AN OPTIMUM ASYMMETRIC PN CODE SEARCH STRATEGY*

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ABSTRACT

A theory is developed which allows one to obtain the optimum asymmetric acquisition search strategy of a PN code despreader when the a priori probability density function is given. The results developed here extend the theory of an optimum symmetric PN code search strategy [1] to the more easily implementable asymmetric search pattern. In the case when the a priori probability density function is Gaussian and for an environment such as the TDRSS (Tracking Data Relay Satellite System), the acquisition time is reduced by about 40% compared to the more standard uniform sweep.

INTRODUCTION

The acquisition circuitry of a despreader (a PN code acquisition and tracking system) is commonly designed so that complete passes are made across the entire code range uncertainty, as shown in Figure 1, during the initial search for the code epoch. The actual search is commonly made in discrete steps one-half a PN code chip apart in time; however, for simplicity in the optimization, we consider "continuous steps" with negligible loss in accuracy. This search, which is commonly implemented by retarding one-half a chip at a time, then integrating and comparing to a threshold (Figure 2), continues until the signal is acquired. This scheme is efficient when the a priori location of the signal in the uncertainty region has a uniform probability density function; however, when the a priori density function is peaked, it is more likely to find the signal in the peaked region than elsewhere, so the full sweep approach may not be the best one.

This paper is concerned with a method that allows one to determine the optimum asymmetric sweep pattern to minimize the acquisition time, while achieving a required probability of signal detection, for a given a priori probability density of the signal location. The calculation is carried out for a Gaussian a priori signal location probability

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density function as illustrated in Figure 3. The approach is general, however, so that it can be applied to any given a priori signal location probability density function.

The basis of this method relies on the fact that any meaningful statistics (see [2], for example) of acquisition time, which is the time required to search the code until acquisition, depends directly upon the number of chips (code symbols) to be searched. Therefore, searching where the likelihood to find the signal first reduces the number of positions and therefore time to search.

A POSTERIORI PROBABILITY AFTER ONE, TWO AND THREE SWEEPS

In this section, we will show how the a posteriori probability density function of the location of the true signal position changes as a function of the number of sweeps across the code phase uncertainty. In Figure 4B, the asymmetric sweep pattern is presented. This scheme, although not symmetric about the midpoint position, is easier to implement than the symmetric scheme (Figure 4A) of reference 1 ([1]) since the retraces do not have to be "jam-set" to the next sweep's code phase position, but just turned around.

Consider an asymmetric search centered at the mean of a symmetric, unimodal, a priori probability density function, as shown in Figure 4B (for the case of N = 4 sweeps). Let L_1, L_2, L_3....L_{N+1} denote the search lengths during the N sweeps (as denoted in Figure 4B for N = 4), and assume that $L_{N+1} \geq L_N \geq L_{N-1} \ldots \geq L_1$. Let p(x) denote the a priori probability density function of the location of the signal. Further, let S_i denote the event that the signal is not detected in anyone of the first i sweeps over regions L_1, L_2,...L_{i+1}. Furthermore, we shall use the notation S_0 to denote the event that the signal is not detected with zero sweeps, which is, of course, a sure event. It is clear that the conditional probability density of the signal location x, given that no sweep has yet occurred, is equal simply to the a priori density function p(x), i.e.,

$$p(x|S_0) = p(x) \tag{1}$$

This density is sketched in Figure 3 for a Gaussian distribution function, although the theory applies to all symmetric, unimodal density functions. Suppose that no signal is detected during the first sweep over $L_1 U L_2$ ($L_1 U L_2$ denotes the sum or union of the two line segments) and that the event S_1 has occurred. The conditional density of $p(x\,|S_1)$ is equal to, by use of Bayes' rule,

$$p(x|S_1) = \frac{P(S_1|x)\ p(x)}{P(S_1)} \tag{2}$$

In (2), the conditional probability density function $p(S_1|x)$ is clearly given by

$$p(S_1|x) = \begin{cases} 1 - P_D & \text{if } x \in L_1UL_2 \\ 1 & \text{if } x \notin L_1UL_2 \end{cases} \tag{3}$$

