Mathematical System Theory and the Ecosystem Concept

AN APPROACH TO MODELLING WATERSHED BEHAVIOR

by
James Joseph Rogers

Technical Reports on Hydrology and Water Resources

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Department of Hydrology
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if time is available for the job, one finds that he can only make guesses as to the long-range effects of management practices. This is simply due to the extreme complexity of the system being managed. The guesses are usually biased by the examiner's own interest and experience. In general, past management has either not considered or not been able to predict correctly management effects on a stand. This seems true whether one is considering timber, range, wild life, watershed, or even effects of fire suppression management. It seems that the failure is not with the individuals but rather with the inability of the human mind to deduce the effects of management practices on the behavior of the extremely complex systems being managed. The overall objective of this study is to explore the way in which mathematical system theory may be used to extend our ability to understand and to predict the behavior of the complex natural systems we are managing.
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ABSTRACT

This study explores the possible role of mathematical system theory in integrating existing ecological knowledge within the existing concepts of the structure of the biosphere. The objective of this integration is a theory of ecosystems which must include interactions. The basic unit of the biosphere is the biogeocoenose; similar to the ecosystem, but homogeneous with respect to topographic, microclimatic, vegetation, animal, pedalogical, hydrological and geochemical conditions. The role of the biogeocoenose in a theory of ecosystems based on system theory is discussed. The biogeocoenose may serve as the building block for modeling watersheds as ecosystems. The fundamentals of system theory are reviewed. As an example, an analysis and synthesis of the arid zone water balance follows. The water balance is resolved into twenty components which represent the water balance of (1) the canopy, (2) the mulch, (3) the soil surface, (4) the soil, and (5) the plant, including interactions. The twenty components were modeled as separate systems which were later coupled into one overall, complex, well defined ecosystem water balance system. The example illustrates the role of system theory in integrating ecological knowledge. Further discussion indicates the need for explicitly including plant behavior in the water balance model.
CHAPTER 1

INTRODUCTION

Water is essential to all forms of animal and plant life as well as to most human activities. The sensitivity of water to modifications of the environment resulting from human activities creates problems for mankind. These problems result from increasing water demands and the effects of human activities on the water balance and the biosphere. Now as never before, man possesses the knowledge and ability to intervene with great effect in the water balance of the biosphere. It is increasingly important to intervene correctly and avoid mistakes, the long-term consequences of which may be irreversible.

By intervening, man changes the land from one ecological state to another. In the past these changes have occurred so slowly that either the land with its plants and animals or man could adapt to the new conditions (Dubos 1969). However, the rate of intervention of man today may not be compatible with the adaptive processes of the land or of man to the new conditions. Resource managers are or will be faced with the task of managing the environment in such a way as to establish and maintain conditions under which man can develop his most desirable potentialities (Dubos 1970a).

In transforming the land, man is intervening in the behavior of a tremendously complex system. Resource managers should not rely on intuition and empiricisms alone to predict the outcome of intervention.
In the words of Dubos (1969, p. 10):

From now on most of the transformations of the earth's surface will occur so rapidly that we may often create those terrible situations resulting in erosion and destruction of the land. It therefore is urgent that we develop a new kind of ecological knowledge to enable us to predict the likely consequences of massive technological intervention, and to provide rational guides as substitutes for the spontaneous and empirical adjustments that centuries used to make possible.

The type of ecological knowledge which will allow us to predict consequences of intervention in the biosphere must incorporate rather precise and detailed knowledge about the behavior and the interactions of the various components of the biosphere. It must consider not only the circulation, transformation and accumulation of energy and matter by the components of the biosphere, but also the interactions and interrelationships of the various components of the biosphere.

In the development of watershed models, hydrologists have generally considered only the abiotic components of the watershed as influencing water movement. Abiotic components include climate, soil, topography and related non-biotic factors. Hydrologists have generally not considered the influence of biotic components and interactions of biotic and abiotic components.

**Interactions**

The engineering-oriented hydrologist may question the need to consider biotic interactions in modeling watersheds. Suffice it to say that he is probably not ecologically oriented and cannot see the watershed because of the water. It is easier not to consider interactions and interplay.
Dubos (1968, 1970b) has stressed that consideration of interactions and interplay between component parts of a system is as important as the study of any single component of the system. He points out that the common reductionist approach resulting from specialization assumes that knowledge of complex natural phenomena and living systems can be gained from studying much simpler systems or by studying their individual components. This approach, however, generally neglects interaction between components.

In his book So Human an Animal, Dubos (1968) shows that all living things including man are deeply affected by the environmental conditions of life. He points out that the destructive mismanagement of natural resources as well as of human lives is more a result of our neglect of the interplay between the components of and forces in the environment and modern society than it is to ignorance concerning these components and forces themselves.

Dubos (1970b) discusses the way in which science today permits us to exploit natural resources without regard for genuine human needs or long-range consequences of scientific and technological intervention into both nature and man's life. This attitude toward nature will be socially destructive in the long run. He suggests that another neglected but more important aspect of science is "the development of knowledge and attitudes that would help man to examine objectively, rationally, and creatively the problems that are emerging as a result of social evolution." (Dubos 1970b, p. xi). We need this knowledge because "...not only does man survive and function in his environment, he is shaped by
it, biologically, mentally and socially. To be healthy in the full sense of the word, the environment must provide conditions that favor the development of desirable human characteristics." (Dubos 1970b, p. 171). Hence, the objective of resource managers should be to establish and maintain these conditions.

Dubos (1968) shows that there are certain principles based on certain unchangeable aspects of man's nature which hold true for all environmental planning. Man's genetic endowment, and biological and mental needs not only place limitations on alterations to the environment but also determine what is desirable. Hence, "scientific knowledge of environmental management will contribute little to health and happiness if it continues to neglect the human values symbolized by phrases such as the good earth, a brilliant sky, sparkling waters, a place of one's own." (Dubos 1970b, p. 168). Many of man's potentialities are realized only when stimulated by favorable environmental conditions, especially for the young child. Therefore, environmental planning should also provide a diversified environment. "Science and technology can play a crucial role in the shaping of mental attributes by making it possible to create environments more diversified and thereby more favorable for the expression of a wider range of human potentialities." (Dubos 1970b, p. 172).

Perhaps the previous discussion is best summarized in the following phrases. "When man truly enters the age of science he will abandon his crude and destructive attempts to conquer nature. He will instead learn to insert himself in the environment in such a manner that
his ways of life and technologies make him once more at harmony with nature." (Dubos 1970b, p. xvii).

Obviously, the type of ecological knowledge which will allow us to predict consequences of intervention must embody formal mathematics if only to be able to handle the complexity and communicate precisely about the system. Recently, there has been much discussion about the development of mathematical models of whole ecosystems. The purpose of these models would be to predict consequences of intervention by man by implementing the models on computers and simulating the effects of changes. Some approaches to the development of these models are outlined by Van Dyne (1969a, 1969b), Bledsoe and Jameson (1969), Cooper (1969), Schultz (1969) and Swartzman (1970).

Cooper (1969) gives an excellent discussion of the possible uses of ecosystem models in watershed management. Cooper's entire discussion is based upon a suitable theory of ecosystems which would allow the development of an ecosystem model. He feels that it is not possible at the present time to create adequate ecosystem models. His justification for this is that it is not possible to quantitatively translate results of hydrological observations on small plots to either small or large watersheds. However, he does not comment on whether it might be possible to predict the water balance using existing theory or even to construct ecosystem models using existing theory.

Peterson (1969, p. 46) indicated that hydrology is becoming more ecologically oriented when he summarized probable trends in hydrology with the statement: "Clearly both the problems of the biosphere and of
the human environment are intimately related to the problems of hydrology. An increased emphasis on hydrology as it relates to resource use and conservation, biological effects and environmental quality will be the direction in the future."

That the water balance must be an integral part of any watershed model is apparent because of the dependence of plants and animals on water. This dependence and interaction of plants and animals with water is discussed throughout this thesis. However, examination of existing watershed or water balance models shows that none of these explicitly include the biota (either plant or animal) as interacting components of the watershed which effects water movement. The USDAHL-70 model (Holtan and Lopez 1970) developed by the United States Department of Agriculture Hydrograph Laboratory does attempt to include effects of land use in an empirical manner. It seems that existing watershed or water balance models are essentially abiotic in nature, since evapotranspiration is generally included in a strictly empirical manner.

On the other hand, examination of existing ecosystem or partial ecosystem models reveals that these generally ignore the water balance and interactions or handle it in a completely empirical manner. These include the models of Bledsoe and Van Dyne (1970), Van Dyne (1969b), Bledsoe and Jameson (1969) and Byrne and Tognetti (1969). An exception to this is Paltridge (1970). These are further discussed in Chapter 5.

Obviously, if a theory of ecosystems is to be developed, it must be able to kludge the water balance with other determinants of ecosystem behavior. It must also be able to do this in a realistic manner. Hence, it must be broader in scope than the theory of linear differential
equations which is generally used in current ecosystem models. Van Dyne (1969a, 1969b, 1969c), Cooper (1969), Dale (1970), Patten (1966) and Watt (1968) indicate that a theory of ecosystems or systems ecology must incorporate the ideas of cybernetics, operations research, general systems theory, set theory, automata, differential equations, probability and statistics, stochastic processes, information theory and control theory among others.

These are the ideas which will be needed to integrate our collection of known facts concerning the behavior of components of the biosphere within the realm of fundamental concepts concerning the way in which the biosphere is constructed. It seems that Kisiel (1969, p. 366) summarizes this very well in the following:

Yet more basic than facts and language is integration in the realm of fundamental concepts because 'it is in the nature of concepts to be extensible, while facts can only fill dead spaces in knowledge. Concepts guide the researcher when he is looking for facts; they provide the matrix in which facts are embedded.' (Margenau, 1967, p. 37). The future of hydrology rests on our ability and willingness to undertake the last integrative effort on a continuing and adaptable basis. This effort is particularly urgent if one accepts the thesis that each watershed or basin is a law unto itself. Transferability of laboratory knowledge to the field, and of knowledge from one watershed to another or from one climate to another rests inexplicably on our ability to provide a mathematical foundation to the cycle of model building and its parts.

Mathematical System Theory

However, it seems that Wymore (1967, 1969, 1971a, 1971b) has provided the mathematical foundation for the cycle of model building in his mathematical system theory. The theory relies on mathematical rigor rather than on intuition. It is structured in such a way as to cope
with three basic problems. The first is to define precisely what constitutes a system. The second problem is to determine what it means to couple systems into complex systems and to deduce the behavior of the resulting complex system. A related problem is to resolve a complex system into simpler components. The third problem is to create a class of models sufficiently powerful to model any existing real world or engineering phenomenon of interest. Real world phenomenon in this sense includes those vast, complicated and interrelated ecologic, social, economic and political systems which exist in the real world. Engineering here is taken in its broadest sense to include the design and analysis of any sort of system which will be needed to design solutions to social problems including broad ecological management systems (Wymore 1971b). The fundamental concepts of this theory are briefly reviewed in Chapter 3.

Any theory of ecosystems must be capable of handling the extreme complexity which will result from the integration of existing knowledge. Mathematical systems theory is designed to deal with this complexity. It also provides a basis for precise communication between those developing different parts of a theory of ecosystems. Since it is structured to cope with the problem of coupling simpler systems into more complex systems, it provides a basis for developing a theory of ecosystems.

The Ecosystem Concept

A problem arises in how to structure a theory of ecosystems around the fundamental concepts of the structure of the biosphere. The ecosystem has been defined both as a concept and as a basic unit of the biosphere. Tansley (1935) defined an ecosystem as "a system resulting
from the integration of all living and nonliving factors of the environment." The term "ecosystem" is also used to describe the concept or approach of studying abiotic-biotic complexes. The ecosystem concept implies that the emphasis is on going beyond one's professional specialty to consider interactions between various biological components and their environment (Van Dyne 1969a).

Hence, the ecosystem concept may be applied in modeling a watershed as an ecosystem. However, the watershed ecosystem may actually be a set of heterogeneous smaller ecosystems. The coupling of the smaller ecosystems would result in the larger watershed ecosystem. Hence it is necessary to more explicitly define the smaller ecosystems in terms of basic structural units which may be modeled, analyzed and managed using the ecosystem concept. This is done utilizing the Russian term "biogeocoenose," which is considered the elementary primary structural unit of the biosphere (Sukachev and Dylis 1968, Kovda 1970). The ecosystem and biogeocoenose are commonly considered as synonyms. However, biogeocoenose has a much more specific definition. This is given in Chapter 2 in a further informal discussion of the ecosystem concept, its role in modeling watersheds and some implications for the structure of a theory of ecosystems.

Theoretically and physically sound models of each component of an ecosystem can be developed by integrating existing knowledge of each component (where available) into a system model of that component. These individual component models can then be coupled into a more complex model while, at the same time, providing for interactions between each
component. This approach is illustrated in Chapters 5 and 6 in an analysis and synthesis of the water balance of an ecosystem. The analysis and synthesis attempt to consider the interactions of the plant component and the water balance in a quantitative manner. The treatment does not attempt to model surface runoff at this time.

In the process of modeling the ecosystem water balance, the author has attempted to incorporate into the model insights as to the behavior of each component. These insights have been gained by experts in the various fields and are well documented in the literature. The author is not an expert in all these fields and does not claim that these insights are all valid. However, the validity of these insights can be tested by incorporating them into a model and simulating the process under study. This in turn may yield new insights.

**Objectives**

The objective of this thesis is to illustrate the way in which mathematical system theory may form the basic mathematical foundation for a theory of ecosystems. This is done by (1) reviewing some basic concepts concerning the structure of the biosphere, (2) reviewing some concepts of mathematical system theory, (3) using mathematical system theory to model the water balance of an ecosystem, and (4) discussing the way in which mathematical system theory may be used in integrating existing knowledge of the water balance within existing concepts of the structure of the biosphere.
CHAPTER 2

THE STRUCTURE OF THE BIOSPHERE: FUNDAMENTAL CONCEPTS

It was indicated in Chapter 1 that a watershed may be viewed as an ecosystem composed of smaller ecosystems. However, it was also indicated that there is a need to define more specifically what is the basic unit of a watershed or, for that matter, of the biosphere.

Sources and Reaches

Many investigators have recognized a hierarchical system of stream patterns in catchments. This system is characterized by source areas which give rise to first-order streams. First-order streams join to form second-order streams, etc. Within this hierarchical structure we can delineate two distinct substructures with respect to function in surface runoff. These are (1) source-catchments (or sources) which consist of the area enclosed by the surface runoff system (overland flow system) of a first-order channel; and (2) a reach-catchment (or reaches) which consists of the area enclosed by the overland flow system of a portion of a higher than first-order channel assumed to be homogeneous in some sense, and which begins at the confluence of a tributary or a part in the channel where the homogeneous assumptions are no longer valid and terminates at the beginning of a new reach. The primary distinction in the above is in the existence or nonexistence of concentrated tributary inflow in addition to lateral inflow (Eagleson 1970).
Such a classification allows the derivation of criteria for decomposing a watershed by delineating sources and reaches. Going one step further, suppose it is possible to model consistently each source and reach as a system. Then system theory tells us that these systems may be coupled to form a new system (WyMORE 1967). It would seem that the source and the reach are fundamental concepts of watershed structure.

The question arises, "How to model sources and reaches as systems?" The catchment area of both sources and reaches will undoubtedly exhibit heterogeneity with respect to soils, vegetation, microclimate, etc., although reach channels could be considered homogeneous by definition of a reach. These would be difficult systems to model in a consistent manner. Hence we ask, "What is fundamental to the source and reach?"

The Biogeocoenose

It is usually apparent to foresters and other natural resource managers that source and reach catchments can be subdivided into areas homogeneous with respect to type of land use and/or vegetative cover (or lack thereof). But in order to simplify physical models of hydrologic processes it is useful to consider homogeneity of soils, microclimate, etc. Hence, if we could delineate homogeneous units (in some sense) which could be consistently modeled as a system, then these system models could be coupled to form system models of source and reach. These concepts are to some extent incorporated in the work of England and Holtan (1969), Stephenson and England (1969), and England and Stephenson (1970) in developing the concept of hydrologic response.
units used in the USDAHL-70 watershed model described by Holtan and Lopez (1970). The concept is also implicit in the unit-source watershed (Kincaid, Osborn, and Gardner 1966) and in the Utah Simulation Model II (Riley, Chadwick, and Bagley 1966).

Sukachev and Dylis (1968, p. 26) give the following definition of biogeocoenose, which is considered the basic unit of the biosphere by Russian ecologists:

A biogeocoenose is a combination on a specific area of the earth's surface of homogeneous natural phenomena (atmosphere, mineral strata, vegetable, animal and microbic life, soil and water conditions) possessing its own specific type of interaction of these components and a definite type of interchange of this matter and energy among themselves and with other natural phenomena....

Note that the phrase "specific type of interaction of these components and a definite type of interchange of this matter and energy" implies that a biogeocoenose exhibits system-like behavior.

In Figure 2.1 the major interactions of the components (vegetation, animals, microorganisms, atmosphere, and soil and mineral strata) of a biogeocoenose are shown diagrammatically. Soil and subsurface mineral strata extend as far below soil surface and atmosphere as far above soil surface as influence exists between them and other components of the biogeocoenose. All interactions of the components are connected and are interdependent. The better these interactions are understood, the sounder will be our management planning.

The Type

Another useful concept in forest management is that of type. A similar concept in range management is the range site. These concepts
Figure 2.1 Interactions of Components of a Biogeocoenose.
(Sukachev and Dylis 1968)
are essentially used to group landscape units which are homogeneous with respect to certain site factors. Sukachev and Dylis (1968, p. 48) give the following definition of type:

A type is a union of separate biogeocoenoses homogeneous in the species composition of their layers of vegetation, in fauna, in microbial population, in average climate, soil, and water conditions, in the interrelationships between plants and environment in intrabiogeocoenotic exchange of matter and energy, in regeneration processes, and in the direction of change in them. This homogeneity in features of components of, and the whole of biogeocoenoses united as a type requires the application of identical management measures in identical socioeconomic conditions.

In a broad sense, type as defined by Sukachev and Dylis (1968) is similar to the hydrologic response units of England and Stephenson (1970).

**Implications for a Theory of Ecosystems**

The notions of the biogeocoenose and type as defined and used by Sukachev and Dylis (1968), however, have a more fundamental meaning to the possible development of a theory of ecosystems. A type represents a set of biogeocoenoses for which the development of a system model for one member of the set would imply the existence of a model for other members of the set. That is, it should only be necessary to develop one general model for each type. Having developed one general systems model for the set of biogeocoenoses represented by a type, one should be able to apply this model to all members of the type.

Let's assume that each of the types in some watershed ecosystem has been modeled as a system using techniques to be described in later chapters. Then this should imply the existence of a system model for each biogeocoenose in the ecosystem. It would then be possible to couple the
systems representing adjacent biogeocoenoses into an overall system representing the ecosystem watershed.

