ERROR PERFORMANCE BOUNDS FOR M-ARY DIGITAL FM WITH PREDETECTION SAMPLING

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ABSTRACT

Coherent detection of full response M-ary digital FM corrupted by additive white gaussian noise is studied. Prior to detection processing the signal plus noise is bandpass filtered and sampled. Upper error bounds which are applicable to the sampled system are given. With these bounds some comparisons of the effects of system parameter selection on the error performance can be made. These system parameters include deviation ratio, baseband pulse shape, sampling rate, number of levels (M), and signal-to-noise ratio.

INTRODUCTION

As data rates increase to meet the demand of higher information transfer the consideration or use of complex modulation schemes is needed. One such modulation type which has enjoyed some inspirational development as of late is M-ary digital FM, that is the digital FM scheme which uses a larger symbol alphabet than the simple binary case. Since most of the digital data is binary in its original form the larger alphabet sizes are usually powers of two owing to the fact that the symbols are generated by grouping n bit blocks into words. Generated in this manner, these words take on one of the M different values available in the alphabet where $M = 2^n$. For high complexity systems the detection problem is one of determining which one of the M symbols was sent in a particular symbol period which can be a sizable task even for moderate n values. For $n = 8$ the detection process must identify each received symbol as one of a possible 256. The implementation problems this suggests in the analog domain can be addressed more satisfactorily in the digital domain leading to the notion of a digital detection processor. Certainly, the employment of a digital processor introduces some performance characteristics not found in the analog detectors, such as quantization and sampling effects, so it is necessary to develop the performance analysis to incorporate these effects. As a precursor to a full analysis, this study concerns the effects of predetection sampling without regard to quantization effects or arithmetic round-off errors.

The performance bounds presented herein are based on statistically optimal receiver structures which detect the baseband data by processing the modulated signal directly. This differs greatly, in concept and implementation, from the baseband detection processing at the output of a limiter-discriminator which may be considered the more common or conventional approach. With this in mind the following assumptions and system constraints are listed for which the ensuing development is valid.

1) Random input data, uniformly distributed
2) Full response baseband signalling
3) Representative waveforms: rectangular, half cycle sinusoid, and raised cosine
4) Additive white gaussian noise channel
5) Fully coherent detection
6) Detection interval length of one symbol period
7) Ideal bandpass presampling filter with a bandwidth of one-half of the sampling rate

The first assumption is that the digital input data is random and uniformly distributed. For a continuous sequence of M-ary symbols this means that any symbol in any location in that sequence will take on one of the M distinct values of the alphabet with probability 1/M. Let a(j) denote the data symbol existing during the j^{th} time slot, then the M-ary data sequence at time t will be the sum of the current symbol and all previous symbols, expressed as

$$\sum_{j=-\infty}^{k} a(j) \tag{1}$$

where

$$kT_s \le t < (k+1)T_s \tag{2}$$

$$T_s = \text{the symbol period} \tag{3}$$

For convenience, the values that the a(j)'s may take are a_i for i = 1, 2, . . . , M where a_i = 2i - (M+1). This convention provides equally spaced baseband symbols that are symmetric about zero.

The digital FM signal studied herein is a generalized form of the continuous phase frequency shift keying (CPFSK) signal. The properties of phase continuity of the modulated signal over the symbol boundaries and a single valued deviation ratio are common, but the baseband pulse shaping allowed in the digital FM case permits frequency variations of the FM signal during the symbol period. This added degree of freedom affects the range of achievable performance beyond that of the CPFSK, and it yields another system parameter for control of power spectral characteristics in addition to the deviation ratio and value of M. The full response baseband signalling constraint means that the pulse shaping function, f(t), has an existence interval of $[0,T_s)$ which avoids any baseband symbol overlap. Since the frequency modulation process involves an integration of the baseband then a normalized phase function is introduced. This is

$$g\left(\frac{t}{T_s}\right) = \frac{1}{T_s}\int_0^t f(u)\,du \tag{4}$$

By applying an additional constraint that the pulse shape function, f(t), has an absolute maximum of 1 somewhere in the interval $[0,T_s)$ then the deviation ratio, h, can be defined as the ratio of the difference of two adjacent signalling frequencies to the symbol rate. With this nomenclature introduced the digital FM signal can be expressed in the time interval $[kT_s,(k+1)T_s)$ as

