}^r \widetilde{\mathcal{M}}))$ by definition (4.2.5), it follows that

$$I(\sigma_{\mathcal{M}}(\widetilde{\sigma}_{\mathcal{M}}^{-1}(\operatorname{Fil}^r \widetilde{\mathcal{M}}))) = \widetilde{\sigma}_{\mathcal{M}}(\widetilde{\sigma}_{\mathcal{M}}^{-1}(\operatorname{Fil}^r \widetilde{\mathcal{M}})) \subseteq \operatorname{Fil}^r \widetilde{\mathcal{M}}.$$

For $y \in \operatorname{Fil}^r T_r(T_r^{-1}(\widetilde{\mathcal{M}}))$, we may choose $x \in \operatorname{Fil}^r T_r^{-1}(\widetilde{\mathcal{M}})$ with $y = \sigma_{\mathcal{M}}(x)$ and then

$$I(y) = I(\sigma_{\mathcal{M}}(x)) = \widetilde{\sigma}_{\mathcal{M}}(x),$$

so

$$\begin{aligned} I(\varphi_{\widetilde{\mathcal{M}},r}(y)) &= I(\sigma_{\mathcal{M}}(\varphi_{\mathcal{M},r}(x))) \\ &= \widetilde{\sigma}_{\mathcal{M}}(\varphi_{\mathcal{M},r}(x)) \\ &= \widetilde{\sigma}_{\mathcal{M}}(1 \otimes \varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(x))) \\ &= \varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(x)) \\ &= \varphi_{\widetilde{\mathcal{M}},r}(I(y)). \end{aligned}$$

Thus, $T_r(T_r^{-1}(\widetilde{\mathcal{M}})) \cong \widetilde{\mathcal{M}}$ in $\operatorname{Mod}_{\widetilde{S}_1}^{\varphi,r}$.

Now suppose $\mathcal{M} \in \operatorname{Mod}_{S_1}^{\varphi,r}$ and set $\widetilde{\mathcal{M}} = T_r(\mathcal{M})$. Define the map

$$\hat{\sigma}_{\mathcal{M}} : \varphi_{\widetilde{\mathcal{M}},r}(\operatorname{Fil}^r \widetilde{\mathcal{M}}) \to \mathcal{M}$$

by

$$\hat{\sigma}_{\mathcal{M}}(\varphi_{\widetilde{\mathcal{M}},r}(z)) = \varphi_{\mathcal{M},r}(\hat{z})$$

where $\hat{z}$ is any lift of z in $\operatorname{Fil}^r \mathcal{M}$ in the sense that $\sigma_{\mathcal{M}}(\hat{z}) = z$. Since two lifts differ by an element of $\operatorname{Fil}^q S_1 \cdot \mathcal{M}$, and $\varphi_{\mathcal{M},r}\big|_{\operatorname{Fil}^q S_1 \cdot \mathcal{M}} = 0$, the map $\hat{\sigma}_{\mathfrak{M}}$ is well-defined and $k[u^q]/u^{eq}$-linear. We will show that the map

$$\operatorname{id} \otimes \hat{\sigma}_{\mathcal{M}} : T_r^{-1}(T_r(\mathcal{M})) \to \mathcal{M}$$

is an isomorphism of S_1-modules that respects Fil^r and commutes with φ_r. The image of $\hat{\sigma}_{\mathcal{M}}$ is $\varphi_{\mathcal{M},r}(\operatorname{Fil}^r \mathcal{M})$ which generates $\mathcal{M}$ as an S_1-module, and thus $\operatorname{id} \otimes \hat{\sigma}_{\mathcal{M}}$ is a surjective S_1-linear map of equal rank (see the first lines of the proofs of Propositions 4.2.10 and 4.2.12) S_1-modules. So by Lemma 4.1.15, $\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}}$ is an isomorphism of S_1-modules. It remains to prove that it respects Fil^r and commutes with φ_r. We claim that the following diagram is commutative:

$$\begin{array}{ccc} T_r^{-1}(T_r(\mathcal{M})) & \xrightarrow{\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle \widetilde{\sigma}_{\mathcal{M}}} \downarrow & & \downarrow {\scriptstyle \sigma_{\mathcal{M}}} \\ \widetilde{\mathcal{M}} & \xrightarrow{\mathrm{id}} & \widetilde{\mathcal{M}} \end{array}$$

To see this, suppose $x \in \mathrm{Fil}^r\, T_r^{-1}(T_r(\mathcal{M}))$ and write $x = \sum s_i \otimes \varphi_{\widetilde{\mathcal{M}},r}(y_i)$ with $s_i \in S_1$ and $y_i \in \mathrm{Fil}^r\, \widetilde{\mathcal{M}}$. Then,

$$\begin{aligned} \sigma_{\mathcal{M}}(\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}}(x)) &= \sigma_{\mathcal{M}}(\sum s_i \varphi_{\mathcal{M},r}(\hat{y}_i)) \\ &= \sum \sigma(s_i) \varphi_{\widetilde{\mathcal{M}},r}(y_i) \\ &= \widetilde{\sigma}_{\mathcal{M}}(x). \end{aligned}$$

Now, if $x \in \mathrm{Fil}^r\, T_r^{-1}(T_r(\mathcal{M}))$, we have $\widetilde{\sigma}_{\mathcal{M}}(x) \in \mathrm{Fil}^r\, T_r(\mathcal{M})$ and by the above diagram, $(\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}})(x)$ is a lift of $\widetilde{\sigma}_{\mathcal{M}}(x)$. Two lifts differ by an element of $\mathrm{Fil}^q\, S_1 \cdot \mathcal{M} \subseteq \mathrm{Fil}^r\, \mathcal{M}$ and by the definition of $\widetilde{\sigma}_{\mathcal{M}}(x) \in \mathrm{Fil}^r\, T_r(\mathcal{M})$, there is some lift of $\widetilde{\sigma}_{\mathcal{M}}(x)$ in $\mathrm{Fil}^r\, \mathcal{M}$. Therefore, it follows that $(\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}})(x)$ is in fact in $\mathrm{Fil}^r\, \mathcal{M}$. Moreover, in this case,

$$\begin{aligned} (\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}})(\varphi_{\mathcal{M},r}(x)) &= (\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}})(1 \otimes \varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(x))) \\ &= \hat{\sigma}_{\mathcal{M}}(\varphi_{\widetilde{\mathcal{M}},r}(\widetilde{\sigma}_{\mathcal{M}}(x))) \\ &= \varphi_{\mathcal{M},r}((\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}})(x)), \end{aligned}$$

since $(\mathrm{id} \otimes \hat{\sigma}_{\mathcal{M}})(x)$ is a lift of $\widetilde{\sigma}_{\mathcal{M}}(x)$. This completes the proof. □

We will now work to show that the functor Θ_r of 4.1.21 induces an equivalence between the categories $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and $\mathrm{Mod}_{S_1}^{\varphi,r}$. The first result shows that for objects $\mathcal{M}$ of $\mathrm{Mod}_{S_1}^{\varphi,r}$, the submodule $\mathrm{Fil}^r\, \mathcal{M}$ is, up to $\mathrm{Fil}^q\, S \cdot \mathcal{M}$, finitely generated by as many elements as the S_1-rank of $\mathcal{M}$.

Proposition 4.2.14.

1. *If* $\widetilde{\mathcal{M}}$ *has rank* d *in* $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$, *then there exists a basis* $\{e_1, ..., e_d\}$ *of* $\widetilde{\mathcal{M}}$ *and integers* $0 \leq r_1, ..., r_d \leq er$ *such that*

$$\mathrm{Fil}^r \widetilde{\mathcal{M}} = \bigoplus_{i=1}^{d} \widetilde{S}_1 \alpha_i,$$

where $\alpha_i = u^{r_i} e_i$.

2. *If* $\mathcal{M}$ *has rank* d *in* $\mathrm{Mod}_{S_1}^{\varphi,r}$, *then there exists a basis* $\{e_1, ..., e_d\}$ *of* $\mathcal{M}$ *and integers* $0 \leq r_1, ..., r_d \leq er$ *such that*

$$\mathrm{Fil}^r \mathcal{M} = \bigoplus_{i=1}^{d} S_1 \alpha_i + \mathrm{Fil}^q S_1 \cdot \mathcal{M},$$

where $\alpha_i = u^{r_i} e_i$.

Proof. The proof mimics [6] 2.1.2.5. By the equivalence of categories given in Theorem 4.2.13, it suffices to show (1). To be more clear, if $T_r(\mathcal{M})$ satisfies (1) for some $\mathcal{M} \in \mathrm{Mod}_{S_1}^{\varphi,r}$, choose $\hat{\alpha}_i \in \mathrm{Fil}^r \mathcal{M}$ such that $\sigma_{\mathcal{M}}(\hat{\alpha}_i) = \alpha_i \in T_r(\mathcal{M})$. Then for any $y \in \mathrm{Fil}^r \mathcal{M}$ there is an $x \in \sum S_1 \hat{\alpha}_i$ such that $\sigma_{\mathcal{M}}(y) = \sigma_{\mathcal{M}}(x)$. But then this implies that $y - x \in \mathrm{Fil}^q S_1 \cdot \mathcal{M}$ (as shown in 4.2.10).

To see (1), let $\{g_1, ..., g_d\}$ be elements of $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ whose images in $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^e \mathrm{Fil}^r \widetilde{\mathcal{M}}$ form a basis for the free $k[u]/u^e$-module, as guaranteed by Lemma 4.2.7. Write $g_i = u^{r_i} f_i$ where $f_i \in \widetilde{\mathcal{M}} \backslash u\widetilde{\mathcal{M}}$. Reorder if necessary so that $r_i \leq r_{i+1}$. Denote by $\overline{x}$ the image of an element x of $\widetilde{\mathcal{M}}$ in the k-vector space $\widetilde{\mathcal{M}}/u\widetilde{\mathcal{M}}$. By Nakayama's Lemma, $\{u^{r_1} f_1, ..., u^{r_d} f_d\}$ generates $\mathrm{Fil}^r \widetilde{\mathcal{M}}$. If the set $\{\overline{f}_1, \overline{f}_2\}$ is linearly dependent, then since $\overline{f}_2 \neq 0$, there is some $k_1 \in k^\times$ such that $k_1 f_1 + f_2 = u^\varepsilon f_2'$ with $\varepsilon \geq 1$ and $f_2' \in \widetilde{\mathcal{M}} \backslash u\widetilde{\mathcal{M}}$. Then since $r_1 \leq r_2$, we see that $u^{r_2} f_2$ is in the span of $\{u^{r_1} f_1, u^{r_2+\varepsilon} f_2'\}$ and so we deduce that $\{u^{r_1} f_1, u^{r_2+\varepsilon} f_2', ..., u^{r_d} f_d\}$ generates $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ over $\widetilde{S}_1$.

Now, it may be that $\{\overline{f}_1, \overline{f'}_2\}$ is still linearly dependent, in which case we repeat the argument. This will stop before $r_2 + \varepsilon > er$ because otherwise, since $u^{er} f_2'$ is in

$\mathrm{Fil}^r \widetilde{\mathcal{M}}$, we would have for some $s_i \in S$,

$$u^{er} f_2' = s_2 u^{r_2+\varepsilon} f_2' + \sum_{i \neq 2} s_i u^{r_i} f_i.$$

Therefore,

$$u^{er} f_2'(1 - s_2 u^{r_2+\varepsilon-er}) = \sum_{i \neq 2} s_i u^{r_i} f_i,$$

and if $r_2 + \varepsilon - er > 0$, we would have that $1 - s_2 u^{r_2+\varepsilon-er}$ is a unit in S_1 and this would give $d-1$ generators for $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ and hence $d-1$ generators for $\mathrm{Fil}^r \widetilde{\mathcal{M}}/u^e \mathrm{Fil}^r \widetilde{\mathcal{M}}$, which is a contradiction. Thus, without loss of generality we may assume that $\{\overline{f}_1, \overline{f}_2\}$ is linearly independent and $r_1, r_2 \leq er$. If $r_2 > r_3$, then repeat the above argument for $\{\overline{f}_1, \overline{f}_3\}$ and reorder. Eventually, we will have a generating family $\{u^{r_i} f_i\}$ for $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ with $r_i \leq r_{i+1}$ and $\{\overline{f}_1, \overline{f}_2\}$ linearly independent. Now, if $\{\overline{f}_1, \overline{f}_2, \overline{f}_3\}$ is linearly dependent, then there exist $k_1, k_2 \in k$, not both zero, such that

$$k_1 f_1 + k_2 f_2 + f_3 = u^{\varepsilon} f_3' \text{ with } \varepsilon \geq 1 \text{ and } f_3' \in \widetilde{\mathcal{M}} \backslash u\widetilde{\mathcal{M}}.$$

Again, $\{u^{r_1} f_1, u^{r_2} f_2, u^{r_3+\varepsilon} f_3, ..., u^{r_d} f_d\}$ generates $\mathrm{Fil}^r \widetilde{\mathcal{M}}$. We can continue in the above fashion until we can guarantee that we have a generating family $\{u^{r_i} f_i\}$ for $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ with $r_i \leq r_{i+1}$ and $\{\overline{f}_1, \overline{f}_2, \overline{f}_3\}$ linearly independent. Continuing in this way, we are assured a generating family $\{u^{r_i} f_i\}$ for $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ with $\{\overline{f}_i\}$ a basis for $\widetilde{\mathcal{M}}/u\widetilde{\mathcal{M}}$. Then by Nakayama's Lemma, $e_i = f_i$ forms the desired basis for $\widetilde{\mathcal{M}}$. □

The technical lemma and its corollary that follow are adapted from [6], immediately following 2.1.2.6.

Lemma 4.2.15. *Suppose that* $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ *is an injective morphism in* $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ *and identify* $\widetilde{\mathcal{M}}$ *with its image as a sub-object of* $\widetilde{\mathcal{N}}$. *Then*

$$\mathrm{Fil}^r \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}} \cap \mathrm{Fil}^r \widetilde{\mathcal{N}}.$$

Proof. By definition of morphisms in $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ we know that

$$\mathrm{Fil}^r \widetilde{\mathcal{M}} \subseteq \widetilde{\mathcal{M}} \cap \mathrm{Fil}^r \widetilde{\mathcal{N}}.$$

Suppose that $x \in \widetilde{\mathcal{M}} \cap \mathrm{Fil}^r \widetilde{\mathcal{N}}$. Let n be the smallest nonnegative integer such that $u^{en}x \in \mathrm{Fil}^r \widetilde{\mathcal{M}}$, noting that n is at most r, and suppose that $n > 0$. Thinking of x as an element of $\mathrm{Fil}^r \widetilde{\mathcal{N}}$ and again using the definition of morphisms in $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$, we have $\varphi_{\widetilde{\mathcal{M}},r}(u^{en}x) = u^{enq}\varphi_{\widetilde{\mathcal{N}},r}(x) = 0$ since $u^{eq} = 0$ in $\widetilde{S}_1$. By injectivity, this must mean that $\varphi_{\widetilde{\mathcal{M}},r}(u^{en}x) = 0$ in $\widetilde{\mathcal{M}}$. By the isomorphism given in (4.2.3), this means that $u^{en}x \in u^e \mathrm{Fil}^r \widetilde{\mathcal{M}}$. But then for some $y \in \mathrm{Fil}^r \widetilde{\mathcal{M}}$ we have $u^e \cdot (u^{e(n-1)}x - y) = 0$, and so by Lemma 4.2.6, we know that $u^{e(n-1)}x - y \in u^{e(q-1)}\widetilde{\mathcal{M}}$. Now, since $r \leq q-1$, we know that $u^{e(q-1)}\widetilde{\mathcal{M}} \subseteq \mathrm{Fil}^r \widetilde{\mathcal{M}}$, hence $u^{e(n-1)}x \in \mathrm{Fil}^r \widetilde{\mathcal{M}}$, a contradiction that $n > 0$ was the smallest such integer. Therefore, we conclude that $x \in \mathrm{Fil}^r \widetilde{\mathcal{M}}$. □

Corollary 4.2.16. *Suppose that $\mathcal{M} \to \mathcal{N}$ is an injective morphism in $\mathrm{Mod}_{S_1}^{\varphi,r}$ and identify $\mathcal{M}$ with its image as a subobject of $\mathcal{N}$. Then*

$$\mathrm{Fil}^r \mathcal{M} = \mathcal{M} \cap \mathrm{Fil}^r \mathcal{N}.$$

Proof. This just uses the previous lemma and the definitions of 4.2.9. Suppose that $x \in \mathcal{M} \cap \mathrm{Fil}^r \mathcal{N}$. Then

$$\sigma_{\mathcal{N}}(x) \in \widetilde{\mathcal{M}} \cap \mathrm{Fil}^r \widetilde{\mathcal{N}} = \mathrm{Fil}^r \widetilde{\mathcal{M}}.$$

Hence, $\sigma_{\mathcal{M}}(x) = \sigma_{\mathcal{M}}(y)$ for some $y \in \mathrm{Fil}^r \mathcal{M}$. But then x and y differ by an element of $\mathrm{Fil}^q S \cdot \mathcal{M} \subset \mathrm{Fil}^r \mathcal{M}$, so $x \in \mathrm{Fil}^r \mathcal{M}$. □

Corollary 4.2.17. *Suppose that $h : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ is a morphism of objects in $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ such that h induces an isomorphism of $\widetilde{S}_1$-modules, $\widetilde{\mathcal{M}} \cong \widetilde{\mathcal{M}}'$. Then h gives an isomorphism $\widetilde{\mathcal{M}} \cong \widetilde{\mathcal{M}}'$ as objects in $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$. Furthermore, the analogous result holds in $\mathrm{Mod}_{S_1}^{\varphi,r}$.*

Proof. It suffices to show that the map $h : \mathrm{Fil}^r \widetilde{\mathcal{M}} \to \mathrm{Fil}^r \widetilde{\mathcal{M}}'$ is surjective. Since h is an isomorphism of $\widetilde{S}_1$-modules, it is injective, so identifying $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ with its image in

$\widetilde{\mathcal{M}}'$, and applying Lemma 4.2.15 gives

$$\mathrm{Fil}^r\,\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}} \cap \mathrm{Fil}^r\,\widetilde{\mathcal{M}}' = \widetilde{\mathcal{M}}' \cap \mathrm{Fil}^r\,\widetilde{\mathcal{M}}' = \mathrm{Fil}^r\,\widetilde{\mathcal{M}}'.$$

So h is surjective on the Fil^r and this means that $h : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ is an isomorphism of objects in $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$. The analogous result in $\mathrm{Mod}_{S_1}^{\varphi,r}$ just follows from Corollary 4.2.16 using the same argument. □

Recall the functor $\Theta_r : \text{Mod-FI}_{\mathfrak{S}}^{\varphi,r} \to \text{Mod-FI}_{S}^{\varphi,r}$ defined in 4.1.20. When Θ_r is restricted to objects of $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$, it clearly takes values in $\mathrm{Mod}_{S_1}^{\varphi,r}$. Now denote by $\widetilde{\Theta}_r : \mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r} \to \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ the composite $T_r \circ \Theta_r$.

To aid with notation, given a ring R and $v_1, ..., v_d$ in an R-module N, denote by

$$\underline{v} := \text{ column vector of } v_1, ..., v_d \text{ in } N^d. \tag{4.2.9}$$

Also, for a map f on N, denote by $f(\underline{v})$ the vector of $f(v_1), ..., f(v_d)$. Finally denote by $M_d(R)$ the set of $d \times d$ matrices with coefficients in R.

Proposition 4.2.18. *The functors Θ_r and $\widetilde{\Theta}_r$ induce equivalences of categories between $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and $\mathrm{Mod}_{S_1}^{\varphi,r}$ and between $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$, respectively.*

Proof. We again adapt a proof of Breuil's, [5] 4.1.1. Since $T_r : \mathrm{Mod}_{S_1}^{\varphi,r} \to \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ is an equivalence of categories, it suffices to check that $\widetilde{\Theta}_r$ is fully faithful and essentially surjective. We begin by showing the functor is fully faithful.

Let $\mathfrak{M}, \mathfrak{N} \in \mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and write $\mathcal{M} = \Theta_r(\mathfrak{M})$ and $\mathcal{N} = \Theta_r(\mathfrak{N})$, and likewise write $\widetilde{\mathcal{M}} = \widetilde{\Theta}_r(\mathfrak{M})$ and $\widetilde{\mathcal{N}} = \widetilde{\Theta}_r(\mathfrak{N})$. Suppose that $\widetilde{g} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ is a morphism in $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$. We need to show that there exists a unique morphism $g : \mathfrak{M} \to \mathfrak{N}$ in $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ such that $\widetilde{\Theta}_r(g) = \widetilde{g}$.

Notice that

$$\begin{aligned} \widetilde{\mathcal{M}} &= \widetilde{S}_1 \underset{\sigma\circ\varphi,\mathfrak{S}_1}{\otimes} \mathfrak{M} \\ &= \widetilde{S}_1 \underset{\sigma\circ\varphi,\mathfrak{S}_1}{\otimes} \mathfrak{M}/u^e\mathfrak{M}, \end{aligned} \tag{4.2.10}$$

since $\sigma(\varphi(u^e)) = 0$ in $\widetilde{S}_1$. The same holds for $\mathfrak{N}$ and $\widetilde{\mathcal{N}}$.