Hence, it seems that the biogeocoenose and type as well as the ecosystem are fundamental units in the structure of the biosphere. The definitions of biogeocoenose and type are somewhat similar to that of the stand and type as used in intensive forest management. The definitions are extendable to any area, whether urban, rural or natural. However, the ecosystem as a unit is still rather elusive because it has no fixed boundaries by definition. It seems that it would be more useful to define an ecosystem in terms of one or more biogeocoenoses. That is, if there was only one biogeocoenose under study, then the ecosystem would be that biogeocoenose. However, if there were more than one biogeocoenose in the unit under study, then the ecosystem would be the resultant of the coupling of those biogeocoenoses in the unit.

**Modeling the Biogeocoenose**

The greatest problem facing the development of a whole ecosystem model is modeling a biogeocoenose.

The question of what to include in the model must be answered. Wymore (1967) takes the viewpoint that the model should be as complete and accurate as it is possible to make it. This allows one to quantitatively assess effects of errors arising from later approximations. It also provides deeper insight into the problem under study and allows one to make better decisions at a later date. It seems that it is always easier to simplify a complex model through approximations than it is to expand a simple model.
Dubos (1970b) repeatedly stresses the need to begin integrating our knowledge so as not only to understand the environment in which we live, but also for creating and maintaining a dynamic balance with the environment which man needs to develop his full potentials.

Hence the approach should be to pick a major process as a suitable starting point and begin integrating what is currently known about that process so as to develop a complete and accurate model of that process. It should always be kept in mind that one is working at the level of a biogeocoenose and is attempting to integrate information concerning the behavior of the major components of the system. He should not allow himself to be drawn into the details of the physiological processes occurring in a single plant or cell. This would only lead away from the integrative process and into the reductionist approach.

The integrative approach will be illustrated here by considering the water balance of a biogeocoenose or ecosystem. The development considers interactions of plants, soil, atmosphere and litter in determining the water status of the ecosystem. It also illustrates the value of mathematical system theory in organizing and handling the complexity arising from integration of knowledge into complex models. This development will follow a review of the fundamental concepts of system theory.
CHAPTER 3

FUNDAMENTAL CONCEPTS OF MATHEMATICAL SYSTEM THEORY

System theory as proposed by Wymore (1967, 1969, 1971b) is not to be confused with the theory of differential equations. System theory subsumes the theory of continuous systems represented by differential equations and the theory of discrete automata. It includes the ideas of information and communication theory, linear control theory, topological algebra, operations research, modern control theory, cybernetics, decision theory and the theory of stochastic processes. Klir (1969), Bertalanffy (1968) and Mesarovic, Macko and Takahara (1970) have presented or discussed approaches to a general theory of systems. However, Wymore's theory seems particularly useful for modeling large-scale complex systems.

System theory is particularly useful because it provides a means of communicating precisely when developing and discussing a complex system model. In order to talk about complicated things in a precise manner the language is going to be complicated. System theory embodies the mathematical language necessary for getting a notational or symbolic handle on complexity. Such a handle allows the modeler to communicate precise descriptions of complex systems, to prove theorems concerning the behavior of the system, and to deduce in some precise way the behavior of the interconnection of systems.
Physically based hydrologic models of watersheds are typically water balance models. The water balance is of importance in many fields and hence models of the water balance are of interest to researchers and managers in these fields. If one accepts the thesis that the water balance will be a key component of a realistic and complete ecosystem model, then the need to communicate precise descriptions of these models should be apparent. Development of models of complex ecosystems will require interdisciplinary teamwork and communication. System theory provides the language for precise communication between those developing separate models of components of the system. It also provides the necessary theory for coupling these models into an overall model of the system.

A definition of system for modeling purposes should satisfy one's intuition as to systemlike behavior, should allow one to model almost anything of interest, and should be sufficiently simple to give some interesting mathematical theory (Wymore 1971c).

The following discussions, definitions, and the methodology of modeling in Chapter 5 are strongly influenced by the lectures and manuscripts of Wymore (1967, 1969, 1971b, 1971c).

Admissible Sets of Input Functions

Intuitively, the notion of a system may be represented by a box with inputs and outputs. The inputs enter the system at points called input ports and consist of the kinds of things in the system's environment to which the system responds.
The system exists in real time and operates along some time scale. The time scale may be continuous, discrete or some combination of both.

At any point in real time some member of the set of possible inputs is presented at an input port of the system. The system samples these inputs at points along its time scale. In order to discuss the arrangement of these inputs in time it is necessary to talk about input functions, which are defined everywhere over real time.

Since systems exist in real time it is necessary that a set of input functions possess certain characteristics. An experimenter must be able to select the origin of the system time scale arbitrarily. Hence, he must be able to translate an input function to a selected point, say r, on the real time scale, and the translation of the function by the amount r must be a valid input function.

If f is any input function, then the translation of f by the amount r evaluated at some time t is the same as the value of the input function f evaluated at the time t + r.

The experimenter must be able to select the input function to be applied at the start of the experiment, time 0, irregardless of the input functions the system was receiving before time 0. Hence, the functions which can be obtained by hooking up old input functions to new input functions must be valid input functions.

If f and g are legitimate input functions to a system, the segmentation of f and g is defined as follows: the segmentation of f and g is an input function defined at time t as being f(t) if t is negative and g(t) if t is not negative.
The set of functions which possesses the above characteristics is called an admissible set of input functions. The concepts associated with admissible sets of input functions are formally defined in definition 3.2. Definition 3.2 follows a collection of miscellaneous definitions given in definition 3.1.

Miscellaneous Concepts (Definition 3.1)

Understanding of system theory rests in an ability to understand the symbolic language. The symbolic notation adopted by the author for modeling is meant to convey the meaning and hence aid understanding of the language as suggested by Wymore (1971c).

The symbols {} will always enclose the definition of a set. The symbol \( \cup \) will denote "union." The symbol \( \cap \) will denote "intersection." The symbol \( \subseteq \) will denote "subset." The symbol \( \subset \) will denote "containment." The symbol \( \times \) will denote the "Cartesian product operator."

If \( A \) is a set and if \( b \) is a member of \( A \), then this is denoted by \( b \in A \).

The empty set is denoted \( \emptyset \).

The set of all real numbers is denoted REALS and the set of all integers is denoted INTEGERS. Subsets of these may be denoted by appropriate prefixes such as POSITIVEREALS, NONNEGATIVEREALS, where

\[
\text{POSITIVEREALS} = \{t: 0 < t \leq \infty\}, \quad \text{and} \quad \text{NONNEGATIVEREALS} = \text{POSITIVEREALS} \cup \{0\}.
\]

Smaller subsets may be denoted by the following definitions:

\[
\text{REALS}[a,b] = \{t: a \leq t < b\},
\]

\[
\text{INTEGERS}[a,b] = \{t: a \leq t < b\},
\]
and other similar combinations of the symbols \([, ]\), \((, )\) may be used as suffixes to REALS and INTEGERS or subsets of each.

A function \(f\) defined on a set \(A\) with values in a set \(B\) is a set such that: \(f\) is a subset of \(\{(a,b) : a \in A, b \in B\}\), if \(a \in A\), then there exists \(b = f(a) \in B\) such that \((a,b) \in f\), if \((a_1,b_1) \in f\) and \((a_2,b_2) \in f\) and \(a_1 = a_2\), then \(b_1 = b_2\).

The set of all functions defined on \(A\) with values in \(B\) is denoted \(\text{FUNCTIONS}(A,B)\).

If \(A\) and \(B\) are sets, and \(f \in \text{FUNCTIONS}(A,B)\) and \(A' \subseteq A\), then the restriction of \(f\) to \(A'\) is denoted \(\text{RESTRICTION}(f,A')\) and is defined as follows:

\[
\text{RESTRICTION}(f,A') = \{(a,b) : a \in A', (a,b) \in f\}.
\]

If \(A\) is a set, then the identity mapping defined on \(A\) with values in \(A\) is denoted \(\text{IDENTITY}\) and is defined as follows:

\[
\text{IDENTITY}(A) = \{(a,a) : a \in A\}.
\]

If \(f \in \text{FUNCTIONS}(A,B)\), then the domain of \(f\) is \(A\) and is denoted \(\text{DOMAIN}(f)\). The range of \(f\) is a subset of \(B\) and is denoted \(\text{RANGE}(f)\) and defined as follows:

\[
\text{RANGE}(f) = \{b : b \in B, (a,b) \in f \text{ for } a \in A\}.
\]

If \(A\) is a set with only a finite number of elements then \(\#(A)\) will denote the number of elements in \(A\).

Let \(B\) be a finite set not empty of sets not empty. Let \(n = \#(B)\). Let \(B = \{A_1, \ldots, A_n\}\). Then the vector product of the sets in \(B\) is denoted \(A_1 \times A_2 \times \cdots \times A_n\), and is defined as follows:
\( A_1 \times A_2 \times \ldots \times A_N = \{(a_1, a_2, \ldots, a_n) : \text{for all } i \in \text{INTEGERS}[1,n], a_i \in A_i\}. \)

If \( f \in \text{FUNCTIONS}(A,B) \) such that for every \( b \in B \), there exists \( a \in A \) such that \((a,b) \in f\) then \( f \) is said to be "onto \( B\)," and may be denoted \( f \in \text{FUNCTIONS}(A, \text{onto}, B)\).

A function \( f \) defined on a set \( A \) with values in a set \( B \) is 1 to 1, if and only if, if \((a_1,b_1) \in f\) and \((a_2,b_2) \in f\) and \( b_1 = b_2\), then \( a_1 = a_2\). We may say \( f \in \text{FUNCTIONS}(A,1 \text{ to } 1,B)\).

The sets \( A \) and \( B \) are said to be set isomorphic if and only if \( \text{FUNCTIONS}(a,1 \text{ to } 1, \text{onto}, B) \neq \emptyset\). If \( f \in \text{FUNCTIONS}(A,1 \text{ to } 1, \text{onto}, B)\) then we may say that \( A \) is \( \text{ISOMORPHIC}(f) \) to \( B\).

Let \( f \in \text{FUNCTIONS}(A,B), \ g \in \text{FUNCTIONS}(B,C)\), then the composition of \( f \) and \( g \) is denoted \( gf \) and defined as follows:

\[ gf = \{(a,c) : a \in A, c \in C, \text{there exists } b \in B, \text{such that } (a,b) \in f \text{ and } (b,c) \in g\}. \]

Let \( A \) and \( B \) be sets not empty and \( b \in B \). The function which is constant on \( A \) and equal to \( b \) is denoted \( \text{CONSTANT}(A,b) \) and defined as follows:

\[ \text{CONSTANT}(A,b) = \{(a,b) : a \in A\}. \]

If \( B \) is a set, then \((B)^n\) denotes the set of all \( n \)-tuples \((b_1,\ldots,b_n)\) where \( b_i \in B, i \in \text{INTEGERS}[1,n]\).

Let \( B \) be a set of sets not empty. Then the Cartesian product of the sets in \( B \) is denoted \( \prod B \) and is defined as follows:

\[ \prod B = \{f : f \in \text{FUNCTIONS}(B,\cup B), f(A) \in A \text{ for all } A \in B\}. \]
If $B$ is a set of sets and $A \in B$ and $\chi \in \mathcal{X}_B$, then $\chi(A)$ is called the $A$ coordinate or the $A$ component of $\chi$. If $\mathcal{C} \subseteq B$ then $\text{PROJECTION}(C) \in \text{FUNCTIONS}(\mathcal{X}_B, \mathcal{X}_C)$ defined as follows:

if $y \in \mathcal{X}_C$, $((\text{PROJECTION}(C))(y))(D) = y(D)$ for every $D \in C$. If $A \in B$, then $\text{PROJECTION}({A})$ is called the projection into the $A$ coordinate.

If $B = \{A_1, \ldots, A_n\}$ is a set of sets not empty then $\mathcal{X}_B$ is $\text{ISOMORPHIC}(g)$ for $A_1 \mathcal{X} \ldots A_n$ where for every $f \in \mathcal{X}_B$, $g(f) = (f(A_1), \ldots, f(A_n))$.

Admissible Sets of Input Functions (Definition 3.2)

Let $P$ be a set not empty, $f, g \in \text{FUNCTIONS}(\text{REALS}, P)$, $r \in \text{REALS}$, then $\text{TRANSLATION}(f,r) = \{(t,f(r+t)): t \in \text{REALS}\}$ and $\text{SEGMENTATION}(f,g) = \{(t,f(t)): t \in \text{NEGATIVE}\text{REALS}\} \cup \{(t,g(t)): t \in \text{NONNEGATIVE}\text{REALS}\}$.

A set $F$ is an admissible set of input functions with values in $P$ if and only if: $f \in \text{FUNCTIONS}(\text{REALS}, P)$, $f \neq \emptyset$, and if $f, g \in F$ and $r \in \text{REALS}$, then $\text{TRANSLATION}(f,r) \in F$ and $\text{SEGMENTATION}(f,g) \in F$.

The set of all admissible input functions with values in $P$ is denoted $\text{ADMISSIBLES}(P)$.

If $G \in \text{FUNCTIONS}(\text{REALS}, P)$, then the smallest admissible set of input functions which contains $G$ is denoted $\text{ADMISSIBLESET}(G)$ and is defined as follows:

$\text{ADMISSIBLESET}(G) = \bigcap \{F : F \in \text{ADMISSIBLES}(P), F \supseteq G\}$. 
Assemblages and Systems

A system may be viewed as a box which receives inputs at a number of input ports. Intuitively, the inputs cause activity inside the box. A set of states is used to represent what goes on inside the box. The states are analogous to dials and gauges which allow one to tell how a system is performing.

The fundamental idea behind the definition of a set of states of a system is that if we know the state of the system at some time $t$, and we know all the inputs the system experienced from time $t$ to some time $s + t$, then we must be able to compute the state of the system at time $s + t$.

A state transition function in essence tells how to predict the behavior of the system given an input function, the initial state, and the time scale. Hence, the set of inputs, states, input functions, and time scale determine the totality or range of behavior available to the system, from the given state transition function.

These concepts are formally summarized in the following definition of an assemblage.

Assemblages (Definition 3.3)

An assemblage is a 6-tuple $Z = (S, P, F, M, T, \sigma)$ where:

- $S$ is a set not empty;
- $P$ is a set not empty;
- $F \in \text{ADMISSIBLES}(P)$;
- $\text{FUNCTIONS}(S, S), \text{IDENTITY}(S) \in M$;
TCREALS, $0 \in T$;

$\sigma \in \text{FUNCTIONS} (F \times T, \text{onto, } M)$.

If $Z$ is an assemblage and $Z = (S, P, F, M, T, \sigma)$ then $\text{STATES}(Z) = S$, $\text{INPUTS}(Z) = P$, $\text{INPUTFUNCTIONS}(Z) = F$, $\text{BEHAVIOR}(Z) = M$, $\text{TIMESCALE}(Z) = T$, $\text{MOTION}(Z) = \sigma$, and $\text{INPUTPORTS}(Z) = I$ if $P = X_I$ for some set $I$ not empty of sets not empty and $#(I) > 2$, $= \{P\}$ otherwise.

The symbols $\text{STATES}(Z)$, $\text{INPUTS}(Z)$, $\text{INPUTPORTS}(Z)$, $\text{INPUTFUNCTIONS}(Z)$, $\text{BEHAVIOR}(Z)$, $\text{TIMESCALE}(Z)$, and $\text{MOTION}(Z)$ denote the states, inputs, set of input ports, set of input functions, set of state transitions, timescale, and the state transition function of an assemblage $Z$, respectively.

If $f \in F$, $t \in T$, $x \in S$, then the state of the assemblage at time $t$ given the input function $f$ and the initial state $x$ is $(\sigma(f, t))(x)$.

An output function for an assemblage is any $\xi \in \text{FUNCTIONS}(S, Q)$ where $Q$ is a set not empty.

The concept of an assemblage includes many possible mathematical constructs which may exhibit behavior which is intuitively not system-like. These include things whose behavior depends upon future states or inputs. This does not say a system cannot attempt to predict the future. Also, the state of the system at time $0$ as determined by the state transition function must be the same as the initial state of the system. Since the origin of the time scale is arbitrary, it is necessary to eliminate those assemblages whose behavior changes with translations of the time scale. These systems are considered time-varying.
Also, the behavior of a system should be the same over a time interval for which two input functions agree if the initial state at the start of the time interval agrees.

These notions are summarized in the following definition of a system.

Systems (Definition 3.4)

A system is an assemblage \( Z = (S,P,F,M,T,\sigma) \) such that:

1) \( \sigma(f,0) = \text{IDENTITY}(S) \) for every \( f \in F \);
2) \( \sigma(\text{TRANSOLUTION}(f,s),t)(f,s) = (f,s+t) \) for every \( f \in F \), \( s, t \in T \) such that \( s+t \in T \);
3) \( \sigma(f,t) = \sigma(g,t) \) for every \( f, g \in F \) and \( t \in \text{REALS} \) if \( \text{RESTRICTION}(f,T[0,t)) = \text{RESTRICTION}(g,T[0,t)) \) when \( t \geq 0 \).

Discrete Systems and Differentiable Systems

Discrete systems and differentiable systems represent two archetypal subclasses of systems respectively. These subclasses are extensively discussed by Wymore (1969).

Discrete Systems (Definition 3.5)

A system \( Z = (S,P,M,F,T,\sigma) \) is a discrete system if and only if:

\[ F \supset \{ \text{CONSTANT(REALS, p)} : p \in P \} \];
\[ T = \text{NONNEGATIVEINTEGERS} \];
and for every \( f \in F , t \in T , x \in S \):
\[
\sigma(f,t)(x) = \begin{cases} 
\chi & \text{if } t = 0, \\
\sigma(\text{CONSTANT}(\text{REALS}, f(t-1)), \text{CONS}(\text{REALS}, f(t-1)))(\chi) & \text{if } t \neq 0.
\end{cases}
\]

A discrete system is completely determined by specifying \(\sigma(\text{CONSTANT}(\text{REALS}, p), 1)(\chi)\) for every \(p \in P, \chi \in S\).

Differentiable Systems (Definition 3.6)

A system \(Z = (S, P, F, M, T, \sigma)\) is a differentiable system if and only if:

- \(S = (\text{REALS})^n\) for some \(n \in \text{POSITIVEINTEGERS}\);
- \(P = (\text{REALS})^m\) for some \(m \in \text{POSITIVEINTEGERS}\);
- \(F \supset \{\text{CONSTANT}(\text{REALS}, p) : p \in P\}\);
- \(T = \text{NONNEGATIVEREALS}\);

and for every \(f \in F, t \in T, \chi \in S\),

\[
\frac{d(\sigma(f,t)(\chi))}{dt}(t) = \frac{d(\sigma(\text{CONSTANT}(\text{REALS}, f(t)), \text{CONS}(\text{REALS}, f(t)))(\sigma(f,t)(\chi))}{dt} (0)
\]

or for all \(f \in F, t \in T, \chi \in S\), \(\sigma(f,t)(\chi)\) is the value of the solution \(y\) at time \(t\) of the differential equation

\[
\frac{dy}{dt}(\epsilon) = \frac{d\sigma(\text{CONSTANT}(\text{REALS}, f(s)), \text{CONS}(\text{REALS}, f(s)))(y(s)}{dt} (0)
\]

with \(y(0) = \chi\) and \(s \in T\).