where P_D is the probability of detection given that the signal is present. The notation $x \in L_1$ and $x \notin L_1$ denotes the fact that the location of the signal is within the set L_1 or not in L_1, respectively, and $P(S_1)$ is the probability of the event S_1:

$$P(S_1) = 1 - P_D\, P(L_1UL_2) \tag{4}$$

where $P(L_1UL_2)$ denotes the probability that the signal location x is with the set L_1UL_2:

$$P(L_1UL_2) = \int_{-L_1}^{L_2} p(x)\, dx \tag{5}$$

Substituting (3) and (4) into (2), we thus obtain, after the first sweep,

$$\begin{aligned} p(x|S_1) &= \frac{(1 - P_D)\, p(x)}{1 - P_D\, P(L_1UL_2)} & x \in (L_1UL_2) \\ &= \frac{p(x)}{1 - P_D\, P(L_1UL_2)} & x \notin (L_1UL_2) \end{aligned} \tag{6}$$

A sketch of (6) is shown in Figure 5A. Notice that the a posteriori density function is smaller inside the region L_1UL_2, but greater outside the region L_1UL_2- It is easy to show that

$$\int_{-\infty}^{\infty} p(x|S_1)\, dx = 1 \tag{7}$$

as, of course, it should. For two sweeps (N = 2), it is easy to show by the same reasoning that the a posteriori density function of the location of the signal is given by

$$p(x|S_3) = \begin{cases} \dfrac{(1 - P_D)^2\, p(x)}{P(S_2)} & x \in (L_1UL_2) \\ \dfrac{(1 - P_D)\, p(x)}{P(S_2)} & x \in (L_3 - L_1) \\ \dfrac{p(x)}{P(S_2)} & x \notin (L_3UL_2) \end{cases} \tag{8}$$

where it will be shown later that

$$P(S_2) = 1 - P_D \, P(L_1 \cup L_2) - P_D(1 - P_D) \, P(L_1 \cup L_2) - P_D(L_3 - L_1) \tag{9}$$

The notation $L_3 - L_1$ denotes the region in L_3 that does not include L_1. A sketch of the a posteriori density function after two sweeps is shown in Figure 5B. Extending the a posteriori density function results to the case of three sweeps leads us to the result

$$p(x|S_3) \begin{cases} = \dfrac{(1 - P_D)^3 \, p(x)}{P(S_3)} & x \in (L_1 \cup L_2) \\[2ex] = \dfrac{(1 - P_D)^2 \, p(x)}{P(S_3)} & x \in (L_3 - L_1) \\[2ex] = \dfrac{(1 - P_D) \, p(x)}{P(S_3)} & x \in (L_4 - L_3) \\[2ex] = \dfrac{p(x)}{P(S_3)} & x \notin (L_3 \cup L_4) \end{cases} \tag{10}$$

where shortly it will be seen that

$$\begin{aligned} P(S_3) = {} & 1 - P_D \, P(L_1 \cup L_3 - P_D(1 - P_D) \, P(L_1 \cup L_2) - P_D \, P(L_3 - L_2) \\ & - P_D \, (1 - P_D)^2 \, P(L_1 \cup L_2) - P_D(1 - P_D) \, P(L_3 - L_1) - P_D \, P(L_4 - L_2) \end{aligned} \tag{11}$$

Again it can be shown that $p(x|S_3)$, integrates to one. The a posteriori density function, $p(x|S_3)$, is sketched in Figure 5C. We see that, as the number of sweeps increases, the a posteriori density function $p(x|S_i)$, approaches a uniform distribution.