$$s(t) = A\cos\left[\omega_c(t-kT_s) + \pi h a(k) g\left(\frac{t}{T_s} - k\right) + \phi_k\right] \tag{5}$$

where

$$\omega_c = \text{the carrier frequency} \tag{6}$$

$$\phi_k = \pi h g(1)\sum_{j=-\infty}^{k-1} a(j) + \omega_c k T_s + \theta_0 \tag{7}$$

$$\theta_0 = \text{the initial carrier phase} \tag{8}$$

CHANNEL MODELLING

Equation 5 is an expression for the digital FM signal which is applied to the channel. In this study the channel is assumed to be an additive white gaussian noise channel so the received signal prior to the presampling filter is simply the digital FM signal plus white gaussian noise with double sided power spectral density of $N_0/2$. Prior to detection processing this received signal is first order bandpass sampled with a presampling filter specified as an ideal bandpass filter (rectangular frequency response) with a bandwidth of one-half of the sampling rate. It is convenient to define the sampling rate in terms of the symbol period, so for a system which has N samples per symbol the sampling rate is N/T_s. After the filter and sampler the received signal can be represented in vector form as **r = s** + **n** where the components of the signal vector are the time samples of s(t) when the discrete time index defines t as

$$t = \left(\frac{n}{N} + k\right)T_s \tag{9}$$

The elements of the noise vector are the samples of the white gaussian noise process taken after the presampling filter at the time instants defined by Equation 9. The statistics of these noise samples depend not only on the shape and relative bandwidth of the presampling filter, but also on its position relative to the sampling frequency. Assuming that the lower bandedge of is set to some integer multiple of one-half of the sampling rate then these noise samples are independent, identically distributed normal variates with zero mean and variance of $N\mathcal{N}_0/2T_s$.

PERFORMANCE BOUNDS

Any development of an explicit performance measure is characterized by the constraints placed upon the detection processor. The detection interval length is specified as one symbol period so the detector processes each received vector of length N and decides which one of the M symbols was most likely transmitted. Fully coherent detection has been assumed here which means that the signal amplitude, A, the carrier frequency, ω_c, and the phase prior to the detection interval, ϕ_k, are all known to the detector. Since the symbols are equiprobable the optimum receiver structure for a maximum likelihood detector can be implemented, for the sampled case, as a bank of M finite impulse response (FIR) digital filters matched to each of the M possible signals. A decision circuit selects the symbol corresponding to the filter with the largest biased[1] output at the end of the symbol period. Of course, this representation of the optimum detection processor tacitly assumes that the digital FM signal is unaltered by the presampling filter which is strictly true only when the signal has a finite bandwidth less than the bandwidth of the filter. However, if almost all of the signal power falls within the filter bandwidth then this approach is fairly accurate.

The measure of performance for this system is the average symbol error probability. This average is first taken over all possible transmitted symbols. Cox (1) has developed an

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[1] Although digital FM is of a class of constant amplitude signals, it does not have constant energy over all possible symbols. Thus, each matched filter output must be biased by a term which reduces it by the amount of one-half of the square of the modulus of the signal vector.

expression for an upper bound on the average symbol error probability using the union bound. This is

$$P_e(\phi_k) \le \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{j=1 \\ (j \ne i)}}^{M} Q\left(\sqrt{\frac{A^2 T_s}{2N_0}}\; d_{i,j}(\phi_k)\right) \tag{10}$$

where

$$d_{i,j}(\phi_k) = \frac{1}{A\sqrt{N}} \left| s_i(\phi_k) - s_j(\phi_k) \right| \tag{11}$$

The term $d_{i,j}(\phi_k)$ is the normalized Euclidian distance metric which is just the vector difference of the signal vector for the symbol value ai and the signal vector for the symbol value a_j. From Equation 5 it can be seen that this distance is a function of the signal phase at the beginning of the detection interval, ϕ_k. To obtain an overall average on the symbol error bound Cox (1) derived the probability functions for ϕ_k and then used the following upper bound to the Q-function to average over ϕ_k.