A differentiable system is completely determined by specifying

\[
\frac{d(\sigma(\text{CONSTANT}(\text{REALS}, p), \text{CONS}(\text{REALS}, p)))(\chi))}{dt} (0)
\]

for every \(p \in P, \chi \in S\).
Coupling Recipes and System Coupling

The second problem for a theory of systems is to determine what it means to couple systems into a complex system and determine the behavior of the resulting complex system.

The definition of a coupling recipe tells precisely what it means to couple systems. A coupling recipe tells (1) which systems are involved in the coupling, (2) which of the input ports of each system are assigned to receive their inputs from which other systems, and (3) what are the output functions from systems which provide inputs to the input ports of receiving systems.

Coupling Recipes (Definition 3.7)

A coupling recipe is a triple $C = (\xi, a, o)$ where: $\xi$ is a set not empty of systems such that for every $Z, Z^1 \in \xi$: $\text{STATES}(Z) \neq \text{STATES}(Z^1)$, $\text{TIMESCALE}(Z^1)$, \{(\text{PROJECTION}(\text{INPUTPORTS}(Z) \cap \text{INPUTPORTS}(Z^1)))f : f \in \text{INPUTFUNCTIONS}(Z)\} = \{(\text{PROJECTION}(\text{INPUTPORTS}(Z) \cap \text{INPUTPORTS}(Z^1)))f : f \in \text{INPUTFUNCTIONS}(Z^1)\}$; $a$ is a function defined on $\xi$ such that for every $Z, Z^1, Z^{11} \in \xi$, $a(Z, Z^1) \cap \text{INPUTPORTS}(Z)$ and $a(Z, Z^{11}) \cap \text{INPUTPORTS}(Z) = \emptyset$ if $Z^1 \neq Z^{11}$; $o(Z, Z^1) = \emptyset$ if $a(Z^1, Z) = \emptyset$, but $o(Z, Z^1) \in \text{FUNCTIONS}(\text{STATES}(Z), Xa(Z^1, A))$ if $a(Z^1, Z) \neq \emptyset$.

If $C = (\xi, a, o)$ is a couple and $Z, Z^1 \in \xi$ then:
- the input port assignments of $C$ is the function $a$;
- the output port assignments of $C$ is the function $o$;
- the set of system components of the resultant determined by $C$ is $\text{COMPONENTS}(C) = \xi$;
- the set of input ports of $Z$ which are occupied by $C$ is $\text{OCCUPIEDPORTS}(Z, C)$.
\[ \cup \{a(Z, Z_1): Z^1 \in \xi \} \]

the set of input ports of \( Z \) left unoccupied by \( C \) is \( \text{UNOCCUPIEDPORTS}(Z,C) \)

= \( \text{INPUTPORTS}(Z) \sim \text{OCCUPIEDPORTS}(Z,C) \);

the set of states of the resultant determined by \( C \) is \( \text{RESULTANTSTATESET}(C) \)

= \( \text{X}_{\text{STATES}}(Z^1); Z^1 \in \xi \);

the total set of input ports managed by \( C \) is \( \text{TOTALINPUTPORTS}(C) \)

= \( \cup \{\text{INPUTPORTS}(Z^1): Z^1 \in \xi \} \);

the total input determined by \( C \) is \( \text{TOTALINPUT}(C) = \text{XTOTALINPUTPORTS}(C) \);

the total set of input functions determined by \( C \) is \( \text{TOTALINPUTFUNCTIONS}(C) \)

= \{ f: f \in \text{FUNCTIONS}(\text{REALS}, \text{TOTALINPUT}(C)), (\text{PROJECTION}(\text{INPUTPORTS}(Z^1))) \}

f \in \text{INPUTFUNCTIONS}(Z^1) \text{ for all } Z^1 \in \xi \};

the set of unoccupied input ports of the resultant determined by \( C \) is \( \text{TOTALUNOCCUPIEDPORTS}(C) = \cup \{\text{UNOCCUPIEDPORTS}(Z^1,C): Z^1 \in \xi \} \);

the input of the resultant determined by \( C \) is \( \text{RESULTANTINPUT}(C) \)

= \( \text{X}_{\text{TOTALUNOCCUPIEDPORTS}}(C) \);

the set of input functions of the resultant determined by \( C \) is \( \text{RESULTANTINPUTFUNCTIONS}(C) = \{f: f \in \text{FUNCTIONS}(\text{REALS}, \text{RESULTANTINPUTS}(C)), \text{there exists } g \in \text{TOTALINPUTFUNCTIONS}(C) \text{ such that } f = \text{PROJECTION}-\) \( (\text{TOTALUNOCCUPIEDPORTS}(C))g) \}\);

and the time scale of the resultant determined by \( C \) is \( \text{RESULTANTTIME-SCALE}(C) = \text{TIMESCALE}(Z) \).

We want the resultant of a general coupling recipe to be a system. If the coupling has been done correctly then the resultant should be a system. We need to know what it means to do it right. We also want to
be able to deduce the behavior of the resultant from the specification of the operation of each of the individual components.

In order to determine the behavior of the resultant from the behavior of each component we must know what is going into every input port at every instant of time. Then the initial states and the functions going into the input ports will completely determine the resultant. However, the state transition function of the resulting system can only be defined if the coupling recipe determines a coupling function \( \kappa \) which satisfies certain relations. The function \( \kappa(f, \chi) \) is the function which tells us what the inputs are that are being supplied to each of the input ports of all of the components of the system, whether free or occupied, where \( \chi \) is the initial state of the resultant and \( f \) is supplying inputs to all unoccupied input ports.

These notions are made explicit in the following definitions of the coupling function and the resultant of a system couple.

The Coupling Function (Definition 3.8)

Let \( C = (\xi, a, o) \) be a coupling recipe. Then a function \( \kappa \) is a coupling function with respect to \( C \) if and only if:

\[
\kappa \in \text{FUNCTIONS(RESULTANTINPUTFUNCTIONS(C) \times RESULTANTSTATESET(C)},
\]

\[
\text{TOTALINPUTFUNCTIONS(C)}; \text{ and for every } f \in \text{RESULTANTINPUTFUNCTIONS(C)},
\]

\[
t \in \text{RESULTANTTIMESCALE(C)} \ \chi \in \text{RESULTANTSTATESET(C)}, \text{ and } V \subseteq \text{TOTALINPUTPORTS(C)}:
\]

\[
(\kappa(f, \chi))(t)(V)
\]

\[
= (f(t))(V) \text{ if } V \subseteq \text{TOTALUNOCCUPIEDPORTS(C)},
\]

\[
= ((o(Z,Z^1))(((MOTION(Z))((\text{PROJECTION(INPUTPORTS(Z))))\kappa(f, \chi), t))
\]
The Resultant of a System Couple (Definition 3.9)

If $C = (\xi, a, o)$ is a coupling recipe and $\kappa$ is a coupling function with respect to $C$, then the resultant of $C$ is an assemblage denoted $\text{RESULTANT}(C)$ and is defined as follows:

- $\text{STATES}(\text{RESULTANT}(C)) = \text{RESULTANTSTATESET}(C)$;
- $\text{INPUTS}(\text{RESULTANT}(C)) = \text{RESULTANTINPUTS}(C)$;
- $\text{INPUTFUNCTIONS}(\text{RESULTANT}(C)) = \text{RESULTANTINPUTFUNCTIONS}(C)$;
- $\text{BEHAVIOR}(\text{RESULTANT}(C)) = \text{RANGE}(\text{MOTION}(\text{RESULTANT}(C)))$;
- $\text{TIMESCALE}(\text{RESULTANT}(C)) = \text{RESULTANTTIMESCALE}(C)$;

and if $f \in \text{INPUTFUNCTIONS}(\text{RESULTANT}(C))$, $t \in \text{TIMESCALE}(\text{RESULTANT}(C))$, and $\chi \in \text{STATES}(\text{RESULTANT}(C))$, then

\[
(((\text{MOTION}(\text{RESULTANT}(C)))(f,t)(\chi))(\text{STATES}(Z)) = (((\text{MOTION}(Z))((\text{PROJECTION}(\text{INPUTPORTS}(Z))))\kappa(f,\chi), t))(\chi(\text{STATES}(Z)))
\]

for every $Z \in \xi$.

A coupling recipe $C$ is a system couple if and only if there exists a coupling function $\kappa$ with respect to $C$. Coupling functions generally exist when the components all consist of discrete systems or all are differentiable systems (Wymore 1971b, 1971c).

**Components and Subsystems**

There is a lot of confusion in the literature as to the meaning of the ideas of subsystem and component. These are defined explicitly in the following definitions.
Subsystem (Definition 3.10)

Let $Z$ and $Z^1$ be two systems. Then the system $Z$ is a subsystem of the system $Z^1$ if and only if:

- $\text{STATES}(Z) \subseteq \text{STATES}(Z^1)$,
- $\text{INPUTS}(Z) \subseteq \text{INPUTS}(Z^1)$,
- $\text{INPUTFUNCTIONS}(Z) \subseteq \text{INPUTFUNCTIONS}(Z^1)$,
- $\text{TIMESCALE}(Z) \subseteq \text{TIMESCALE}(Z^1)$,

and for every $f \in \text{INPUTFUNCTIONS}(Z)$, $t \in \text{TIMESCALE}(t)$, $(\text{MOTION}(Z))(f,t) = \text{RESTRICTION}((\text{MOTION}(Z^1))(f,t), \text{STATES}(Z))$.

Component (Definition 3.11)

Let $Z$ and $Z^1$ be systems. Then the system $Z$ is a component of the system $Z^1$ if and only if there exists a system couple $C$ such that $Z \in \text{COMPONENTS}(C)$ and $Z^1$ is isomorphic to $\text{RESULTANT}(C)$.

There are many aspects of system theory not relevant to this development which are not included here. These include the ideas of what it means to decompose a system into its isolated subsystems and also to resolve a system into its independent components. It also includes the notions of system homomorphism and its specialization system isomorphism which are basic tools in the development of the theory of systems. The definition of these notions permits the definition of what it means to simulate a system and to implement a system.
CHAPTER 4

THE APPROACH TO ANALYSIS AND SYNTHESIS

The discussion in Chapter 2 focused on some fundamental concepts concerning the structure of a watershed. The fundamental unit of the watershed may be taken as the biogeocenose which may be considered as an ecosystem.

The analysis of the ecosystem water balance will involve the resolution of the complex system into simpler components, the development of mathematical models of these components, and also modeling the interfaces or interconnections between these simpler components. The synthesis of an ecosystem will then involve the coupling or interconnecting of the set of simpler components into an overall complicated system model of the ecosystem.

It seems logical that the starting point in the analysis of the water balance of the ecosystem lies in dividing the ecosystem into its principal components. As discussed earlier in Chapter 2, these are the ecotope which includes the climate and soil, and the biocoenose which includes the vegetation, animal population and microorganisms. The components of the ecotope are termed abiotic while those of the biocoenose are termed biotic.

Now consider the role of each component in the storage of water. In the static state the important storage elements are in the soil and climatic components. However, in the dynamic state (in which the system
is subject to inputs of mass and energy) it is necessary to consider possible temporary storage at the interfaces of the components receiving the input to those transmitting the input. In the case of water input, significant temporary storage may exist at the soil surface-air interface, as retention storage, and during overland flow, as detention storage. This will lead to modeling of interception, retention and detention storage, as interface water storage components. One can also consider soil-plant and soil-animal interfaces as potential storage sites during fluxes since water has a tendency to follow roots and other holes from the soil surface into the soil.

Considering the fact that there exist at least three interface water storage components, it is necessary to determine the water balance for each in terms of the processes causing additions and depletions. This leads to consideration of the interception process including stemflow in terms of a probability of stemflow or interception and the depletion of a potential canopy storage.

In considering water fluxes and resulting storage at the soil surface it is necessary to realize that there may exist a soil-plant-animal-decomposer-air interface which is represented in the mulch or litter layer. This component represents a potential storage layer which is treated in this analysis.

The depletion of total storage on the soil surface by infiltration is treated here as being dependent on properties of soil matrix-soil water interface which are in turn dependent on the soil water content. The soil matrix-soil water interface properties at the start of a storm together with storm rainfall intensity and duration properties also
determine the time at which storage on the soil surface begins as well as the infiltration rate after surface storage begins.

The depletion of detention storage by lateral flow is also treated as being affected by the vegetative cover as well as the air-soil interface.

The depletion of surface storage by lateral flow is allowed for but is not explicitly modeled. The author feels that this process can best be treated by coupling biogeocoenose or ecosystem models.

The distribution of soil water storage is treated as being controlled by soil properties of capillary conductivity and moisture potential. These properties affect surface fluxes of infiltration and evaporation as well as transpiration (which is discussed later).

In all cases, soil, soil surface, interception and mulch layer water storage are subject to depletion by evaporation which results from energy inputs from the climate. No attempt is made to model the energy balance.

Perhaps the dominant interaction occurs at the interface of the soil, plant, and climatic components which is expressed through the interrelated processes of evaporation, transpiration and plant growth. Other interactions occur at the soil-climate interface in the evaporation of water from the soil surface, mulch layer, and retention and detention storage. Evaporation also occurs at the plant-climate interface but is treated here as being coupled to the transpiration process through the plant.
The treatment of the transpiration process is based on the assumption that depletion of soil moisture by transpiration is controlled by inter-relationships of soil, plant and climatic conditions. This is a rather unusual approach in watershed modeling but is a key to studying effects of manipulation of watersheds by man.

Transpiration losses are distributed over the soil profile as a negative source term in the soil water balance equations. The distribution is done by weighting the total transpiration by a fraction proportional to root properties, soil properties and soil water content in the layer under consideration.

There is no attempt made to model producer, consumer or decomposer components. Interactions of the plant with light, soil water and also the cover density are considered insofar as they affect transpiration. This is done primarily to show how to incorporate these components and the interactions.

The synthesis will involve the stepwise coupling of each component. The order of coupling will follow the path of precipitation through the watershed. The result of the synthesis will be a water balance model of an ecosystem. This model will allow one to study the behavior of the ecosystem or to couple to other similar models in order to model a watershed.
CHAPTER 5

ANALYSIS OF THE WATER BALANCE

The objective of this chapter is to attempt to resolve the ecosystem water balance into simpler components, develop mathematical models of these components, and model the interfaces or interconnections between these components. The individual components will be modeled as systems after a general discussion of their behavior.

As indicated in Chapter 1, the author does not claim to be an expert in all the fields represented by the different components which are modeled in this chapter. An extensive literature review revealed the insights which experts in each field have gained about the behavior of each component. These insights, together with intuitive hypothesis where necessary, were used in developing the system model of each component.

In the discussions, no attempt will be made to review all the literature concerned with each component. The literature which this author feels to be most significant will be included and in many cases this must be limited.

It should be noted that the individual component models are achieved by concentrating upon only one component at a time and ignoring all other components. In modelling it is simply assumed that the inputs are available with no consideration explicitly given as to where these inputs come from.
The analysis of the ecosystem water balance will proceed along the line of the water cycle of an ecosystem based upon the general concepts as indicated in Chapter 4. Hence, it will begin with interception and end with transpiration. Then the interactions of water with other components which influence the water cycle will be considered.

Throughout the discussions, the notation used in the literature being referenced will be used for discussion purposes. Hence, no special meaning should be attached to notation outside of the particular discussion in which it is used, unless the notation has been formally defined in the definition of a system or within a discussion.

The Canopy Water Balance

Interception is the process of stopping, deterring, or changing the continuity of path of fall of precipitation before reaching the mulch or soil surface. Interception affects the water balance by partitioning precipitation entering the ecosystem. Both snow and rain are subject to interception; however, this discussion will treat only rainfall interception and the resulting canopy water balance.

The interception process has been under study for nearly a century. Reported work has been mostly with respect to forest interception and is well summarized by Wicht (1941), Kittredge (1948, 1962), Zinke (1967) and Penman (1963). There has been little work done on interception by plants in arid and semi-arid areas. Published literature includes reports by Slatyer (1965), Hamilton and Rowe (1949), Specht (1957, 1963), Aldridge and Jackson (1968), Aldridge (1968), Skau (1964), Corbett and Crouse (1968) and Collings (1966). These studies in general
show that the interception process may play a significant ecological role in the water balance of plants as a result of the redistribution of partitioned precipitation (Voight 1960), Slatyer (1965), Specht (1957).

Leonard (1967) gives a comprehensive discussion of the mechanics of the interception process. Initially, raindrops striking the leaves and stems spatter, but the largest part is usually held on the surfaces until the storage capacity is reached. The stem or leaf storage capacity is a function of surface area and surface tension forces. As storage capacity is reached and exceeded, drops form and fall to lower leaves or the litter layers or form rivulets which run down the stem. That which runs down the stem to ground will be termed stemflow and that which falls or otherwise passes through the canopy (excluding stemflow) to the ground will be termed throughfall or effective precipitation. Voight (1960) found that bark texture affects stem storage capacity and quantity of stemflow. Shaking of the canopy by wind will decrease storage capacity. Canopy storage capacity is the amount of moisture the canopy can hold and is a function of leaf and stem area, surface tension and storm intensities and characteristics.

Leonard (1967) also considers evaporation from canopy storage as contributing to total interception losses during a storm. He proposes a general equation for interception losses but does not consider inter-storm recovery periods.

A simple canopy water balance model will now be developed. This and the mulch layer water balance will be the most hypothetical models of the entire analysis. However, a justification (on an ecological
basis) for including at least a hypothetical model of the canopy water balance follows the development of the model.

**Symbol Definitions**—Define the following notation relevant to system modeling:

ISTOR(t) is the equivalent depth of storage of intercepted precipitation over the projected canopy area at time t; such that

ISTOR(t) ∈ REALS[0, IMAX(t)] where:

IMAX(t) is the maximum possible value of ISTOR(t) at time t;

IEVAP(t) is the rate of evaporation from interception storage at time t;

INTR(t) is the rate of interception at time t;

STMFLW(t) is the rate of stemflow at time t;

PRECIPRATE(t) is the rainfall intensity at time t;

COVERDENSITY(t) is the projected canopy density at time t;

KSF(t) is the value of a constant dependent on the geometry and characteristics of plant stem at time t;

EFFPRECPRECIPRATE(t) is the effective precipitation rate beneath the projected canopy at time t and is the difference between precipitation rate and interception rate;

Δt is the length of time interval under consideration;

EWET(t) is the potential evaporation rate from the wetted canopy surface at time t.