Probability of Detection After N Sweeps

In this section, we determine the probability of acquisition after N sweeps. Let P_i , i = 1,2,3,...N, denote, respectively, the probabilities that the signal is acquired during the ith sweep, but not in the lst, 2^{nd},... or (i-1)th sweeps. Therefore, Q_N, the probability of acquiring the signal in N sweeps is given by

$$Q_N = P_1 + P_2 + P_3 + \ldots P_N \, . \tag{12}$$

First consider the value of P_1. The probability of acquiring after the first sweep is the probability of the signal being in the region L_1UL_2 times the probability of obtaining a hit P_D, given that the signal is located in L_1UL_2. Hence,

$$P_1 = P_D \, P(L_1UL_2) \tag{13}$$

The probability P_2 is, by definition, the joint probability of acquiring in the second sweep and not acquiring in the first sweep. So we have

$$P_2 = P_D \, P(L_3 - L_1) + P_D(1 - P_D) \left[P(L_2UL_1)\right] \tag{14}$$

For P_3, we have

$$P_3 = P_D \, P(L_4 - L_2) + P_D(1 - P_D) \, P(L_3 - L_1) + P_D(1 - P_D)^2 \, P(L_2UL_1) \tag{15}$$

In the same manner, P_4 and P_5 are given by (extending Figure 4B in the obvious way)

$$P_4 = P_D \, P(L_6 - L_4) + P_D(1 - P_D) \, P(L_4 - L_2) + P_D(1 - P_D)^2 \, P(L_3 - L_1) + P_D(1 - P_D)^3 \, P(L_1UL_2) \tag{16}$$

and

$$P_5 = P_D \, P(L_6 - L_4) + P_D(1 - P_D) \, P(L_5 - L_3) + P_D(1 - P_D)^2 \, P(L_4 - L_2) \; (1 - P_D)^3 \, P(L_3 - L_1) + P_D(1 - P_D)^4 \, P(L_1UL_2) \tag{17}$$

It therefore follows that the probability of detection in one sweep, Q_1, is given, from (12) and (13)

$$Q_1 = P_D \, P(L_1UL_2) \; . \tag{18}$$

The interpretation of Q_N is the acquisition probability accumulated after N sweeps. Typically Q_N would be 0.5 or 0.9 in many applications. When Q_N is 0.9, it means that the probability of acquisition is 0.9 at the end of the Nth sweep. Now, to find Q_2, we add P_1 to P_2 . From (12, (13) and (14), we have

$$Q_2 = P_D \, P(L_1UL_2) + P_D \, P(L_3 - L_1) + P_D(1 - P_D) \, P(L_2UL_1) \tag{19}$$

Notice that $P(S_2) = 1 - Q_2$ and, in general, $P(S_N) = 1 - Q_N$. Since the probabilities are additive, we have

$$P(L_1 U L_2) = P(L_1) + P(L_2) \tag{20}$$

and

$$P(L_3 - L_1) = P(L_3) - P(L_1) \tag{21}$$

Using (20) and (21) in (19) leads us to

$$Q_2 = P_D(1 - P_D)\ P(L_1) + \left[P_D + P_D(1 - P_D)\right] + P_D\ P(L_3) \tag{22}$$

In the same way, it can be shown that Q_3 and Q_4 are given by

$$Q_3 = P_D(1 - P_D)^2\ P(L_1) + P_D\left[(1 - P_D)^2 + (L - P_D)\right] P(L_2) + P_D(2 - P_D)\ P(L_3) + P_D\ P(L_4) \tag{23}$$

and

$$Q_4 = P_D(1 - P_D)^3\ P(L_1) + P_D\left[(1 - P_D)^2 + (1 - P_D)^3\right] P(L_2) + P_D\left[(1 - P_D) + (1 - P_D)^2\right] P(L_3) + P_D(2 - P_D)\ P(L_4) \tag{24}$$

The Q_i's will be used to obtain the optimum sweep length.

OPTIMUM ASYMMETRIC SEARCH STRATEGY

In this section, we specify the optimum lengths $\{L_i\}$ so that the total search length, given by the sum of all the individaul sweep segments, is minimized.

Define T_{Q_N} as the time required to complete N sweeps with probability Q_N. It is assumed that T_{Q_N} is proportional to the sum of the individual sweep times. The proportionality factor depends upon the false alarm probability, the dwell time, etc.