First, we claim that that the $k[u^q]/u^{eq}$-modules $\varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}})$ and

$$\widetilde{c}^r \otimes \mathfrak{M} := \{x \in \widetilde{\mathcal{M}} \mid x = \widetilde{c}^r \otimes 1 \otimes y \text{ for some } y \in \mathfrak{M}\} \tag{4.2.11}$$

are equal in $\widetilde{\mathcal{M}} = \widetilde{S}_1 \otimes_\sigma S_1 \otimes_{\varphi,\mathfrak{S}_1} \mathfrak{M}$, and the same will be true for $\widetilde{N}$. For the containment $\widetilde{c}^r \otimes \mathfrak{M} \subseteq \varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}})$, note that for any $m \in \mathfrak{M}$, as we saw in (4.1.5), there is some $x \in \mathrm{Fil}^r \mathcal{M}$ such that $\varphi_{\mathcal{M},r}(x) = \widetilde{c}^r \otimes m$. Then $\varphi_{\widetilde{\mathcal{M}},r}(\sigma_{\mathfrak{M}}(x)) = \widetilde{c}^r \otimes m$. Now let $\widetilde{x} \in \mathrm{Fil}^r \widetilde{\mathcal{M}}$. By Definition 4.2.9, that means there is some $x \in \mathrm{Fil}^r \mathcal{M}$ with $\sigma_{\mathcal{M}}(x) = \widetilde{x}$. Recall that the definition of $\mathrm{Fil}^r \mathcal{M}$ means that $\mathrm{id} \otimes \varphi_{\mathfrak{M}}(x) \in \mathrm{Fil}^r S \otimes \mathfrak{M}$. So write

$$(\mathrm{id} \otimes \varphi_{\mathfrak{M}})(x) = \sum_{i=1}^{n} t_i \otimes m_i$$

with $t_i \in \mathrm{Fil}^r S_1$ and $m_i \in \mathfrak{M}$. Let $\widetilde{t}_i = \sigma(t_i) \in \mathrm{Fil}^r \widetilde{S}_1$. Then $\widetilde{t}_i = u^{er} p_i(u)$ for some $p_i(u) \in k[u]/u^{eq}$. Using the fact that $\sigma \circ \varphi_r = \varphi_r \circ \sigma$ and $\iota \circ \varphi = \varphi \circ \iota$, and writing $\widetilde{\mathcal{M}}$ as $\widetilde{S}_1 \otimes_\sigma S_1 \otimes_{\varphi,\mathfrak{S}_1} \mathfrak{M}$, we get that

$$\begin{aligned}
\varphi_{\widetilde{\mathcal{M}},r}(\widetilde{x}) &= \sigma_{\mathcal{M}}(\varphi_{\mathcal{M},r}(x)) \\
&= \sum_{i=1}^{n} 1 \otimes (\varphi_r(t_i) \otimes m_i) \\
&= \sum_{i=1}^{n} \sigma(\varphi_r(t_i)) \otimes (1 \otimes m_i) \\
&= \sum_{i=1}^{n} \widetilde{c}^r \varphi(p_i(u)) \otimes (1 \otimes m_i) \\
&= \sum_{i=1}^{n} \widetilde{c}^r \otimes \iota(\varphi(p_i(u))) \otimes m_i \\
&= \sum_{i=1}^{n} \widetilde{c}^r \otimes (1 \otimes \iota(p_i(u)) m_i).
\end{aligned}$$

Therefore, we see that $\varphi_{\widetilde{\mathcal{M}},r}(\widetilde{x}) \in \widetilde{c}^r \otimes \mathfrak{M}$. So $\varphi_{\widetilde{\mathcal{M}},r}(\mathrm{Fil}^r \widetilde{\mathcal{M}}) = \widetilde{c}^r \otimes \mathfrak{M}$. Furthermore, since $\widetilde{g}$ commutes with the φ_r and $\widetilde{g}(\mathrm{Fil}^r \widetilde{\mathcal{M}}) \subseteq \mathrm{Fil}^r \widetilde{\mathcal{N}}$, we see that $\widetilde{g}$ sends $\widetilde{c}^r \otimes \mathfrak{M}$ into $\widetilde{c}^r \otimes \mathfrak{N}$. Then by $\widetilde{S}_1$-linearity, $\widetilde{g}$ sends the set $1 \otimes \mathfrak{M}$ into $1 \otimes \mathfrak{N}$.

Let $\{e_1, ..., e_d\}$ be a basis for $\mathfrak{M}$. Write $\widetilde{g}(\widetilde{c}^r \otimes e_i) = \widetilde{c}^r \otimes f_i$ for some $f_i \in \mathfrak{N}$ and note by (4.2.10) that there is a choice in the f_i with different choices differing by $u^e\mathfrak{N}$. At this point, we are able to construct an $\mathfrak{S}_1$-linear map $g : \mathfrak{M} \to \mathfrak{N}$ given by $g(e_i) = f_i$ and we would know that $\mathrm{id} \otimes g = \widetilde{g}$. We will show that there is a unique choice of $f_i \in \mathfrak{N}$ such that $g \circ \varphi_{\mathfrak{M}} = \varphi_{\mathfrak{N}} \circ g$. Start by fixing some choice of f_i.

Let A denote the matrix of $\varphi_{\mathfrak{M}}$ with respect to the basis $\{e_1, ..., e_d\}$. That is, $A = (a_{ij})$ where $\varphi_{\mathfrak{M}}(e_j) = \sum_i a_{ij}e_i$ for $a_{ij} \in \mathfrak{S}_1 = k[\![u]\!]$. Because $\mathfrak{M}$ has E-height $h \leq r$ (see Definition 4.1.7), we know that for $1 \leq k \leq d$, we get $u^{er}e_k = \sum_j b_{jk}\varphi_{\mathfrak{M}}(e_j)$ for some $b_{jk} \in \mathfrak{S}$, and so the matrix $B = (b_{jk}) \in M_d(k[\![u]\!])$ satisfies $u^{er}I = AB$. Then $A^{-1} \in M_d(k(\!(u)\!))$, and $u^{er}A^{-1} \in M_d(k[\![u]\!])$. We claim that there exist $y_1, ..., y_d \in \mathfrak{N}$ such that

$$u^{er}A^{-1}\varphi_{\mathfrak{N}}(\underline{f}) = u^{er} \cdot \underline{f + u^e y} \tag{4.2.12}$$

To see this, let b_{ij} be the (i,j)-th entry of $B = u^{er}A^{-1}$ so that $u^{er}e_i = \sum_{j=1}^d b_{ij}\varphi_{\mathfrak{M}}(e_j)$. Using (4.1.5), this means that $\varphi_{\mathcal{M},r}(\sum_{j=1}^d b_{ij} \otimes e_j) = \overline{c}^r \otimes e_i$ in $\mathcal{M}$. Applying $\sigma_{\mathcal{M}}$ followed by $\widetilde{g}$ then gives that

$$\begin{aligned}
\widetilde{c}^r \otimes f_i &= \widetilde{g}(\varphi_{\widetilde{\mathcal{M}},r}(1 \otimes (\sum_{j=1}^d b_{ij} \otimes e_j))) \\
&= \varphi_{\widetilde{\mathcal{N}},r}(\widetilde{g}(1 \otimes (\sum_{j=1}^d b_{ij} \otimes e_j))) \\
&= \varphi_{\widetilde{\mathcal{N}},r}(1 \otimes (\sum_{j=1}^d b_{ij} \otimes f_j)) \\
&= \sigma_{\mathcal{N}}(\varphi_{\mathcal{N},r}(\sum_{j=1}^d b_{ij} \otimes f_j)).
\end{aligned}$$

Since $\sum_{j=1}^d b_{ij} \otimes f_j \in \mathrm{Fil}^r \mathcal{N}$, we know that $(\mathrm{id} \otimes \varphi_{\mathfrak{N}})(\sum_{j=1}^d b_{ij} \otimes f_j) \in \mathrm{Fil}^r S_1 \otimes_{\mathfrak{S}_1} \mathfrak{N}$. Write $(\mathrm{id} \otimes \varphi_{\mathfrak{N}})(\sum_{j=1}^d m_{ij} \otimes f_j) = \sum_{k=1}^l s_k \otimes n_k$ with $s_k \in \mathrm{Fil}^r S_1$. However, we can also say that $(\mathrm{id} \otimes \varphi_{\mathfrak{N}})(\sum_{j=1}^d b_{ij} \otimes f_j) \in (k[u]/u^{eq}) \otimes_{\mathfrak{S}_1} \mathfrak{N}$ as $k[u]/u^{eq}$ is the image of

$\mathfrak{S}_1$ in S_1. By Lemma 4.1.14,

$$(\mathrm{Fil}^r S_1 \underset{\mathfrak{S}_1}{\otimes} \mathfrak{M}) \cap ((k[u]/u^{eq}) \underset{\mathfrak{S}_1}{\otimes} \mathfrak{M}) = (\mathrm{Fil}^r S_1 \cap (k[u]/u^{eq})) \underset{\mathfrak{S}_1}{\otimes} \mathfrak{M},$$

so we can assume $s_k \in \mathrm{Fil}^r S_1 \cap (k[u]/u^{eq})$. By Proposition 4.1.3, we know

$$\mathrm{Fil}^r S \cap \mathfrak{S} = E(u)^r \mathfrak{S},$$

and since $\mathrm{Fil}^r S_1$ is the image of $\mathrm{Fil}^r S$ in S_1 and $k[u]/u^{eq}$ is the image of $\mathfrak{S}$ in S_1, we can write $s_k = u^{er} t_k$ for some $t_k \in k[u]/u^{eq}$. Thus, we have

$$\sum_{k=1}^{l} u^{er} t_k n_k = \sum_{j=1}^{d} b_{ij} \varphi_{\mathfrak{N}}(f_j).$$

Using that $\varphi_r(s_k) = \overline{c}^r \varphi(t_k)$, it is a simple calculation to see that

$$\begin{aligned} \sigma_{\mathcal{N}}(\varphi_{\mathcal{N},r}(\sum_{j=1}^{d} b_{ij} \otimes f_j)) &= \sum_{k=1}^{l} \widetilde{c^r} \otimes (\varphi(t_k) \underset{\varphi}{\otimes} n_k) \\ &= \sum_{k=1}^{l} \widetilde{c^r} \otimes t_k n_k. \end{aligned}$$

So $\widetilde{c^r} \otimes f_i = \widetilde{c^r} \otimes (\sum_{k=1}^{l} t_k n_k)$ and thus by (4.2.10), there exists some $y_i \in \mathfrak{N}$ such that $\sum_{k=1}^{l} t_k n_k = f_i + u^e y_i$. Multiplying both sides by u^{er} gives the desired result (4.2.12) since $u^{er} \sum_{k=1}^{l} t_k n_k = \sum_{j=1}^{d} b_{ij} \varphi_{\mathfrak{N}}(f_j)$.

We then have, in $\mathfrak{N}^d$, that

$$\varphi_{\mathfrak{N}}(\underline{f}) = A\underline{f + u^e y} \tag{4.2.13}$$

which gives (4.2.12). We now seek $\beta_1, ..., \beta_d \in \mathfrak{N}$ such that by setting $g(e_i) = f_i + u^e \beta_i$, we get an $\mathfrak{S}_1$-linear map with $g \circ \varphi_{\mathfrak{M}} = \varphi_{\mathfrak{N}} \circ g$ and $\Theta_r(g) = \widetilde{g}$. Written another way, we want

$$Ag(\underline{e}) = \varphi_{\mathfrak{N}}(g(\underline{e})),$$

where the multiplication on the left-hand side is $\mathfrak{S}_1$-linear. Expanding $g(e_i)$, we want

$$A\underline{g} + u^e A\underline{\beta} = \varphi_{\mathfrak{N}}(\underline{f}) + u^{eq} \varphi_{\mathfrak{N}}(\underline{\beta})$$

Using (4.2.13), we see that the β_i must satisfy

$$\underline{\beta} = \underline{y} + u^{e(q-1)}A^{-1}\varphi_{\mathfrak{N}}(\underline{\beta}). \tag{4.2.14}$$

Since $r < q-1$, and we emphasize that this inequality is strict, the coefficients of $u^{e(q-1)}A^{-1} = u^{e(q-1-r)}B$ are in $uk[\![u]\!]$ and so iterating (4.2.14) yields

$$\underline{\beta} = \sum_{k=0}^{\infty} u^{e(q-1)}A^{-1}\varphi(u^{e(q-1)}A^{-1})\cdots\varphi^{k-1}(u^{e(q-1)}A^{-1})\varphi_{\mathfrak{N}}^{k}(\underline{y}).$$

This solution for $\beta_1, ..., \beta_d$ is unique after the choice of f_i above. Thus, if we made another choice $f_i' = f_i + u^e c_i$ and got the solution $f_i' + u^e\beta_i'$, then

$$f_i' + u^e\beta_i' = f_i + u^e(c_i + \beta_i') = f_i + u^e\beta_i,$$

so it follows that the morphism g is unique, and by construction, $\Theta_r(g) = \widetilde{g}$. We therefore conclude that Θ_r is fully faithful.

We will now show that $\widetilde{\Theta}_r$ is essentially surjective. Let $\widetilde{\mathcal{M}} \in \mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ be free of rank d. By Lemma 4.2.8, there exist $y_1, ..., y_d \in \mathrm{Fil}^r\widetilde{\mathcal{M}}$ such that

$$\{\widetilde{c}^{-r} \otimes \varphi_{\widetilde{\mathcal{M}},r}(y_1), ..., \widetilde{c}^{-r} \otimes \varphi_{\widetilde{\mathcal{M}},r}(y_d)\}$$

forms a basis for $\widetilde{\mathcal{M}}$. We claim that $y_1, ..., y_d$ must span $\mathrm{Fil}^r\widetilde{\mathcal{M}}$. By Lemma 4.2.8, we know that the images of the y_i in $\mathrm{Fil}^r\widetilde{\mathcal{M}}/u^e\widetilde{\mathcal{M}}$ form a basis for the $k[u]/u^e$-module $\mathrm{Fil}^r\widetilde{\mathcal{M}}/u^e\widetilde{\mathcal{M}}$, and so by Nakayama's lemma, the y_i must span $\mathrm{Fil}^r\widetilde{\mathcal{M}}$. For each i, write $\varphi_{\widetilde{\mathcal{M}},r}(y_i) = \widetilde{c}^r e_i$ and let $\overline{B} \in M_d(k[u]/u^{eq})$ give the matrix of (y_i) in the basis of $\{e_1, ..., e_d\}$. Since $u^{er}e_i \in \mathrm{Fil}^r\widetilde{\mathcal{M}}$ for all i, it follows that there is some matrix $\overline{D} \in M_d(k[u]/u^{eq})$ such that $\overline{D}\cdot\overline{B} = u^{er}I$. Let $B, D \in M_d(\mathfrak{S}_1)$ be any lifts of $\overline{B}$ and $\overline{D}$, and let $C \in M_d(\mathfrak{S}_1)$ be such that $DB = u^{er}I + u^{eq}C = u^{er}(I + u^{e(q-r)}C)$. Since $r < q$, we get that $I + u^{e(q-r)}C$ is invertible in $\mathfrak{S}_1$. Let $A = (I + u^{e(q-r)}C)^{-1}D$ so that we have $AB = u^{er}I$.

Let $\mathfrak{M}$ be the $\mathfrak{S}_1$-module $\mathfrak{M} = \bigoplus_{i=1}^{d}\mathfrak{S}_1 f_i$, and define a Frobenius on $\mathfrak{M}$ by $\varphi_{\mathfrak{M}}(\underline{f}) = A\underline{f}$ and extending φ-semilinearly. Put in terms of the linearization

$$\mathrm{id}\otimes\varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M},$$

the matrix A is the matrix for the $\mathfrak{S}_1$ linear map $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ under the basis $\{1 \otimes f_i\}$ of $\varphi^*\mathfrak{M}$ and the basis $\{f_i\}$ of $\mathfrak{M}$. We first need to check that $\mathfrak{M}$ is in $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$. Clearly, it is a free $\mathfrak{S}_1$-module of rank d, so we only need to check that u^{er} kills the cokernel of the linearization $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$. But we know that $AB = u^{er} I$, and so $u^{er}\mathfrak{M}$ is contained in the image of $\mathrm{id} \otimes \varphi_{\mathfrak{M}}$ and the cokernel is killed by u^{er} as desired.

Let $\widetilde{\mathcal{M}}' := \widetilde{\Theta}_r(\mathfrak{M})$. Our goal is to show that $\widetilde{\mathcal{M}}' \cong \widetilde{\mathcal{M}}$ as objects of $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$. Consider the $\widetilde{S}_1$-module morphism $h : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ induced by $h(e_i) = 1 \otimes f_i$. This takes a basis to a basis, so clearly this is an isomorphism of $\widetilde{S}_1$-modules. By Corollary 4.2.17, we just need to show that $h(\mathrm{Fil}^r \widetilde{\mathcal{M}}) = \mathrm{Fil}^r \widetilde{\mathcal{M}}'$ and h commutes with the φ_r-maps.

We will first show that $h(\mathrm{Fil}^r \widetilde{\mathcal{M}}) \subset \mathrm{Fil}^r \widetilde{\mathcal{M}}'$ and h commutes with the φ_r-maps. As above, there exists an $\widetilde{S}_1$-spanning set $\{y_1, ..., y_d\}$ of $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ such that we have a basis of $\widetilde{\mathcal{M}}$ given by $e_i = \widetilde{c}^{-r}\varphi_{\widetilde{\mathcal{M}},r}(y_i)$, and $\overline{B}$ is the matrix for $\{y_1, ..., y_d\}$ under this basis. Since h is $\widetilde{S}_1$-linear, we just need to show that $h(y_i) \in \mathrm{Fil}^r \widetilde{\mathcal{M}}'$ for each i. By the definition of $\mathrm{Fil}^r \widetilde{\mathcal{M}}'$ (4.2.9), this is true if and only if $h(y_i)$ is in the image of $\sigma_{\mathcal{M}}$ on $\mathrm{Fil}^r \Theta_r(\mathfrak{M})$. Note that $h(\underline{y}) = \sigma_{\mathcal{M}}(B\underline{1 \otimes f})$ and on $\Theta_r(\mathfrak{M}) = S \otimes_{\varphi} \mathfrak{M}$,

$$\begin{aligned}\mathrm{id} \otimes \varphi_{\mathfrak{M}}(B\underline{1 \otimes f}) &= AB\underline{1 \otimes f} \\ &= \underline{u^{er} \otimes f}.\end{aligned}$$

This result is in $\mathrm{Fil}^r S_1 \otimes_{\mathfrak{S}_1} \mathfrak{M}$, which means by the definition of $\mathrm{Fil}^r \Theta_r(\mathfrak{M})$, we have $B\underline{1 \otimes f} \in (\mathrm{Fil}^r \Theta_r(\mathfrak{M}))^d$. So $h(y_i) \in \mathrm{Fil}^r \widetilde{\mathcal{M}}'$ for every i.

To show that h commutes with the φ_r-maps, first consider the y_i. Using the previous calculation,

$$\varphi_{\widetilde{\mathcal{M}}',r}(h(y_i)) = \sigma_{\mathcal{M}}((\varphi_r \otimes \mathrm{id})(u^{er} \otimes f_i)) = \widetilde{c}^r(1 \otimes f_i).$$

On the other hand, we defined y_i such that $\varphi_{\widetilde{\mathcal{M}},r}(y_i) = \widetilde{c}^r e_i$, so

$$\begin{aligned} h(\varphi_{\widetilde{\mathcal{M}},r}(y_i)) &= h(\widetilde{c}^r e_i) \\ &= \widetilde{c}^r h(e_i) \\ &= \widetilde{c}^r (1 \otimes f_i) \\ &= \varphi_{\mathcal{M}',r}(h(y_i)). \end{aligned}$$

Now, by the S_1-linearity of h and the φ-semilinearity of the φ_r, we know that h commutes with the φ_r on all of $\mathrm{Fil}^r \widetilde{\mathcal{M}}$, and this completes the proof. □

To end this section, we will now show that $\text{Mod-FI}_S^{\varphi,r}$ is stable under the quotient $\mathcal{M}/\varpi^n\mathcal{M}$. In particular, this will imply that $\text{Mod-FI}_S^{\varphi,r}$ is a full sub-category of $\mathrm{Mod}_{S_\infty}^{\varphi,r}$. The technical lemmas that follow are adapted from [6] 2.1.1.3 and [4] 2.3.1.2.

There is an equivalent condition to the property that the image of $\varphi_{\mathcal{M},r}$ generates $\mathcal{M}$, which we now consider. There is a map $S \to W(k)_F$ that extends the map $\mathfrak{S} \to \mathfrak{S}/u\mathfrak{S} = W(k)_F$. Such a map exists since the image of $E(u)$ is in $\varpi W(k)_F$, which has $\mathcal{O}_F$-divided powers. This gives $W(k)_F$ the structure of an S-module, and we have the following result:

Lemma 4.2.19. *Suppose that $\mathcal{M}$ is an object of $'\mathrm{Mod}_S^{\varphi,r}$ and suppose further that $\mathcal{M}$ is a finitely generated S-module. Then the condition that the map*

$$\Phi_{\mathcal{M},r} : \mathrm{Fil}^r \mathcal{M} \to W(k)_F \otimes_S \mathcal{M}, \quad x \mapsto 1 \otimes \varphi_{\mathcal{M},r}(x)$$

is surjective is equivalent to the condition that the image of $\varphi_{\mathcal{M},r}$ on $\mathrm{Fil}^r \mathcal{M}$ generates $\mathcal{M}$ as an S-module.