On the basis of the previous discussion a general water balance equation for the canopy may be written as

\[
\text{ISTOR}(t) = \text{ISTOR}(t-\Delta t) + (\text{INTR}(t-\Delta t) - \text{STMFLW}(t-\Delta t) - \text{IEVAP}(t-\Delta t)) \times \Delta t. \tag{5.1}
\]
The previous equation will be the basis for the canopy water balance component system.

There have been a number of models suggested for interception rate (Penman 1963). Most of these involve an exponential decay term. A rather simple model will be developed here. Assume that if the storage of intercepted precipitation at time $t$ is zero, then

$$\frac{d(\text{ISTOR}(t))}{dt} = \text{PRECIPRATE}(t) \times \text{COVERDENSITY}(t). \quad (5.2)$$

Equation 5.2 is based on the assumptions: (1) if interception storage is zero, then the fraction, $\text{COVERDENSITY}(t)$ of the precipitation will be intercepted, and $1 - \text{COVERDENSITY}(t)$ will pass through, and (2) the rate of change of interception storage with time varies directly with the ratio of available storage ($\text{IMAX}(t) - \text{ISTOR}(t)$) to maximum storage, $\text{IMAX}(t)$, as illustrated in Figure 5.1.

This relationship is:

$$\text{INTR}(t) = \frac{d(\text{ISTOR})}{dt} (t)$$

$$= \text{PRECIPRATE}(t) \times \text{COVERDENSITY}(t) \times \frac{(\text{IMAX}(t) - \text{ISTOR}(t))}{\text{IMAX}(t)}. \quad (5.3)$$

Equation 5.3 will be used to model the rate of interception of precipitation by the canopy.

To estimate stemflow the assumption is made that the rate of stemflow is linearly proportional to the ratio of interception storage to maximum interception storage, $\text{ISTOR}(t)/\text{IMAX}(t)$, such that the stemflow rate is zero when interception storage is zero, and the stemflow rate is a constant proportion of the fraction, $\text{COVERDENSITY}(t)$, of the total precipitation striking the canopy when interception storage
Figure 5.1. Interception Rate as a Function of Interception Storage.
equals maximum interception storage. This relationship is illustrated in Figure 5.2 and is expressed as

\[
STMFLW(t) = KSF(t) \times COVERDENSITY(t) \times PRECIPRATE(t) \\
\times ISTOR(t)/IMAX(t).
\] (5.4)

The constant KSF is to be supplied by other components. Equation 5.4 will be used to model stemflow.

The rate of evaporation from interception storage is modeled by assuming that evaporation of intercepted precipitation proceeds at a rate which is directly proportional to the ratio of interception storage to maximum possible interception storage. If the ratio is equal to 1.0 then the rate cannot exceed potential evaporation EWET and if the ratio is zero then the rate must be zero. This is illustrated in Figure 5.3 and is expressed:

\[
IEVAP(t) = EWET(t) \times ISTOR(t)/IMAX(t).
\] (5.5)

The value of EWET(t) will be supplied by a component to be described in a later section.

Equations 5.1, 5.3-5.5 will be modeled as discrete component systems to determine the canopy water storage, interception rate, stemflow rate, and rate of evaporation from interception storage, respectively.

Figure 5.4 is a diagram illustrating the general interrelationships of the four component systems whose interconnection comprises the general canopy water balance system.

It will be shown later that the plant transpiration component must be coupled to the interception evaporation component in order to determine actual transpiration.
Figure 5.2. Stemflow Rate as a Function of Interception Storage.
Figure 5.3. Evaporation from interception storage as a function of interception storage.
Figure 5.4 The Canopy Water Balance.
The coupling with the plant evapotranspiration component points out one interaction of the interception component with the plant component which affects the behavior of the plant. In general, it seems that in a stationary watershed, interception is more important for its ecological role than it is for its role in the rainfall-runoff process. That is, interception does not significantly affect flood peaks and volumes. However, in a nonstationary watershed, effects of interception will affect the rainfall-runoff process over a period of time because vegetation intercepts not only rainfall mass but also rainfall energy. Changes in the amount of rainfall energy reaching the soil surface will in turn affect soil surface properties (Baird, Richardson, and Knisel 1970; Rauzi, Fly and Dyksterhuis 1968; and Rauzi 1963).

Many prediction equations may adequately describe the gross canopy water balance of an area but not describe the distribution of rainfall reaching the ground. They may not indicate the ecological significance of rainfall partitioning and redistribution by the canopy, particularly if throughfall and stemflow are lumped.

Voight (1950) studied rainfall distribution under beech, red pine, and hemlock canopies. He reported a large portion of rainfall hitting the canopy was concentrated by stems and foliage, and released in a relatively narrow band around the base of the tree. He found that water flowing down the stem may follow major roots into the soil. Voight also reported rainfall concentration by drip around canopy edges, branch stubs and direct throughfall caused by random openings.

Slatyer (1965) reported on interception by an arid zone plant community (Acacia aneura F. Muell). He found that effective precipitation
and stemflow started after 0.8 and 2.8 millimeters of gross rainfall were received, respectively, and that throughfall and stemflow increased rapidly with gross rainfall. When precipitation reached 12.5 millimeters, throughfall and stemflow were approximately 55 and 40 percent of precipitation, respectively. As precipitation increased further, throughfall and stemflow changed toward 60 and 35 percent, respectively. The percentage values of throughfall and stemflow were computed with respect to equivalent surface depth of water over the projected canopy area. The extremely high value of stemflow was concentrated in a narrow band about the stem. The limiting infiltration rate near the stem was approximately 5 to 6 inches per hour, decreasing to 0.7 inches per hour near the canopy edge and 0.3 inches per hour in the openings. The trees were strongly pedastaled and roots, while spreading, were strongly concentrated in the zone beneath the main stem. Throughfall was found to increase with radius from the main stem.

Slatyer's data are similar to that of Hamilton and Rowe (1949) for chaparral in semiarid zones of California. They found stemflow of up to 37 percent based on total stand area and not of canopy area alone. Aldridge and Jackson (1968) reported similar stemflow figures for manuka in New Zealand. Stemflow in manuka averaged 23 percent and reached 30 to 40 percent of gross rainfall, contributing approximately half of net rainfall. No data was given for soil infiltration rates and information on the influence of individual plant species and associated fauna on soil infiltration rates is limited (Lyford 1968). In all three studies, the morphological characteristics of the stems favored channeling of intercepted rainfall.
Specht (1957) studied soil moisture patterns produced by interception and stemflow in dark island heath in South Australia. He showed that interception and stemflow by two dominant species produced patterns of rain shadows and centers of concentrations under these species. For one species (*Banksia ornata*), water flowed down the stems to enter the soil at ground level. In the other species (*Xanthorrhoea australis*), water flowed down between buried leaf bases to enter the soil at the stock (caudex) some 9 to 12 inches below ground level. Both species were nanophyllous and their morphology favored stemflow. Leptophyllous undershrubs did not exhibit stemflow and were eventually suppressed and killed.

Similar mechanisms may act in southwestern United States. A complete study of interception in major arid zone plant communities might well be very useful in understanding soil-plant-water relations in desert ecosystems, particularly if it included infiltration rates.

The Interception Rate System, $Z_1$ (Definition 5.1)

This system models equation 5.3 in discrete form to determine rate of interception and the effective precipitation. Rewriting equation 5.3 gives

$$\text{INTR}(t) = \text{PRECIPRATE}(t-\Delta t) \times \text{COVERDENSITY}(t-\Delta t) \times (\text{IMAX}(t-\Delta t) - \text{ISTOR}(t-\Delta t))/\text{IMAX}(t-\Delta t).$$

(5.6)

Also, by definition,

$$\text{EFFPRECIPRATE}(t) = \text{PRECIPRATE}(t-\Delta t) - \text{INTR}(t-\Delta t).$$

(5.7)

Define the discrete system $Z_1 = (S_1, P_1, F_1, M_1, T_1, \sigma_1)$ where:

$s_1 = x_1(s_1^1, s_1^2)$ where $s_1^1 = \text{INTR}, s_1^2 = \text{EFFPRECIPRATE};$
\( P_1 = X(p_1, p_2, p_3, p_4) \) where \( p_1 \) = PRECIPRATE, \( p_2 \) = COVERDENSITY, \( p_3 \) = IMAX, \( p_4 \) = ISTOR;

and for \( p \in P_1, t \in T_1, x \in S_1: \)

\[
\sigma_1(\text{CONSTANT(REALS, } p), 1)(x)(S_1^1) = p(p_1) \times p(p_2) \times (p(p_3) - p(p_4)) / p(p_4),
\]

\[
\sigma_1(\text{CONSTANT(REALS, } p), 1)(x)(S_2) = p(p_1) - x(S_1^1).
\]

The output functions are defined as \( \zeta_1^1(x) = x(S_1^1), \zeta_1^2(x) = x(S_2). \)

The Stemflow System, \( Z_2 \) (Definition 5.2)

This system models equation 5.4 discretely to determine the stemflow rate. Rewriting equation 5.4 gives

\[
\text{STMFLW}(t) = \text{KSF}(t-\Delta t) \times \text{COVERDENSITY} \times \text{PRECIPRATE}(t-\Delta t) \times \text{ISTOR}(t-\Delta t)/\text{IMAX}(t-\Delta t).
\]

Define the discrete system \( Z_2 = (S_2, P_2, F_2, M_2, T_2, \sigma_2) \) where:

\( S_2 = \text{STMFLOW}; \)

\( P_2 = X(p_2^1, p_2^2, p_2^3, p_2^4, p_2^5) \) where \( p_2^1 \) = COVERDENSITY, \( p_2^2 \) = PRECIPRATE, \( p_2^3 \) = ISTOR, \( p_2^4 \) = IMAX, \( p_2^5 \) = KSF;

and for all \( p \in P_2, t \in T_2, x \in S_2: \)

\[
\sigma_2(\text{CONSTANT(REALS, } p), 1)(x) = p(p_2^1) \times p(p_2^5) \times p(p_2^1) \times p(p_2^3) / p(p_2^4).
\]
The Interception Evaporation System, \( Z_3 \) (Definition 5.3)

This system models equation 5.5 discretely to determine the rate of evaporation of interception storage. Rewriting equation 5.5 gives

\[
\text{IEVAP} (t) = \text{EWET}(t-\Delta t) \times \text{ISTOR}(t-\Delta t)/\text{IMAX}(t-\Delta t). \tag{5.9}
\]

Define the discrete system \( Z_3 = (S_3, P_3, F_3, M_3, T_3, \sigma_3) \) where:

\[
S_3 = \text{IEVAP};
\]

\[
P_3 = X(P_3^1, P_3^2, P_3^3) \text{ where } P_3^1 = \text{EWET}, P_3^2 = \text{ISTOR}, P_3^3 = \text{IMAX};
\]

and for all \( p \in P_3, t \in T_3, \chi \in S_3 \):

\[
(\sigma_3(\text{CONSTANT}(\text{REALS},p),1))(\chi) = p(P_3^1) \times p(P_3^2)/p(P_3^3).
\]

The Canopy Water Storage System, \( Z_4 \) (Definition 5.4)

This system models equation 5.1 to determine the interception storage in the canopy.

Define the discrete system \( Z_4 = (S_4, P_4, F_4, M_4, T_4, \sigma_4) \) where:

\[
S_4 = \text{ISTOR};
\]

\[
P_4 = X(P_4^1, P_4^2, P_4^3, P_4^4) \text{ where } P_4^1 = \text{INTR}, P_4^2 = \text{STMFLW}, P_4^3 = \text{IEVAP}, P_4^4 = \text{IMAX};
\]

and for every \( p \in P_4, t \in T_4, \chi \in S_4 \):

\[
(\sigma_4(\text{CONSTANT}(\text{REALS},p),1))(\chi) = \text{MAX}(0,\text{MIN}(p(P_4^4), \chi + (p(P_4^1) - p(P_4^2) - p(P_4^3)) \times \Delta t)).
\]
The Mulch Layer Water Balance

The soil surface layer, which acts as an interface or buffer between the atmosphere and mineral soil, includes both organic and inorganic materials. The organic portion will be termed mulch (Tomanek 1969) and the inorganic portion will be termed gravel if sufficient to form a definite layer.

The term mulch layer as used here includes what is known as mulch, litter, debris, duff, protection cover, etc., including algae or microfloral growth. It includes the upper layer of undecayed plant residuum and the partially decayed, disintegrated and fragmented fresh mulch forming a layer next to the soil surface (Tomanek 1969). It does not include the standing dead or cured herbage and forage.

Tomanek (1969) gave a very comprehensive review of the role and dynamics of the mulch layer in grassland ecosystems. He reported that the mulch layer increases soil moisture through increased infiltration rates, decreased runoff and decreased evaporation. The mulch also stabilizes soil temperature and this, combined with increased moisture, improves germination and increases green herbage production. Increased herbage production will in turn increase mulch production which may lead to changes in plant species. Hence, Tomanek considers mulch as a vital component in the balance existing between living and dead vegetation, other organisms and the soil.

Sukachev and Dylis (1968) consider the mulch layer as the principal component of an ecosystem because (1) it is formed from dead organic matter and contains large reserves of nutrients and mineral
matter which is released in a regulated way by decomposition, (2) it is the layer of most vigorous biotic activity, both flora and fauna, due to the high concentration of nutrients, and (3) it is the basic source of humus or organic matter formation.

In a sense the mulch layer is the result of the interaction of the climatic, soil, producer, consumer and decomposer components of the ecosystem. This indicates not only the importance of the layer but also the difficulty in modeling the interactions. This part of the analysis will only be concerned with the water balance in a rather simple way while recognizing that the quantity of mulch may change.

Symbol Definitions—Define the following notation:

\( h_{STOR}(t) \) is the water storage in the mulch layer at time \( t \) as equivalent depth over the area of the mulch layer such that

\( h_{STOR}(t) \in \text{REALS}[0, M_{MAX}(t)] \) where:

\( M_{MAX}(t) \) is the maximum water storage capacity of the mulch layer at time \( t \);

\( M_{COVR}(t) \) is the fraction of the soil surface covered by the mulch layer at time \( t \);

\( M_{EVAP}(t) \) is the evaporation rate from the mulch layer at time \( t \);

\( NET_{PRECIPRATE}(t) \) is the net precipitation rate at the soil surface at time \( t \) (defined in equation 5.12).

A general water balance equation for the mulch layer may be written as

\[
M_{STOR}(t) = M_{STOR}(t-\Delta t) + M_{COVR}(t-\Delta t) \times (EFF_{PRECIPRATE}(t-\Delta t) - M_{EVAP}(t-\Delta t)) \times \Delta t. \tag{5.10}
\]
Equation 5.10 assumes that all effective precipitation striking the portion of soil surface covered by the mulch layer is intercepted so long as mulch layer storage is below capacity. It also assumes that stem-flow is not intercepted by the mulch layer but passes directly to the soil surface along the stem.

The assumption is made that evaporation from mulch water storage varies directly with mulch water storage. If mulch storage is zero, evaporation is zero, and if mulch storage is at maximum, then evaporation is at the potential rate. Therefore

\[ \text{MEVAP}(t) = \frac{\text{EWET}(t-\Delta t) \times \text{MSTOR}(t-\Delta t)}{\text{MMAX}(t-\Delta t)}. \]  
(5.11)

The net precipitation rate to the soil surface is taken as the difference between the effective precipitation rate and the rate of change of mulch water storage or

\[ \text{NETPRECIPRATE}(t) = \text{EFFPRECIPRATE}(t-\Delta t) - \left( \frac{\text{MSTOR}(t-\Delta t)}{\Delta t} \right) \]
\[ = 0 \text{ otherwise}. \]  
(5.12)

Equations 5.10-5.12 will be used to model the mulch layer water balance system as three component systems: mulch evaporation, mulch water storage and net precipitation.

The interconnection of these components to form the general mulch layer water balance is shown in Figure 5.5.

The Mulch Evaporation System, \( Z_5 \) (Definition 5.5)

This system models equation 5.11 to determine the rate of evaporation from mulch layer storage.
Figure 5.5 The Mulch Layer Water Balance.
Define the discrete system \( Z_5 = (S_5, P_5, F_5, M_5, T_5, \sigma_5) \) where:

\[
S_5 = \text{MEVAP};
\]

\[
P_5 = X\{P^1_5, P^2_5, P^3_5\} \text{ where } P^1_5 = \text{EWET}, P^2_5 = \text{MSTOR}, P^3_5 = \text{MMAX};
\]

and for all \( p \in P_5, t \in T_5, x \in S_5:\)

\[
(\sigma_5(\text{CONSTANT}(\text{REALS},p),1))(x) = p(P^1_5) \times p(P^2_5)/p(P^3_5).
\]

The Mulch Water Storage System, \( Z_6 \) (Definition 5.6)

This system models equation 5.10 to determine the water storage in the mulch layer.

Define the discrete system \( Z_6 = (S_6, P_6, F_6, M_6, T_6, \sigma_6) \) where:

\[
S_6 = \text{NSTOR};
\]

\[
P_6 = X\{P^1_6, P^2_6, P^3_6, P^4_6\} \text{ where } P^1_6 = \text{MMAX}, P^2_6 = \text{EFFPRECIPRATE}, P^3_6 = \text{MEVAP}, P^4_6 = \text{MCOVR};
\]

and for every \( p \in P_6, t \in T_6, x \in S_6:\)

\[
(\sigma_6(\text{CONSTANT}(\text{REALS},p),1))(x) = \text{MAX}\{0,\text{MIN}\{p(P^1_6), x + (p(P^4_6) - p(P^3_6)) \times \Delta t\}\}. 
\]

The Net Precipitation System, \( Z_7 \) (Definition 5.7)

This system models equation 5.12 to determine the net precipitation rate on the soil surface.

Define the discrete system \( Z_7 = (S_7, P_7, F_7, M_7, T_7, \sigma_7) \) where:
\[ S_7 = \chi(S^1_7, S^2_7) \text{ where } S^1_7 = \text{NETPRECIPRATE}, S^2_7 = \text{MSTOR}; \]

\[ P_7 = \chi(P^1_7, P^2_7) \text{ where } P^1_7 = \text{MSTOR}, P^2_7 = \text{EFFPRECIPRATE}; \]

and for all \( p \in P_7, t \in T_7, x \in S_7: \)

\[ (\sigma_7(\text{CONSTANT(REALS,p),1})(x))(S_7^1) \]

\[ = p(P^2_7) - (p(P^1_7) - \chi(S^2_7))/\Delta t \text{ if } p(P^2_7) > 0, \]

\[ = 0 \text{ if } p(P^2_7) \leq 0; \]

\[ (\sigma_7(\text{CONSTANT(REALS,p),1})(x))(S_7^2) = p(P^1_7). \]

The output is \( \zeta_7 = \chi(S^1_7). \)

**The Soil Surface Water Balance**

The soil surface represents the next possible storage site for water. Soil surface storage is usually considered as two components, detention storage and retention storage. Retention storage is considered to be that portion of surface storage which is not subject to depletion by what is called overland flow or flow over the soil surface. Detention storage is considered to be that portion of surface storage which is subject to flow over the surface. Hence, retention storage is that which occupies small depressions while detention storage is that which is flowing over the surface.