Hence, our problem becomes: determine the optimum search lengths L_1 , L_2,...L_N, L_{N+1} for our N sweep procedure so that Q_N equals the desired acquisition probability and so that

$$T_{Q_N} = K \sum_{i=1}^{N} \left[L_i + L_{i+1}\right] \tag{25}$$

is minimized. The parameter K relates acquisition time to code length search length segments $\{L_i\}$. Our optimization procedure is to use the La Grane multiplier method. Let F be given by

$$F_N = Q_N - \lambda \sum_{i=1}^{N} \left[L_i + L_{i+1}\right] \tag{26}$$

where λ is the unknown La Grange multiplier. Up to this point, the theory is quite general, the only requirement being that the a priori density be unimodal and symmetric and that $P(L_i)$ be differentiable.

Since this problem was initially motivated by the need to improve the acquisition time for the TDRSS multiple-access ground receiver [3] ana since the best estimates for the a priori location of the signal were Gaussian, we shall illustrate the theory by assuming that the a priori density function is Gaussian. With the Gaussian assumption, we have

$$P(L_i) \begin{cases} = \displaystyle\int_0^{L_i} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt & (i \text{ odd}) \\ = \displaystyle\int_{-L_i}^{0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt & (i \text{ even}) \end{cases} \tag{27}$$

For N = 1, it is easy to show that $L_1 = L_2 = L$ and a solution exists if L is large enough that Q_1 is equal to the acquisition probability. A more interesting case occurs for two sweeps (N = 2). From (12), we have

$$Q_2 = (P_D - P_D)^2 P(L_1) + (2P_D - P_D^2) P(L_2) + P_D P(L_3) \tag{28}$$

where

$$P(L_i) = \int_0^{L_i} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = Q\left(\frac{L_i}{\sigma}\right) \tag{29}$$

From our La Grange function F (26), we have

$$F_2 = L_1 + 2L_2 + L_3 - \lambda P_D \left[(1 - P_D)\ P(L_1) + (2 - P_D)\ P(L_2) + P(L_3)\right] \tag{30}$$

Letting $\lambda' = \lambda/\sigma$ and differentiating with respect to L_1 gives us

$$1 = \frac{\lambda'\ P_D(1 - P_D)}{\sqrt{2\pi}} e^{-\frac{L_1^2}{2\sigma^2}} \tag{31}$$

Solving (31) for L_1 produces

$$\ell_1 = \frac{L_1}{\sigma} = \sqrt{2}\ \sqrt{\ell n \left\{\frac{\lambda'\ P_D(1 - P_D)}{\sqrt{2\pi}}\right\}} \tag{32}$$

This equation can be written as

$$\ell_1 = \sqrt{2}\sqrt{C + \ell n\ \left[P_D(1 - P_D)\right]} \tag{33}$$

where C is the constant

$$C = \ell n \left(\frac{\lambda'}{\sqrt{2\pi}}\right) \tag{34}$$

and ℓ_i is the normalized chip uncertainty. In the same manner, we can solve for the optimum value of ℓ_2 by solving

$$\frac{\partial F_2}{\partial L_2} = 2 - \lambda'\ P_D(2 - P_D)\ \frac{\partial Q(L_2/\sigma)}{\partial L_2} = 0 \tag{35}$$

for ℓ_2 . We obtain

$$\ell_2 = \sqrt{2}\sqrt{C + \ell n\ \left[\left(1 - \frac{P_D}{2}\right) P_D\right]} \tag{36}$$

in the manner the optimum value of ℓ_3 satisifes

$$\ell_3 = \sqrt{2}\sqrt{C + \ell n\ [P_D]} \tag{37}$$

Substituting (33), (36) and (37) for ℓ_i back into the equation for Q_2 (29) allows one, in principle, to solve for λ' and therefore C. Unfortunately, the resulting transcendental equation makes it nearly impossible to solve for C analytically. However, the solution can be solved simply on a digital computer by trial and error, choosing values of C so that Q_2 equals the desired value. The actual optimum may occur for values of N > 2. Hence, in general, the solutions must be obtained for all values of N, and the value of N which minimizes the value of T_{Q_N} corresponds to the true optimum under the constraint of an asymmetric search pattern.