Proof. Let $W := W(k)_F$. Denote the kernel of the map $\mathfrak{S} \to W$ by I. Then I is generated by u and $\delta_m(u^e)$ for $m \geq 1$. We have the following commutative diagram:

$$\begin{array}{ccccc} & & S \underset{\varphi,S}{\otimes} \operatorname{Fil}^r \mathcal{M} & \xrightarrow{\operatorname{id} \otimes \varphi_{\mathcal{M},r}} & \mathcal{M} \\ & & \downarrow & & \downarrow \\ \operatorname{Fil}^r \mathcal{M} & \overset{\sigma}{\twoheadrightarrow} & W \underset{\varphi,W}{\otimes} \operatorname{Fil}^r \mathcal{M} & \xrightarrow{\operatorname{id} \otimes \Phi_{\mathcal{M},r}} & W \underset{S}{\otimes} \mathcal{M} \end{array}$$

where the map σ is given by $m \mapsto 1 \otimes m$ and is surjective since φ restricted to W is an automorphism (it is the unique lift of q-power Frobenius on k), and the composite of the bottom row is $\Phi_{\mathcal{M},r}$. If $\operatorname{id} \otimes \varphi_{\mathcal{M},r}$ is surjective, then clearly $\operatorname{id} \otimes \Phi_{\mathcal{M},r}$ is surjective and so $\Phi_{\mathcal{M},r}$ is surjective. Suppose instead that $\Phi_{\mathcal{M},r}$, and hence $\operatorname{id} \otimes \Phi_{\mathcal{M},r}$, is surjective. We claim that $I\mathcal{M}$ is the kernel of $\mathcal{M} \to W \otimes_S \mathcal{M}$, and this can be seen by tensoring the exact sequence of S-modules

$$0 \to I \to S \to W \to 0$$

with $\mathcal{M}$, noting that the image of $I \otimes_S \mathcal{M} \to S \otimes_S \mathcal{M}$ is IM. Let $\mathcal{N}$ be the image of $\operatorname{id} \otimes \varphi_{\mathcal{M},r}$. Then using the diagram we see that $\mathcal{M} = \mathcal{N} + I\mathcal{M}$. Therefore, it suffices to show that I is contained in the Jacobson radical of S, since then Nakayama's Lemma (as $\mathcal{M}$ is finitely generated) would show that $\mathcal{N} = \mathcal{M}$. We claim that I consists of topologically nilpotent elements. We know that I is generated by u and $\delta_m(u^e)$ for $m \geq 1$. As seen in the proof of Proposition 4.1.1, we know that $u^{eq} \in \varpi S$. Moreover, using Proposition 3.1.3 (3), we can see that $\delta_m(u^e)$ raised to a large enough power (q would suffice) becomes divisible by ϖ. This shows that any element of I is topologically nilpotent, and so any element of $1+I$ is a unit is S. Thus, I is contained in the Jacobson radical of S, and this completes the proof. $\square$

Lemma 4.2.20. *If* $\mathcal{M} \in \text{Mod-FI}_S^{\varphi,r}$*, then*

$$\mathcal{M}[\varpi^n] \in \text{Mod-FI}_S^{\varphi,r} \quad \textit{and} \quad \mathcal{M}/\varpi^n\mathcal{M} \in \text{Mod-FI}_S^{\varphi,r}$$

for any n*, where*

$$\operatorname{Fil}^r \mathcal{M}[\varpi^n] := \mathcal{M}[\varpi^n] \cap \operatorname{Fil}^r \mathcal{M} \quad \textit{and} \quad \varphi_{\mathcal{M}[\varpi^n],r} := \varphi_{\mathcal{M},r}\big|_{\mathcal{M}[\varpi^n]}$$

and

$$\mathrm{Fil}^r(\mathcal{M}/\varpi^n\mathcal{M}) \coloneqq \mathrm{Fil}^r\mathcal{M}/\varpi^n\,\mathrm{Fil}^r\mathcal{M} \quad \textit{and} \quad \overline{\varphi}_{\mathcal{M},r} \coloneqq \varphi_{\mathcal{M},r} \textit{ reduced modulo } \varpi^n.$$

Proof. We adapt the proof in [4] Lemma 2.3.1.2. It is easy to check that $\varpi^n\mathcal{M}$ is an object of Mod-$\mathrm{FI}_S^{\varphi,r}$, with filtration defined by $\mathrm{Fil}^r(\varpi^n\mathcal{M}) = \varpi^n\,\mathrm{Fil}^r\mathcal{M}$ and the restriction of $\varphi_{\mathcal{M},r}$ to $\varpi^n\,\mathrm{Fil}^r\mathcal{M}$. To the S-module $\overline{\mathcal{M}} \coloneqq \mathcal{M}/\varpi^n\mathcal{M}$, we want to define the submodule

$$\mathrm{Fil}^r\overline{\mathcal{M}} \coloneqq \mathrm{Fil}^r\mathcal{M}/\varpi^n\,\mathrm{Fil}^r\mathcal{M}$$

and to define $\overline{\varphi}_{\mathcal{M},r}$ to be the reduction of $\varphi_{\mathcal{M},r}$ modulo ϖ^n. However, for $\mathrm{Fil}^r\overline{\mathcal{M}}$ to actually be a submodule of $\overline{\mathcal{M}}$ and for $\overline{\varphi}_{\mathcal{M},r}$ to be well-defined, we need to show that

$$\varpi^n\mathcal{M} \cap \mathrm{Fil}^r\mathcal{M} = \varpi^n\,\mathrm{Fil}^r\mathcal{M}.$$

Furthermore, the S-module $\mathcal{M}[\varpi^n]$ clearly has the submodule

$$\mathrm{Fil}^r\mathcal{M}[\varpi^n] \coloneqq \mathcal{M}[\varpi^n] \cap \mathrm{Fil}^r\mathcal{M}$$

and $\varphi_{\mathcal{M},r}$ takes values in $\mathcal{M}[\varpi^n]$ when restricted to $\mathrm{Fil}^r\mathcal{M}[\varpi^n]$. Then $\mathcal{M}[\varpi]$ is an object of Mod-$\mathrm{FI}_S^{\varphi,r}$ if the image of $\varphi_{\mathcal{M},r}$ on $\mathrm{Fil}^r\mathcal{M}[\varpi^n]$ generates $\mathcal{M}[\varpi^n]$ as an S-module. for this we will use Lemma 4.2.19.

Induct on the smallest power of ϖ that kills $\mathcal{M}$. To be exact, for $m \geq 2$ let $P(m)$ be the statement that for any $\mathcal{M} \in$ Mod-$\mathrm{FI}_S^{\varphi,r}$ such that $\varpi^m\mathcal{M} = 0$ and $\varpi^{m-1}\mathcal{M} \neq 0$, then

$$\varpi^i\mathcal{M} \cap \mathrm{Fil}^r\mathcal{M} = \varpi^i\,\mathrm{Fil}^r\mathcal{M}$$

for $0 \leq i \leq m-1$ and $\mathcal{M}[\varpi^n]$ is in Mod-$\mathrm{FI}_S^{\varphi,r}$ for all n. Note that if ϖ^m kills $\mathcal{M}$, then there is no need to check the claim for $n \geq m$. First suppose that for $\mathcal{M} \in$ Mod-$\mathrm{FI}_S^{\varphi,r}$, we have $\varpi^2\mathcal{M} = 0$ and suppose $\varpi\mathcal{M} \neq 0$. We will first show that $\mathcal{M}[\varpi]$ is an object of $\mathrm{Mod}_{S_1}^{\varphi,r}$. Consider the commutative diagram with exact rows given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^r \mathcal{M}[\varpi] & \longrightarrow & \mathrm{Fil}^r \mathcal{M} & \xrightarrow{\cdot\varpi} & \varpi \mathrm{Fil}^r \mathcal{M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi_{\mathcal{M},r}} & & \downarrow{\scriptstyle \Phi_{\mathcal{M},r}} & & \downarrow{\scriptstyle \Phi_{\mathcal{M},r}} & & \\ 0 & \longrightarrow & W \underset{S}{\otimes} \mathcal{M}[\varpi] & \longrightarrow & W \underset{S}{\otimes} \mathcal{M} & \xrightarrow{\mathrm{id}\otimes\varpi} & W \underset{S}{\otimes} \varpi\mathcal{M} & \longrightarrow & 0 \end{array}$$

We know that the two right-most vertical maps are surjective and we want to show the left vertical map is surjective. Let $x \in W \otimes_S \mathcal{M}[\varpi]$. Then there is some $\hat{x} \in \mathrm{Fil}^r \mathcal{M}$ such that $\Phi_{\mathcal{M},r}(\hat{x}) = x$. Moreover, we know that $\varpi\hat{x}$ is in the kernel of $\Phi_{\mathcal{M},r}$ on $\mathrm{Fil}^r \varpi\mathcal{M}$. Since $\varpi\mathcal{M}$ is killed by ϖ and $\delta_i(\varpi) \in \varpi S$ for $i \geq 1$, we know that $\delta_i(E(u))m = \delta_i(u^e)m$ for any $i \geq 1$ and $m \in \varpi\mathcal{M}$. We claim that the kernel of $\Phi_{\mathcal{M},r}$ on $\varpi \mathrm{Fil}^r \mathcal{M}$ is actually $u \mathrm{Fil}^r \varpi\mathcal{M} + \mathrm{Fil}^q S \cdot (\varpi\mathcal{M})$. To see this, note that elements $s \cdot m$ of $\mathrm{Fil}^q S \cdot (\varpi\mathcal{M})$ are in $\mathrm{Fil}^r \mathcal{M}$, and using (4.1.3) and (4.1.2) we see that

$$\varphi_r(sm) = c^{-r}\varphi_r(s)\varphi_r(u^{er}m) = 0$$

since $\varphi_r(s) \in \varpi S$ and $\varphi_r(u^{er}m) \in \varpi\mathcal{M}$. On the other hand, $\widetilde{\mathcal{M}} := (\varpi\mathcal{M})/(\mathrm{Fil}^q S \cdot \varpi\mathcal{M})$ is naturally an object of $\mathrm{Mod}_{\widetilde{S}_1}^{\varphi,r}$ with $\mathrm{Fil}^r \widetilde{\mathcal{M}} = \mathrm{Fil}^r \varpi\mathcal{M}/(\mathrm{Fil}^q S \cdot \varpi\mathcal{M})$. Then we get the induced map

$$\Phi_{\widetilde{\mathcal{M}},r} : \mathrm{Fil}^r \widetilde{\mathcal{M}} \overset{1\otimes\varphi_{\widetilde{\mathcal{M}},r}}{\longrightarrow} k \underset{k[u]/u^{eq}}{\otimes} \widetilde{\mathcal{M}} \cong \widetilde{\mathcal{M}}/u\widetilde{\mathcal{M}}.$$

Using Proposition 4.2.14 and (4.2.3), we have a set of generators $\alpha_1, ...\alpha_d$ of $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ such that their images under $\varphi_{\widetilde{\mathcal{M}},r}$ form a basis for $\widetilde{\mathcal{M}}/u\widetilde{\mathcal{M}}$, and then it is easy to see that the kernel of $\Phi_{\widetilde{\mathcal{M}},r}$ is $u \mathrm{Fil}^r \widetilde{\mathcal{M}}$. Finally, this shows that the kernel of $\Phi_{\mathcal{M},r}$ on $\mathrm{Fil}^r \varpi\mathcal{M}$ is precisely $u \mathrm{Fil}^r \varpi\mathcal{M} + \mathrm{Fil}^q S \cdot \mathcal{M}$. Using that

$$\mathrm{Fil}^r \varpi\mathcal{M} = \varpi \mathrm{Fil}^r \mathcal{M} \quad \text{and} \quad \mathrm{Fil}^q S \cdot (\varpi\mathcal{M}) = \varpi(\mathrm{Fil}^q S \cdot \mathcal{M}),$$

we can choose $\hat{y} \in u \mathrm{Fil}^r \mathcal{M} + \mathrm{Fil}^q S \cdot \mathcal{M}$ to be such that $\varpi\hat{x} = \varpi\hat{y}$. So in particular, $\hat{x} - \hat{y} \in \mathrm{Fil}^r \mathcal{M}[\varpi]$. It may be that $\Phi_{\mathcal{M},r}(\hat{y}) \neq 0$, but we do know that $\Phi_{\mathcal{M},r}$ is 0 on $u \mathrm{Fil}^r \mathcal{M}$ since $\varphi(u) \in uS$, and $\Phi_{\mathcal{M},r}$ has image in $\varpi(W \otimes_S \mathcal{M})$ on $\mathrm{Fil}^q S \cdot \mathcal{M}$ by (4.1.2).

Therefore, write $\Phi_{\mathcal{M},r}(\hat{y}) = \varpi z$ and let $\hat{z}$ be a lift of z in $\mathrm{Fil}^r \mathcal{M}$ along $\Phi_{\mathcal{M},r}$. Then $\hat{x} - \hat{y} + \varpi\hat{z} \in \mathrm{Fil}^r \mathcal{M}[\varpi]$ since $\mathcal{M}$ is killed by ϖ^2 and applying the additive map $\Phi_{\mathcal{M},r}$, we get that

$$\Phi_{\mathcal{M},r}(\hat{x} - \hat{y} + \varpi\hat{z}) = x.$$

Thus, by Lemma 4.2.19, we get that $\mathcal{M}[\varpi]$ is an object of $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$.

We now naturally have an injection $\varpi\mathcal{M} \hookrightarrow \mathcal{M}[\varpi]$ of objects of $\mathrm{Mod}_{S_1}^{\varphi,r}$, and so by Corollary 4.2.16 and the definitions of $\mathrm{Fil}^r \varpi\mathcal{M}$ and $\mathrm{Fil}^r \mathcal{M}[\varpi]$, we have

$$\varpi \,\mathrm{Fil}^r \mathcal{M} = \varpi\mathcal{M} \cap \mathrm{Fil}^r \mathcal{M}[\varpi] = \varpi\mathcal{M} \cap (\mathrm{Fil}^r \mathcal{M} \cap \mathcal{M}[\varpi]) = \varpi\mathcal{M} \cap \mathrm{Fil}^r \mathcal{M}.$$

This completes the base case.

Suppose that the statement $P(i)$ given above is true for all $i \leq m$ for some m and suppose that $\mathcal{M} \in \text{Mod-FI}_S^{\varphi,r}$ is killed by ϖ^{m+1}. Using a similar argument to the base case, we can show that $\mathcal{M}[\varpi^m] \in \text{Mod-FI}_S^{\varphi,r}$ and so by induction, $\mathcal{M}[\varpi^n] = (\mathcal{M}[\varpi^m])[\varpi^n]$ is an object of $\text{Mod-FI}_S^{\varphi,r}$ for all n, and in particular, $\mathcal{M}[\varpi]$ is an object of $\mathrm{Mod}_{S_1}^{\varphi,r}$. Now, $\varpi^m\mathcal{M} \hookrightarrow \mathcal{M}[\varpi]$ is an injection, so using Corollary 4.2.16 as in the base case, we get that

$$\varpi^m \,\mathrm{Fil}^r \mathcal{M} = \varpi^m\mathcal{M} \cap \mathrm{Fil}^r \mathcal{M}.$$

Now we know that $\mathcal{M}/\varpi^m\mathcal{M}$ is an object of $\text{Mod-FI}_S^{\varphi,r}$ killed by ϖ^m, so by induction,

$$\varpi^i(\mathrm{Fil}^r \mathcal{M}/\varpi^m \,\mathrm{Fil}^r \mathcal{M}) = [\varpi^i(\mathcal{M}/\varpi^m\mathcal{M})] \cap (\mathrm{Fil}^r \mathcal{M}/\varpi^m \,\mathrm{Fil}^r \mathcal{M})$$

for any $0 \leq i \leq m-1$, and then we claim that

$$\varpi^i \,\mathrm{Fil}^r \mathcal{M} = \varpi^i\mathcal{M} \cap \mathrm{Fil}^r \mathcal{M}.$$

Let $x \in \varpi^i\mathcal{M} \cap \mathrm{Fil}^r \mathcal{M}$. Then if $\overline{x}$ denotes the image of x in $\mathcal{M}/\varpi^m\mathcal{M}$, we know that

$$\overline{x} \in [\varpi^i(\mathcal{M}/\varpi^m\mathcal{M})] \cap (\mathrm{Fil}^r \mathcal{M}/\varpi^m \,\mathrm{Fil}^r \mathcal{M})$$

and hence we can write $\overline{x} = \varpi^i \overline{y}$ for $\overline{y} \in \mathrm{Fil}^r \mathcal{M}/\varpi^m \mathrm{Fil}^r \mathcal{M}$. Let $y \in \mathrm{Fil}^r \mathcal{M}$ be a lift of $\overline{y}$. Then we have that $x = \varpi^i y + \varpi^m z$ for some $z \in \mathcal{M}$. But in fact since $x, y \in \mathrm{Fil}^r \mathcal{M}$, we know that $\varpi^m z \in \mathrm{Fil}^r \mathcal{M}$, and so $\varpi^m z \in \varpi^m \mathcal{M} \cap \mathrm{Fil}^r \mathcal{M}$ and hence in $\varpi^m \mathrm{Fil}^r \mathcal{M}$. Thus, $x \in \varpi^i \mathrm{Fil}^r \mathcal{M}$. This completes the proof. □

The same argument as in Corollary 4.1.9 then gives the following result.

Corollary 4.2.21. *The category* $\text{Mod-FI}_S^{\varphi,r}$ *is a full subcategory of* $\mathrm{Mod}_{S_\infty}^{\varphi,r}$.

4.3. An equivalence of categories

We will now prove the equivalence of the categories $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and $\mathrm{Mod}_S^{\varphi,r}$. We start with a lifting of Lemma 4.2.14

Lemma 4.3.1. *Suppose* $\mathcal{M} \in \mathrm{Mod}_S^{\varphi,r}$ *has rank* d. *Then there exist* $\alpha_1, ..., \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ *such that* $\{\varphi_{\mathcal{M},r}(\alpha_1), ..., \varphi_{\mathcal{M},r}(\alpha_d)\}$ *is a basis for* $\mathcal{M}$ *and*

$$\mathrm{Fil}^r \mathcal{M} = \bigoplus_{i=1}^d S\alpha_i + \mathrm{Fil}^q S \cdot \mathcal{M}.$$

Proof. The proof closely follows that of Liu [22] 4.1.2. Consider $\mathcal{M}/\varpi\mathcal{M}$. By Proposition 4.2.14 and Proposition 4.1.19, there exist $\alpha_1, ..., \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ such that

$$\mathrm{Fil}^r(\mathcal{M}/\varpi\mathcal{M}) := \mathrm{Fil}^r \mathcal{M}/\varpi \mathrm{Fil}^r \mathcal{M} = \bigoplus_{i=1}^d S_1\overline{\alpha}_i + \mathrm{Fil}^q S_1 \cdot (\mathcal{M}/\varpi\mathcal{M}), \tag{4.3.1}$$

where $\overline{\alpha}_i$ is the image of α_i in $\mathcal{M}/\varpi\mathcal{M}$. Let $\mathcal{N} = \bigoplus_{i=1}^d S\alpha_i + \mathrm{Fil}^q S \cdot \mathcal{M}$ and note that since by definition $\mathrm{Fil}^r S \cdot \mathcal{M} \subseteq \mathrm{Fil}^r \mathcal{M}$, we have $\mathcal{N} \subseteq \mathrm{Fil}^r \mathcal{M}$. Consider the map

$$f : \mathcal{N}/(\mathrm{Fil}^q S \cdot \mathcal{M}) \to \mathrm{Fil}^r \mathcal{M}/(\mathrm{Fil}^q S \cdot \mathcal{M}).$$

Now, the ring $S/\mathrm{Fil}^q S$ is isomorphic to $\mathfrak{S}/E(u)^q\mathfrak{S}$ by Proposition 4.1.3, and this is a ϖ-adically complete, Noetherian ring. So, by Nakayama's Lemma, the map f is surjective if it is surjective modulo ϖ. But this is given by (4.3.1) above. So, in fact,

$$\operatorname{Fil}^r \mathcal{M} = \bigoplus_{i=1}^{d} S\alpha_i + \operatorname{Fil}^q S \cdot \mathcal{M}.$$

To see that $\{\varphi_{\mathcal{M},r}(\alpha_1), ..., \varphi_{\mathcal{M},r}(\alpha_d)\}$ is a basis for $\mathcal{M}$, first recall that $\varphi_{\mathcal{M},r}(\operatorname{Fil}^r \mathcal{M})$ generates $\mathcal{M}$. Since $\varphi_{\mathcal{M},r}(\operatorname{Fil}^q S \cdot \mathcal{M}) \subseteq \varpi\mathcal{M}$ and since S is complete with respect to the ideal ϖS, it follows from ϖ-adic completeness that $\{\varphi_{\mathcal{M},r}(\alpha_1), ..., \varphi_{\mathcal{M},r}(\alpha_d)\}$ forms a basis for $\mathcal{M}$. □

We can now lift the result of Corollary 4.2.17 to objects in $\operatorname{Mod}_S^{\varphi,r}$.

Lemma 4.3.2. *Suppose that $h : \mathcal{M} \to \mathcal{M}'$ is a morphism of objects in $\operatorname{Mod}_S^{\varphi,r}$ that induces an isomorphism of S-modules, $\mathcal{M} \cong \mathcal{M}'$. Then h gives an isomorphism $\mathcal{M} \cong \mathcal{M}'$ as objects in $\operatorname{Mod}_S^{\varphi,r}$.*

Proof. As in Corollary 4.2.17, it suffices to show that $h : \operatorname{Fil}^r \mathcal{M} \to \operatorname{Fil}^r \mathcal{M}'$ is surjective. Let $\mathcal{N}$ be the image of $\operatorname{Fil}^r \mathcal{M}$ under h. Since $h(\mathcal{M}) = \mathcal{M}'$ is surjective and S-linear and since $\operatorname{Fil}^q S \cdot \mathcal{M} \subset \operatorname{Fil}^r \mathcal{M}$, we see that $\mathcal{N}$ contains $\operatorname{Fil}^q S \cdot \mathcal{M}'$ and so $\operatorname{Fil}^r \mathcal{M}'/(\operatorname{Fil}^q S \cdot \mathcal{M}') \twoheadrightarrow \operatorname{Fil}^r \mathcal{M}'/\mathcal{N}$ is surjective. Now, by Lemma 4.3.1, we know that $\operatorname{Fil}^r \mathcal{M}'/(\operatorname{Fil}^q S \cdot \mathcal{M}')$ is finitely generated, and hence $\mathcal{M}/\mathcal{N}$ is finitely generated. Using Lemma 4.1.18, we know that

$$h \bmod \varpi : \mathcal{M}/\varpi\mathcal{M} \to \mathcal{M}'/\varpi\mathcal{M}'$$

is a morphism in $\operatorname{Mod}_{S_1}^{\varphi,r}$ that induces an S_1-module isomorphism $\mathcal{M}/\varpi\mathcal{M} \cong \mathcal{M}'/\varpi\mathcal{M}'$. So by Corollary 4.2.17,

$$\operatorname{Fil}^r \mathcal{M}/\varpi \operatorname{Fil}^r \mathcal{M} = \operatorname{Fil}^r(\mathcal{M}/\varpi\mathcal{M}) \cong \operatorname{Fil}^r(\mathcal{M}'/\varpi\mathcal{M}') = \operatorname{Fil}^r \mathcal{M}'/(\varpi \operatorname{Fil}^r \mathcal{M}').$$

It follows that $\operatorname{Fil}^r \mathcal{M} = \mathcal{N} + \varpi \operatorname{Fil}^r \mathcal{M}$. Since (ϖ) is contained in the radical of S and $\mathcal{M}/\mathcal{N}$ is finitely generated, we get by Nakayama's Lemma that $h(\operatorname{Fil}^r \mathcal{M}) = \operatorname{Fil}^r \mathcal{M}'$. This completes the proof. □

Notice that for an object of $\mathrm{Mod}_S^{\varphi,r}$, we can define a Frobenius $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ by $\varphi_{\mathcal{M}}(x) = c^{-r}\varphi_{\mathcal{M},r}(E(u)^r x)$, and one can check that this is compatible with $\varphi_{\mathcal{M},r}$ in the sense that for $x \in \mathrm{Fil}^r \mathcal{M}$ we get the identity $\varphi_{\mathcal{M}}(x) = \varpi^r \varphi_{\mathcal{M},r}(x)$. Hence, Lemma 4.3.1 is almost enough to give a descent to $\mathfrak{S}$-modules in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$. For instance, using Lemma 4.3.1, we could define $\mathfrak{M}$ to be the $\mathfrak{S}$-span of $\{\varphi_{\mathcal{M},r}(\alpha_1), ..., \varphi_{\mathcal{M},r}(\alpha_d)\}$. We may then try to define $\varphi_{\mathfrak{M}}$ as above, but we cannot be sure that for $x \in \mathfrak{M}$ the element $c^{-r}\varphi_{\mathcal{M},r}(E(u)^r x)$ is in fact in the $\mathfrak{S}$-span of the basis. However, if we knew that $E(u)^r \varphi_{\mathcal{M},r}(\alpha_j)$ is an $\mathfrak{S}$-linear combination of the α_i, then, up to the unit c^r, the map $\varphi_{\mathfrak{M}}$ would be well-defined. The following key lemma ensures that there exist α_i satisfying this additional condition.