Depletions from surface storage arise from evaporation, infiltration and surface flow. Additions to surface storage arise from net precipitation, stemflow and surface flow. In general, retention
storage over an area is not always filled before detention storage and surface flow appears. The kinematic wave theory may be used to approximate detention storage and surface flow but it assumes a uniformly smooth surface. Hence, it cannot directly account for retention storage. In general, the dynamics of surface storage and surface flow on actual watersheds are poorly understood from the quantitative aspect.

Field studies integrating theory with well-planned experiments as suggested by Schreiber (1970) may prove to be most useful in gaining quantitative insight into the processes involved. Studies involving actual or simulated rainfall on well-instrumented micro-watersheds may also be useful.

For the present purposes, it is assumed that the surface water depth is uniform over the ecosystem at any time. It is further assumed that there is some method of determining surface flow into and out of the ecosystem. Evaporation will be assumed to occur at the potential rate.

Symbol Definitions-- Define the following notation:
SURFACEH2ODEPTH (t) is the depth of surface water storage at time t where SURFACEH2ODEPTH (t) ∈ NONNEGATIVEREALS;
INFIL(t) is the infiltration rate from surface storage at time t;
SRFCFLOWIN(t) is rate of surface flow into the ecosystem at time t;
SRFCFLOWOUT(t) is the rate of surface flow out of the ecosystem at time t.
A general surface water balance equation may be written as

\[
\text{SURFACEH2ODEPTH}(t) = \text{SURFACEH2ODEPTH}(t-\Delta t)
\]
\[
+ (\text{NETPRECIPRATE}(t-\Delta t) + \text{SRFCFLOWIN}(t-\Delta t) + \text{STMFLW}(t-\Delta t)
\]
\[
- \text{SRFCFLOWOUT}(t-\Delta t) - \text{INFIL}(t-\Delta t)
\]
\[
- \text{EWET}(t-\Delta t)) \times \Delta t,
\]

where

\[
\text{SURFACEH2ODEPTH}(t) \geq 0. \tag{5.13}
\]

The above equation 5.13 will be used to determine the surface water balance. The surface water balance will be modeled as a one component system. No attempt is made here to model surface flow using kinematic wave or other equations.

The Surface Water Balance System, \(Z_8\) (Definition 5.8)

This system models equation 5.13 to determine the surface water balance or depth of water on the soil surface.

Define the discrete system \(Z_8 = \{S_8, P_8, F_8, M_8, T_8, \sigma_8\}\) where:

\(S_8 = \text{SURFACEH2ODEPTH};\)

\(P_8 = \{P_8^1, P_8^2, P_8^3, P_8^4, P_8^5, P_8^6\}\) where \(P_8^1 = \text{NETPRECIPRATE}, P_8^2 = \text{SRFCFLOWIN}, P_8^3 = \text{STMFLW}, P_8^4 = \text{SRFCFLOWOUT}, P_8^5 = \text{INFIL}, P_8^6 = \text{EWET};\)

and for all \(p \in P_8, t \in T_8, x \in S_8:\)

\[
(\sigma_8(\text{CONSTANT(REALS, p)}, 1))(x) = \text{MAX}\{0, x + (p(P_8^1) + p(P_8^2) + p(P_8^3)
\]
\[
- p(P_8^4) - p(P_8^5) - p(P_8^6)) \times \Delta t\}.\]
The Soil Water Balance

The availability of soil moisture and its levels of distribution in time and space is often a critical factor in determining the type of vegetation in any climatic zone (Shubert 1969). In arid and semiarid areas availability of soil moisture may be the principal resource limiting vegetation growth (Lewis 1970, Dahl 1963).

The major process by which soil water storage is replenished in arid and semiarid regions is infiltration. Infiltration is defined as the process of entry into the soil of water made available at its surface (Philip 1969). Vegetation in turn influences soil properties controlling infiltration. Other ways in which soil moisture may be replenished are through capillary rise from a water table and lateral inflow from adjoining soil masses. This analysis will assume the water table is too deep for capillary rise to be of importance and that lateral inflow is not significant. These assumptions are usually valid for desert ecosystems.

In areas where vegetation is present, the major process by which soil water depletion occurs is usually evaporation and transpiration. Otherwise, the major process is either evaporation, percolation to the water table, or lateral flow to adjoining soil masses. In arid and semiarid ecosystems, percolation to water tables and lateral flow usually is negligible. The assumption is made that these processes are not important in the soil water balance of the arid and semiarid ecosystems.

Hence, this discussion will treat the water balance of what is often termed the effective soil layer or just the soil layer. The soil
layer will be defined as extending from the soil surface to the maximum depth from which plant roots will be able to extract water. In arid and semiarid ecosystems in general, plant roots do not extend below the zone of deepest penetration of infiltrated water.

It is to be noted that, although the assumptions are made that capillary rise and deep percolation do not occur, this does not preclude incorporation of these processes. This could be done by defining systems to represent the zone between the lower soil boundary and water table, the water table itself, and providing a suitable coupling.

There are two fundamental properties of the soil water interface which have considerable influence on the behavior of the soil water system as well as on the plant component. These properties are the capillary conductivity and the moisture potential. Both are properties of both soil matrix properties and soil water content (Van Bavel 1968).

In general, the soil moisture profile determines the status of the soil water system. The resulting conductivity and potential profiles determine fluxes due to infiltration, evaporation and transpiration and hence the resulting soil water balance.

The Soil Water Interface

As indicated earlier, the state of the soil water interface is determined by the capillary conductivity and moisture potential properties. The relationship between capillary conductivity and moisture content, and also between moisture potential and moisture content is: (1) frequently not single-valued, (2) strongly nonlinear, and (3) dependent upon soil physical properties.
The lack of single-valuedness results from a dependence of moisture potential upon not only water content but also on whether the soil is wetting or drying. This dependence is termed hysteresis.

The physical properties influencing capillary conductivity include porosity, texture, structure, organic matter, temperature, fracturing and texture among others (Wolff 1970; Gray, Norum and Murray 1969). However, the exact relationship between the physical properties, water content and the resulting conductivity and potential may never be known. Generally, it is possible to characterize the microhydrological properties of the soil by empirically determining the capillary conductivity and moisture potential functions (Elrick 1968). However, the relationships may not be constant because the soil physical properties may change, especially in the layers near the surface. These changes are caused by interaction of the soil with plant, animal, microorganisms and climatic components.

The qualitative role of these components in influencing soil properties has been reviewed by many authors. Wolff (1970) presents a comprehensive review of infiltration and factors controlling it in grassland ecosystems.

For a given soil the properties of the soil surface are important in determining infiltration rate. Raindrop splash on the soil surface alters the structure of the surface soil layer and causes the formation of a relatively impermeable surface crust which may limit the infiltration rate (McIntyre 1958a, 1958b; Lemos and Lutz 1957; Sor and Bertrand 1962; Bisal 1967; Ellison 1945). Large clods on the surface
help to protect the surface from beating effects of raindrops by deflecting the drops (Moldenhauer and Koswara 1968; Moldenhauer and Kemper 1969, Moldenhauer 1970). A gravel layer also acts to protect the surface and maintain structure (Epstein, Grant, and Struchtmeyer 1966, Grant and Struchtmeyer 1959). However, the effect of gravel would be rather stationary under natural conditions. The presence of a mulch layer or low vegetation acts to: (1) protect the soil by absorbing the kinetic energy of drops, (2) increase structure of the soil surface through added organic matter and (3) provide channels for water to enter the soil along live and dead plant stems and roots (Knight 1969, Rauzi et al. 1968, Fletcher 1960, Horton 1940, Kincaid, Gardner and Schreiber 1964, Lyford 1968, Pearse and Woolley 1936, Tomanek 1969, and Meeuwig 1970).

Paris (1969) reviews the function of soil fauna in grassland ecosystems and includes a brief literature review showing that soil fauna act to increase the structure and permeability of soils.

Lewis (1970) gives an excellent discussion and literature review of the effects of various biotic activities such as burrowing, construction of ant mounds, and root growth and deterioration on maintaining the permeability and structure of the upper soil horizons.

Gross (1969) discusses the role of small herbivorous mammals in grassland ecosystems and their interactions with the soil.

It would obviously be quite difficult to model these effects. They essentially are involved in the genesis of soils (Jenny 1941, 1958, 1961). It would seem that any attempt to model the effects of
interactions on capillary conductivity and moisture potential relationships must initially be done in an empirical or even hypothetical manner.

For present purposes only one-dimensional, vertical movement in a nonhomogeneous soil in which the flow system is free from hysteresis will be considered. Therefore, for a given soil the capillary conductivity and moisture potential will be functions of the depth from the soil surface and the moisture content.

Symbol Definitions -- Define the following notation:

SOILDEPTH is the maximum depth from which plant roots can extract soil water (measured vertically downward from the soil surface);

MOISTUREDIST(t) is the function which gives the soil moisture profile at time t where

MOISTUREDIST = FUNCTIONS(DEPTHS, SOILH2O);

SOILPROPERTY(t) is the function which gives the profiles of capillary conductivity and moisture potential at time t where

SOILPROPERTY = FUNCTIONS(DEPTHS, CAPCOND X POTNTLS);

SOILH2O(z,t) is the volumetric soil moisture content, where arbitrarily SOILH2O = REALS [0,1], at depth z at time t;

POTNTLS(z,t) is the moisture potential at depth z at time t;

CAPCOND(z,t) is the capillary conductivity at depth z at time t;

KAP is the capillary conductivity - moisture content function such that KAP = FUNCTIONS(REALS[0, SOILDEPTH] X SOILH2O, CAPCOND);

PSI is the moisture potential - moisture content function such that

PSI = FUNCTIONS(REALS[0, SOILDEPTH] X SOILH2O, POTNTLS)
DEPTHS = \{d : d = i \times (\Delta z), i \in \text{INTEGERS}[0,n], \text{and } n \times (\Delta z) = \text{SOILDEPTH}\}; \text{ that is, DEPTHS is a set of } n+1 \text{ equally spaced points between the upper and lower levels of the soil; } \\
\Delta z \text{ is depth interval between the } n+1 \text{ equally spaced points in the } \\
\text{set DEPTHS; } \\
\text{MEANKAP}(t) \text{ is the mean capillary conductivity at time } t \text{ taken over } \\
z \in \text{DEPTHS; } \\
\text{MEANPOT}(t) \text{ is the mean moisture potential at time } t \text{ taken over } \\
z \in \text{DEPTHS. } \\
\text{Hence, for some depth } z \in \text{DEPTHS} \text{ below the soil surface and at } \\
some moisture content } \theta = \text{MOISTDIST}(z), \\
\quad k = \text{KAP}(z,\theta) \quad (5.14) \\
is the capillary conductivity where } k \in \text{CAPCOND, and } \\
\quad \psi = \text{PSI}(z,\theta) \quad (5.15) \\
is the moisture potential where } \psi = \text{POTNTLS}. \\
\text{Note that the functions } \text{KAP} \text{ and } \text{PSI} \text{ are completely arbitrary } \\
\text{functions which must be defined from data analysis. } \\
\text{It will be useful to define a system which will evaluate the } \\
\text{functions } \text{KAP} \text{ and } \text{PSI} \text{ at a finite number of points between the soil sur-} \\
\text{face and the depth } \text{SOILDEPTH} \text{ simultaneously. This is done in the fol-} \\
\text{lowing definition. } \\
\text{The Soil Water Interface System, } Z_9 \text{ (Definition 5.9)} \\
\text{This system determines the capillary conductivity profile, mois-} \\
\text{ture potential profile, and mean conductivity and mean potential from}
the soil moisture profile using relationships of the form given in the definition of KAP and PSI and equations 5.14 and 5.15.

Define the discrete system \( z_g = (S_g, P_g, F_g, M_g, T_g, \varrho_g) \) where:

\[ S_g = \text{SOIL PROPERTY} = \text{FUNCTIONS(DEPTHS, CAPCOND} \times \text{POTNTLS)}; \]

\[ P_g = \mathbf{X}\{P_g^1, P_g^2, P_g^3\} \text{ where } P_g^1 = \text{FUNCTIONS(DEPTHS, SOILH2O)}, \]

\[ P_g^2 = \text{FUNCTIONS(REALS}[0, \text{SOILDEPTH}] \times \text{SOILH2O, CAPCOND)}, \]

\[ P_g^3 = \text{FUNCTIONS(REALS}[0, \text{SOILDEPTH}] \times \text{SOILH2O, POTNTLS)}; \]

and for all \( p \in P_g, t \in T_g, x \in S_g: \)

\[ \text{MOISTUREDIST} = p(P_g^1), \text{KAP} = p(P_g^2), \text{PSI} = p(P_g^3), \]

\[ (\varrho_g(\text{CONSTANT(REALS, p),}1))(x) = \{(d, (\text{KAP}(d, \text{MOISTUREDIST}(d)), \text{PSI}(d, \text{MOISTUREDIST}(d)))) : d \in \text{DEPTHS}\}. \]

The outputs are:

\[ \xi_g^1(x) = (\sum \text{PROJECTION}(x)(\text{CAPCOND}) : d \in \text{DEPTHS})/#(\text{DEPTHS}) \text{ where} \]

\[ \xi_g^1 \in \text{FUNCTIONS}(S_g, \text{MEANKAP}); \]

\[ \xi_g^2(x) = (\sum \text{PROJECTION}(x)(\text{POTNTLS}) : d \in \text{DEPTHS})/#(\text{DEPTHS}) \text{ where} \]

\[ \xi_g^2 \in \text{FUNCTIONS}(S_g, \text{MEANPOT}). \]
Infiltration

It is necessary to consider surface fluxes into and out of the soil mass before attempting to model the soil water balance. This discussion will consider infiltration while a later discussion will consider evaporation. The general theory on infiltration is reviewed by Philip (1969).

As stated earlier capillary conductivity and moisture potential control soil water movement. Philip (1969) gives the general equation for soil water transfer in one dimension in soils free from hysteresis as

\[ \frac{\partial \theta}{\partial t} = \frac{z}{\partial z} \left( K \frac{\partial \phi}{\partial z} \right) \]  (5.16)

where \( \theta \) is volumetric water content, \( K \) is the capillary conductivity, \( \phi \) is the total potential and is the sum of gravity head \( z \) and moisture potential \( \psi \), \( z \) is the vertical distance positive downward such that \( z = 0 \) at the surface, and \( t \) is time. The above equation 5.16 subject to appropriate initial and boundary conditions has been used extensively in studying movement of water in unsaturated soils.

There have been a number of numerical studies of infiltration. Soil water relations during rain infiltration were reported by Rubin and Steinhardt (1963, 1964) and Rubin, Steinhardt and Reiniger (1964) in a three-part series. Rubin (1969) reported on soil water relations during ponded rainfall infiltration. He concluded that theoretical relations between ponded rainfall infiltration rates and infiltration for rains of various intensities cannot be expressed by a simple curve and differ significantly from the flood infiltration capacity-time function as shown
in Figure 5.6. This is in part attributed to the incipient ponding time which varies more with rainfall intensity than the incipient ponding water uptake. Rubin (1969) indicated that there is less discrepancy when the same data are plotted as infiltration rate-cumulative water uptake curves (rate-uptake curves). Hence the use of rate-uptake curves may result in less error than use of rate-time curves. Rate-uptake curves would be more useful for modeling in the case of varying rain intensities.

Other relevant numerical studies include analysis of infiltration into stratified soil columns (Wang and Lakshminarayana 1968, Whisler and Klute 1968), redistribution of infiltrated moisture (Staple 1966, 1969), and simultaneous redistribution and evaporation following infiltration (Bresler, Kemper and Hanks 1969; Gardner, Hillel and Benjamini 1970; Black, Gardner and Thurtell 1969). These studies demonstrate the value of the theory of soil water transfer in modeling the soil water balance.

There are a number of empirical equations available for estimating infiltration rates. These are reviewed and compared or tested in various combinations in reports by Whisler and Bouwer (1970), Childs (1969), Wolff (1970), Skaggs et al. (1968), Philip (1969), and Overton (1964). They will be briefly reviewed here.

Green and Ampt (1911) derived an equation which Philip (1954, 1969) later derived formally and placed on a firm theoretical basis. This equation is of the form

$$\frac{di(t)}{dt} = K_{SAT} \frac{(1 + (e - e_0)(P + H))}{i(t)}$$

(5.17)
Figure 5.6. An Example of the Relation between Infiltration Rate and Infiltration Duration during Water Uptake.

Solid line curves represent relations during rainfall infiltration. The labels indicate relative intensities. Broken line curve represents the relation during infiltration due to flooding (from Rubin 1969).
where $i(t)$ is cumulative infiltration to time $t$, $P$ is the capillary potential at the wetting front, $K_{\text{SAT}}$ is the hydraulic conductivity in the wetted zone, $H$ is the depth of ponded water, $\theta$ and $\theta_0$ are the volumetric water contents in the wetted zone at time $t$ and initially at time zero, respectively. This model assumes that there is a definite sharp wet front behind which the water content is $\theta$ and the conductivity is $K_{\text{SAT}}$, and in front of which the water content retains its initial value $\theta_0$. This model was used by Lichty, Dawdy and Bergmann (1968) in a rainfall-runoff model for small watersheds. (Also see Dawdy, Lichty and Bergmann (1970).)

Gardner and Widstoe (1921) and later Horton (1933) assumed infiltration rate decreased exponentially with time from a maximum value to a limiting value. They proposed

$$f(t) = f_c + (f_0 - f_c) \exp (-kt) \quad (5.18)$$

where $f(t)$ is the potential or maximum possible infiltration rate if the surface is saturated, $f_0$ is the initial and $f_c$ the final infiltration rate, and $k$ is a constant. This is a widely used equation but has no theoretical basis (Philip 1969).

Kostiakov (1932) suggested an equation of the form

$$i(t) = K t^n \quad (5.19)$$

where $i(t)$ is total infiltrated water at time $t$, and where $n$ and $K$ are constants. Philip (1969) suggests that according to the theory of infiltration, $K$ must be sorptivity, $n$ must equal 1/2 and the equation is valid only for short-term prediction.
Philip (1969) derived an algebraic infiltration equation of the form

\[ v_0 = \frac{1}{2} S t^{-1/2} + A \]  \hspace{1cm} (5.20)

or

\[ i(t) = St^{1/2} + At \]  \hspace{1cm} (5.21)

where \( i(t) \) is infiltrated water to time \( t \), \( S \) is sorptivity, \( v_0 \) is infiltration rate at time \( t \), and \( A \) is approximately the value of hydraulic conductivity at the moisture content of the surface. Both \( S \) and \( A \) are functions of the initial moisture content and the soil, and may be determined empirically.