Now consider the solution for N = 3. From (23), (25) and (26), we have

$$F_3 = L_1 + 2L_2 + 2L_3 + L_4 - \lambda P_D\left[(1-P_D)^2 P(L_1) + (2-3P_D+P_D^2)P(L_2) + (2-P_D)P(L_3) + P(L_4)\right] \quad (38)$$

Differentiating F_3 in respect to L_1, L_2, L_3 and L_4, respectively, we arrive at

$$\ell_1 = \sqrt{2}\sqrt{C + \ln\left[P_D(1-P_D)^2\right]} \quad (39)$$

$$\ell_2 = \sqrt{2}\sqrt{C + \ln\left[P_D - \frac{3}{2}P_D^2 + \frac{1}{2}P_D^3\right]} \quad (40)$$

$$\ell_3 = 2\sqrt{C + \ln\left[P_D - \frac{1}{2}P_D^2\right]} \quad (41)$$

$$\ell_4 = \sqrt{2}\sqrt{C + \ln\left[P_D\right]} \quad (42)$$

In general, this procedure can be continued for any desired value of N.

UNIFORM A PRIORI DENSITY SWEEP STRATEGY

The usual strategy for sweeping to obtain acquisition is to start at the end of the uncertainty region where the range delay is minimal, then retard the range in increments of, typically, one-half chips. By sweeping from the minimum delay to the maximum delay, the chances of acquiring a multipath signal are diminished. If the probability of detection, given that the received code and reference code are aligned, is given by P_d and, if the a priori probability density function is Gaussian with zero mean and $6\sigma = \Delta T$, then the cumulative probability of acquisition is as shown in Figure 6. If, for example, a probability of 0.5 is chosen as the desired probability of acquisition, the curve could be read off the abscissa, and the associated time, denoted by $T_{.5}$, would be the time it takes to acquire with a probability of 0.5 (the median acquisition time)

A measure of the improvement of the optimized scheme over the uniform sweep scheme can be measured as follows. Denote T_Q^U as the time to acquire with a probability of Q using the uniform sweep approach. Next, let T_Q^0 denote the time to acquire with the optimized sweep, the improvement factor of the optimized sweep over the uniform sweep is then given by

$$r_Q = \frac{T_Q^U}{T_Q^0} \tag{43}$$

The acquisition time is then $T_Q^U \, r_Q^{-1} = T_Q^0$. Clearly, $r_Q \geq 1$ since unity is achieved with the uniform sweep strategy and therefore the method never increases acquisition time.

NUMERICAL RESULTS

In this section, we present some actual optimizations for a few cases of interest. In what follows, we let $\Delta T = 6\sigma$ and neglect the end effects. In Table 1, the case of $P_D = 0.25$ and $Q = 0.5$ was specified so that the acquisition time was, in fact, the median time.

Table 1. $P_D = 0.25$, $Q = 0.5$

	Optimum Search with Three Sweeps
$\frac{L_i}{\sigma}$	1.273 1.399 1.593 1.665
r_Q	1.63

As can be seen from Table 1 when $P_D = 0.25$ and $Q = 0.5$, a reduction to $1/r_Q = 61.3\%$ was obtained. In Table 2, the parameters used were $P_D = 0.6$ and $Q = 0.9$.

Table 2. $P_D = 0.6$ and $Q = 0.9$

	Optimum Search with Three Sweeps
$\frac{L_i}{\sigma}$	1.529
	1.850
	2.281
	2.50
r_Q	1.27

As can be seen from Table 2, when Q increases, the improvement factor decreases; in this case, a reduction to only 78% was obtained. A subtle point pertaining to the relationship between r_Q P_D and Q is best illustrated in Figure 7 based on the theory given here [4]. A can be seen from the figure, only certain values of P_D and Q give a reduction in acquisition time.

CONCLUSIONS

A general method has been presented that can be used to optimize (minimize) the acquisition time for a PN-type spread spectrum system when the a priori probability density function is not uniform by utilizing an asymmetric sweep.

Specifically, we have calculated for an assumed a priori Gaussian density function that the acquisition time, when the 0.5 probability acquisition time (median) was used as a measure of acquisition time, was reduced by 39% for a cell detection probability of 0.25 and when three sweeps were used. When the acquisition time probability was set to 0.9 instead of 0.5, the reduction was only 22% of the uniform sweep acquisition time.