We will use the notation of (4.2.9).

Lemma 4.3.3. *Let $\mathcal{M} \in \mathrm{Mod}_S^{\varphi,r}$ have rank d. Then there exist $\alpha_1, ..., \alpha_d \in \mathrm{Fil}^r \mathcal{M}$ and a basis $\{e_1, ..., e_d\}$ of $\mathcal{M}$ such that $e_i = c^{-r}\varphi_{\mathcal{M},r}(\alpha_i)$ and $\underline{\alpha} = B\underline{e}$, where B is a $d \times d$ matrix with coefficients in $\mathfrak{S}$, and*

$$\mathrm{Fil}^r \mathcal{M} = \bigoplus_{i=1}^{d} S\alpha_i + \mathrm{Fil}^q S \cdot \mathcal{M}.$$

Proof. This proof is an adaptation of Lemma 2.2.2 of [9]. We prove inductively that we can construct $\alpha_1^{(n)}, ..., \alpha_d^{(n)} \in \mathrm{Fil}^r \mathcal{M}$ such that $e_i^{(n)} = c^{-r}\varphi_{\mathcal{M},r}(\alpha_i^{(n)})$ forms a basis of $\mathcal{M}$ and there exist matrices $B^{(n)} \in M_d(\mathfrak{S})$ and $C^{(n)} \in M_d(\varpi^n \mathrm{Fil}^{n+q} S)$ such that

$$\underline{\alpha}^{(n)} = (B^{(n)} + C^{(n)})\underline{e}^{(n)}. \tag{4.3.2}$$

For the base case, $n = 0$, the $\underline{\alpha}^{(0)}$ are given by Lemma 4.3.1, and $\underline{e}^{(0)}$ is defined after multiplication by the unit c^{-r}. The matrices $B^{(0)}$ and $C^{(0)}$ follow from that fact that $S = \mathfrak{S} + \mathrm{Fil}^q S$. Now suppose that we have $\alpha_1^{(n)}, ..., \alpha_d^{(n)}$ and put $e_i^{(n)} = c^{-r}\varphi_{\mathcal{M},r}(\alpha_i^{(n)})$. Set

$$\underline{\alpha}^{(n+1)} := B^{(n)}\underline{e}^{(n)}$$

and define

$$\begin{aligned}\underline{e}^{(n+1)} &:= c^{-r}\varphi_{\mathcal{M},r}(\underline{\alpha}^{(n+1)})\\ &= c^{-r}\varphi_{\mathcal{M},r}(\underline{\alpha}^{(n)} - C^{(n)}\underline{e}^{(n)})\\ &= (I - D^{(n)})\underline{e}^{(n)}, \end{aligned} \tag{4.3.3}$$

for the matrix $D^{(n)} \in M_d(S)$ given by $c^{-r}\varphi_{\mathcal{M},r}(C^{(n)}\underline{e}^{(n)}) = D^{(n)}\underline{e}^{(n)}$. We will show that $D^{(n)}$ actually has coefficients in $\varpi^{\lambda_n+n}S$ where

$$\lambda_n = n + q - r - \left\lfloor \frac{n+q-1}{q-1} \right\rfloor.$$

For $s \in \operatorname{Fil}^r S$ and $x \in \mathcal{M}$, by (4.1.3),

$$\varphi_{\mathcal{M},r}(sx) = c^{-r}\varphi_r(s)\varphi_{\mathcal{M},r}(E(u)^r x).$$

Because $C^{(n)}$ has coefficients in $\varpi^n \operatorname{Fil}^{n+q} S$, it suffices then to show that $\varphi_r(s) \in \varpi^{\lambda_n} S$ for any $s \in \operatorname{Fil}^{n+q} S$. Now, recall that $\operatorname{Fil}^{n+q} S$ is topologically generated by $\delta_m(E(u))$ for $m \geq n+q$ and so such an s can be represented by

$$s = \sum_{m=n+q}^{\infty} a_m(u)\delta_m(E(u)), \quad a_m(u) \in \mathfrak{S}, a_m(u) \to 0 \ \varpi\text{-adically},$$

Let $m = b_0 + b_1 q + \cdots + b_l q^l$ with $0 \leq b_i < q$ and recall from (3.1.2) that

$$m_{(q)} = \frac{1}{q-1}[m - (b_0 + \cdots + b_l)].$$

Since $m_{(q)}$ is an integer and is less than or equal to $\frac{m-1}{q-1}$, we can say

$$m_{(q)} \leq \left\lfloor \frac{m-1}{q-1} \right\rfloor. \tag{4.3.4}$$

Note that $m - \left\lfloor \frac{m-1}{q-1} \right\rfloor$ is a non-decreasing function since if m increases by 1, then $\left\lfloor \frac{m-1}{q-1} \right\rfloor$ increases by at most 1. To see this, let m and n be positive integers. Thus,

$$\begin{aligned}(m+n) - \left\lfloor \frac{m+n-1}{q-1} \right\rfloor &\geq (m+n) - \left(\left\lfloor \frac{m-1}{q-1} \right\rfloor + n\right)\\ &= m - \left\lfloor \frac{m-1}{q-1} \right\rfloor. \end{aligned} \tag{4.3.5}$$

Then for $m \geq n+q$, we have

$$\begin{aligned} m - r - m_{(q)} &\geq m - r - \left\lfloor \frac{m-1}{q-1} \right\rfloor && \text{by (4.3.4)} \\ &\geq n + q - r - \left\lfloor \frac{n+q-1}{q-1} \right\rfloor && \text{by (4.3.5)} \\ &= \lambda_n. \end{aligned}$$

Now, because $\varphi(E(u)) \in \varpi S$, we know that

$$\varphi_r(\delta_m(E(u))) = \varphi(E(u)^m)/\varpi^{r+m_{(q)}} \in \varpi^{\lambda_n} S$$

for any $m \geq n+q$, and so $\varphi_r(s) \in \varpi^{\lambda_n} S$ as desired.

Now, for any integer $n \geq 0$, since $r < q-1$, we find

$$\begin{aligned} \lambda_n &= n + q - r - \left\lfloor \frac{n+q-1}{q-1} \right\rfloor \\ &\geq (q-2)\frac{n+q}{q-1} - r + \frac{1}{q-1} \\ &\geq \frac{q^2-2q+1}{q-1} - r \\ &> 0. \end{aligned}$$

Since λ_n is an integer, we know that $\lambda_n \geq 1$. We just showed that $D^{(n)}$ has coefficients in $\varpi^{n+\lambda_n} S$, so for any $n \geq 0$, we know ϖ divides the coefficients of $D^{(n)}$, so $I - D^{(n)}$ is invertible with

$$(I - D^{(n)})^{-1} = I + D^{(n)} + (D^{(n)})^2 + ...$$

Therefore (4.3.3) shows that $\{e_1^{(n+1)}, ..., e_d^{(n+1)}\}$ forms a basis of $\mathcal{M}$ and

$$\underline{\alpha}^{(n+1)} = B^{(n)}(I - D^{(n)})^{-1}\underline{e}^{(n+1)}.$$

Let $A = B^{(n)}(I - D^{(n)})^{-1}$. We want to write A as $B^{(n+1)} + C^{(n+1)}$ for matrices $B^{(n+1)} \in M_d(\mathfrak{S})$ and $C^{(n+1)} \in M_d(\varpi^{n+1} \operatorname{Fil}^{n+1+q} S)$. By what we have seen above, write $D^{(n)} = \varpi^{\lambda_n+n} F^{(n)}$ and $F^{(n)} = F_1^{(n)} + F_2^{(n)}$ where the coefficients of $F_1^{(n)}$ can all be written in the form

$$\sum_{m=0}^{n+q} a_m(u)\delta_m(E(u)) \tag{4.3.6}$$

and all of the coefficients of $F_2^{(n)}$ can be written in the form

$$\sum_{m=n+q+1}^{\infty} a_m(u)\delta_m(E(u)),$$

with all $a_m(u) \in \mathfrak{S}$. For $m \le n+q$, we compute

$$\begin{aligned}
\lambda_n + n - m_{(q)} &\ge 2n + q - r - 2 \cdot \frac{n+q-1}{q-1} \\
&= (2n - \frac{2n}{q-1}) + (q-2) - r \\
&\ge q - 2 - r \\
&\ge 0,
\end{aligned} \tag{4.3.7}$$

and note that $\lambda_n + n - m_{(q)} > 0$ when $m = 0$. So anything of the form (4.3.6) is in $(\varpi, E(u))\mathfrak{S} = (\varpi, u)\mathfrak{S}$ after multiplying by ϖ^{λ_n+n}. Therefore,

$$\varpi^{\lambda_n+n} F_1^{(n)} \in M_d((u,\varpi)\mathfrak{S}) \quad \text{and} \quad \varpi^{\lambda_n+n} F_2^{(n)} \in M_d(\varpi^{n+1} \operatorname{Fil}^{n+1+q} S).$$

Write $D^{(n)} = D_1^{(n)} + D_2^{(n)}$ with $D_1^{(n)} = \varpi^{\lambda_n+n} F_1^{(n)}$ and $D_2^{(n)} = \varpi^{\lambda_n+n} F_2^{(n)}$. Then

$$\begin{aligned}
B^{(n)}(I - D^{(n)})^{-1} &= \sum_{i=0}^{\infty} B^{(n)}(D_1^{(n)} + D_2^{(n)})^i \\
&= B^{(n+1)} + C^{(n+1)},
\end{aligned}$$

where $B^{(n+1)} = \sum_{i=0}^{\infty} B^{(n)}(D_1^{(n)})^i$ converges in $\mathfrak{S}$ since $\mathfrak{S}$ is (ϖ, u)-adically complete, and $C^{(n+1)} = D_3^{(n)} D_2^{(n)}$ for a matrix $D_3^{(n)} \in M_d(S)$ and so

$$C^{(n+1)} \in M_d(\varpi^{n+1} \operatorname{Fil}^{n+1+q} S).$$

This completes the induction.

For every n we defined $\underline{\alpha}^{(n+1)} = B^{(n)} \underline{e}^{(n)}$, and so by (4.3.2) we get that

$$\underline{\alpha}^{(n+1)} - \underline{\alpha}^{(n)} = -C^{(n)} \underline{e}^{(n)}. \tag{4.3.8}$$

But the coefficients of $C^{(n)}$ are divisible by ϖ^n, so the sequence $\underline{\alpha}^{(n)}$ has a limit $\underline{\alpha}$. Moreover, by (4.3.3),

$$\underline{e}^{(n+1)} - \underline{e}^{(n)} = D^{(n)} \underline{e}^{(n)},$$

so $\underline{e}^{(n+1)} - \underline{e}^{(n)}$ is divisible by ϖ^n and so the sequence $\underline{e}^{(n)}$ has a limit $\underline{e}$.

The calculation in (4.3.7) actually shows that for $m \leq n+q$ the integer

$$\lambda_n + n - m_{(q)} \geq \frac{(q-2)(2n)}{q-1}$$

and hence

$$\min_{m \leq n+q} (\lambda_n + n - m_{(q)}) \to \infty \quad \text{as} \quad n \to \infty.$$

Then $D_1^{(n)} = \varpi^{\lambda_n + n} F_1^{(n)}$, where the coefficients of $F_1^{(n)}$ were defined in (4.3.6), will have coefficients in $\varpi^{\lambda_n + n - (n+q)_{(q)}} \mathfrak{S}$. The $B^{(n)}$ therefore converge in $\mathfrak{S}$ to a matrix B since by construction $B^{(n+1)} - B^{(n)} = \sum_{i=1}^{\infty} B^{(n)} {D_1^{(n)}}^i$ and we now see that $D_1^{(n)}$ becomes increasingly more divisible by ϖ in $\mathfrak{S}$ as $n \to \infty$.

As $\alpha_i - \alpha_i^{(n)} \in \varpi^n \operatorname{Fil}^r \mathcal{M}$ and $\varphi_{\mathcal{M},r}$ is φ-semilinear and $\varphi(\varpi) = \varpi$, we know

$$c^{-r} \varphi_{\mathcal{M},r}(\alpha_i) - c^{-r} \varphi_{\mathcal{M},r}(\alpha_i^{(n)}) \in \varpi^n \mathcal{M}.$$

But we also know that $e_i - e_i^{(n)} \in \varpi^n \mathcal{M}$ and $e_i^{(n)} = c^{-r} \varphi_{\mathcal{M},r}(\alpha_i^{(n)})$, so

$$c^{-r} \varphi_{\mathcal{M},r}(\alpha_i) - e_i \in \varpi^n \mathcal{M}$$

for every n. By the ϖ-adic completeness of $\mathfrak{S}$, we get that $\underline{e} = c^{-r} \varphi_{\mathcal{M},r}(\underline{\alpha})$. Similarly, for any $n \geq 0$ we have

$$\underline{\alpha} - B^{(n)} \underline{e} = (\underline{\alpha} - \underline{\alpha}^{(n+1)}) + B^{(n)}(\underline{e}^{(n)} - \underline{e}) \in \varpi^n \mathcal{M}$$

and so

$$\underline{\alpha} - B\underline{e} = (\underline{\alpha} - B^{(n)} \underline{e}) + (B^{(n)} - B)(\underline{e}),$$

and this converges to 0 in $\mathcal{M}$ as $n \to \infty$, so $\underline{\alpha} = B\underline{e}$.

Finally, we know that $\underline{\alpha}^{(0)}$ generates $\operatorname{Fil}^r \mathcal{M}$ up to $\operatorname{Fil}^q S \cdot \mathcal{M}$ and $\alpha_i^{(0)} = \alpha_i^{(1)} + m_i$, where $m_i \in \operatorname{Fil}^q S \cdot \mathcal{M}$ by (4.3.8), so $\underline{\alpha}^{(1)}$ generates $\operatorname{Fil}^r \mathcal{M}$ up to $\operatorname{Fil}^q S \cdot \mathcal{M}$. Now, α_i

is congruent to $\alpha_i^{(1)}$ modulo ϖ by (4.3.8), so using the same argument as in Lemma 4.3.1, it follows that

$$\operatorname{Fil}^r \mathcal{M} = \bigoplus_{i=1}^{d} S\alpha_i + \operatorname{Fil}^q S\mathcal{M},$$

which completes the proof. □

Proposition 4.3.4. *The functor* $\Theta_r : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \to \mathrm{Mod}_S^{\varphi,r}$ *is essentially surjective.*

Proof. Let $\mathcal{M} \in \mathrm{Mod}_S^{\varphi,r}$ have rank d. Let $\{e_1, ..., e_d\}$ and $B \in M_d(\mathfrak{S})$ be as in Lemma 4.3.3. We will show that there is an $A \in M_d(\mathfrak{S})$ such that $AB = E(u)^r I$. Since $E(u)^r e_i \in \operatorname{Fil}^r \mathcal{M}$ for all i, we know by Lemma 4.3.3 that there exist matrices $A' \in M_d(S)$ and $C' \in M_d(\operatorname{Fil}^q S)$ such that $A'B + C' = E(u)^r I$. But since any element of S can be written as the sum of an element of $\mathfrak{S}$ and an element of $\operatorname{Fil}^q S$, we can assume $A' \in M_d(\mathfrak{S})$. Then $C' = E(u)^r I - A'B$ has coefficients in $\mathfrak{S} \cap \operatorname{Fil}^q S = E(u)^q \mathfrak{S}$ by Proposition 4.1.3. So we can write $C' = E(u)^q C$ with $C \in M_d(\mathfrak{S})$. Thus,

$$A'B = E(u)^r (I - E(u)^{q-r} C)$$

and $E(u)^{q-r} C \in M_d((\varpi, u)\mathfrak{S})$ so, as $r < q$, we have that $(I - E(u)^{(q-r)} C)$ is invertible in $M_d(\mathfrak{S})$. Let $A = (I - E(u)^{q-r} C)^{-1} A' \in M_d(\mathfrak{S})$ and then we have $AB = E(u)^r I$.

Now, define $\mathfrak{M} := \bigoplus_{i=1}^d \mathfrak{S} f_i$ with Frobenius $\varphi_{\mathfrak{M}}$ defined using A:

$$\varphi_{\mathfrak{M}}(\underline{f}) = A\underline{f},$$

and extending φ-semilinearly. The argument used in the proof of Proposition 4.2.18 on page 71 comes over *mutatis mutandis* to show that $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$. Let $\mathcal{M}' = \Theta_r(\mathfrak{M})$ and define $h : \mathcal{M} \to \mathcal{M}'$ by $h(e_i) = 1 \otimes f_i$ and extend S-linearly. We want to show that the S-module isomorphism h is an isomorphism in $\mathrm{Mod}_S^{\varphi,r}$. It is clearly an isomorphism of S-modules, so by Lemma 4.3.2, it suffices to show that h is a morphism in $\mathrm{Mod}_S^{\varphi,r}$.

Using Lemma 4.3.3, there are $\alpha_1, ..., \alpha_d \in \operatorname{Fil}^r \mathcal{M}$ such that $e_i = c^{-r}\varphi_r(\alpha_i)$, $\underline{\alpha} = B\underline{e}$, and

$$\operatorname{Fil}^r \mathcal{M} = \bigoplus_{i=1}^{d} S\alpha_i + \operatorname{Fil}^q S \cdot \mathcal{M}.$$

Since h is S-linear and $\operatorname{Fil}^r S \cdot \mathcal{M}' \subseteq \operatorname{Fil}^r \mathcal{M}'$, we just need to show that $h(\alpha_i) \in \operatorname{Fil}^r \mathcal{M}'$ for each i. The argument is the same as in the proof of Proposition 4.2.18 on page 72. Carrying that argument further, h commutes with the φ_r-maps on the α_i. That is, $h(\varphi_{\mathcal{M},r}(\alpha_i)) = \varphi_{\mathcal{M}',r}(h(\alpha_i))$. Now, by the S-linearity of h and the φ-semilinearity of the φ_r-maps, we know that h commutes with φ_r on $\bigoplus_{i=1}^{d} S\alpha_i$. Consider sx with $s \in \operatorname{Fil}^q S$ and $x \in \mathcal{M}$. Then by (4.1.3),

$$\varphi_{\mathcal{M},r}(sx) = c^{-r}\varphi_r(s)\varphi_{\mathcal{M},r}(E(u)^r x).$$

But since $AB = E(u)^r I$, we know that $E(u)^r x \in \bigoplus_{i=1}^{d} S\alpha_i$, and so we have that $\varphi_{\mathcal{M}',r}(h(E(u)^r x)) = h(\varphi_{\mathcal{M},r}(E(u)^r x))$ and then a simple calculation shows that

$$\varphi_{\mathcal{M}',r}(h(sx)) = h(\varphi_{\mathcal{M},r}(sx)),$$

and therefore h commutes with the φ_r on all of $\operatorname{Fil}^r \mathfrak{M}$. $\square$

The next step is to show that Θ_r is fully faithful. To prove that Θ_r is fully faithful, we will show that there is an isomorphism on Hom-sets. For simplicity of notation, let $\operatorname{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N})$ denote the set of morphisms $\mathfrak{M} \to \mathfrak{N}$ in $'\operatorname{Mod}_{\mathfrak{S}}^{\varphi,r}$. That is, $\mathfrak{S}$-linear homomorphisms that intertwine $\varphi_{\mathfrak{M}}$ and $\varphi_{\mathfrak{N}}$. Likewise, let $\operatorname{Hom}_S(\mathcal{M}, \mathcal{N})$ denote the set of morphisms $\mathcal{M} \to \mathcal{N}$ in $'\operatorname{Mod}_S^{\varphi,r}$. That is, S-linear homomorphisms that respect the submodules $\operatorname{Fil}^r \mathcal{M}$ and $\operatorname{Fil}^r \mathcal{N}$ and that intertwine $\varphi_{\mathcal{M},r}$ and $\varphi_{\mathcal{N},r}$. In the proof we will need to use that the categories $\operatorname{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ and $\operatorname{Mod}_{S_\infty}^{\varphi,r}$ are *exact categories* using the definition of exact category formalized by Quillen [25]. In short, these categories are exact categories because they are stable under extensions and contained in the abelian categories $\operatorname{Mod}_{\mathfrak{S}}$ and Mod_S. Importantly, exact categories may not be abelian themselves, but in such categories, applying the Hom functor gives rise to long exact sequences in the usual way, where the sets $\mathrm{Ext}^n_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N})$ are derived functors of $\mathrm{Hom}_{\mathfrak{S}}(-,\mathfrak{N}) \to \mathrm{Ab}$. Moreover, Buchsbaum [7] showed that in exact categories, the derived functors are equivalent to the sets $\mathrm{Ext}^n_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N})$ in the manner of Yoneda [28]. In particular, $\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N})$ is the set of extensions of $\mathfrak{M}$ by $\mathfrak{N}$ in $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_\infty}$ up to isomorphism. Likewise for Ext^1_S.