$$Q(u) \le \frac{1}{2} \exp\left\{-\frac{u^2}{2}\right\} \tag{12}$$

The bound resulting from this averaging process can be expressed as an equivalent bit error probability as a function of the CNR, which is the ratio of the carrier power to the noise power in a bandwidth equal to the bit rate. In this form it is given by

$$P_e \le \frac{\log_2 M}{M} \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \exp\left\{-\frac{\mathrm{CNR}\ \log_2 M}{2}\, c1_{i,j}\right\} I_0\left(\frac{\mathrm{CNR}\,\log_2 M}{2}\, c2_{i,j}\right) \tag{13}$$

where

$$c1_{i,j} = 1 - \frac{1}{N} \sum_{n=0}^{N-1} \cos[\pi h(a_i - a_j)] \tag{14}$$

$$c2_{i,j} = \left| \mathcal{S}_N \right| \tag{15}$$

$$\mathcal{S}_N = \frac{1}{2N} \sum_{n=0}^{N-1} \left[\exp j\left(\frac{2\omega_c T_s}{N} n + 2\pi h a_i g\left(\frac{n}{N}\right) \right) \right.$$

$$+ \exp j\left(\frac{2\omega_c T_s}{N} n + 2\pi h a_j g\left(\frac{n}{N}\right) \right)$$

$$\left. -2 \exp j\left(\frac{2\omega_c T_s}{N} n + \pi h (a_i + a_j) g\left(\frac{n}{N}\right) \right) \right] \qquad (16)$$

The function I_0 in Equation 13 is the modified Bessel function of the first kind, order zero. Equation 13 represents the upper bound on the equivalent bit error probability averaged over all possible transmitted symbols and all possible values of the phase, ϕ_k, subject to the constraints and assumptions presented herein. This bound can be used to determine specific performance measures for given parameters h, f(t), N, M, and CNR. Some numerical examples will follow after the specification of these parameters.

BASEBAND PULSE SHAPES

Three representative full response pulse shapes will be used in the examples. These are the rectangular, half cycle sinusoid, and raised cosine shapes. The rectangular shape is the most common in literature and practice since it can be considered as the natural output of an M-ary symbol generator. It maintains a constant amplitude of 1 over the existence interval, thus it is defined as

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t < T_s \\ 0 & \text{otherwise} \end{cases} \qquad (17)$$

The baseband phase function corresponding to this f(t) is found using Equation 4 and is given as

$$g(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (18)$$

Using the nomenclature presented by Anderson, Aulin, and Sundberg (2), the pulse defined in Equation 17 will be called a 1 REC frequency pulse. The “1” indicates the existence interval of f(t) is one full symbol period and the “REC” is an abbreviation of rectangular.

The half cycle sinusoid pulse shape, denoted 1HCS, is defined as

$$f(t) = \begin{cases} \sin\left(\frac{\pi t}{T_s}\right) & \text{for } 0 \le t < T_s \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

The corresponding phase function for 1HCS is

$$g(x) = \begin{cases} \frac{1}{\pi}\,[1 - \cos(\pi x)] & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

The raised cosine pulse shape, denoted 1RC, is defined as

$$f(t) = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi t}{T_s}\right)\right] & \text{for } 0 \le t < T_s \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

The corresponding phase function for 1RC is

$$g(x) = \begin{cases} \frac{x}{2} - \frac{\sin(2\pi x)}{4\pi} & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

NUMERICAL EXAMPLES

Some numerical examples are presented here to indicate the utility of the bound in Equation 13 and to show the effects of parameter variation on the error performance. Figure 1 is a plot of bit error probability versus CNR for three deviation ratio values of binary (M=2) 1REC digital FM with 4 samples per symbol period. The curve for h = 0.5 corresponds to the bound for the MSK type of signalling, and the curve for h = 0.823 corresponds to the bound for the case in which the c1 term given in Equation 14 was maximized. Since the c1 term is the dominant h dependent term in Equation 13, then the h value which maximizes that term is very nearly the optimum h value.