Although the calculations were for Gaussian a priori density functions, the theory is directly applicable to unimodal, symmetric a priori density functions and $P(L_i)$ is differentiable. Extensions to more general a priori density functions could also be made.

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Figure 1 Uniform Sweep Strategy.

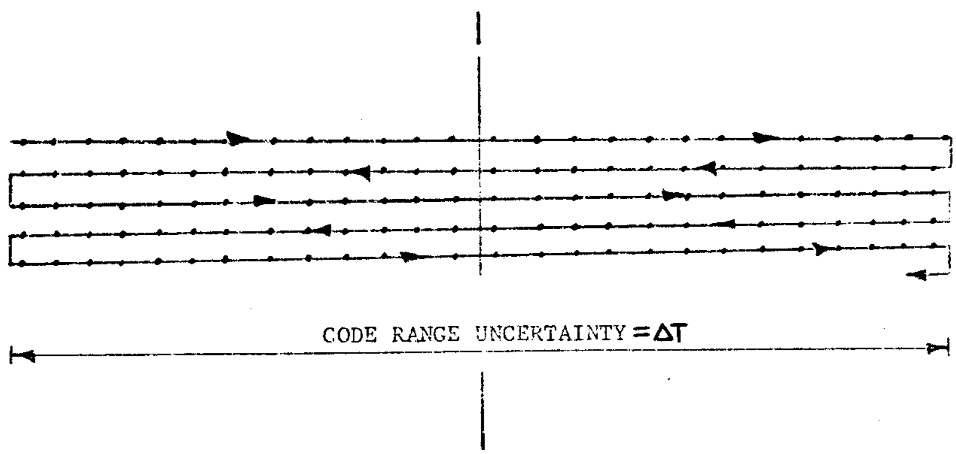
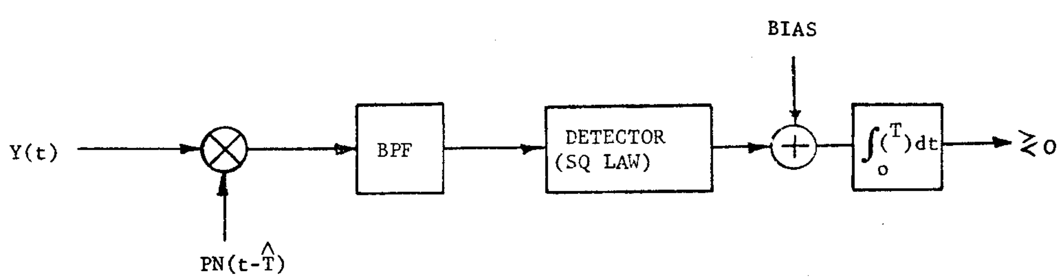


FIGURE 2 TYPICAL SIMPLIFIED FIXED DWELL TIME ACQUISITION SYSTEM



$$Y(t) = \sqrt{2P}\, d(t) PN(t-T) \cos(\omega_0 t + \theta_0) + N(t)$$

Figure 3 Gaussian Location of The Signal.