We start with a lemma that establishes a base case for an induction (see Proposition 4.1.10) that will be used in the proof of Proposition 4.3.6.

Lemma 4.3.5. *Suppose that* $\mathfrak{N} \in \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_1}$. *Then for any* $\mathfrak{M} \in \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_\infty}$, *we have*

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) \cong \mathrm{Hom}_S(\Theta_r(\mathfrak{M}),\Theta_r(\mathfrak{N})), \tag{4.3.9}$$

$$\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) \to \mathrm{Ext}^1_S(\Theta_r(\mathfrak{M}),\Theta_r(\mathfrak{N})) \quad \textit{is injective.} \tag{4.3.10}$$

Proof. We will prove these statements simultaneously for any $\mathfrak{M} \in \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_\infty}$ using the induction-style argument introduced in Proposition 4.1.10. Let $\mathcal{N} := \Theta_r(\mathfrak{N})$ and for $\mathfrak{M} \in \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_\infty}$ write $\mathcal{M} := \Theta_r(\mathfrak{M})$. First suppose that $\mathfrak{M} \in \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_1}$. Then by Proposition 4.2.18, we have

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) \cong \mathrm{Hom}_S(\mathcal{M},\mathcal{N})$$

and so (4.3.9) is satisfied for any $\mathfrak{M} \in \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_1}$.

To show (4.3.10) when $\mathfrak{M} \in \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_1}$, let

$$0 \to \mathfrak{M} \xrightarrow{f} \mathfrak{E} \xrightarrow{g} \mathfrak{N} \to 0$$

be an extension of $\mathfrak{M}$ by $\mathfrak{N}$ and suppose that after applying the functor Θ_r, which by Proposition 4.1.23 is exact on $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_\infty}$, we get $\Theta_r(\mathfrak{E}) \cong \mathcal{M} \oplus \mathcal{N}$. That is, suppose we get the trivial extension of $\mathcal{M}$ by $\mathcal{N}$. Note that $\Theta_r(\mathfrak{E})$ is in $\mathrm{Mod}^{\varphi,r}_{S_1}$ since it is the direct sum of $\mathcal{M}$ and $\mathcal{N}$, and hence by Proposition 4.2.18, $\mathfrak{E} \cong \Theta_r^{-1} \circ \Theta_r(\mathfrak{E})$ is an object of $\mathrm{Mod}_{\mathfrak{S}_1}$. Since $\Theta_r(\mathfrak{E}) \cong \mathcal{M} \oplus \mathcal{N}$, there exists a morphism $h : \mathcal{N} \to \Theta_r(\mathfrak{E})$

such that $\Theta_r(g) \circ h$ is the identity map on $\mathcal{N}$. Now by Proposition 4.2.18, Θ_r is fully faithful for objects of $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$, so there exists a morphism $\mathfrak{h} : \mathfrak{N} \to \mathfrak{E}$ such that $\Theta_r(\mathfrak{h}) = h$. Now, $\Theta_r(g \circ \mathfrak{h}) = \mathrm{id}$, and since there is a unique endomorphism of $\mathfrak{N}$ that maps to the identity on $\mathcal{N}$ via Θ_r, it follows that $g \circ \mathfrak{h} = \mathrm{id}$. This then shows that $\mathfrak{E} \cong \ker g \oplus \mathrm{im}\ \mathfrak{h} \cong \mathfrak{M} \oplus \mathfrak{N}$, the split extension of. So $\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Ext}^1_S(\mathcal{M}, \mathcal{N})$ is injective.

Now suppose that for some $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ we have an exact sequence

$$0 \longrightarrow \mathfrak{M}' \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}'' \longrightarrow 0$$

and that (4.3.9) and (4.3.10) hold for $\mathfrak{M}'$ and $\mathfrak{M}''$. Then by applying $\mathrm{Hom}_{\mathfrak{S}}(-, \mathfrak{N})$ and Θ_r to this short exact sequence, using that Θ_r is exact on $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ by Proposition 4.1.23, we get the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}'', \mathfrak{N}) & \to & \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) & \to & \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{N}) & \longrightarrow & \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \mathrm{Hom}_S(\mathcal{M}'', \mathcal{N}) & \to & \mathrm{Hom}_S(\mathcal{M}, \mathcal{N}) & \to & \mathrm{Hom}_S(\mathcal{M}', \mathcal{N}) & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}'', \mathfrak{N}) & \to & \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) & \to & \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{N}) \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & \mathrm{Ext}^1_S(\mathcal{M}'', \mathcal{N}) & \to & \mathrm{Ext}^1_S(\mathcal{M}, \mathcal{N}) & \to & \mathrm{Ext}^1_S(\mathcal{M}', \mathcal{N}) \end{array}$$

By the assumptions on $\mathfrak{M}'$ and $\mathfrak{M}''$, we know that

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}'', \mathfrak{N}) \cong \mathrm{Hom}_S(\mathcal{M}'', \mathcal{N}) \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{N}) \cong \mathrm{Hom}_S(\mathcal{M}', \mathcal{N})$$

are isomorphisms and $\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}'', \mathfrak{N}) \to \mathrm{Ext}^1_S(\mathcal{M}'', \mathcal{N})$ is injective. These together with a diagram chase shows that $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) \cong \mathrm{Hom}_S(\mathcal{M}, \mathcal{N})$. Moreover, the facts that

$$\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}'', \mathfrak{N}) \to \mathrm{Ext}^1_S(\mathcal{M}'', \mathcal{N}) \quad \text{and} \quad \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{N}) \to \mathrm{Ext}^1_S(\mathcal{M}', \mathcal{N})$$

are injective and $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{N}) \cong \mathrm{Hom}_S(\mathcal{M}', \mathcal{N})$ can be used in a diagram chase to show that $\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Ext}^1_S(\mathcal{M}, \mathcal{N})$ is injective, and this completes the proof. □

Proposition 4.3.6. *The functor Θ_r is fully faithful on $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$.*

Proof. The proof follows Kisin [18] 1.1.11. Let $\mathfrak{M}, \mathfrak{N} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and $\mathcal{M} = \Theta_r(\mathfrak{M})$ and $\mathcal{N} = \Theta_r(\mathfrak{N})$. We wish to show that

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) \cong \mathrm{Hom}_S(\mathcal{M}, \mathcal{N}).$$

Suppose it was the case that

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}/\varpi^n\mathfrak{M}, \mathfrak{N}/\varpi^n\mathfrak{N}) \cong \mathrm{Hom}_S(\Theta_r(\mathfrak{M}/\varpi^n\mathfrak{M}), \Theta_r(\mathfrak{N}/\varpi^n\mathfrak{N}))$$

for all n and recall that by Lemma 4.1.22, we get that $\Theta_r(\mathfrak{M}/\varpi^n\mathfrak{M}) \cong \mathcal{M}/\varpi^n\mathcal{M}$, and likewise for $\mathfrak{N}$ and $\mathcal{N}$. Then the desired result on $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ follows from a dévissage: given $g : \mathcal{M} \to \mathcal{N}$, for every n there exists a unique $f_n : \mathfrak{M}/\varpi^n\mathfrak{M} \to \mathfrak{N}/\varpi^n\mathfrak{N}$ with $\mathrm{id} \otimes f_n \equiv g$ modulo ϖ^n. Furthermore, these f_n are compatible modulo ϖ and so define $f : \mathfrak{M} \to \mathfrak{N}$ as

$$f = \varprojlim_n f_n.$$

Then modulo any power of ϖ, we get that f commutes with $\varphi_{\mathfrak{M}}$ and $\varphi_{\mathfrak{N}}$, and so we know $f \circ \varphi_{\mathfrak{M}} = \varphi_{\mathfrak{N}} \circ f$. Also, $\Theta_r(f) = \mathrm{id} \otimes f$ is congruent to g modulo any power of ϖ, so by ϖ-adic completeness in S, we know $\Theta_r(f) = g$. Since each of the f_n's were unique, f is the unique morphism $\mathfrak{M} \to \mathfrak{N}$ in $\mathrm{Mod}_S^{\varphi,r}$ such that $\Theta_r(f) = g$.

Therefore, using Proposition 4.1.8 and Corollary 4.1.9, we have reduced to showing that Θ_r is fully faithful on $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$. We will show this using Proposition 4.1.10 for objects $\mathfrak{N} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ as follows: Let $P(\mathfrak{N})$ be the statement that for any $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$, we have (4.3.9) and (4.3.10) as in Lemma 4.3.5:

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) \cong \mathrm{Hom}_S(\mathcal{M}, \mathcal{N}),$$

$$\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Ext}^1_S(\mathcal{M}, \mathcal{N}) \quad \text{is injective.}$$

If $\mathfrak{N} \in \mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$, then this is precisely Lemma 4.3.5. Otherwise, suppose that for some $\mathfrak{N} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$, we have a short exact sequence

$$0 \to \mathfrak{N}' \to \mathfrak{N} \to \mathfrak{N}'' \to 0$$

in $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ such that $P(\mathfrak{N}')$ and $P(\mathfrak{N}'')$ are true. Then by applying $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},-)$ and Θ_r to this short exact sequence, using that Θ_r is exact on $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ by Proposition 4.1.23, we get the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}') & \to & \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) & \to & \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}'') & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N}') & \to & \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N}) & \to & \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N}'') & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}') & \longrightarrow & \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) & \longrightarrow & \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}'') \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathrm{Ext}^1_{S}(\mathcal{M},\mathcal{N}') & \longrightarrow & \mathrm{Ext}^1_{S}(\mathcal{M},\mathcal{N}) & \longrightarrow & \mathrm{Ext}^1_{S}(\mathcal{M},\mathcal{N}'')
\end{array}$$

By assumption, we have that

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}') \cong \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N}') \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}'') \cong \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N}'').$$

Using the diagram, this shows that

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) \to \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N})$$

is injective. This map is also surjective if the map from $\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}')$ to $\mathrm{Ext}^1_{S}(\mathcal{M},\mathcal{N}')$ is injective. But this follows from our assumption on $\mathfrak{N}'$. This shows that

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) \cong \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N}).$$

Finally, we must show that $\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}) \to \mathrm{Ext}^1_{S}(\mathcal{M},\mathcal{N})$ is injective. But this follows from a diagram chase using the facts that

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}'') \cong \mathrm{Hom}_{S}(\mathcal{M},\mathcal{N}'')$$

and

$$\mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}') \to \mathrm{Ext}^1_{S}(\mathcal{M},\mathcal{N}') \quad \text{and} \quad \mathrm{Ext}^1_{\mathfrak{S}}(\mathfrak{M},\mathfrak{N}'') \to \mathrm{Ext}^1_{S}(\mathcal{M},\mathcal{N}'')$$

are both injective. This completes the proof. $\square$

The desired equivalence now follows by Propositions 4.1.23, 4.3.4, and 4.3.6, which we summarize below.

Theorem 4.3.7. *The functor* $\Theta_r : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \to \mathrm{Mod}_{S}^{\varphi,r}$ *induces an exact equivalence of categories.*

5. Application to ϖ-divisible $\mathcal{O}$-modules

5.1. The results of Kisin-Ren

We now give an application of Theorem 4.3.7, following the constructions of Kisin and Ren [19] and Faltings [12]. Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$, uniformizer ϖ and residue field k_F with $|k_F| = q$. Let $\mathcal{G}$ be a Lubin-Tate group over F corresponding to the uniformizer ϖ, as described in Section 2.1. This means that for $a \in \mathcal{O}_F$, there is a corresponding power series $[a] \in \mathcal{O}_F[\![X]\!]$ whose first term is aX and where $[\varpi](X) \equiv X^q \bmod \varpi$ such that the map

$$a \mapsto [a] : \mathcal{O}_F \to \mathrm{End}(\mathcal{G})$$

is a ring homomorphism. In particular, this says that $[ab](X) = [a] \circ [b](X)$ for any $a, b \in \mathcal{O}_F$. The power series $[a](X)$ are determined by the choice of $[\varpi](X)$ satisfying the conditions of Definition 2.2.1, but different choices are isomorphic in the sense of Corollary 2.2.7. Hence, we can and do assume that $[\varpi](X)$ is a polynomial of degree q.

Let k be a finite field containing k_F and $K_0 = W(k)[1/p]$. Denote by $K_{0,F}$ the compositum of F and K_0. It is an unramified extension of F with uniformizer ϖ, residue field k and ring of integers $W(k)_F = \mathcal{O}_F \otimes_{W(k_F)} W(k)$. For $n \geq 1$, let $K_{n,F}$ be the field generated over $K_{0,F}$ by the ϖ^n-torsion points of $\mathcal{G}$, and set $K_\infty = \bigcup K_n$. We will suppose that $K = K_{1,F}$ and write $\Gamma = \mathrm{Gal}(K_\infty/K)$.

Remark 5.1.1. The results given in [19] occur in the generality where $K \subset K_\infty$, where $K_\infty = \bigcup K_{n,F}$ as defined above (and with $\Gamma = \mathrm{Gal}(K_\infty/K)$). We have chosen to only consider the case where $K = K_{1,F}$. By adapting proofs of Cais-Lau [8], the results should follow more generally.

If $T\mathcal{G}$ is the ϖ-adic Tate module of $\mathcal{G}$, then $T\mathcal{G}$ is a rank one $\mathcal{O}_F$-module generated by some $v = (v_n)_{n\geq 0}$ with $v_n \in \mathcal{O}_{\overline{K}}$ characterized by

$$v_0 = 0, v_1 \neq 0, \text{ and } [\varpi](v_{n+1}) = v_n.$$

The action of $\mathcal{O}_F$ on v is given by $a \cdot v = ([a](v_n))_{n\geq 0}$. Moreover, if $\sigma \in \Gamma$, then $\sigma(v_n)$ remains in $\mathcal{G}[\varpi^n](\mathcal{O}_{\overline{K}})$ and so $(\sigma(v_n))_{n\geq 0} \in T\mathcal{G}$ and is in fact a generator of $T\mathcal{G}$. Hence, we get a character $\chi : \Gamma \to \mathcal{O}_F^\times$ such that $(\sigma(v_n))_{n\geq 0} = \chi(\sigma) \cdot (v_n)_{n\geq 0}$. This generalizes the situation where $F = \mathbb{Q}_p$, $\varpi = p$, $\mathcal{G}$ is the multiplicative group scheme $\hat{\mathbb{G}}_m$, and χ is the cyclotomic character.

Let $\mathfrak{S} := W(k)_F[\![u]\!]$. On $W(k)_F$, there is a natural lifting, φ_q, of the q-power Frobenius map on k, and this lifting is trivial on $\mathcal{O}_F$. Define the endomorphism φ on $\mathfrak{S}$ by $\varphi(u) = [\varpi](u)$ and extending φ_q-linearly. Let $E(u) = \varphi(u)/u$, which is an Eisenstein polynomial of degree $e = q - 1$. Then $E(u)$ is the minimal polynomial for a uniformizer $\pi \in K$ over $K_{0,F}$. We further define an action of Γ on $\mathfrak{S}$ that is trivial on $W(k)_F$ and for $\gamma \in \Gamma$ sends u to $[\chi(\gamma)](u)$. Notice that for $\gamma \in \Gamma$ we have

$$\begin{aligned}
\varphi(\gamma(u)) &= \varphi([\chi(\gamma)](u)) \\
&= [\chi(\gamma)](\varphi(u)) \\
&= [\chi(\gamma)]([\varpi](u)) \\
&= [\varpi\chi(\gamma)](u) \\
&= [\varpi]([\chi(\gamma)](u)) \\
&= \gamma([\varpi](u)) \\
&= \gamma(\varphi(u)).
\end{aligned}$$

Thus the actions of φ and Γ commute on $\mathfrak{S}$. Also note that by the definition of $[\chi(\gamma)](X)$, the element $[\chi(\gamma)](u)$ is divisible by u in $\mathfrak{S}$ and in fact $[\chi(\gamma)](u)/u$ is a unit in $\mathfrak{S}$ as the constant term of the resulting power series is $\chi(\gamma) \in \mathcal{O}_F^\times$. Likewise, the element $[\chi(\gamma)](\varphi(u))/\varphi(u)$, which is φ applied to the previous unit, is hence a unit in $\mathfrak{S}$, and therefore we see that

$$\begin{aligned}\gamma(E(u)) &= \varphi([\chi(\gamma)](u))/[\chi(\gamma)](u) \\ &= [\chi(\gamma)](\varphi(u))/[\chi(\gamma)](u) \\ &= s \cdot \varphi(u)/u \\ &= sE(u), \qquad (5.1.1)\end{aligned}$$

for s a unit in $\mathfrak{S}$. So $\gamma(E(u))$ is a unit multiple of $E(u)$ in $\mathfrak{S}$.

We now define the period rings of Fontaine [14]. Let $R = \varprojlim \mathcal{O}_{\overline{K}}/\varpi$, where the transition maps are given by the q-power Frobenius map $x \mapsto x^q$. We note, as in [19] 1.1, that R can be identified with $\varprojlim \mathcal{O}_{\overline{K}}/p$, and so R is well-known from the theory of norm fields [27]. Since $[\varpi](X) \equiv X^q$ modulo ϖ, we have a map

$$\iota : T\mathcal{G} \to R, \quad (v_n)_{n\geq 0} \mapsto (\overline{v}_n)_{n\geq 0}. \qquad (5.1.2)$$

Let $W(R)$ be the Witt vector ring of R and consider $W(R)_F = \mathcal{O}_F \otimes_{W(k_F)} W(R)$. For $y \in W(R)_F$, since $[\varpi](X)$ is a polynomial with coefficients in $\mathcal{O}_F$, we can write $[\varpi](y) \in W(R)_F$ to mean the evaluation of $[\varpi]$ at y. On $W(R)_F$ there is a natural lifting of the q-power Frobenius map, which we denote φ_q. Explicitly, any element of $W(R)_F$ can be uniquely written as $\sum_{n\geq 0} \varpi^n [x_n]_T$ where $[\cdot]_T : R \to W(R) \hookrightarrow W(R)_F$ is the Teichmüller lifting. Then φ_q is given as

$$\varphi_q(\sum_{n\geq 0} \varpi^n [x_n]_T) = \sum_{n\geq 0} \varpi^n [x_n^q]_T.$$

We seek a lifting $\{\cdot\} : R \to W(R)_F$ on which $\varphi_q(\{x\}) = [\varpi](\{x\})$. The existence and uniqueness of such a lifting is demonstrated by Colmez [11], and is given by

$$\{x\} = \lim_{n\to\infty} ([\varpi] \circ \varphi_q^{-1})^n (\tilde{x}),$$

where $\tilde{x}$ is any lift of x and $([\varpi] \circ \varphi_q^{-1})^n$ is n-fold composition of $[\varpi] \circ \varphi_q^{-1}$. We show that $\{\cdot\}$ is well-defined and summarize its properties in the following lemma, which is given in part by [11], 8.3, and by [19], 1.2.

Lemma 5.1.2. *The map (5.1) is the unique map* $\{\cdot\} : R \to W(R)_F$ *where* $\{x\}$ *is a lifting of* x *such that the lift of* q*-power Frobenius satisfies* $\varphi_q(\{x\}) = [\varpi](\{x\})$*. The map* $\{\cdot\}$ *further has the property that for* $v \in T\mathcal{G}$ *and* $a \in \mathcal{O}_F$ *one has*

$$[a](\{\iota(v)\}) = \{\iota(av)\}$$

and $\{\cdot\}$ *respects the actions of* G_K *on* R *and* $W(R)_F$*. On* $\{\iota(T\mathcal{G})\}$*, this action factors through* Γ *with*

$$\gamma \cdot \{\iota(v)\} = \{\gamma \cdot \iota(v)\} = \{\iota(\gamma v)\} = [\chi(\gamma)](\{\iota(v)\}).$$

Proof. In short, the reason $\{x\}$ exists and is unique is because the map $[\varpi] \circ \varphi_q^{-1}$ is not only stable on the coset $\tilde{x} + \varpi W(R)_F$ but the image is a coset of $\varpi^2 W(R)_F$, and repeated application of $[\varpi] \circ \varphi_q^{-1}$ shrinks the size of the image. To be more exact, since $[\varpi]$ is the q-power Frobenius modulo ϖ, we know that $[\varpi] \circ \varphi_q^{-1}(\tilde{x})$ remains a lift of x. Furthermore, if $y \equiv z \bmod \varpi^k$, then $[\varpi](y) \equiv [\varpi](z) \bmod \varpi^{k+1}$. Now, φ_q is trivial on $\mathcal{O}_F$ and so $[\varpi]$ and φ_q commute as the coefficients of $[\varpi](X)$ lie in $\mathcal{O}_F$. We can therefore view $([\varpi] \circ \varphi_q^{-1})^n$ to be n-fold composition of φ_q^{-1} followed by n-fold composition of $[\varpi]$. Hence $([\varpi] \circ \varphi_q^{-1})^n(\tilde{x}) \equiv ([\varpi] \circ \varphi_q^{-1})^{n-1}(\tilde{x}) \bmod \varpi^n$, and so the limit exists and is unique.

Furthermore, the action of G_K is compatible with $\{\cdot\}$ by uniqueness of $\{\cdot\}$: if $g \in G_K$, then g fixes $\mathcal{O}_F$ and commutes with φ_q, so $\varphi_q(g\{x\}) = [\varpi](g\{x\})$ and $g\{x\}$ is a lifting of gx, thus $g\{x\} = \{gx\}$. Moreover, the action of G_K restricted to the image of ι as given in (5.1.2) factors through Γ. If $v \in T\mathcal{G}$ and $a \in \mathcal{O}_F$, then since φ_q is trivial on $\mathcal{O}_F$,

$$\begin{aligned}\varphi_q([a](\{\iota(v)\})) &= [a](\varphi_q(\{\iota(v)\})) \\ &= [a] \circ [\varpi](\{\iota(v)\}) \\ &= [\varpi] \circ [a](\{\iota(v)\}),\end{aligned}$$

and so by the uniqueness of $\{\cdot\}$, as the image of $[a](\{\iota(v)\})$ is $[a](\iota(v)) = \iota(av)$ in R, we know $[a](\{\iota(v)\}) = \{\iota(av)\}$. □

Remark 5.1.3. By Lemma 5.1.2, we get an injection $\mathfrak{S} \hookrightarrow W(R)_F$ by sending u to $\{\iota(v)\}$ and this image is stable under φ_q and G_K and the action of G_K on the image factors through Γ. We therefore identify $\mathfrak{S}$ with its image, which is a φ_q and G_K-stable subring. Furthermore, we will now write φ for φ_q on all of $W(R)_F$, as there is no cause for confusion.