It should be noted that the above equations 5.17-5.21 are for uniform soils where the soil surface is maintained in a saturated state from time zero to time \( t \). Also, the equations 5.17 to 5.21 do not incorporate rainfall intensity, except to the extent that they assume the intensity is sufficient to maintain the surface in a saturated state. The equations 5.18 to 5.21 are essentially infiltration rate-time equations and for reasons given by Rubin (1969), as discussed earlier, are subject to error. Equation 5.17 is an infiltration rate-cumulative uptake equation and hence is subject to less error than equations 5.18 to 5.21. However, it does not account for intensity or variations in properties due to biotic effects. Holtan (1961, 1969) suggested an empirical equation in which infiltration is viewed as an exhaustion of the available porosity of the upper soil horizons. The infiltration rate converges to the relatively stable transmission potential (permeability) of the lower soil horizons. The equation is of the form
\[ f = a S_a^n + f_c \]  \hspace{1cm} (5.22)

where \( f \) is the potential infiltration rate, \( S_a \) is the storage potential of the upper soil horizons (total porosity minus the antecedent soil moisture in units of length), \( f_c \) is the transmission potential of the lower zones or impeding strata (the limiting rate of infiltration after prolonged wetting), \( n \) is a constant for a given soil type and may be taken as approximately 1.4, \( a \) is a constant which depends on soil and vegetation conditions. Holtan, England and Allen (1967) suggest that the constant \( a \) is definitely related to percent basal area of the plant. Equation 5.22 in a sense overcomes some of the drawbacks of equations 5.17 to 5.21. It does not include time explicitly as the independent variable but rather storage and hence is applicable during periods of intermittent and low intensity precipitation. The parameters \( n \) and \( a \) may be related to soil and vegetation conditions respectively and hence it empirically accounts for biotic effects. Soil survey data may be used to estimate the limiting rate \( f_c \) and the maximum storage potential (Holtan 1969). The equation provides a basis for utilizing readily available soil and vegetation data (Holtan, England and Whelan 1967, England and Holtan 1969).

None of the equations 5.17 to 5.22 explicitly recognize capillary conductivity and moisture potential distributions as determining infiltration rates. They also do not provide any information about the resulting soil moisture, conductivity or potential distributions. The importance of these distributions in determining plant behavior,
especially transpiration and the distribution of transpiration losses is reviewed in later discussions.

In this respect it is interesting to look at infiltration and depletion equations used in some models of ecosystem behavior. Bledsoe and Van Dyne (1970) in a compartment model of secondary succession do not consider soil moisture at all. They do suggest that a soil moisture compartment would be useful in providing a more realistic and explicit way of modeling growth and succession.

In the nine-compartment model of energy flow in a semiarid steppe ecosystem (Van Dyne 1969b), soil moisture is included in a rather simple way. It is not clear in the program listing, but apparently all precipitation is assumed to infiltrate. Loss from soil moisture is a function of mean temperature for each of the five previous days.

In the simulation of a pasture-environment interaction (Byrne and Tognetti 1969), the soil is assumed to be homogeneous and is divided into 10 layers. No runoff is allowed and infiltrated water simply fills each layer completely, starting from the top, before draining to the next. Surface evaporation is dependent on tank evaporation, surface cover and surface layer moisture content. Transpiration is determined for each layer by distributing tank evaporation over the layers in ratios proportional to root density and length.

In a model for simulating soil water regimes in alternating fallow crop systems (Fitzpatrick and Nix 1969), it was assumed that all rainfall was effective in recharging soil moisture up a maximum depending on the crop. Depletion was then determined as empirical
functions of cover, type and growth rate, meteorological conditions and soil water conditions.

In a model of a growing pasture, Paltridge (1970) simply increases the water content of the soil to study effects of irrigation. However, the depletion is dependent on growth, soil water status, root density, length and radius, and potential evaporation in a physically based manner. The model is realistic and very dependent on the soil water status including conductivity, potential and moisture content profiles.

Finally, in the Mark I model of a grassland ecosystem (Bledsoe and Jameson 1969), infiltration is allowed to decline exponentially as soil moisture increases. They suggest an equation of the form

\[ I_n = P \left( q_1 e^{-K \times S_m} + q_2 \right) \text{ if } P < P_{\text{pmax}}, \]

\[ = P_{\text{pmax}} \left( q_1 e^{-K \times S_m} + q_2 \right) \text{ if } P > P_{\text{pmax}} \]  

(5.23)

where \( q_1, q_2 \in \text{REALS}[0,1] \) such that \( q_1 + q_2 = 1 \), \( q_2 \) represents the fraction which infiltrates when soil is saturated. \( P_{\text{pmax}} \) is the maximum infiltration rate under given soil conditions, \( S_m \) is soil moisture, \( K \) is a constant, and \( P_p \) is precipitation rate. Depletion sources are evaporation, transpiration and percolation.

Infiltration equations 5.17 to 5.22 have been tested by several investigators. Whisler and Bouwer (1970) tested equation 5.17 and 5.21 and a numerical solution of the one-dimensional flow equation. The parameters of the equations were determined experimentally. The numerical solution was found to agree very well with experimental results and
was chosen as the standard for comparing other equations. Equation 5.17 agreed very well with the numerical solution whereas equation 5.21 was very difficult to fit. This was due to the difficulties in estimating sorptivity which is a function of water content. They concluded that the Green and Ampt equation is the simplest to use and gives sufficient accuracy for many field problems.

Skaggs et al. (1968) reported an experimental test of infiltration equations. They tested the Green and Ampt, Philip, Horton and Holtan equations, 5.17, 5.20, 5.18 and 5.22, respectively. They observed that equations 5.18 and 5.22 gave good fits to the measured infiltration rates from sprinkled plots while the other two equations were not as accurate, and as time proceeded, gave estimates which were too low. Errors were magnified in repeated wet runs.

Although empirical equations could be used to predict infiltration, they cannot be used to model soil water distribution. Hence, the theoretical equations for one-dimensional soil water transfer in non-homogeneous soils will be used to model the soil water balance. It is also assumed that the flow system is free from hysteresis, salt, and thermal or vapor effects.

The system model will be based on equation 5.16 with appropriate boundary conditions to describe fluxes due to evaporation or infiltration at the surface and fluxes or other conditions below the root zone. However, equation 5.16 does not provide for transpiration losses.

It is possible to include transpiration losses in the equation by simply adding a negative source or extraction term (Whisler, Klute
and Millington 1968; Molz and Remson 1970). The flow equation then becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( K \frac{\partial \psi}{\partial z} \right) - \frac{\partial K}{\partial z} - S$$  \hspace{1cm} (5.24)

where $S$ is the extraction term and may be a function of space, time and water content.

Wang and Lakshminarayana (1968) write equation 5.16 more explicitly as

$$\frac{\partial \theta}{\partial t} \left|_z \right. = \left( \frac{\partial}{\partial z} \left( K \frac{\partial \psi}{\partial z} \right) \right)_t - \left. \frac{\partial K}{\partial z} \right|_t$$

where the symbol $\left|_z \right.$ or $\left|_t \right.$ indicates evaluated at $z$ or $t$, respectively. They then show that the above equation may be written as

$$\left( \frac{\partial \theta}{\partial t} \right)_z = \left( K \frac{\partial^2 \psi}{\partial z^2} \right) + \left( \frac{\partial K}{\partial \theta} \right)_\theta \left( \frac{\partial \psi}{\partial z} \right)_\theta - 1) \right)

+ \left( \frac{\partial \theta}{\partial t} \right)_t \left( \frac{\partial K}{\partial \theta} \right)_z \left( \frac{\partial \psi}{\partial z} \right)_\theta - 1) \right)

+ \left. \frac{\partial K}{\partial \theta} \right|_\theta \left. \frac{\partial \psi}{\partial \theta} \right|_z + 2K \frac{\partial^2 \psi}{\partial \theta \partial z} \left( \theta, z \right) \right)

+ K \frac{\partial^2 \psi}{\partial \theta \partial z} \left|_z \right. \frac{\partial^2 \theta}{\partial z^2} \left|_t \right. \right).$$  \hspace{1cm} (5.26)

The above equation 5.26 is a nonlinear, parabolic type with no analytical solution.

Let the right-hand side of equation 5.26 equal PARTIALS, then from equation 5.24 one can write

$$\left( \frac{\partial \theta}{\partial t} \right)_z = \text{PARTIALS} - S \left|_z, \theta, t \right.$$  \hspace{1cm} (5.27)
Wang and Lakshminarayana (1968) use an explicit implicit difference scheme to solve equation 5.26 numerically. They write equation 5.26 as a difference equation as follows:

\[
\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = \frac{\psi_{i+1,j,p} - 2\psi_{i,j,p} + \psi_{i-1,j,p}}{(\Delta z)^2} + \frac{K_{i+1,j,p} - K_{i,j,p}}{\Delta z} (\frac{\psi_{i+1,j,p} - \psi_{i,j,p}}{\Delta z} - 1) \\
+ \frac{\theta_{i+1,j} - \theta_{i,j}}{\Delta z} \left( \frac{K_{i,j+1,p+1} - K_{i,j,p}}{\Delta z} \theta_{i,j+1} - \theta_{i,j} \right) \\
+ \frac{\psi_{i+1,j,p} - \psi_{i,j,p}}{\Delta z} - 1 \\
+ \frac{K_{i+1,j,p} - K_{i,j,p}}{\Delta z} (\psi_{i,j+1,p+1} - \psi_{i,j,p}) \\
+ 2(K_{i,j,p}) \left( \frac{1}{\theta_{i,j+1} - \theta_{i,j}} \right) \left( \frac{\psi_{i,j+1,p+1} - \psi_{i,j,p}}{\Delta z} - \frac{\psi_{i+1,j,p} - \psi_{i,j,p}}{\Delta z} \right) \\
+ K_{i,j,p} \left( \frac{\psi_{i,j+1,p+1} - \psi_{i,j,p}}{\theta_{i,j+1} - \theta_{i,j}} \right) \left( \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta z)^2} \right)
\]

(5.28)
where \( i \) is the index along the space coordinate, \( j \) is the index along the time coordinate, \( p \) is the index for soil water content, and \( l \) is the iteration index. The notation is depicted in Figure 5.7. They then introduce:

\[
\text{BRACK1} = \theta_{i,j} \\
+ \frac{\Delta t}{(\Delta z)^2} \left\{ (K_{i,j,p}) (\psi_{i+1,j,p} - 2\psi_{i,j,p} + \psi_{i-1,j,p}) \right\} \\
+ \frac{\Delta t}{(\Delta z)^2} (K_{i+1,j,p} - K_{i,j,p}) (\psi_{i+1,j,p} - \psi_{i,j,p} - \Delta z)
\]

(5.29)

and

\[
\text{BRACK 2} = \frac{\Delta t}{(\Delta z)^2} \times \frac{1}{\theta_{i,j+1} - \theta_{i,j}} \times \\
\left\{ (\theta_{i+1,j} - \theta_{i,j}) (K_{i,j+1,p+1} - K_{i,j,p}) (\psi_{i+1,j,p} - \Delta z) \right\} \\
+ (K_{i+1,j,p} - K_{i,j,p}) (\psi_{i,j+1,p+1} - \psi_{i,j,p}) \\
+ 2(K_{i,j,p}) (\psi_{i+1,j+1,p+1} - \psi_{i,j+1,p+1} - \psi_{i+1,j,p} + \psi_{i,j,p})
\]
Figure 5.7. Net for the Explicit-Implicit Evaluation of Derivatives.

(Wang and Lakshminarayana 1968)
+ (K_{i,j,p}) (\psi_{i,j+1,p+1}^j - \psi_{i,j,p}^j) (\theta_{i+1,j}
- 2\theta_{i,j} + \theta_{i-1,j})\].

Hence equation 5.28 becomes

\[\theta_{i,j+1} = \text{BRACK1} + \text{BRACK2},\] (5.31)

and equation 5.27 becomes

\[\theta_{i,j+1} = \text{BRACK1} - S_{i,j} + \text{BRACK2}.\] (5.32)

To solve equations 5.29, 5.30 and 5.31 Wang and Lakshminarayana use an iterative scheme. The algorithm is of the form:

1. Let \(i = 1\) and estimate \(\theta_{i,j+1}^{i-1}\);

2. compute \(K_{i,j+1,p+1}^j\) and \(\psi_{i,j+1,p+1}^j\) using \(\theta_{i,j+1}^{i-1}\);

3. compute \(\theta_{i,j+1}^j\) using 5.29 to 5.31;

4. let \(i = i + 1\);

5. compute \(K_{i,j+1,p+1}^j\) and \(\psi_{i,j+1,p+1}^j\) using \(\theta_{i,j+1}^{i-1}\);

6. compute \(\theta_{i,j+1}^j\) using 5.29 to 5.31;

7. if \(\max |\theta_{i,j+1}^j - \theta_{i,j+1}^{i-1}|\) is less than an arbitrary tolerance limit \(\epsilon > 0\) or \(i\) is equal to some maximum number of iterations, go to 9;

8. go to 4;

9. accept \(\theta_{i,j+1}^j\) as giving the new soil-water profile.

The same algorithm can be applied to equation 5.29, 5.30, and 5.32.
Symbol Definitions -- Define the following notation:

SURFACEFLUX(t) is the flux of water through the soil surface at time t, and is taken as positive if into the soil and negative if out of the soil;

LOWERH2OCON(t) is the soil water content at depth SOILDEPTH at time t;

TRANSDIST(t) is the function giving the distribution of transpiration loss over the soil profile at time t where TRANSDIST ∈ FUNCTIONS (DEPTHS={0, SOILDEPTH}, TRANSLOSS);

TRANSLOSS(z,t) is the transpiration loss from the soil profile at time t at depth z;

SOILMOISTURE(t) is the total soil moisture content of the soil profile at time t;

DSOILMOISTDT(t) is the rate of change of total soil moisture at time t;

SOILEVAP(t) is the rate of evaporation from soil water storage at time t;

TRANS(t) is the rate of transpiration from soil water storage at time t.

The distribution of soil moisture at some time t will be given by a function MOISTUREDIST ∈ FUNCTIONS (DEPTHS, SOILH2O) which satisfies the equation 5.29, 5.30, and 5.32 subject to the initial and boundary conditions.

The boundary conditions which may be encountered at the surface are those during evaporation and infiltration under ponded and nonponded rainfall conditions.
For infiltration the boundary condition at the surface will be:

\[
\text{SURFACEFLUX}(t-1) = K_{0,j,p} x \left\{ \left( \frac{\psi_{1,j,p} - \psi_{0,j,p}}{\Delta z} \right) \right. \\
+ \left. \left( \frac{\psi_{0,j+1,p+1} - \psi_{0,j,p}}{\Delta z} \right) x \left[ \left( \frac{\theta_{1,j} - \theta_{0,j}}{\Delta z} \right) + 1 \right] \right. \\
\left. \theta_{0,j+1} - \theta_{0,j} \right\}
\]

if \( \psi_{0,j,p} < 0; \) \( (5.33) \)

\[
\psi_{0,j+1,p+1} = \text{SURFACEH2ODEPTH}(t-1)
\]

if \( \psi_{0,j,p} > 0. \) \( (5.34) \)

The notation on the right-hand side of equation 5.33 and the left-hand side of equation 5.34 is the same as that used in 5.29 - 5.32.

The surface boundary conditions must be determined from prevailing microclimatological conditions and the soil water conditions.

For the lower boundary it is assumed that the soil water content remains fairly constant at

\[
\theta_{d,j+1} = \text{LOWERH2OCON}(t-1)
\]

where subscript \( d \) indicates maximum soil depth. This boundary condition does not preclude fluxes through the lower boundary provided a suitable coupling is available.

Equation 5.29, 5.30, 5.32, 5.33, 5.34, and 5.35 are used to model the soil water balance. These equations are rewritten, in the definition 5.10 of the soil water distribution system \( \zeta_{10} \), using the defined
system notation. In the definition of $Z_{10}$ equation 5.38 incorporates 5.29, 5.30 and 5.32, equation 5.39 represents 5.35, equation 5.40 represents 5.33 and equation 5.41 represents 5.34.

A simple soil water balance equation for the soil layer may now be written as

$$SOILMOISTURE(t) = SOILMOISTURE(t-\Delta t) + (INFIL(t-\Delta t)) - SOILEVAP(t-\Delta t) - TRANS(t-\Delta t) \times \Delta t.$$  \hspace{1cm} (5.36)

This will be a rather simple but necessary system. It should be noted that either $INFIL(t)$ or $SOILEVAP(t)$ is determined by soil-microclimate interaction. $SOILMOISTURE(t)$ may be determined by an output function of the state of the soil water distribution system. However, $INFIL(t)$ must be determined from the above equation because the surface boundary conditions may change during infiltration. Therefore

$$INFIL(t - \Delta t) = ((SOILMOISTURE(t) - SOILMOISTURE(t - \Delta t))/\Delta t) + TRANS(t - \Delta t)$$

if $SOILEVAP(t-\Delta t) = 0$,

$$= 0 \text{ otherwise.}$$  \hspace{1cm} (5.37)

The infiltration system $Z_{11}$ will use equation 5.37 to determine infiltration.