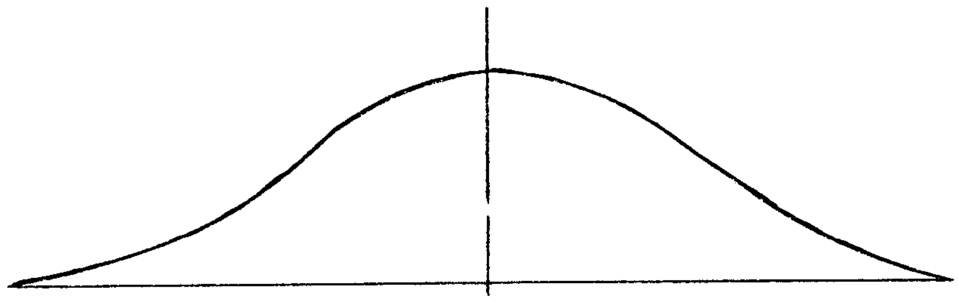


Figure 4A Symmetric Search Pattern

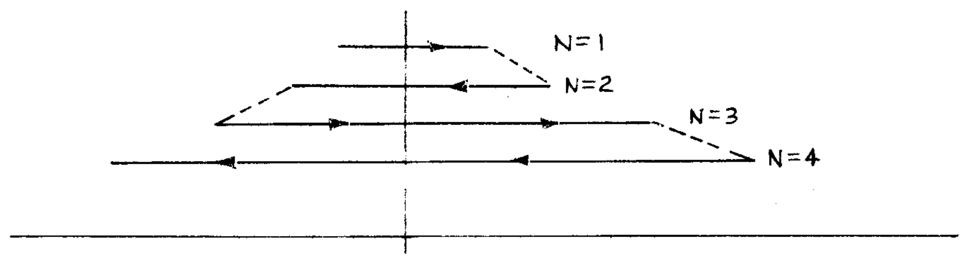
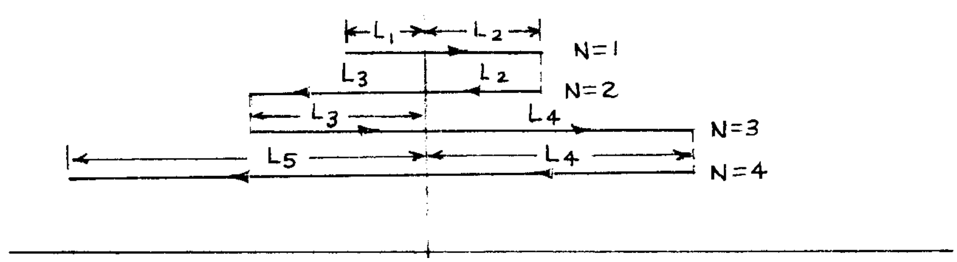