We define the following rings as in [19] 1.3, and one checks that the properties hold.

Definition 5.1.4. Let $\mathcal{O}_{\mathcal{E}}$ be the ϖ-adic completion of $\mathfrak{S}[1/u]$. This is a complete discrete valuation ring with uniformizer ϖ and residue field $k(\!(u)\!)$. The embedding $\mathfrak{S} \hookrightarrow W(R)_F$ extends uniquely to a φ-equivariant and G_K-equivariant embedding $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\text{Frac } R)_F$. Moreover, the action of G_K on $\mathcal{O}_{\mathcal{E}}$ factors through Γ and as $\mathcal{O}_{\mathcal{E}}$ is the completion of a localization of $\mathfrak{S}$, it is flat over $\mathfrak{S}$.

The field Frac R is algebraically closed, and so $k(\!(u)\!)$ has a unique separable closure $k(\!(u)\!)^{\text{sep}}$ in Frac R.

Definition 5.1.5.

- Let $\mathcal{O}_{\mathcal{E}^{\text{ur}}} \subseteq W(\text{Frac } R)_F$ be the strict Henselization of $\mathcal{O}_{\mathcal{E}}$ with residue field $k(\!(u)\!)^{\text{sep}}$. The ring $\mathcal{O}_{\mathcal{E}^{\text{ur}}}$ is also a φ-stable and G_K-stable subring of $W(\text{Frac } R)_F$ by the universal property of strict Henselization.

- Denote by $\mathcal{O}_{\widehat{\mathcal{E}}^{\text{ur}}}$ the ϖ-adic completion of $\mathcal{O}_{\mathcal{E}^{\text{ur}}}$. The actions of φ and G_K extend to $\mathcal{O}_{\widehat{\mathcal{E}}^{\text{ur}}}$.

- We let $\mathcal{E}$, $\mathcal{E}^{\text{ur}}$ and $\widehat{\mathcal{E}}^{\text{ur}}$ be the fraction fields of $\mathcal{O}_{\mathcal{E}}$, $\mathcal{O}_{\mathcal{E}^{\text{ur}}}$, and $\mathcal{O}_{\widehat{\mathcal{E}}^{\text{ur}}}$, respectively.

- Let $\mathfrak{S}^{\text{ur}} := \mathcal{O}_{\widehat{\mathcal{E}}^{\text{ur}}} \cap W(R)_F$. This is also stable under φ and G_K.

In addition, using the theory of norm fields [27] (see [19] Lemma 1.4 for a proof in this case), we have that $(\mathcal{O}_{\widehat{\mathcal{E}}^{\text{ur}}})^{G_{K_\infty}} = \mathcal{O}_{\mathcal{E}}$.

Definition 5.1.6. Denote by $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\varphi,\Gamma}$ the category of (φ,Γ)-*modules over* $\mathcal{O}_{\mathcal{E}}$ consisting of finite free $\mathcal{O}_{\mathcal{E}}$-modules M equipped with a φ-semilinear endomorphism φ_M and a continuous semilinear action of Γ such that:

- The $\mathcal{O}_{\mathcal{E}}$-linear map $\mathrm{id}\otimes\varphi_M : \varphi^*M \to M$ is an isomorphism, and
- The action of Γ commutes with φ_M.

Morphisms are $\mathcal{O}_{\mathcal{E}}$-module homomorphisms that commute with the actions of φ and Γ.

Definition 5.1.7.

- Denote by $\mathrm{Rep}_{\mathcal{O}_F}(G_K)$ the category of finite free $\mathcal{O}_F$-modules equipped with a continuous linear action of G_K.
- Define the category of $\mathcal{O}_F$-lattices of F-*crystalline* representations of G_K, denoted $\mathrm{Rep}_{\mathcal{O}_F}^{F-\mathrm{cris}}(G_K)$, to be the full subcategory of $\mathrm{Rep}_{\mathcal{O}_F}(G_K)$ consisting of representations V such that $V_F := V\otimes_{\mathcal{O}_F} F$ is crystalline as a $\mathbb{Q}_p$-representation and the filtration of $D_{\mathrm{dR}}(V_F)_{\mathfrak{m}}$, where

  $$D_{\mathrm{dR}}(V_F) = (B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V_F)^{G_K},$$

  is trivial for any maximal ideal $\mathfrak{m}$ of $K\otimes_{\mathbb{Q}_p} F$, except when $\mathfrak{m}$ is given by the kernel of $K\otimes_{\mathbb{Q}_p} F\to K$ corresponding to $F\hookrightarrow K$. See also [19] 3.3.7.

For $M\in \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\varphi,\Gamma}$, set

$$V(M) := (\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \underset{\mathcal{O}_{\mathcal{E}}}{\otimes} M)^{\varphi=1},$$

where the action of φ on $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}\otimes_{\mathcal{O}_{\mathcal{E}}} M$ is given by the action of φ on each tensor. For $V\in\mathrm{Rep}_{\mathcal{O}_F}(G_K)$, set

$$M(V) := (\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \underset{\mathcal{O}_K}{\otimes} V)^{G_{K_\infty}},$$

where the action of G_K on $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}\otimes_{\mathcal{O}_K} V$ is given by the action of G_K on each tensor.

Kisin-Ren show the following analogue of Fontaine's theory, c.f. [19] Theorem 1.6.

Theorem 5.1.8. *V and M are quasi-inverse equivalences between the exact tensor categories* $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\varphi,\Gamma}$ *and* $\mathrm{Rep}_{\mathcal{O}_F}(G_K)$*.*

Recall from Definition 4.1.7 that $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ is the category of finite free $\mathfrak{S}$-modules of (any) finite E-height.

Definition 5.1.9.

- Denote by $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ the subcategory of $\mathfrak{S}$-modules $\mathfrak{M}$ in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ equipped with a continuous semilinear action of Γ that commutes with $\varphi_{\mathfrak{M}}$ and such that Γ is trivial on $\mathfrak{M}/u\mathfrak{M}$. The morphisms of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ are morphisms in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi}$ that also commute with the action of Γ.

- Let $\mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$ denote the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ where the cokernel of the $\mathfrak{S}$-linear map $\mathrm{id}\otimes\varphi_{\mathfrak{M}}$ is killed by $E(u)$, not just some power.

- Denote by $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma,\mathrm{an}}$ the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ of objects $\mathfrak{M}$ with an action of Γ that is $\mathcal{O}_F$*-analytic*, meaning the $\mathbb{Z}_p$-linear map

$$d\Gamma : \mathrm{Lie}\,\Gamma \to \mathrm{End}_F(\mathfrak{M} \otimes_{\mathfrak{S}} K[\![u]\!])$$

  is actually $\mathcal{O}_F$-linear. See [19] (2.1.3) and (2.4.3).

Then, we naturally have a functor from $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ to $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\varphi,\Gamma}$ given by

$$\mathfrak{M} \mapsto M := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$$

and $\varphi_M = \varphi \otimes \varphi_{\mathfrak{M}}$. To see this, notice that $E(u)$ in $\mathcal{O}_{\mathcal{E}}$ is of the form $u^q + b\varpi$ and u^q is a unit in $\mathcal{O}_{\mathcal{E}}$. Then $E(u)$ is a unit since $\mathcal{O}_{\mathcal{E}}$ is ϖ-adically complete. Therefore, the condition of finite E-height on $\mathfrak{M}$ ensures that the $\mathcal{O}_{\mathcal{E}}$-linear map $\mathrm{id}\otimes\varphi_M$ on φ^*M is an isomorphism.

Then by Theorem 5.1.8, we have a functor $T : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma} \to \mathrm{Rep}_{\mathcal{O}_F}(G_K)$, and its contravariant version is given by

$$T^*(\mathfrak{M}) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}),$$

with G_K-action $f \mapsto g \circ f \circ g^{-1}$ for $g \in G_K$. In fact, with $\mathfrak{S}^{\mathrm{ur}}$ as defined in 5.1.5, Kisin-Ren show (Lemma 3.2.2 of [19]) in our case that

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}},\varphi}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}, \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}), \tag{5.1.3}$$

and both are free $\mathcal{O}_F$-modules of rank equal to the $\mathfrak{S}$-rank of $\mathfrak{M}$. Finally, if we restrict T^* to the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma,\mathrm{an}}$ of $\mathcal{O}_F$-analytic modules and we consider the category $\mathrm{Rep}_{\mathcal{O}_F}^{F-\mathrm{cris}}(G_K)$ of F-crystalline representations of G_K, then [19] 3.3.8 proves the following theorem:

Theorem 5.1.10 (Kisin-Ren)**.** *The functor* $T^*(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{ur})$ *gives an exact equivalence of* $\otimes$*-categories*

$$\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma,\mathrm{an}} \to \mathrm{Rep}_{\mathcal{O}_F}^{F-\mathrm{cris}}(G_K).$$

This provides a generalization of the theory of Wach modules in, for example, [26], [10], and [1]. If G is a π-divisible $\mathcal{O}_F$-module, as we will define in 5.2.2, the Tate module $T_\varpi G$ is an $\mathcal{O}_F$-lattice of a G_K-representation. If $T_\varpi G \in \mathrm{Rep}_{\mathcal{O}_F}^{F-\mathrm{cris}}(G_K)$, then Theorem 5.1.10 says there exists an associated object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma,\mathrm{an}}$ unique up to isomorphism, which we write $\mathfrak{M}_{\mathrm{KR}}(G)$ and in fact $\mathfrak{M}_{\mathrm{KR}}(G) \in \mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$. We want to give a geometric description of this association in the spirit of Kisin [17] Theorem 2.2.7 and Cais-Lau [8] pg 28.

Let S be the ϖ-adic completion of the $\mathcal{O}_F$-divided power envelope of $\mathfrak{S}$ with respect to the ideal $E(u)\mathfrak{S}$. By 4.1.1, we know that φ on $\mathfrak{S}$ extends to S. Furthermore, the action of Γ extends to S since by (5.1.1), we have that $\gamma(E(u))$ is a unit multiple of $E(u)$ and moreover this means that $\mathrm{Fil}^i S$ is stable under Γ.

Definition 5.1.11. Let $\mathrm{BT}_S^{\varphi,\Gamma}$ denote the subcategory of $\mathrm{Mod}_S^{\varphi,1}$ of finite free S-modules $\mathcal{M}$ equipped with a continuous semilinear action of Γ that commutes with φ_1 and such that the induced action of Γ is trivial on $\mathcal{M} \otimes_S W(k)_F$, where $W(k)_F$ is an S-module by the extension of the map

$$\mathfrak{S} \to \mathfrak{S}/u\mathfrak{S} = W(k)_F.$$

We get such a map $S \to W(k)_F$ since the image of $E(u)$ is in (ϖ), which has $\mathcal{O}_F$-divided powers. Morphisms in $\mathrm{BT}_S^{\varphi,\Gamma}$ are morphisms of $\mathrm{Mod}_S^{\varphi,1}$ that also commute with the actions of Γ.

We have the following Corollary to Theorem 4.3.7:

Corollary 5.1.12. *The equivalence* $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,1} \to \mathrm{Mod}_S^{\varphi,1}$ *of Theorem* 4.3.7 *induces an equivalence between the categories* $\mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$ *and* $\mathrm{BT}_S^{\varphi,\Gamma}$*, where the action of* Γ *on* $S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ *is given by* $\gamma \otimes \gamma$ *for any* $\gamma \in \Gamma$.

Proof. Let us write $\Theta_1 : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,1} \to \mathrm{Mod}\, S^{\varphi,1}$ for the equivalence of Theorem 4.3.7 induced by $\Theta_1(\mathfrak{M}) = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. For $\mathfrak{M} \in \mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$, give $\Theta_1(\mathfrak{M})$ an action of Γ by $\gamma \otimes \gamma$ for any $\gamma \in \Gamma$ and this is well-defined since γ commutes with φ.

For an object $\mathcal{M}$ of $\mathrm{BT}_S^{\varphi,\Gamma}$, let $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,1}$ be its descent as an object of $\mathrm{Mod}_S^{\varphi,1}$. For $\gamma \in \Gamma$, the S-module $\gamma^*\mathcal{M} := S \otimes_{\gamma,S} \mathcal{M}$ is an object of $\mathrm{Mod}_S^{\varphi,1}$ by setting

$$\mathrm{Fil}^1 \gamma^*\mathcal{M} := \gamma^* \mathrm{Fil}^1 \mathcal{M}, \quad \text{and} \quad \varphi_{\gamma^*\mathcal{M},1} := \varphi \otimes \varphi_{\mathcal{M},1},$$

and this can be checked using the definitions and the fact that $\varphi_{\mathcal{M},r}$ on $\mathrm{Fil}^r \mathcal{M}$ and φ on S commute with γ. Furthermore, it can be checked by the definitions that the descent of $\gamma^*\mathcal{M}$ to $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,1}$ is $\gamma^*\mathfrak{M} := \mathfrak{S} \otimes_{\gamma,\mathfrak{S}} \mathfrak{M}$. Furthermore, since the action of Γ commutes with the φ_r, it is easy to check that associated to any $\gamma \in \Gamma$ and $\mathcal{M} \in \mathrm{BT}_S^{\varphi,\Gamma}$ is a morphism $c_{\gamma,\mathcal{M}} \in \mathrm{Hom}_{S,\varphi}(\gamma^*\mathcal{M}, \mathcal{M})$ given by $c_\gamma = 1 \otimes \gamma : \gamma^*\mathcal{M} \to \mathcal{M}$. Since the action of Γ is a group action, we have that

$$c_{\alpha\beta,\mathcal{M}} = c_{\beta,\mathcal{M}} \circ c_{\alpha,\beta^*\mathcal{M}} \quad \text{and} \quad c_{1,\mathcal{M}} = \mathrm{id}\,. \tag{5.1.4}$$

By the full-faithfulness of Θ_1, for any $\gamma \in \Gamma$ we get a morphism $c_{\gamma,\mathfrak{M}}$ of $\mathrm{Mod}_S^{\varphi,1}$, that is, a morphism in $\mathrm{Hom}_{\mathfrak{S},\varphi}(\gamma^*\mathfrak{M}, \mathfrak{M})$. These morphisms satisfy the properties of 5.1.4) by full-faithfulness and so gives a semilinear action of Γ on $\mathfrak{M}$ that commutes with $\varphi_{\mathfrak{M}}$. This gives $\mathfrak{M}$ the structure of an object of $\mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$ and shows that the induced functor is essentially surjective.

To show that the induced functor is also fully-faithful follows similar reasoning. One sees that given a morphism $g : \mathcal{M} \to \mathcal{N}$ of $\mathrm{BT}_S^{\varphi,\Gamma}$, and for any $\gamma \in \Gamma$, the morphism g induces a morphism $\gamma^* g : \gamma^*\mathcal{M} \to \gamma^*\mathcal{N}$ and a commutative diagram in $\mathrm{Mod}_S^{\varphi,1}$:

$$\begin{array}{ccc} \gamma^*\mathcal{M} & \xrightarrow{\gamma^* g} & \gamma^*\mathcal{N} \\ \downarrow{\scriptstyle \mathrm{id}\otimes\gamma} & & \downarrow{\scriptstyle \mathrm{id}\otimes\gamma} \\ \mathcal{M} & \xrightarrow{g} & \mathcal{N} \end{array}$$

and this follows since g commutes with the actions of Γ. If $f : \mathfrak{M} \to \mathfrak{N}$ is the descent of g as a morphism in $\mathrm{Mod}_S^{\varphi,1}$, then the above diagram also descends and this shows that f commutes with the action of γ for any $\gamma \in \Gamma$. Therefore, f is in fact a morphism in $\mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$, and this completes the proof.

□

5.2. ϖ-divisible $\mathcal{O}$-modules and the results of Faltings

Throughout this section, let $\mathcal{O} := \mathcal{O}_F$. We will give the definition of ϖ-divisible $\mathcal{O}$-modules as in Fargues [13] B.2, and this matches the description of ϖ-divisible groups in Faltings [12].

Definition 5.2.1. A *p-divisible group of height h* over a scheme X is a directed system $G = \{G_n,\ \iota_n : G_n \to G_{n+1}\}_{n\geq 1}$ of finite flat group schemes over X such that G_n is p^n-torsion of order p^{nh} and such that

$$0 \to G_n \xrightarrow{\iota_n} G_{n+1} \xrightarrow{\cdot p^n} G_{n+1}[p] \to 0$$

is a short exact sequence (in particular, ι_n gives an isomorphism between G_n and $G_{n+1}[p^n]$). A *morphism* of p-divisible groups, $f : G \to H$, is a compatible system of X-group maps $f_n : G_n \to H_n$. Compatibility means that for every $n \geq 1$ the following diagram commutes

$$\begin{array}{ccc} G_n & \xrightarrow{f_n} & H_n \\ {\scriptstyle \iota_n}\downarrow & & \downarrow{\scriptstyle \iota_n} \\ G_{n+1} & \xrightarrow{f_{n+1}} & H_{n+1} \end{array}$$

For a p-divisible group G over $\mathcal{O}$, the Lie group of G is (see section 1 of [21])

$$\mathrm{Lie}(G) = \ker(G(\mathcal{O}[\varepsilon]/\varepsilon^2) \to G(\mathcal{O})),$$

where the map $G(\mathcal{O}[\varepsilon]/\varepsilon^2) \to G(\mathcal{O})$ corresponds to the map $\mathcal{O}[\varepsilon]/\varepsilon^2 \to \mathcal{O}$ given by $b + c\varepsilon \mapsto b$. The Lie group $\mathrm{Lie}(G)$ has a natural action of $\mathcal{O}$ as follows: for $a \in \mathcal{O}$, there is a ring homomorphism

$$u_a : \mathcal{O}[\varepsilon]/\varepsilon^2 \to \mathcal{O}[\varepsilon]/\varepsilon^2, \quad b + c\varepsilon \mapsto b + ca\varepsilon.$$

That this is a ring homomorphism is easy to check, though we show multiplication. On the one hand we have

$$(b + c\varepsilon)(d + g\varepsilon) = bd + (bg + cd)\varepsilon \mapsto bd + (bg + cd)a\varepsilon$$

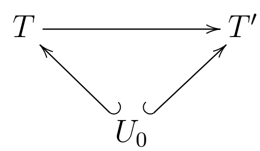
and on the other we have

$$(b + ca\varepsilon)(d + ga\varepsilon) = bd + (bga + cda)\varepsilon = bd + (bg + cd)a\varepsilon.$$

Then this gives an action on $G(\mathcal{O}[\varepsilon]/\varepsilon^2)$ as $a \cdot h = G(u_a)(h)$ for any $h \in G(\mathcal{O}[\varepsilon]/\varepsilon^2)$. Furthermore, the action is stable on the kernel, $\mathrm{Lie}(G)$, of $G(\mathcal{O}[\varepsilon]/\varepsilon^2) \to G(\mathcal{O})$. Thus, we get an induced action of $\mathcal{O}$ on $\mathrm{Lie}(G)$.

Definition 5.2.2. A *ϖ-divisible $\mathcal{O}$-module* over an $\mathcal{O}$-scheme X is a p-divisible group G over X equipped with an action of $\mathcal{O}$ on G that induces the natural action of $\mathcal{O}$ on $\mathrm{Lie}(G)$. That is, for any $a \in \mathcal{O}$, there is a morphism $\{a\} : G \to G$ with $\{aa'\} = \{a\} \circ \{a'\}$ and $\{a + a'\} = \{a\} + \{a'\}$ and $\{1\} = \mathrm{id}_G$ and such that the induced action on $\mathrm{Lie}(G)$ is the action described above.

Let $U_0 = \mathrm{Spec}(\mathcal{O}_K/(\varpi))$ and $\Sigma = \mathrm{Spec}(\mathcal{O})$. We consider a category of $\mathcal{O}$-P.D.-thickenings of $\mathcal{O}_K/(\varpi)$ consisting of pairs (T, δ) where $U_0 \hookrightarrow T$ is a closed immersion such that δ gives the ideal of definition the structure of an $\mathcal{O}$-P.D. ideal as in Definition 3.1.4. Morphisms $(T, \delta) \to (T', \delta')$ are morphisms $T \to T'$ over Σ that commute with the divided powers and such that the following diagram commutes:



For G a ϖ-divisible $\mathcal{O}$-module over $\mathcal{O}_K$, write

$$G_0 \coloneqq G \times_{\mathcal{O}_K} (\mathcal{O}_K/(\varpi)). \tag{5.2.1}$$

Faltings [12] associates to G_0 a functor $\mathbb{D}_{\mathcal{O}}(G_0)$ (written $M(G_0)$ in [12] section 8) on the category of $\mathcal{O}$-P.D.-thickenings (T, δ) with the property that $\mathbb{D}_{\mathcal{O}}(G)(T)$ is a locally free $\mathcal{O}_T$-module. Moreover, Faltings shows that this association is functorial in G_0 and compatible with base change. On page 278 of [12], Faltings gives a connection between $\mathbb{D}_{\mathcal{O}}$ and classical Dieudonné theory. Using the fact that the kernel of the surjection $\mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/(\varpi)$ has topologically $\mathcal{O}$-P.D.-nilpotent divided powers when $p > 2$, it follows from Faltings' construction of $\mathbb{D}_{\mathcal{O}}(G_0)$ that $\mathbb{D}_{\mathcal{O}}(G_0)(\mathcal{O}_K)$ is the Lie algebra of the universal $\mathcal{O}$-vector extension of $G/\mathcal{O}_K$ in the sense of Fargues [13] B.3.3.