There are M-1 distinct $c1_{i,j}$ terms for any given M, so, for M greater than 2, the optimum h value selection criterion can be approximated by selecting the h value which maximizes the minimum $c1_{i,j}$ terms over all i and j. This procedure was followed to determine the h values used for the performance comparison of quaternary (M=4) systems of 1REC, 1HCS, and 1RC, with 8 samples per symbol period. This performance data is given in Figure 2.

Since Equations 14 and 16 are summations of N terms it is natural to assume that the performance of a specific pulse shape is affected by the number of samples per symbol period. Figure 3 illustrates that effect for an octonary (M=8) 1HCS system with N values of 12, 18, and 24. Optimum values for h corresponding to these three sampling rates are found as in the previous example.

As a final example of the effects of parameter selection on the error performance bound a 1RC system is optimized for three values of M: 2, 16, and 256. These three cases are calculated in the same predetection bandwidth, that is four times the bit rate. To maintain this constant bandwidth the number of samples per symbol must be scaled up from the binary value by a factor of log_2M. Thus, N = 8, 32, and 64 for the M = 2, 16, and 256 cases, respectively. This comparison is given in Figure 4.

CONCLUSION

A concise expression for an upper bound on the equivalent bit error probability has been used to investigate the effects of parameter selection on the performance of full response M-ary digital FM with predetection sampling. Through the use of specific numerical examples some performance improvements have been demonstrated by the proper selection of deviation ratio, baseband pulse shape, sampling rate, and number of levels (M).

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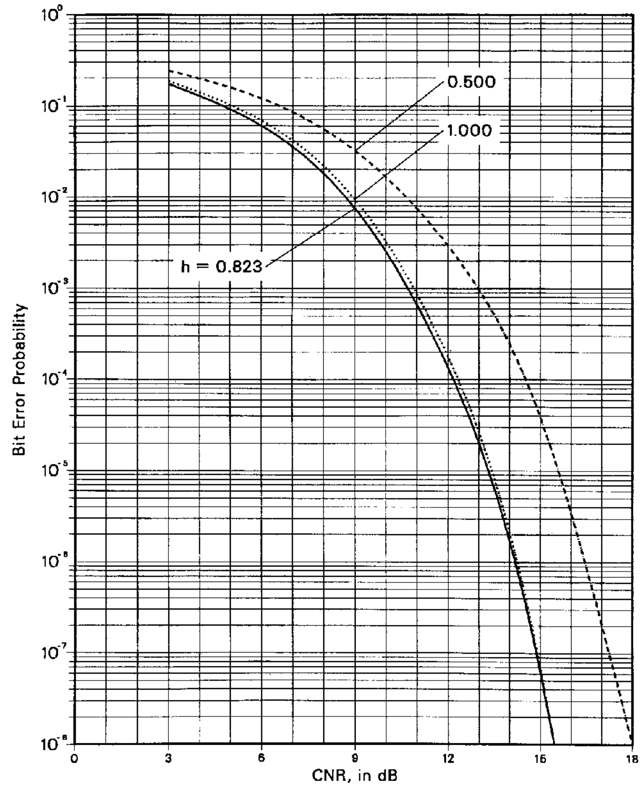

Figure 1. Error performance for three deviation ratios of 1REC, M=2, N=4.