The Soil Water Distribution System, $Z_{10}$ (Definition 5.10)

This system models the set of equations 5.29, 5.30, and 5.32-5.35 to determine the distribution of soil water. Define the discrete system $Z_{10} = (S_{10}, P_{10}, F_{10}, M_{10}, T_{10}, c_{10})$ where:
\[ S_{10} = \text{FUNCTIONS(DEPTHS, SOILH2O)}; \]

\[ P_{10} = X \{ P_{10}^1, P_{10}^2, P_{10}^3, P_{10}^4, P_{10}^5, P_{10}^6 \} \text{ where } P_{10}^1 = \text{SURFACEFLUX}, \]

\[ P_{10}^2 = \text{LOWERH2OCON}, P_{10}^3 = \text{FUNCTIONS(DEPTHS\{0, SOILDEPTH\}, TRANSLOSS)}, \]

\[ P_{10}^4 = \text{SURFACEH2ODEPTH}, \]

\[ P_{10}^5 = \text{FUNCTIONS(REALS[0, SOILDEPTH] X SOILH2O, CAPCOND)}, \]

\[ P_{10}^6 = \text{FUNCTIONS(REALS[0, SOILDEPTH] X SOILH2O, POTNTLS)}; \]

and for all \( p \in P_{10} \), \( t \in T_{10} \), \( x \in S_{10} \):

\[ (a_{10}(\text{CONSTANT(REALS, p),1)}(x) = x' \text{ such that for } \Psi = p(P_{10}^1), \]

\[ \text{KAP} = p(P_{10}^1): \]

\[ x'(d) = x(d) + \frac{\Delta t}{(\Delta z)^2} \times \left( \text{(KAP}(d, x(d)) \times \text{(PSI}(d+\Delta z, x(d+\Delta z)) \right. \]

\[ - 2 \times \text{PSI}(d, x(d)) + \text{PSI}(d-\Delta z, x(d-\Delta z))) \times \text{(KAP}(d+\Delta z, x(d+\Delta z)) \right. \]

\[ - \text{KAP}(d, x(d)) \times \text{(PSI}(d+\Delta z, (d+xz)) - \text{PSI}(d, x(d)) - \Delta z)) \right. \]

\[ x \left( \frac{1}{x'(d) - x(d)} \right) \times \left( \text{((KAP}(d, x'(d)) \times \text{(KAP}(d, x'(d)) \right. \]

\[ - \text{KAP}(d, x(d)) \times \text{(PSI}(d+\Delta z, x(d+\Delta z)) - \text{PSI}(d, x(d)) - \Delta z)) \right. \]

\[ + \text{KAP}(d+\Delta z, (d+\Delta z)) - \text{KAP}(d, x(d)) \times \text{(PSI}(d, x'(d)) - \text{PSI}(d, x'(d)) \right. \]

\[ + 2 \times \text{KAP}(d, x(d)) \times \text{PSI}(d+\Delta z, x'(d+\Delta z)) - \text{PSI}(d, x'(d)) \right. \]

\[ - \text{PSI}(d+\Delta z, x(d+\Delta z)) + \text{PSI}(d, x(d))) + \text{KAP}(d, x(d)) \right. \]

\[ x \left( \text{PSI}(d, x'(d)) - \text{PSI}(d, x(d)) \right) \times \left( \text{((x}(d+\Delta z) - 2 \times x(d) + x(d-\Delta z)) \right. \]

\[ - p(P_{10}^3)(d) \]

\[ \text{if } d \in \text{DEPTHS\{0, SOILDEPTH\}}, \quad (5.38) \]

\[ x'(d) = p(P_{10}^4) \text{ if } d = \text{SOILDEPTH}, \quad (5.39) \]

and \( x'(d) \) is such that
\[ p(P_{10}^1) = KAP(d, \chi(d)) \times ((\Psi(d+\Delta z, (d+\Delta z)) - \Psi(d, \chi(d))) \\
+ ((\Delta z) \times (\Psi(d, \chi'(d)) - \Psi(d, \chi(d))) \times (\chi(d+\Delta z) - \chi(d)) \\
/(\chi'(d) - \chi(d))) + \Delta z)/(\Delta z) \]

if \( d = 0 \) and \( \Psi(0,\chi(0)) = 0 \),

\[ \Psi(0,\chi'(0)) = p(P_{10}^4) \]

if \( d = 0 \) and \( \Psi(0, (0)) > 0 \) \( (5.41) \)

The output is defined as

\[ \zeta_{10}^1(x) = \Sigma\{((x(0) + x(SOILDEPTH)) \times (\Delta z) \\
\cup \{x(\Delta z) : d \in DEPTHS\sim\{0, SOILDEPTH\}\}}, \]

where \( \zeta_{10}^1 \in FUNCTIONS(S_{10}, SOILMOISTURE) \).

The Infiltration System, \( Z_{11} \) (Definition 5.11)

This system uses equation 5.37 to determine infiltration.

Define the discrete system \( Z_{11} = (S_{11}, P_{11}, F_{11}, M_{11}, T_{11}, \sigma_{11}) \)

where:

\[ S_{11} = X\{s_{11}^1, s_{11}^2, s_{11}^3\} \text{ where } s_{11}^1 = SOILMOISTURE, s_{11}^2 = INFIL, \]

\[ s_{11}^3 = DSOILMOISTUD; P_{11} = X\{p_{11}^1, p_{11}^2, p_{11}^3\} \text{ where } p_{11}^1 = SOILMOISTURE, \]

\[ p_{11}^2 = SOILEVAP, p_{11}^3 = TRANS; \]

and for all \( p \in P_{11}, t \in T_{11}, x \in S_{11} \):

\[ (\sigma_{11}(\text{CONSTANT}(\text{REALS}, p),1)(x))(s_{11}^1) = p(p_{11}^1); \]

\[ (\sigma_{11}(\text{CONSTANT}(\text{REALS}, p),1)(x))(s_{11}^2) = 0 \text{ if } p(p_{11}^2) \neq 0, \]

\[ = ((p(p_{11}^1) - x(s_{11}^1))/\Delta t) + p(p_{11}^3) \text{ if } p(p_{11}^3) = 0; \]
\((\sigma_{11}(\text{CONSTANT(REALS, p),1})(x))(S_{11}^3) = (p(P_{11}^1) - x(S_{11}^1))/\Delta t.\)

The output is \(\zeta_{11}^1(x) = x(S_{11}^2), \zeta_{11}^2(x) = x(S_{11}^3).\)

The Soil-Microclimate Interface

As mentioned briefly in discussion 5.6, the surface boundary conditions are determined by the microclimate and soil conditions. Hence, they represent the soil-microclimate interface. It will be treated here as a component coupling the soil surface water balance system to the soil water balance system.

As state earlier, this analysis does not consider the energy balance. Hence, it will be assumed that soil evaporation is known or is provided by another system.

In order to determine the surface flux (flux of water through the soil surface) during rainfall events it is assumed that the surface flux is equal to net precipitation rate minus potential evaporation rate if greater than zero, and zero otherwise. These conditions would only apply during the time the moisture potential at the surface is less than zero. After this time the conditions are not considered. Therefore the conditions may be expressed as:

\[
\text{SURFACEFLUX}(t) = \max(\text{NETPRECIPRATE}(t - \Delta t) - \text{EWET}(t - \Delta t), 0)
\]

if \(\text{NETPRECIPRATE}(t - \Delta t) > 0,\)

\[= \text{SOILEVAP}(t - \Delta t) \text{ otherwise.} \]  \( (5.42) \)

The above equation is used in the definition 5.12 of the surface boundary condition system \(Z_{12}.\)
Figure 5.8 represents the intuitive coupling of systems $Z_8$, $Z_9$, $Z_{10}$, $Z_{11}$ and $Z_{12}$ to form the soil water balance system.

The Surface Boundary Condition System, $Z_{12}$ (Definition 5.12)

This system uses equation 5.42 to determine the flux conditions at the soil surface during non-ponded rainfall conditions.

Define the discrete system $Z_{12} = (S_{12}, P_{12}, F_{12}, M_{12}, T_{12}, \sigma_{12})$

where:

$S_{12} = \text{SURFACEFLUX}$;

$P_{12} = X(P_{12}', P_{12}'' P_{12}''')$ where $P_{12}' = \text{NETPRECIPRATE}, P_{12}'' = \text{EWET}$,

$P_{12}''' = \text{SOILEVAP}$;

and for all $p \in P_{12}$, $t \in T_{12}$, $x \in S_{12}$:

$(\psi_{12}(\text{CONSTANT(REALS, p),1}))(x) = \text{MAX}(0, p(P_{12}') - p(P_{12}''))$ if $p(P_{12}') > 0$,

$= -p(P_{12}''')$ if $p(P_{12}') = 0$.

The Plant Water Balance

Existing watershed models in general lump evaporation and transpiration losses under the heading of evapotranspiration. Evaporation losses include that water lost from soil and water surfaces to the atmosphere by direct vaporization. Transpiration is the process of water movement from the soil to the atmosphere by transport through the plants.

One may view evapotranspiration as controlled only by meteorological conditions. This may be satisfactory when considering only
Figure 5.8. The Soil and Surface Water Balance.
evaporation losses. However, when considering transpiration losses it is necessary to realize that transpiration is controlled by interrelationships of climate, vegetation and soil conditions (Rijtema 1968, Penman 1968).

The following analysis of plant water balance assumes that fluxes through the plant are far more important than the actual storage of water in the plant. Hence, an attempt is made to model the processes controlling transpiration rather than model the actual plant water balance. It is also assumed that transpiration is controlled by interrelationships of soil, plant and climatic components while evaporation is controlled primarily by the climatic and soil components. Hence, transpiration is a result of the interaction between soil, plant and climatic components and is dependent on the state of each component.

Potential Evapotranspiration

Potential evaporation is the maximum possible evaporation from the free water surface under consideration under the existing climatic conditions. Potential evapotranspiration is the rate of combined evaporation from the soil and transpiration by plants from a vegetated surface, actively growing, when the supply of water is not limiting (Penman 1948, Van Bavel 1966).

There are a number of methods of estimating potential evapotranspiration. These include energy balance methods, mass transport methods, combination methods and empirical methods, using radiation, air temperature or pan evaporation. Reviews of these methods may be found in
proceedings of a conference entitled "Evapotranspiration and its role in water resources management" (ASAE 1966), Sellers (1964), Rijtema (1965), and Harrold (1969). An improved version of the original combination method developed by Penman (1948) was selected for use here.

Penman (1948) combined aerodynamic and energy balance methods to derive a formula for estimating evaporation from open water surfaces. Rijtema (1965) reviews attempts of Penman (1948) and others (Penman 1956, Penman and Schofield 1951) to apply a reduction factor to this equation to estimate potential evapotranspiration from grass cut very short. Penman's formula for potential evaporation from a free water surface is

\[ E_o = \frac{\Delta H_{NT}}{L} + \gamma E_a \]

(5.43)

where \( \Delta \) is the slope of the temperature-vapor pressure curve, \( H_{NT} \) the net radiation, \( L \) the latent heat of vaporization, \( \gamma \) the psychrometer constant, and

\[ E_a = 0.35 (0.50 + 0.54u)(e_a - e_a) \]

(5.44)

where \( u \), \( e_a \), and \( e_a \) are, respectively, wind speed, saturated vapor pressure at existing temperature, and actual vapor pressure, all measured at two meter height.

Van Bavel (1966) proposed improvements to Penman's (1948) combination approach. These improvements eliminated empiricisms in Penman's method. The empiricisms arise in estimate of net radiation, omission of soil heat flux, the wind function and use of daily or other values for humidity, air temperature and wind speed where instantaneous values are
necessary. Following Penman's (1948) method, Van Bavel derived the instantaneous evaporation rate as

\[ E_o = \frac{\Delta H/L + \gamma B_v d_a}{\Delta + \gamma} \]  \hspace{1cm} (5.45)

where \( \Delta, \gamma, \) and \( L \) are as before, \( d_a \) the vapor pressure deficit at elevation \( z_a \), \( H \) is the algebraic sum of all energy input rates at the surface other than those of sensible heat flux and of latent heat. \( B_v \) is a transfer coefficient for water vapor, and

\[ B_v = \frac{\rho \varepsilon k^2}{p} \frac{u_a}{(\ln(z_a/z_0))^2} \]  \hspace{1cm} (5.46)

where \( \rho \) is air density, \( \varepsilon \) the water-air molecular weight ratio, \( k \) the Von Karman constant, \( p \) the ambient pressure, \( u_a \) the wind speed at elevation \( z_a \) above the surface, and \( z_0 \) the roughness parameter. Equation 5.46 is for adiabatic conditions only.

Van Bavel (1966) tested the improved version of Penman's method using lysimeter data and showed that the method gives accurate estimates of potential evapotranspiration even when there is a large proportion of advection of sensible heat. He also suggests that the combination method is not restricted to a short grass surface as suggested by Penman's (1948) original definition. Potential evaporation can be defined for any condition in terms of the appropriate meteorological variables and the aerodynamic and radiative properties of the surface.

Actual Transpiration

Van Bavel's method gives estimates of potential evapotranspiration and as such considers evapotranspiration as being controlled by the
climate. Rijtema (1965, 1968) and Enrodi and Rijtema (1969) developed a general equation for calculating real evapotranspiration which takes into account properties of the crop, soil and climate. It combines both evaporation of intercepted precipitation and real transpiration and hence the term real evapotranspiration. It is of the form

$$\text{E}_{\text{re}} = \text{E}_{T}^{\text{re}} + \text{E}_{I}$$

$$\text{E}_{T}^{\text{re}} = \frac{(\Delta H_{\text{NT}}/L) + \gamma(E'_{a} + f(z_{o}, d) u R_{c} E_{I})}{\Delta + \gamma(1 + f(z_{o}, d) u R_{c})}$$  \hspace{1cm} (5.47)

where $\Delta, L, \gamma, z_{o}, H_{\text{NT}}$ are as before, $\text{E}_{\text{re}}$ is the real evapotranspiration, $\text{E}_{T}^{\text{re}}$ is the real transpiration of the crop, $E'_{a} = f(z_{o}, d) u(\varepsilon_{a} - e_{a})$; $u$, $\varepsilon_{a}$, $e_{a}$ are as before, but at the 2-meter height, $f(z_{o}, d)$ is a function of the roughness length $z_{o}$ and the zero plane displacement $d$ of the evaporating surface, $R_{c}$ is the diffusion resistance of the crop, and $E_{I}$ is the evaporation of precipitation intercepted by the vegetation.

When $R_{c}$ is zero equation 5.47 gives the potential evapotranspiration from the surface under consideration as

$$E_{\text{wet}} = \frac{\Delta H_{\text{NT}}/L + \gamma E'_{a}}{\Delta + \gamma}.$$  \hspace{1cm} (5.48)

Rijtema (1968) assumes that $E_{I}$ cannot exceed $E_{\text{wet}}$ and combines equations 5.47 and 5.48 to get

$$\text{E}_{\text{re}} = \text{E}_{T}^{\text{re}} + \text{E}_{I} = \frac{(\Delta + \gamma)(E_{\text{wet}} - E_{I})}{\Delta + \gamma(1 + f(z_{o}, d) u R_{c})} + E_{I}.$$  \hspace{1cm} (5.49)
In the above equation 5.49, real transpiration is the term
\[ E_T^{re} = \frac{(\Delta + \gamma)(E_{\text{wet}} - E_T)}{\Delta + \gamma(1 + f(z_o,d) \cdot uR_c)}. \] \hspace{1cm} (5.50)

The real evapotranspiration term does not include evaporation from the soil surface or other surface water.

The Roughness Function

A problem in using equation 5.49 arises in determining \( f(z_o,d) \), \( E_T \) and \( R_c \).

The value of the roughness function depends on crop height and density and on wind velocity. Rijtema (1965) gives the following for a grass crop:

\[ f(z_o,d) = g(1) \times h(u) \] \hspace{1cm} (5.51)

where \( g(1) \) is a function of crop height \( (1) \) and \( h(u) \) is a function of wind velocity.

Another expression can be derived for the roughness function by comparing equations 5.45 and 5.46 (Van Bavel's model) with equation 5.48. This suggests that

\[ E'_a = B_v \cdot d_a = f(z_o,d) \cdot u(\varepsilon_a - e_a), \] \hspace{1cm} (5.52)

and since \( d_a = (\varepsilon_a - e_a) \), then

\[ f(z_o,d) = \frac{\rho \varepsilon_k^2}{P(\ln(z_a/z_o))^2}. \] \hspace{1cm} (5.53)

This expression eliminates empiricism in deriving the roughness function but is valid for adiabatic conditions only.
The Crop Diffusion Resistance

Difficulties arise in determining the diffusion resistance, $R_c$, of the crop. It takes into account geometry of the evaporating surface such as soil cover and leaf area, stomatal opening as influenced by light intensity and transport resistances in the water flow path as influenced by soil water availability and plant condition. Rijtema (1968) assumes that the combined effect may be given by

$$R_c = R^L_c + R^C_c + R^w_c,$$

(5.54)

where $R^L_c$ is the diffusion resistance term depending on light intensity, $R^C_c$ is the contribution to $R_c$ depending on soil cover, and $R^w_c$ is the factor giving the effect of soil moisture. Each of the terms on the right-hand side of equation 5.54 must be estimated and these values will be supplied by other components.

The evaporation of intercepted precipitation, $E_I$, is required in equations 5.49 and 5.50. In all cases where transpiration is reduced below the potential rate, interception will increase total evapotranspiration loss while decreasing transpiration (Rijtema 1968, Rutter 1967, Burgy and Pomeroy 1958, McMillan and Burgy 1960).

In summary, methods have been given for calculating instantaneous values of potential evapotranspiration from the wetted surface with equation 5.48 and real transpiration with equation 5.50. These in turn require the value of the roughness function which may be estimated using equation 5.51 or 5.53, the diffusion resistance of the crop from equation 5.54, evaporation from interception, net radiation and other climatic inputs and surface parameters.
The methods for calculating potential evapotranspiration, real transpiration, the value of the roughness function and the diffusion resistance of the crop will each be modeled as a system. The coupling of these systems in Chapter 6 will constitute the plant water balance component as illustrated in Figure 5.9 which includes interactions to be discussed later.

**Symbol Definitions**—Define the following notation:

- \( \text{WIND}(t) \) is the wind velocity at MEASELV height at time \( t \);
- \( \text{CROPDIFFRES}(t) \) is the crop diffusion resistance at time \( t \);
- \( \text{ROUGHNESS}(t) \) is the value of the roughness function at time \( t \);
- \( \text{DELTA}(t) \) is the slope of the temperature-saturated vapor pressure curve at time \( t \);
- \( \text{PSYCHOON}(t) \) is the psychrometer constant at time \( t \);
- \( \text{NETRAD}(t) \) is the net radiation at time \( t \);
- \( \text{ROUGHNESSLENGTH}(t) \) is the roughness length at time \( t \);
- \( \text{SOILPLANTRES}(t) \) is the soil plant resistance \( \left( R_c^W \right) \) in equation 5.54 at time \( t \);
- \( \text{COVERRES}(t) \) is the crop cover resistance \( \left( R_c^C \right) \) in equation 5.54 at time \( t \);
- \( \text{LIGHTRES}(t) \) is the resistance due to light \( \left( R_c^L \right) \) in equation 5.54 at time \( t \);
- \( \text{VAPPRESDEF}(t) \) is the vapor pressure deficit at time \( t \);
- MEASELV is the elevation at which climatic measurements are taken, a constant;
- \( \text{PARTLCROPRES}(t) \) is the crop diffusion resistance at time \( t \) calculated using equation 5.50 with \( R_c^C \) equal to \( \text{PARTLCROPRES}(t) \);
Figure 5.9. The Plant Water Balance with Interactions.
LATENTHEAT is the latent heat of vaporization, a constant;
POINTLTRANS(t) is the potential transpiration at time t, calculated
using equation 5.50 with \( R_C^C \) equal to PARTLCROPRES(t);
ROUGHNESS is the value of the term \( \rho_e K^2 / \rho \) in equation 5.53
at standard temperature and pressure. At 25° C, 1000 mb., the
value is 0.122 g cm\(^{-3} \) mb\(^{-1} \) (Van Bavel 1966).