Now consider the $\mathcal{O}$-P.D.-thickening $S \twoheadrightarrow \mathcal{O}_K/(\varpi)$ with kernel $(\varpi, \mathrm{Fil}^1 S)$. Set $\mathcal{M}(G) \coloneqq \mathbb{D}_{\mathcal{O}}(G_0)(S)$, which is a free S-module of finite rank.

Proposition 5.2.3. *The finite free S-module $\mathcal{M}(G) = \mathbb{D}_{\mathcal{O}}(G_0)(S)$ can be given the structure of an object of $\mathrm{BT}_S^{\varphi,\Gamma}$.*

Proof. The q-power Frobenius on $\mathcal{O}_K/(\varpi)$ is a map $F : U_0 \to U_0$. Let

$$F_q : G_0 \to G_0^{(q)} \coloneqq G \times_{U_0, F} U_0$$

be the q-power relative Frobenius map of G_0. Now, since $\varphi : S \to S$ is a lift of the q-power Frobenius map, the compatibility of $\mathbb{D}_{\mathcal{O}}$ with base change gives that

$$\begin{aligned}\mathbb{D}_{\mathcal{O}}(G_0^{(q)})(S) &= \mathbb{D}_{\mathcal{O}}(G_0 \times_{U_0,F} U_0)(S)\\ &\cong \mathbb{D}_{\mathcal{O}}(G_0)(S) \underset{S,\varphi}{\otimes} S = \varphi^*\mathcal{M}.\end{aligned}$$

By the functoriality of $\mathbb{D}_{\mathcal{O}}$, we have that F_q induces a map $\varphi^*\mathcal{M} \to \mathcal{M}$ which defines a φ-linear map $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$. Since the surjection $S \twoheadrightarrow \mathcal{O}_K$ is a $\mathcal{O}$-P.D.-morphism of $\mathcal{O}$-P.D. thickenings of $\mathcal{O}_K/(\varpi)$ with kernel $\operatorname{Fil}^1 S$, the compatibility of $\mathbb{D}_{\mathcal{O}}$ with base change gives that

$$\mathcal{M}/(\operatorname{Fil}^1 S \cdot \mathcal{M}) = \mathcal{M} \otimes_S \mathcal{O}_K \cong \mathbb{D}_{\mathcal{O}}(G_0)(\mathcal{O}_K). \tag{5.2.2}$$

By Fargues [13] Section B.3, since $\mathbb{D}_{\mathcal{O}}(G_0)(\mathcal{O}_K)$ is the Lie algebra of the universal $\mathcal{O}$-vector extension of $G/\mathcal{O}_K$, it lies in a short exact sequence

$$0 \to V_{\mathcal{O}}(G) \to \mathbb{D}_{\mathcal{O}}(G_0)(\mathcal{O}_K) \to \operatorname{Lie}(G) \to 0.$$

Define $\operatorname{Fil}^1 \mathcal{M}$ to be the preimage of $V_{\mathcal{O}}(G)$ in $\mathcal{M}$ under the canonical surjection $\mathcal{M} \twoheadrightarrow \mathcal{M}/(\operatorname{Fil}^1 S \cdot \mathcal{M})$, and this contains $\operatorname{Fil}^1 S \cdot \mathcal{M}$ by (5.2.2). By its very construction (see B.3.3 of [13]), $V_{\mathcal{O}}(G)$ is a quotient of the invariant differentials ω_{G^*} on the dual of G. It follows as in Kisin [17] A.2 that $\varphi_{\mathcal{M}}$ carries $\operatorname{Fil}^1 \mathcal{M}$ into $\varpi\mathcal{M}$. Define then the φ-semilinear map $\varphi_{\mathcal{M},1} : \operatorname{Fil}^1 \mathcal{M} \to \mathcal{M}$ by $\varphi_{\mathcal{M},1} = \varphi_{\mathcal{M}}/\varpi$. Since $\mathcal{M} \otimes_S W(k)$ is the classical $\mathcal{O}$-Dieudonné module of $G \times_{\mathcal{O}_K/(\varpi)} k = \overline{G}$ in the sense of Fargues [13] section B.8, the argument of Kisin [17] A.2 shows that the induced map

$$\operatorname{id} \otimes \varphi_{\mathcal{M},1} : \varphi^* \operatorname{Fil}^1 \mathcal{M} \to \mathcal{M}$$

is surjective. Finally, the action of Γ on S, which commutes with $S \twoheadrightarrow \mathcal{O}_K/(\varpi)$ since Γ acts trivially on $\mathcal{O}_K$, is therefore a morphism in the category of $\mathcal{O}$-P.D.-thickenings of $\mathcal{O}_K/(\varpi)$ and thus by functoriality of $\mathbb{D}_{\mathcal{O}}$, we get, by arguing as in Cais-Lau [8] 3.2.1, an action of Γ on $\mathcal{M}$. Finally, this gives $\mathcal{M}$ the structure of an object of $\mathrm{BT}_S^{\varphi,\Gamma}$. $\square$

We construct a ring $A_{\mathrm{cris},F}$ much like in the classical case. This ring is also defined in [12] section 9. Given $(x_n)_{n\geq 0} \in R$, choose a lift $\hat{x}_n \in \mathcal{O}_{\overline{K}}$. The sequence $(\hat{x}_{m+n})^{q^n}$ converges to a $x^{(m)} \in \mathcal{O}_{\overline{K}}$ that does not depend on the choice of lifts. So for any $(x_n)_{n\geq 0} \in R$ we get a sequence $(x^{(n)})_{n\geq 0}$ in $\mathcal{O}_{\overline{K}}$ with $(x^{(n+1)})^q = x^{(n)}$. Let $\widehat{\overline{K}}$ be the completion of $\overline{K}$, and let $\mathcal{O}_{\widehat{\overline{K}}}$ be its ring of integers. Colmez [11] 7.3 defines a map $\Theta : W(R)_F \to \mathcal{O}_{\widehat{\overline{K}}}$ by

$$\Theta\left(\sum_{n=0}^{\infty} \varpi^n [x_n]_T\right) = \sum_{n=0}^{\infty} \varpi^n x_n^{(0)},$$

and Colmez further shows that this is a surjective $W(\overline{k})$-algebra homomorphism. Furthermore, in Proposition 8.6 of [11], Colmez shows that $\ker\Theta$ is principally generated by $E(u) = \varphi(u)/u$, thinking of $\mathfrak{S}$ as a subring of $W(R)_F$ as in 5.1.3.

We define $A_{\mathrm{cris},F}$ to be the ϖ-adic completion of the $\mathcal{O}_F$-divided power envelope of $W(R)_F$ with respect to $\ker\Theta$. Then $A_{\mathrm{cris},F}$ is stable under φ, which can be seen using the same argument as in Proposition 4.1.1. Moreover, $A_{\mathrm{cris},F}$ has a filtration topologically generated by the $\mathcal{O}_F$-divided powers, as defined in 3.2.3. Then $A_{\mathrm{cris},F}$ is stable under G_K since G_{K_∞} fixes $E(u)$ and $\gamma(E(u))$ is a unit multiple of $E(u)$ for any $\gamma \in \Gamma$ as in (5.1.1). Because $\ker\Theta$ is generated by $E(u)$, we have an inclusion $S \hookrightarrow A_{\mathrm{cris},F}$ extending $\mathfrak{S} \hookrightarrow W(R)_F$ that respects the filtrations and actions of Frobenius and G_K. We can therefore consider $A_{\mathrm{cris},F}$ as an S-algebra.

Then given $\mathcal{M} \in \mathrm{BT}_S^{\varphi,\Gamma}$, the contravariant functor

$$T^*_{\mathrm{cris}}(\mathcal{M}) := \mathrm{Hom}_{S,\mathrm{Fil}^1,\varphi_1}(\mathcal{M}, A_{\mathrm{cris},F})$$

gives an $\mathcal{O}_F$-representation of G_K by the G_K-action defined as $f \mapsto \gamma f \gamma^{-1}$.

By the work of Faltings in section 9 of [12] one gets the following connection to the Tate module of a ϖ-divisible $\mathcal{O}_F$-module over $\mathcal{O}_K$.

Theorem 5.2.4. *If G is a ϖ-divisible $\mathcal{O}_F$-module over $\mathcal{O}_K$, let $\mathcal{M}(G) = \mathbb{D}_{\mathcal{O}}(G)(S)$ be the associated S-module in $\mathrm{BT}_S^{\varphi,\Gamma}$, as in Proposition* 5.2.3. *On the other hand, let*

$T_\varpi G$ be the Tate module of G, which is in $\mathrm{Rep}_{\mathcal{O}_F}(G_K)$. Then

$$T^*_{cris}(\mathcal{M}(G)) \cong T_\varpi G$$

as $\mathcal{O}_F[G_K]$-modules. In particular, $T^*_{cris}(\mathcal{M}(G))$ has $\mathcal{O}_F$-rank equal to the $\mathcal{O}_F$-rank of $T_\varpi G$ which is equal to the S-rank of $\mathcal{M}$.

For a π-divisible $\mathcal{O}_F$-module G, if we are in the setting where the Tate module $T_\varpi G$ is in $\mathrm{Rep}^{F-\mathrm{cris}}_{\mathcal{O}_F}(G_K)$, then we have both $\mathfrak{M}_{\mathrm{KR}}(G) \in \mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}}$ and $\mathcal{M}(G) \in \mathrm{BT}^{\varphi,\Gamma}_S$. Using Corollary 5.1.12, we can descend $\mathcal{M}(G)$ to an $\mathfrak{S}$-module $\mathfrak{M}(G)$ in $\mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}}$. We will show that $\mathfrak{M}_{\mathrm{KR}}(G) \cong \mathfrak{M}(G)$.

5.3. Kisin-Ren classification of ϖ-divisible $\mathcal{O}$-modules

We want to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}} & \xrightarrow{T^*} & \mathrm{Rep}_{\mathcal{O}_F}(G_K) \\ {\scriptstyle S\otimes_{\varphi,\mathfrak{S}}(-)}\downarrow & \nearrow{\scriptstyle T^*_{\mathrm{cris}}} & \\ \mathrm{BT}^{\varphi,\Gamma}_S & & \end{array}$$

We know that $T^*(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})$ is a free $\mathcal{O}_F$-module of rank equal to the $\mathfrak{S}$-rank of $\mathfrak{M}$ by (5.1.3). We further know by Theorem 5.2.4 that the $\mathcal{O}_F$-rank of $\mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{\mathrm{cris},F})$ is the S-rank of $\mathcal{M} = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ which is the $\mathfrak{S}$-rank of $\mathfrak{M}$. Viewing $\mathfrak{S}^{\mathrm{ur}}$ as a (φ, G_F)-stable subring of $A_{\mathrm{cris},F}$ by $\mathfrak{S}^{\mathrm{ur}} \hookrightarrow W(R)_F \hookrightarrow A_{\mathrm{cris},F}$, we can consider then the G_K-equivariant morphism of $\mathcal{O}_F$-modules of the same rank,

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{\mathrm{cris},F}) \tag{5.3.1}$$

given by composing a morphism $\mathfrak{M} \to \mathfrak{S}^{\mathrm{ur}}$ with the inclusion $\mathfrak{S}^{\mathrm{ur}} \xrightarrow{\varphi} \mathfrak{S}^{\mathrm{ur}} \to A_{\mathrm{cris},F}$. That is, if ι_φ is this composition, we get a morphism $\mathcal{M} \to A_{\mathrm{cris},F}$ by

$$S \underset{\varphi,\mathfrak{S}}{\otimes} \mathfrak{M} \xrightarrow{\mathrm{id}\otimes\iota_\varphi} A_{\mathrm{cris},F}.$$

This is G_K-equivariant since the morphisms $\mathfrak{M} \to \mathfrak{S}^{\mathrm{ur}}$, $\mathcal{M} \to A_{\mathrm{cris},F}$, and φ all commute with the actions of G_K.

Proposition 5.3.1. *Let* $\mathfrak{M} \in \mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$ *and let* $\mathcal{M} = S \otimes_{\mathfrak{S},\varphi} \mathfrak{M}$*. Then there is a natural,* G_K*-equivariant isomorphism*

$$T^*(\mathfrak{M}) \to T^*_{\mathrm{cris}}(\mathcal{M}).$$

given by (5.3.1).

Proof. The proof follows the argument given by Kisin in [17] Theorem 2.2.7 and Cais-Lau in [8] 4.2.1, and we also give credit to [16] for help with the details of Kisin's proof. Since $\iota_\varphi : \mathfrak{S}^{\mathrm{ur}} \to A_{\mathrm{cris},F}$ is injective, the map (5.3.1) is an injective map of $\mathcal{O}_F$-modules of the same rank. Therefore, it suffices to show that modulo ϖ, the map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})/(\varpi) \to \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{\mathrm{cris},F})/(\varpi)$$

is an isomorphism of k_F-vector spaces. Since these are k_F-vector spaces of the same rank, it it enough to show that this map is injective. Now, consider the short exact sequence

$$0 \to \mathfrak{S}^{\mathrm{ur}} \xrightarrow{\cdot\varpi} \mathfrak{S}^{\mathrm{ur}} \to \mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}} \to 0.$$

Applying $\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, -)$, we get the exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \xrightarrow{\cdot\varpi} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}/\varpi\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}}).$$

This shows that we naturally have an injection

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}})/(\varpi) \hookrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}/\varpi\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}}).$$

Likewise, we have an injection

$$\mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}, A_{\mathrm{cris},F})/(\varpi) \hookrightarrow \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}/\varpi\mathcal{M}, A_{\mathrm{cris},F}/\varpi A_{\mathrm{cris},F}).$$

Therefore, it suffices to show that the map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}/(\varpi), \mathfrak{S}^{\mathrm{ur}}/(\varpi)) \to \mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathcal{M}/(\varpi), A_{\mathrm{cris},F}/(\varpi))$$

is an injection.

Suppose that for $f : \mathfrak{M}/(\varpi) \to \mathfrak{S}^{\mathrm{ur}}/(\varpi)$ we get that the image of

$$\varphi \circ f : \mathcal{M}/(\varpi) \to \mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}}$$

is 0 in $A_{\mathrm{cris},F}/\varpi A_{\mathrm{cris},F}$. We claim that the kernel of $\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}} \to A_{\mathrm{cris},F}/\varpi A_{\mathrm{cris},F}$ is $u^{eq}(\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}})$. To see this, we first note the analogues of Proposition 4.1.3 and Remark 4.2.2 for $A_{\mathrm{cris},F}$, the $\mathcal{O}_F$-divided power envelope of $W(R)_F$ with respect to $\ker\Theta$, which is principally generated by $E(u)$. That is, we have that the following composition is the identity

$$\begin{aligned} R/u^{eq}R \to A_{\mathrm{cris},F}/\varpi A_{\mathrm{cris},F} &\to A_{\mathrm{cris},F}/(\varpi, \mathrm{Fil}^q A_{\mathrm{cris},F}) \\ &\cong W(R)_F/(\varpi, E(u)^q) \\ &\cong R/u^{eq}R, \end{aligned}$$

and so the first map is injective. Moreover, $\mathfrak{S}^{\mathrm{ur}} \cap (u^{eq}W(R)_F) = u^{eq}\mathfrak{S}^{\mathrm{ur}}$ since

$$\mathfrak{S}^{\mathrm{ur}} = \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \cap W(R)_F$$

and u^{eq} is a unit in $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$. Thus, the kernel of the map

$$\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}} \to A_{\mathrm{cris},F}/\varpi A_{\mathrm{cris},F}$$

is $u^{eq}(\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}})$.

Therefore, the image of $\varphi \circ f$ is in $u^{eq}(\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}})$, so we must have that $f(m)$ is in $u^{e}(\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}})$ for any $m \in \mathfrak{M}/\varpi\mathfrak{M}$. But $\mathfrak{M}$ has E-height 1 and so for any $m \in \mathfrak{M}/\varpi\mathfrak{M}$, writing $u^e m = \sum s_i \varphi_{\mathfrak{M}}(m_i)$ for some $s_i \in \mathfrak{S}$ and $m_i \in \mathfrak{M}$, we can see that

$$\begin{aligned} u^e f(m) &= f(u^e m) \\ &= f(\sum_i s_i \varphi_{\mathfrak{M}}(m_i)) \\ &= \sum_i s_i \varphi(f(m_i)) \in u^{eq}(\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}}). \end{aligned}$$

Therefore, as u is not a zero divisor on $\mathfrak{M}/\varpi\mathfrak{M}$, we have $f(m) \in u^{e(q-1)}(\mathfrak{S}^{\mathrm{ur}}/\varpi\mathfrak{S}^{\mathrm{ur}})$. But we also know $q > 2$ and so we can repeat this process and hence conclude that $f(m) = 0$ for all $m \in \mathfrak{M}/\varpi\mathfrak{M}$.

□

By Proposition 5.2.3 and Corollary 5.1.12, if G is a ϖ-divisible $\mathcal{O}_F$-module and $\mathcal{M}(G) = \mathbb{D}_{\mathcal{O}}(G_0)(S)$, we can descend to an $\mathfrak{S}$-module $\mathfrak{M}(G) \in \mathrm{BT}_{\mathfrak{S}}^{\varphi,\Gamma}$ and we now know by Proposition 5.3.1, that

$$T^*(\mathfrak{M}(G)) \cong T^*_{\mathrm{cris}}(\mathcal{M}(G)) \cong T_\varpi G.$$

If it happens that $T_\varpi G \in \mathrm{Rep}_{\mathcal{O}_F}^{F-\mathrm{cris}}(G_K)$, then we get $\mathfrak{M}_{\mathrm{KR}}(G) \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ with $T^*(\mathfrak{M}_{\mathrm{KR}}(G)) \cong T_\varpi G$. Our goal is to show that $\mathfrak{M}(G) \cong \mathfrak{M}_{\mathrm{KR}}(G)$. To do so, we will show that the functor $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\varphi,\Gamma}$ is fully faithful. The equivalence of categories in Theorem 5.1.8 will show that $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}(G) \cong \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_{\mathrm{KR}}(G)$ and so if the scalar extension is fully faithful, we have the desired isomorphism as objects of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$.

We begin with some properties of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ that we will need.

Lemma 5.3.2. *If $\mathfrak{N}$ is an $\mathfrak{S}$-module of finite type that is torsion-free, then*

$$N := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}$$

is a free $\mathcal{O}_{\mathcal{E}}$-module of finite rank with $\mathfrak{N} \to N$ injective. Viewing $\mathfrak{N}$ as a sub $\mathfrak{S}$-module of N, the module $F(\mathfrak{N}) = \mathfrak{N}[1/\varpi] \cap N$, where the intersection can be thought of in $\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{N}$, is a free $\mathfrak{S}$-module of finite rank. Furthermore, if $\mathfrak{N}$ has a φ-semilinear endomorphism, is of finite E-height, and has an action of Γ that is trivial on $\mathfrak{N}/u\mathfrak{N}$, then $F(\mathfrak{N}) \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$.

Proof. The first part of the statement follows essentially verbatim from Fontaine [14] B1.2.4, but since our ring $\mathfrak{S}$ is defined a bit more generally, we will repeat the arguments here. We begin with a result due to Serre and given in [20] Chapter 5 Theorem 3.1 which holds for any ring of the form $\mathfrak{o}[\![X]\!]$ where $\mathfrak{o}$ is a complete discrete valuation ring, and hence holds for $\mathfrak{S}$. It says that for a finitely generated $\mathfrak{S}$-module $\mathfrak{N}$, there exists a quasi-isomorphism $\eta : \mathfrak{N} \to \mathfrak{M}$ with $\mathfrak{M}$ a finitely generated elementary $\mathfrak{S}$-module. Recall that a quasi-isomorphism is a morphism in which the kernel and cokernel are $\mathfrak{S}$-modules of finite length, and an elementary module $\mathfrak{M}$ is a direct sum of a free $\mathfrak{S}$-module and modules of the form $\mathfrak{S}/(\varpi^{n_i})$ for some integers n_i and modules of the form $\mathfrak{S}/(f_j^{m_j})$ for distinguished polynomials f_j (monic, irreducible with non-leading coefficients divisible by ϖ) and integers m_j.

If $\mathfrak{N}$ has no torsion, then the kernel of η, which is finite length and so must be killed by some power of ϖ, must be 0 and hence η is injective. Moreover, if we write $\mathfrak{M} = \mathfrak{M}' \oplus \mathfrak{M}''$ where $\mathfrak{M}'$ has no ϖ-torsion and $\mathfrak{M}''$ is killed by some power of ϖ, then the composite $\mathfrak{N} \to \mathfrak{M} \to \mathfrak{M}'$ must also be injective and has cokernel killed by a power of ϖ. So we can replace $\mathfrak{M}$ by $\mathfrak{M}'$ and η by the composite and assume $\mathfrak{M}$ has no ϖ-torsion. If ϖ^h kills the cokernel of η, then we can define a map from $\mathfrak{M}$ to $\mathfrak{N}$ by sending $x \in \mathfrak{M}$ to $y \in \mathfrak{N}$ where $\eta(y) = \varpi^h x$. The map is well-defined since η is injective, and, because $\mathfrak{M}$ has no ϖ-torsion, this map is injective and so gives an identification of $\mathfrak{M}$ with a submodule $\mathfrak{L}$ of $\mathfrak{N}$ and $\mathfrak{N}/\mathfrak{L}$ is killed by ϖ^h. Since $\mathfrak{N}$ has no torsion whatsoever, $\mathfrak{L}$ and $\mathfrak{M}$ are elementary $\mathfrak{S}$-modules without torsion, hence free. We therefore have injective maps $\mathfrak{L} \to \mathfrak{N}$ and $\mathfrak{N} \to \mathfrak{M}$ with $\mathfrak{L}$, and $\mathfrak{M}$ free and the cokernels of the maps are killed by a power of ϖ.