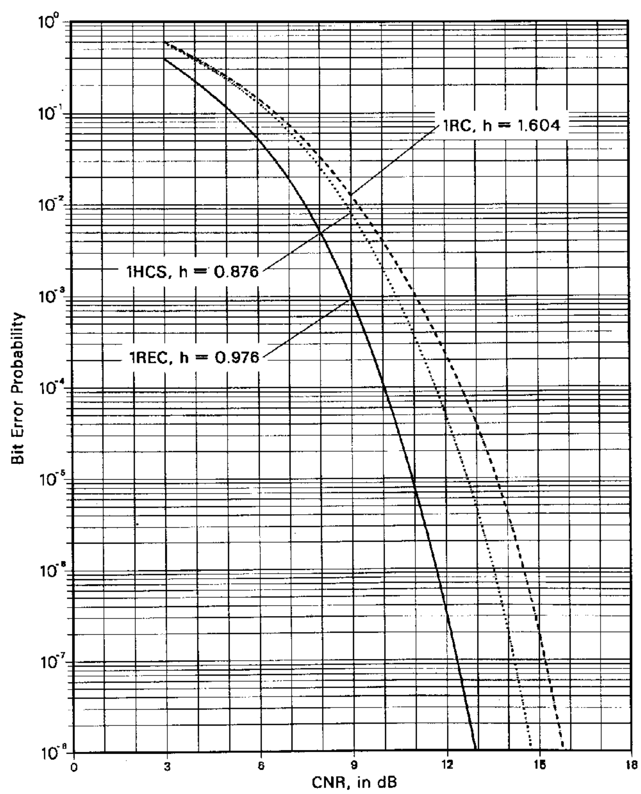

Figure 2. Error performance for three different pulse shapes with M = 4 and N = 8.

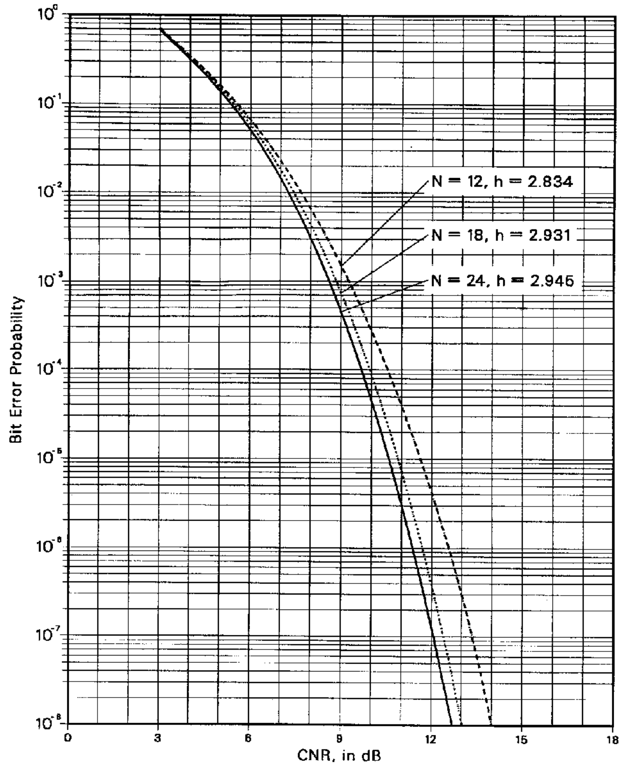

Figure 3. Error performance for three sampling rates of 1HCS, M=8.

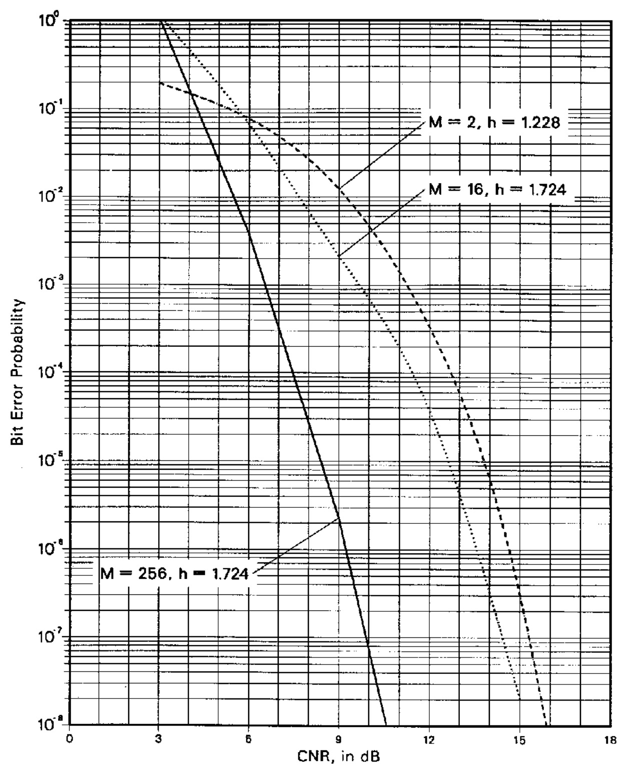

Figure 4. Error performance for three values of M (2,16,256) for 1RC with corresponding values of N (8,32,64).