Figure 4B Asymmetric Search Pattern



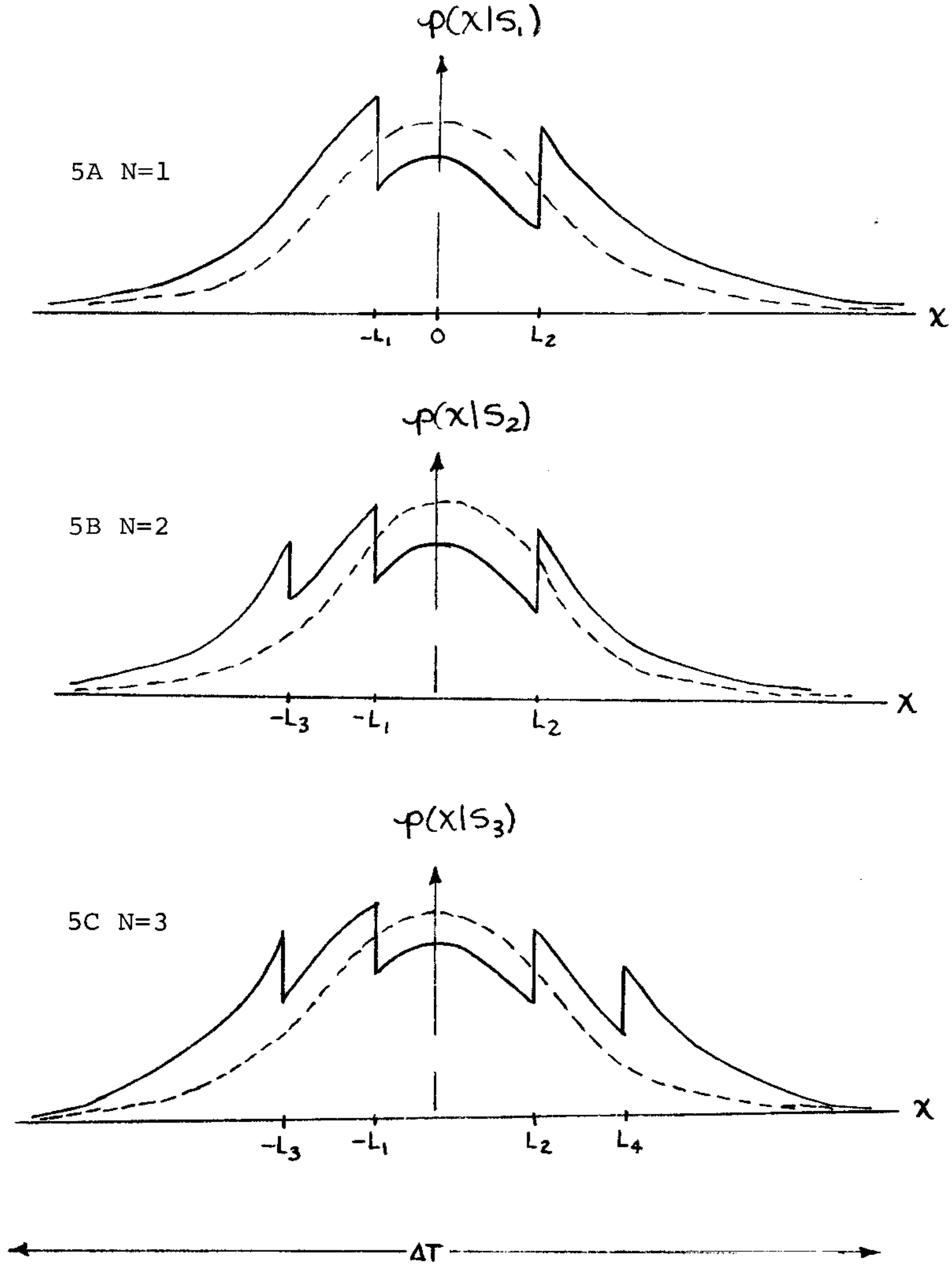


Figure 5. Aposteriori Density Function After One, Two, and Three Sweeps.

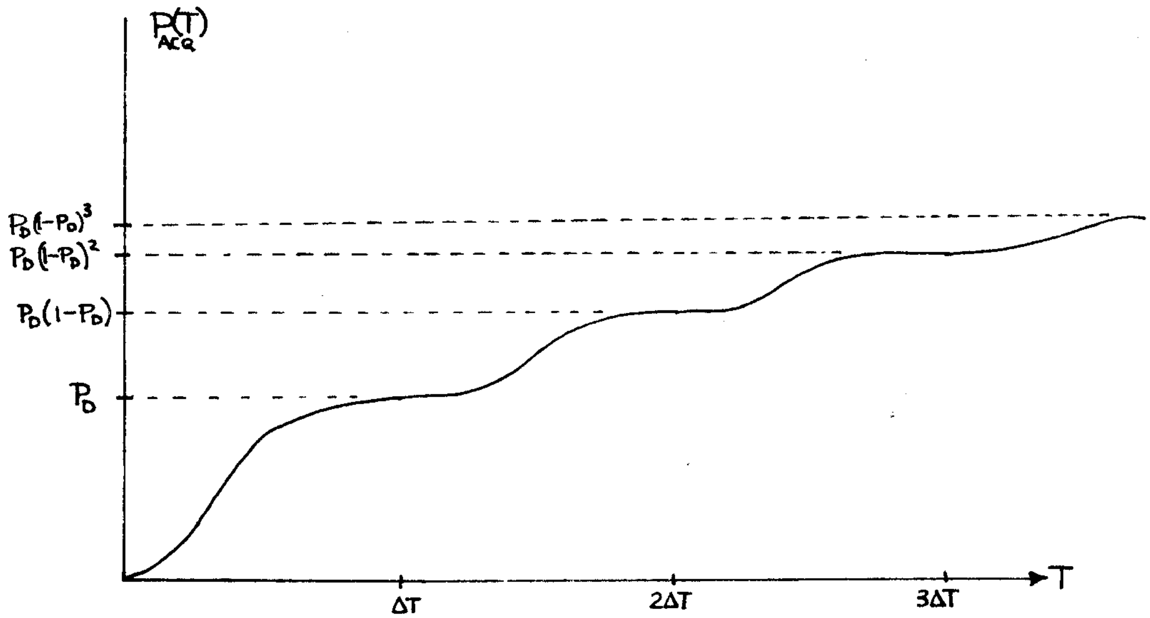


Figure 6. Cumultive Acquisition Time probability versus Acquisition Time, T.