Now consider the commutative (but not necessarily exact) diagram

$$\begin{array}{ccccc} \mathfrak{L} & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{M} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{L} & \longrightarrow & \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N} & \longrightarrow & \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \end{array}$$

The top row consists of injective maps as established above, and so the bottom row of maps is also injective since $\mathcal{O}_{\mathcal{E}}$ is a flat $\mathfrak{S}$-module (see Definition 5.1.5). The outer vertical maps are injective since $\mathfrak{L}$ and $\mathfrak{M}$ are free, and so in particular the right square of the diagram gives that $\mathfrak{N} \to \mathcal{O}_{\mathcal{E}} \otimes \mathfrak{N}$ is injective. Also, $N = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}$ has no ϖ-torsion because $\mathfrak{N}$ has no ϖ-torsion. To see this, note that the map $\mathfrak{N} \to \mathfrak{N}$ given by multiplication by ϖ must be injective and so the map $N \to N$ given by multiplication by ϖ is injective by flatness of $\mathcal{O}_{\mathcal{E}}$, hence N has no ϖ-torsion. Thus, N is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module that is torsion-free. Since $\mathcal{O}_{\mathcal{E}}$ is a PID, this means N must be free. Because $\mathfrak{M}$ and $\mathfrak{L}$ are isomorphic, they have the same rank d as $\mathfrak{S}$-modules and it follows from the diagram that N is an $\mathcal{O}_{\mathcal{E}}$-module of rank d. Due to injectivity, we can consider all of the modules as sub-modules of $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$.

Define $F(\mathfrak{N}) = \mathfrak{N}[1/\varpi] \cap N$ where we can think of the intersection occurring inside of $\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{N}$. Because the cokernel of the injective map $\mathfrak{N} \to \mathfrak{M}$ is killed by a power of ϖ, we get that $\mathfrak{N}[1/\varpi] = \mathfrak{M}[1/\varpi]$. Also, note that $\mathfrak{S}[1/\varpi] \cap \mathcal{O}_{\mathcal{E}} = \mathfrak{S}$ and so because $\mathfrak{M}$ is free, $F(\mathfrak{M}) = \mathfrak{M}[1/\varpi] \cap (\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}) = \mathfrak{M}$. Therefore, $F(\mathfrak{N}) = \mathfrak{M}[1/\varpi] \cap N \subseteq \mathfrak{M}$, and so is a finitely generated $\mathfrak{S}$-module.

We now know that $F(\mathfrak{N})$ is a finitely generated $\mathfrak{S}$-module with no torsion and we want to show that $F(\mathfrak{N})$ is free. To see this, first consider $F(\mathfrak{N})/\varpi F(\mathfrak{N})$. This is a finitely generated $k[\![u]\!]$-module, and $k[\![u]\!]$ is a PID, and so $F(\mathfrak{N})/\varpi F(\mathfrak{N})$ is free if it has no torsion. We have a map $F(\mathfrak{N})/\varpi F(\mathfrak{N}) \to N/\varpi N$ and this map is injective by the construction of $F(\mathfrak{N})$. That is, if $x \in F(\mathfrak{N}) \cap \varpi N$, meaning $x = \varpi y$ for some $y \in N$, then $y = (1/\varpi) \cdot x$ is in both $\mathfrak{N}[1/\varpi]$ and N, hence $y \in F(\mathfrak{N})$ and $x \in \varpi F(\mathfrak{N})$. But N is a free $\mathcal{O}_{\mathcal{E}}$-module and so $N/\varpi N$ is torsion-free as an $\mathcal{O}_{\mathcal{E}}/(\varpi)$-module, so $F(\mathfrak{N})/\varpi F(\mathfrak{N})$ must be torsion-free as a $k[\![u]\!]$-module and thus free. By Nakayama's Lemma, we can lift a basis of $F(\mathfrak{N})/\varpi F(\mathfrak{N})$ to a generating set for $F(\mathfrak{N})$ and this set is linearly independent since $F(\mathfrak{N})$ has no ϖ-torsion. This shows $F(\mathfrak{N})$ is free.

Finally, if $\mathfrak{N}$ has a Frobenius $\varphi_{\mathfrak{N}}$, then $\varphi_{\mathfrak{N}}$ can be extended to both $\mathfrak{N}[1/\varpi]$ and $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}$ and thus $F(\mathfrak{N})$ is stable under $\varphi_{\mathfrak{N}}$ on $\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{N}$. Likewise, an action of Γ can be extended to $F(\mathfrak{N})$. We just need to check that the properties of the actions of Frobenius and Γ hold on $F(\mathfrak{N})$ if they hold on $\mathfrak{N}$.

Suppose that $\mathfrak{N}$ has finite E-height at most r and let $\{e_1, ..., e_d\}$ be a basis for

$F(\mathfrak{N})$. We know that for $\varphi_{\mathfrak{N}}$ on the $\mathcal{O}_{\mathcal{E}}$-module N, we have that $\mathrm{id}\otimes\varphi : \varphi^* N \to N$ is an isomorphism, so for any $1 \leq i \leq d$, we can write $e_i = \sum_j s_{ij}\varphi_N(n_{ij})$ for $s_{ij} \in \mathcal{O}_{\mathcal{E}}$ and $n_{ij} \in N$. In fact, since $\mathfrak{N}$ is finitely generated, we can assume that $n_{ij} \in \mathfrak{N}$. Now, there is some power of u, say u^{m_i}, such that $u^{m_i}s_{ij} \in \mathfrak{S}$ for all j. On the other hand, $e_i \in \mathfrak{N}[1/\varpi]$, so there is some power of ϖ, say ϖ^{n_i}, such that $\varpi^{n_i}e_i \in \mathfrak{N}$ and then we have that $\varpi^{n_i}E(u)^r e_i = \sum t_{ik}\varphi(n'_{ik})$ for some $t_{ik} \in \mathfrak{S}$ and $n'_{ik} \in \mathfrak{N}$. Now, for an integer h, any term of $E(u)^h$ is divisible by either $u^{eh'}$ or $\varpi^{h'}$ where $h' = \lfloor h/2 \rfloor$. Therefore, let $h_i = r + 2\cdot\max\{m_i, n_i\}$, and then it is easy to see that $E(u)^{h_i}e_i$ is in the $\mathfrak{S}$-span of the image of $\varphi_{\mathfrak{N}}$ on $\mathfrak{N} \subseteq F(\mathfrak{N})$ and thus $F(\mathfrak{N})$ has finite E-height at most $h = \max\{h_1, ..., h_d\}$.

If Γ is trivial on $\mathfrak{N}/u\mathfrak{N}$, then Γ is trivial on $\mathfrak{N}[1/\varpi]/u\mathfrak{N}[1/\varpi]$. We note that it is easy to see that $uF(\mathfrak{N}) = (u\mathfrak{N}[1/\varpi]) \cap N$ since $uN = N$, and thus for $x \in F(\mathfrak{N})$, it follows that $\gamma(x) - x$ is in both $u\mathfrak{N}[1/\varpi]$ and N and so Γ is trivial on $F(\mathfrak{N})/uF(\mathfrak{N})$. □

Lemma 5.3.3. *Suppose that I is an ideal of $\mathfrak{S}[1/\varpi]$ that is stable by Γ. Then I is generated by an element of the form $\lambda = u^{j_0}\prod_{i=1}^{n}\left(\varphi^{i-1}(E(u))\right)^{j_i}$ for some nonnegative integers $n, j_0, ..., j_n$.*

Proof. This is a generalization of Lemma III.8 in [10]. The ring $\mathfrak{S}[1/\varpi]$, which is the localization of $\mathfrak{S}$ at the prime ideal (u), is a principal ideal domain, so let $f(u)$ be a generator of I. By p-adic Weierstrass preparation theorem, [3] Section 7 Proposition 6, the power series $f(u) = \varpi^m c(u)p(u)$, where m is an integer, $c(u)$ is a unit, and $p(u)$ is a distinguished polynomial. So the zeros of $f(u)$ are precisely the zeros of $p(u)$ and this is a finite subset of $\mathcal{O}_{\overline{K}}$. Denote by V this set of zeros. Since I is stable under the action of Γ, we get that $\gamma f(u) = f([\chi(\gamma)](u))$ is divisible by $f(u)$. But the roots of $\gamma f(u)$ are precisely those of the form $[\chi(\gamma^{-1})](z)$ where z is a root of $f(u)$. Hence, V is stable under the action of Γ given by $\gamma \cdot z = [\chi(\gamma)](z)$.

Since V is finite, there exists a $\gamma \in \Gamma$ with $\gamma \neq 1$ such that if $z \in V$, then $[\chi(\gamma)](z) = z$. But then $[\chi(\gamma) - 1](z) = 0$ and we know $\chi(\gamma) - 1 = a\varpi^n$ for some

$n \geq 1$ and $a \in \mathcal{O}_F^\times$. Therefore, for any $z \in V$, we have that $[a\varpi^n](z) = 0$. But $[a\varpi^n] = [a] \circ [\varpi^n]$ and $[a]$ is an automorphism on $\mathcal{O}_{\overline{K}}$ since a is a unit. Therefore, $[\varpi^n](z) = 0$. If $z = 0$, then the minimal polynomial for z is u. Otherwise, there is a smallest integer $i \geq 1$ such that $[\varpi^i](z) = 0$, in which case the minimal polynomial for z is $[\varpi^i](u)/[\varpi^{i-1}(u)] = \varphi^{i-1}(E(u))$. Therefore, let j_0 be the multiplicity of 0 as a root of f and for $i \geq 1$, let j_i be the sum of the multiplicities of roots $z \in V$ of $f(u)$ such that i is the smallest integer with $[\varpi^i](z) = 0$. Then we have that

$$\lambda \coloneqq u^{j_0} \prod_{i=1}^{n} \left(\varphi^{i-1}(E(u))\right)^{j_i}$$

and, at least up to a unit, λ is $p(u)$, so we can write $f(u) = \varpi^n c(u)\lambda$ for a unit $c(u)$, possibly different from before. Since $\varpi^m c(u)$ is a unit in $\mathfrak{S}[1/\varpi]$, it follows that λ is a generator for I. □

Proposition 5.3.4. *The functor* $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\varphi,\Gamma}$ *is fully faithful.*

Proof. The arguments given here are largely taken from [17]. However, after reducing to the rank 1 case, we take a different approach as described by Cais-Lau in [8]. Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ and set $M_i = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_i$. Let $f : M_1 \to M_2$ be a morphism of (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$. Since $\mathcal{O}_{\mathcal{E}}$ is flat over $\mathfrak{S}$, we can view $\mathfrak{M}_i$ as a sub $\mathfrak{S}$-module of M_i and φ_{M_i} is $\varphi_{\mathfrak{M}_i}$ when restricted to $\mathfrak{M}_i$. We therefore have to show that $f(\mathfrak{M}_1) \subseteq \mathfrak{M}_2$. The proof involves a set of reductions.

First, let $\mathfrak{N}' = f(\mathfrak{M}_1) + \mathfrak{M}_2 \subseteq M_2$. This is a finitely generated $\mathfrak{S}$-module with no torsion. Further, $\mathfrak{N}'$ is stable under both φ_{M_2} and Γ (since the actions are compatible with f). If h is an integer at least as large as the E-heights of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and $x \in \mathfrak{N}'$, then write $x = f(y) + z$ and note that for some $s_i, t_j \in \mathfrak{S}$ and $m_i \in \mathfrak{M}_1$ and $m'_j \in \mathfrak{M}_2$, we get

$$\begin{aligned}E(u)^h x &= f(E(u)^h y) + E(u)^h z \\ &= f(\sum s_i \varphi_{\mathfrak{M}_1}(m_i)) + \sum t_j \varphi_{\mathfrak{M}_2}(m'_j) \\ &= \sum s_i \varphi_{M_2}(f(m_i)) + \sum t_j \varphi_{\mathfrak{M}_2}(m'_j),\end{aligned}$$

which is in the $\mathfrak{S}$-span of $\varphi_{M_2}(\mathfrak{N}')$. So $\mathfrak{N}'$ has finite E-height. Also, since Γ is trivial on $\mathfrak{M}_1/u\mathfrak{M}_1$ and $\mathfrak{M}_2/u\mathfrak{M}_2$, we have that for $x = f(y)+z \in \mathfrak{N}'$ and $\gamma \in \Gamma$, there exist $y' \in \mathfrak{M}_1$ and $z' \in \mathfrak{M}_2$ such that

$$\begin{aligned}\gamma(x) &= \gamma(f(y)) + \gamma(z) \\ &= f(\gamma(y)) + \gamma(z) \\ &= f(y + uy') + z + z' \\ &= (f(y) + z) + u(f(y') + z').\end{aligned}$$

We then conclude that Γ is trivial on $\mathfrak{N}'/u\mathfrak{N}'$. So using Lemma 5.3.2, the $\mathfrak{S}$-module

$$\mathfrak{N} := F(\mathfrak{N}') = \mathfrak{N}'[1/\varpi] \cap (\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}')$$

is in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$. Note that since $\mathfrak{M}_2 \subseteq \mathfrak{N}'$, we get $F(\mathfrak{M}_2) \subseteq \mathfrak{N}$. Also, $\mathfrak{M}_2$ is a free $\mathfrak{S}$-module and so $F(\mathfrak{M}_2) = \mathfrak{M}_2$. Since $\mathfrak{N}' \subseteq M_2$,

$$\mathfrak{N} \subseteq \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}' \subseteq \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} M_2 = M_2.$$

The last equality relies on the equality $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}$ which is easy to see since $\mathcal{O}_{\mathcal{E}}$ is the ϖ-adic completion of $\mathfrak{S}[1/u]$. Thus, we have

$$M_2 = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_2 \subseteq \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N} \subseteq \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} M_2 = M_2.$$

So we get $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N} = M_2$. If we can show that $\mathfrak{N} = \mathfrak{M}_2$, then this would imply that $f(\mathfrak{M}_1) + \mathfrak{M}_2 \subseteq \mathfrak{N} \subseteq \mathfrak{M}_2$, or that $f(\mathfrak{M}_1) \subseteq \mathfrak{M}_2$ as desired.

By replacing $\mathfrak{M}_1$ with $\mathfrak{N}$, we have therefore reduced to the case where $\mathfrak{M}_2 \subseteq \mathfrak{M}_1$, and $M_1 = M_2$ with $f : M_1 \to M_2$ the identity. We need only show that $\mathfrak{M}_1 = \mathfrak{M}_2$.

Since $M_1 = M_2$, we know $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have the same rank as $\mathfrak{S}$-modules. Let ι be the inclusion $\mathfrak{M}_2 \hookrightarrow \mathfrak{M}_1$. If we fix bases $\{e_1, ..., e_d\}$ and $\{f_1, ..., f_d\}$ for $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then we just need to show that the determinant of the matrix of ι under these bases is a unit in $\mathfrak{S}$. For $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$, the wedge product $\mathfrak{M} \mapsto \Lambda^d \mathfrak{M}$ is functorial in $\mathfrak{M}$. Moreover, we have $\Lambda^d \varphi^* \mathfrak{M} = \varphi^* \Lambda^d \mathfrak{M}$ and $\Lambda^d \gamma^* \mathfrak{M} = \gamma^* \Lambda^d \mathfrak{M}$ for $\gamma \in \mathfrak{M}$ since Λ^d is a quotient of a tensor product. Then $\Lambda^d \mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$ by considering the $\mathfrak{S}$-linear maps $\mathrm{id} \otimes \varphi$ and $\mathrm{id} \otimes \gamma$ under Λ^d. We therefore have $\det \iota : \Lambda^d \mathfrak{M}_2 \hookrightarrow \Lambda^d \mathfrak{M}_1$ and $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \Lambda^d \mathfrak{M}_1 = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \Lambda^d \mathfrak{M}_2$. Thus, it suffices to just consider the case that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are rank one modules of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$.

In fact, because $\mathfrak{M}_i = F(\mathfrak{M}_i) = \mathfrak{M}_i[1/\varpi] \cap M_i$ and $M_1 = M_2$, we need only show that $\mathfrak{M}_1[1/\varpi] = \mathfrak{M}_2[1/\varpi]$. Let I be the ideal of all $\lambda \in \mathfrak{S}[1/\varpi]$ such that $\lambda \mathfrak{M}_1[1/\varpi] \subseteq \mathfrak{M}_2[1/\varpi]$. Now, I is non-zero since $\mathfrak{M}_2[1/\varpi] \subseteq \mathfrak{M}_1[1/\varpi]$ and the two modules have rank 1 over $\mathfrak{S}[1/\varpi] = K_{0,F}[\![u]\!]$. Furthermore, if $\lambda \in I$ and $\gamma \in \Gamma$, then $\gamma(\lambda) \in I$ since $\gamma(\lambda)m = \gamma(\lambda\gamma^{-1}(m)) \in \mathfrak{M}_2[1/\varpi]$. So I is stable under Γ. We can therefore apply Lemma 5.3.3 to conclude that I is principally generated by an element of the form

$$\lambda = u^{j_0} \prod_{j=1}^{n} \varphi^{i-1}(E(u))^{j_i} \tag{5.3.2}$$

for some n and nonnegative integers $j_0, ..., j_n$. If $\{e\}$ is a basis for $\mathfrak{M}_1$ and $\{e'\}$ is a basis for $\mathfrak{M}_2$, then $e' = se$ for some $s \in \mathfrak{S}$ since $\mathfrak{M}_2[1/\varpi] \subseteq \mathfrak{M}_1[1/\varpi]$. But then this means $s \in I$, so s is divisible by λ. Therefore we can see that $\{\lambda e\}$ is a basis for $\mathfrak{M}_2[1/\varpi]$. Because $\mathfrak{M}_1$ is one-dimensional, we can write $\varphi_{\mathfrak{M}_1}(e) = s_1 e$ for some $s_1 \in \mathfrak{S}$, and since $\mathfrak{M}$ has finite E-height, we can write $E(u)^d = t_1 \varphi_{\mathfrak{M}_1}(e)$ for some $t_1 \in \mathfrak{S}$ and nonnegative integer d. Therefore, $E(u)^d = t_1 s_1$, and since $E(u)$ is irreducible in $\mathfrak{S}$, we can say that $\varphi_{\mathfrak{M}_1}(e) = c_1 E(u)^{d_1} e$ for some integer $d_1 \geq 0$ and c_1 a unit in $\mathfrak{S}$. Likewise, $\varphi_{\mathfrak{M}_2}(\lambda e) = c_2 E(u)^{d_2}(\lambda e)$ for some integer $d_2 \geq 0$ and $c_2 \in \mathfrak{S}^\times$. But $\mathfrak{M}_2 \subseteq \mathfrak{M}_1$, so we can also say that

$$\varphi_{\mathfrak{M}_2}(\lambda e) = \varphi_{\mathfrak{M}_1}(\lambda e) = c_1 \varphi(\lambda) E(u)^{d_1} e.$$

Therefore, we have

$$c_1 E(u)^{d_1} \varphi(\lambda) = c_2 E(u)^{d_2} \lambda.$$

Let $h = d_1 - d_2$. We will show that $\lambda = u^h$. If $d_1 < d_2$, then since $E(u)$ is irreducible, we have that $c_2 c_1^{-1} \lambda = E(u)^{-h} \varphi(\lambda)$. But using the fact that $\varphi(u) = uE(u)$, we calculate from (5.3.2) that $\varphi(\lambda) = u^{j_0} \prod_{i=1}^{n+1} \varphi^{i-1}(E(u))^{j_{i-1}}$. We have thus written both λ and $\varphi(\lambda)$ as a product of irreducibles, so by comparing terms it has to be the case that $n = 1$, $j_1 = 0$, and $j_1 = j_0 - h$. So we would have $\lambda = u^h$, but $\lambda \in \mathfrak{S}[1/\varpi]$, so this would be a contradiction if $h < 0$. Therefore, $d_2 \leq d_1$ and we know that $c_2 c_1^{-1} E(u)^h \lambda = \varphi(\lambda)$. Similarly in this case, we conclude that $n = 1, j_1 = 0$ and $j_0 = j_1 + h = h$, so that $\lambda = u^h$.

Now consider a $\gamma \in \Gamma$ and note that $\gamma(\lambda) = ([\chi(\gamma)](u))^h \equiv \chi(\gamma)^h \lambda$ modulo u^{h+1}. Under the bases of $\{e\}$ and $\{\lambda e\}$ for $\mathfrak{M}_1[1/\varpi]$ and $\mathfrak{M}_2[1/\varpi]$, respectively, we can write the action of γ as $\gamma(e) = b_1 e$ and $\gamma(\lambda e) = b_2 \lambda e$ for $b_1, b_2 \in \mathfrak{S}$. Now, using that $\mathfrak{M}_2[1/\varpi] \subseteq \mathfrak{M}_1[1/\varpi]$, we also get that $\gamma(\lambda e) = \gamma(\lambda) b_1 e$ and so we have that

$$b_2 \lambda = b_1 \gamma(\lambda). \tag{5.3.3}$$

The fact that Γ acts trivially on $\mathfrak{M}_i / u\mathfrak{M}_i$ forces b_i to be congruent to 1 modulo u. Therefore, comparing the u^h terms of (5.3.3) shows that $\lambda = \chi(\gamma)^h \lambda$. So either $\chi(\gamma) = 1$ or $h = 0$, but since this must be true for all γ, we conclude that $h = 0$. Hence, $\lambda = 1$ and $\mathfrak{M}_1[1/\varpi] = \mathfrak{M}_2[1/\varpi]$. □

We therefore have the following result.

Theorem 5.3.5. *If G is a ϖ-divisible $\mathcal{O}_F$-module such that $T_\varpi G \in \mathrm{Rep}_{\mathcal{O}_F}^{F-\mathrm{cris}}(G_K)$, then the $\mathfrak{S}$-modules $\mathfrak{M}(G)$ and $\mathfrak{M}_{KR}(G)$ are naturally isomorphic.*

Proof. By Proposition 5.3.1, we know that $T^*(\mathfrak{M}(G)) \cong T^*(\mathfrak{M}_{\mathrm{KR}}(G)) \cong T_\varpi G$. Therefore, by Theorem 5.1.8, the (φ, Γ)-modules $\mathcal{O}_\mathcal{E} \otimes_\mathfrak{S} \mathfrak{M}(G)$ and $\mathcal{O}_\mathcal{E} \otimes_\mathfrak{S} \mathfrak{M}_{\mathrm{KR}}(G)$

are isomorphic. But then by Proposition 5.3.4, we know that $\mathfrak{M}(G)$ and $\mathfrak{M}_{\mathrm{KR}}(G)$ are isomorphic as objects of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma}$. $\square$

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