The Potential Evapotranspiration System, \( Z_{13} \)

This system models equation 5.48 in discrete form to determine
the potential evapotranspiration. Rewriting equation 5.48 gives

\[
EWET(t) = \frac{((\text{DELTA}(t-1) \times \text{NETRAD}(t-1))/\text{LATENTHEAT})
+ (\text{PSYCHCON}(t-1) \times \text{WIND}(t-1) \times \text{VAPPRESDEF}(t-1))}{(\text{DELTA}(t-1) + \text{PSYCHCON}(t-1))}.
\]

(5.55)

Define the discrete system \( Z_{13} = (S_{13}, P_{13}, F_{13}, M_{13}, T_{13}, \sigma_{13}) \) where, \( S_{13} = EWET; \)
\( P_{13} = X \{P_{13}^1, P_{13}^2, P_{13}^3, P_{13}^4, P_{13}^5, P_{13}^6\} \) where \( P_{13}^1 = \text{DELTA}, P_{13}^2 = \text{NETRAD}, \)
\( P_{13}^3 = \text{PSYCHCON}, P_{13}^4 = \text{WIND}, P_{13}^5 = \text{VAPPRESDEF}, P_{13}^6 = \text{ROUGHNESS}; \)
and for all \( p \in P_{13}, t \in T_{13}, \sigma \in S_{13}: \)

\[
\sigma_{13}(\text{CONSTANT(REALS, p),1})(\chi)
= ((p(P_{13}^1) \times p(P_{13}^2)/\text{LATENTHEAT}) + p(P_{13}^3) \times p(P_{13}^4) \times p(P_{13}^5)
\times p(P_{13}^6))/(p(P_{13}^1) + p(P_{13}^3)).
\]
The Transpiration System, $Z_{14}$ (Definition 5.14)

This system models equation 5.50 in discrete form to determine actual transpiration and potential transpiration. Rewriting equation 5.50 gives

$$\text{TRANS}(t) = \frac{(\Delta(t-1) + \text{PSYCHCON}(t-1)) \times (\text{EWET}(t-1)
- \text{IEVAP}(t-1)) / (\Delta(t-1) + \text{PSYCHCON}(t-1))
\times (1 + (\text{ROUGHNESS}(t-1) \times \text{WIND}(t-1)
\times \text{CROPDIFFRES}(t-1))))}{(\Delta(t-1) + \text{PSYCHCON}(t-1) \times (1 + (\text{ROUGHNESS}(t-1) \times \text{WIND}(t-1)
\times \text{CROPDIFFRES}(t-1)))})).$$

(5.56)

Define the discrete system $Z_{14} = (S_{14}', P_{14}', F_{14}', T_{14}', M_{14}')$

$s_{14}$) where

$S_{14} = X \{S_{14}', S_{14}''\}$ where $S_{14}' = \text{TRANS}, S_{14}'' = \text{POINTLTRANS};$

$P_{14} = X \{P_{14}', P_{14}'' P_{14}''' P_{14}'''' P_{14}'''', P_{14}''''', P_{14}''''''\}$ where $P_{14}' = \Delta,$

$P_{14}'' = \text{PSYCHCON}, P_{14}''' = \text{EWET}, P_{14}'''' = \text{IEVAP}, P_{14}''''' = \text{ROUGHNESS}, P_{14}'''''' = \text{WIND},$

$P_{14}'''''' = \text{CROPDIFFRES}, P_{14}'''''''' = \text{PARTLCROPRES};$

and for all $p \in P_{14}, t \in T_{14}, x \in S_{14}:$

$$(\sigma_{14}(\text{CONSTANT(REALS, p)},1)(x))(S_{14}');$$

$$= (p(P_{14}') + p(P_{14}'')) \times (p(P_{14}''') - p(P_{14}'') / (p(P_{14}') + (p(P_{14}''))
\times (1+(p(P_{14}'') \times p(P_{14}''') \times p(P_{14}'')))),$$

$$(\sigma_{14}(\text{CONSTANT REALS, p}),1)(x))(S_{14}'');$$

$$= (p(P_{14}') + p(P_{14}'')) \times (p(P_{14}''') - p(P_{14}'') / (p(P_{14}') + (p(P_{14}'')).$$
\[ x \left(1 + (p(P_{14}^5) \times p(P_{14}^6) \times p(P_{14}^8))\right) \].

The output functions are \( \zeta_{14}^1(x) = (S_{14}^1), \zeta_{14}^2(x) = x(S_{14}^2) \). 

The Roughness System, \( Z_{15} \) (Definition 5.15)

This system models equation 5.53 (Van Bavel's (1966) form of the roughness function) in discrete form to determine the value of the roughness function. Rewriting equation 5.53 gives

\[ \text{ROUGHNESS}(t) = \text{ROUGHNESS\_FNCON}/((\ln(\text{MEASELV}/\text{ROUGHNESS\_LENGTH(t-1)}))^2). \tag{5.57} \]

Define the discrete system \( Z_{15} = (S_{15}, P_{15}, F_{15}, T_{15}, M_{15}, c_{15}) \) where:

\[ S_{15} = \text{ROUGHNESS}, \]
\[ P_{15} = \text{ROUGHNESS\_LENGTH}, \]

and for all \( p \in P_{15}, t \in T_{15}, x \in S_{15} \):
\[ (c_{15}(\text{CONSTANT(REALS, p),1}))(x) = \text{ROUGHNESS\_FNCON}/((\text{NATURALLOG(MEASELV/p)})^2). \]

The Crop Diffusion Resistance System, \( Z_{15} \) (Definition 5.16)

This system models equation 5.54 in discrete form to determine the crop diffusion resistance. Rewriting equation 5.54 gives

\[ \text{CROPDIFFRES}(t) = \text{SOILPLANTRES}(t-1) + \text{COVERRES}(t-1) + \text{LIGHTRES}(t-1). \tag{5.58} \]

Define the discrete system \( Z_{16} = (S_{16}, P_{16}, F_{16}, T_{16}, M_{16}, c_{16}) \) where:
\[ S_{16} = X(S_{16}^1, S_{16}^2) \text{ where } S_{16}^1 = \text{CROPDIFFRES}, S_{16}^2 = \text{PARTLCROPRES}; \]
\[ P_{16} = X \{ p^1_{16}, p^2_{16}, p^3_{16} \} \text{ where } p^1_{16} = \text{COVERRES}, p^2_{16} = \text{SOILPLANTRES}, p^3_{16} = \text{LIGHTRES}; \]

and for all \( p \in P_{16}, t \in T_{16}, x \in S_{16} : \)

\[
(\sigma_{16}(\text{CONSTANT(REALS, p),1})(x))(S^1_{16}) = p(p^1_{16}) + p(p^2_{16}) + p(p^3_{16}),
\]

\[
(\sigma_{16}(\text{CONSTANT(REALS, p),1})(x))(S^2_{16}) = p(p^1_{16}) + p(p^3_{16}).
\]

The outputs are \( \xi^1_{16}(x) = x(S^1_{16}) \), \( \xi^2_{16}(x) = x(S^2_{16}) \).

**Interactions**

There are a number of interactions between the various water balance components which should be modeled. The most significant of these are those affecting plant growth which in turn affects values of crop diffusion resistance, transpiration and related factors.

It is beyond the scope of this analysis to model plant growth. However, an attempt is made to model those factors affecting crop diffusion resistance to the extent that these factors are understood. The analysis of these factors is based mostly upon the work of Rijtema (1965, 1968) and Enrodi and Rijtema (1969), and partially upon the work of Cowan and Milthorpe (1968).

The distribution of total transpiration loss over the soil profile is also modeled. This is based primarily on the work of Gardner and Ehlig (1962).

In general, the nature of these interactions seems to be poorly understood at the present. This author feels that if the analysis were
carried into plant growth using only the present state of knowledge, a stronger base would be provided for modeling of interactions.

Soil Plant Transport Resistance

The objective of this discussion is to determine the contribution to the crop diffusion resistance from factors such as the availability of soil moisture and transport resistances in the plant.

The particular functional relationships used are from work by Rijtema (1965, 1968), Enrodi and Rijtema (1969) and Cowan and Milthorpe (1968), and are based on the concept of the potential suction in the leaf tissue. The potential suction in the leaf tissue is defined as the theoretical suction necessary in the leaf tissue to insure potential transpiration at the soil physical conditions prevailing in the effective root zone of the crop.

Rijtema (1965) related the effect of soil moisture conditions and plant transport resistance on diffusion resistance to the potential suction in the leaves. Enrodi and Rijtema (1969) give the relationship as

\[ R_C^\psi = f(\psi_1^{pot}) = f(E_{pot}^T (R_{pl} + (b/K)) + \psi) \] (5.59)

where \( \psi_1^{pot} \) is the potential suction in the leaves, \( E_{pot}^T \) is the potential transpiration rate calculated with equation 5.50 assuming \( R_C^\psi = 0 \), \( R_{pl} \) is the transport resistance for flow in the plant, \( b \) is a geometry factor of the root system depending on rooting depth, root intensity and root activity, \( K \) is the capillary conductivity at mean suction \( \psi \) in the effective root zone of the crop, and \( R_C^\psi \) is the factor giving the effect of soil moisture conditions on the value of crop diffusion
resistance. This relationship is based on the fact that a relation exists between the suction in the leaf tissue and the stomatal opening. The suction in the leaf tissue is in turn related to transpiration, root system geometry, transport resistance in the plant, and the moisture potential and capillary conductivity in the root zone.

Rijtema (1965) based the derivation of equation 5.59 on the work of Gardner and Ehlig (1962) together with data reported by other workers.

Cowan and Milthorpe (1968) derived a relationship of the form

\[ E_w = f(E_w^0, \psi_{Lw}, a/d, \tau_s) \]  \hspace{1cm} (5.60)

where \( d \) is the depth of root penetration, \( \tau_s \) is the potential in the effective root zone, \( \psi_{Lw} \) is the critical value of leaf water potential at which stomatal aperture is affected, \( E_w^0 \) is the transpiration rate when \( \tau_s \) is equal to zero, and \( E_w \) is a measure of the ability of the soil-plant system to supply water to leaves and is the transpiration rate if leaf water potential is at \( \psi_{Lw} \). They also defined:

\[ E_w = (\tau_r - \psi_{Lw})/z, \]  \hspace{1cm} (5.61)

where \( \tau_r \) is the potential of water at the soil surface, \( z \) is the internal impedance of plants to water flow; and

\[ E_w = \frac{\tau_r - \psi_L}{z} \]  \hspace{1cm} (5.62)

where \( \psi_L \) is the potential of water in leaves; and

\[ \tau_r = \tau_s + a\delta/dt, \]  \hspace{1cm} (5.63)

where \( a \) is a function of the number and size of roots;
\[ \alpha = \frac{1}{8}\pi y(\ln(1/v) - 3) \] where \( y \) is total length and \( v \) is effective volume of roots per unit volume of soil, and \( s \) is the reciprocal of capillary conductivity.

Solving for \( \psi_{Lw} \) by combining equations 5.61-5.63 gives

\[ \psi_{Lw} = \tau_r - z E^0_w = \tau_s + \alpha s \frac{d\theta}{dt} - z E^0_w \quad (5.64) \]

Comparing Rijtema's notation with Cowan and Milthorpe shows that

\[ \psi_{\text{pot}} = -\psi_{Lw}, \quad \psi = -\tau_s, \quad k = 1/s, \quad z = R_{pl}, \quad E^0_w = E^{\text{pot}}_T \quad \text{and} \quad b = \alpha \]

Substituting Rijtema's notation in 5.64 gives

\[ \psi^1_{\text{pot}} = E^{\text{pot}}_T R_{pl} - (b/k) d\theta/dt + \psi. \quad (5.65) \]

Hence the basic differences between equation 5.59 from Rijtema and equation 5.65 based on the equations 5.61 to 5.63 due to Cowan and Milthorpe are the terms \(-d\theta/dt\) rather than \(E^{\text{pot}}_T\) and the \(=\) sign rather than the \(\Rightarrow\) sign.

Cowan and Milthorpe suggest the assumptions in this model restrict its use to soil where density of rooting and rate of extraction are uniform over the profile. However, they were attempting to predict the supply function \(E_w\) and not the leaf potential \(\psi^1_{\text{pot}}\). A basic assumption in both Rijtema's and Cowan and Milthorpe's derivations was that of steady state flow. Rijtema used this assumption primarily for purposes of data analysis over balance periods of several days where the transpiration rate was considered constant and neither daily cycles in transpiration nor redistribution of moisture in root zone were considered. However, since the rest of the equations are of an
instantaneous nature, equation 5.65 will be used here in modeling the leaf water potential after replacing the $=$ sign by an $=$ sign.

The soil plant transport resistance component will be modeled by two systems. The first will determine the leaf water potential using equation 5.65. The second will determine the soil plant transport resistance using a functional relationship of the form $R_c^\nu = f(\nu_1^{\text{pot}})$ as in equation 5.59. This relationship is arbitrary and must be determined from data. The primary reason for modeling this component by two component systems, as well as not including an explicit form of the root geometry factor, is to allow greater freedom and ease in changing the model, if necessary, at a later time.

**Symbol Definitions** - Define the following notation:

- $\text{PLANTRES}(t)$ is the plant resistance at time $t$;
- $\text{LEAFPOTENTL}(t)$ is the leaf water potential at time $t$;
- $\text{ROOTGEOMFACT}(t)$ is the value of the root geometry factor at time $t$;
- $\text{SPTRESFCON}$ is the soil plant transport resistance function where $\text{SPTRESFCON} \in \text{FUNCTIONS(LEAFPOTENTL, SOILPLANTRES)}$.

The Leaf Water Potential System, $Z_{17}$ (Definition 5.17)

This system models equation 5.65 in discrete form to determine the leaf water potential. Rewriting equation 5.65 gives

$$\text{LEAFPOTNTL}(t) = \text{POINTLTRANS}(t-1) \times \text{PLANTRES}(t-1) - \frac{(\text{ROOTGEOMFACT}(t-1) \times \text{DSOILMOISTDT}(t-1))}{\text{MEANCAP}(t-1)) + \text{MEANPOT}(t-1)}. \quad (5.66)$$
Define the discrete system $Z_{17} = (S_{17}, P_{17}, F_{17}, M_{17}, T_{17}, c_{17})$ where:

$S_{17} = \text{LEAFPOTNTL};$

$P_{17} = X \{p_1^{17}, p_2^{17}, p_3^{17}, p_4^{17}, p_5^{17}, p_6^{17}\}$ where $p_1^{17} = \text{POINTLTRANS},$

$p_2^{17} = \text{PLANTRES}, p_3^{17} = \text{ROOTGEOMFACT}, p_4^{17} = \text{DSOILDISTDT}, p_5^{17} = \text{MEANCAP},$

$p_6^{17} = \text{MEANPOT};$

and for all $p \in P_{17}, t \in T_{17}, x \in S_{17}:$

$(c_{17}(\text{CONSTANT(REALS, p)},1))(x) = (p(P_1^{17}) \times p(P_2^{17})) - (p(P_3^{17})

\times p(P_4^{17})/p(P_5^{17})) + p(P_6^{17}).$

The Soil Plant Transport Resistance System, $Z_{18}$ (Definition 5.18)

This system models the general relationship given in equation 5.59 to determine the soil plant transport resistance. Rewriting the relationship gives

$$\text{SOILPLANTRES}(t) = \text{SPTRESFCN(LEAFPOTNTL}(t - 1)). \quad (5.67)$$

Define the discrete system $Z_{18} = (S_{18}, P_{18}, F_{18}, M_{18}, T_{18}, c_{18})$ where:

$S_{18} = \text{SOILPLANTRES};$

$P_{18} = X \{p_1^{18}, p_2^{18}\}$ where $p_1^{18} = \text{LEAFPOTNTL},$

$p_2^{18} = \text{FUNCTIONS(LEAFPOTNTL, SOILPLANTRES)};$

and for all $p \in P_{18}, t \in T_{18}, x \in S_{18}, \text{SPTRESFCN} = p(P_2^{18});$
\((c_{18}(\text{CONSTANT(REALS, p),1}))(x)\)

\[= \text{SPTRESFCN}(p(P_{18}^1)).\]

Light and Soil Cover Resistance

Rijtema (1965) uses mean radiation intensity as a measure for the light dependent factor controlling stomatal opening under field conditions. The balance periods which Rijtema (1965, 1968) and Enrodi and Rijtema (1969) use are one day or more. He bases his work on that of Kuiper (1961) and other workers who showed effects of light on stomatal opening and diffusion resistance. However, under field conditions these relationships are much more difficult to establish.

Hence it is assumed that the portion of stomatal resistance dependent on light intensity is a function of the mean radiation intensity where the mean radiation intensity is averaged over an undetermined prior period of time.

The effects of partial soil cover on crop diffusion resistance are also difficult to establish. Rijtema (1968) and Enrodi and Rijtema (1969) give some data on these for several crops. Here it is assumed that the soil cover diffusion resistance for a specific crop is dependent only on the density of vegetative cover.

The effects of light and cover resistance will be modeled by one system in a rather general manner. The primary reason for including this is simply to demonstrate the way in which interactions with the plant component may be included if the functional relationships are known.
Symbol Definitions. Define the following notation:

\( \text{LIQ}_1 \text{TRESFCN} \) is the light resistance function where

\( \text{LIQ}_1 \text{TRESFCN} \in \text{FUNCTIONS}(\text{MEANNETRAD}, \text{LIQ}_1 \text{TRES}); \)

\( \text{MEANNETRAD}(t) \) is the mean net radiation at time \( t \) averaged over some period prior to time \( t; \)

\( \text{COVERRESFCN} \) is the soil cover resistance function where

\( \text{COVERRESFCN} \in \text{FUNCTIONS}(\text{COVERDENSITY}, \text{COVERRES}); \)

\( \text{COVERDENSITY}(t) \) is the density of the vegetative cover at time \( t. \)

The Light and Cover Resistance System, \( Z_{19} \) (Definition 5.19)

This system determines the contributions of light and cover resistance to crop diffusion resistance in a rather general way as presented in the prior discussion.

Define the discrete system \( Z_{19} = (S_{19}, P_{19}, F_{19}, T_{19}, M_{19}, o_{19}) \) where:

\( S_{19} = X \{ S_{19}^1, S_{19}^2 \} \) where \( S_{19}^1 = \text{LIQ}_1 \text{TRES}, S_{19}^2 = \text{COVERRES}; \)

\( P_{19} = X \{ P_{19}^1, P_{19}^2, P_{19}^3, P_{19}^4 \} \) where \( P_{19}^1 = \text{COVERDENSITY}, \)

\( P_{19}^2 = \text{MEANNETRAD}, P_{19}^3 = \text{FUNCTIONS}(\text{MEANNETRAD}, \text{LIQ}_1 \text{TRES}), \)

\( P_{19}^4 = \text{FUNCTIONS}(\text{COVERDENSITY}, \text{COVERRES}); \)

and for all \( p \in P_{19}, t \in T_{19}, x \in S_{19}, \) \( \text{LIQ}_1 \text{TRESFCN} = p(P_{19}^3), \)

\( \text{COVERRESFCN} = p(P_{19}^4); \)

\( (o_{19}(\text{CONSTANT}(\text{REALS}, p), 1)(x))(S_{19}^1) = \text{LIQ}_1 \text{TRESFCN}(p(P_{19}^2)) \)
\[ (\sigma_{19}(\text{CONSTANT(REALS, p,1)}(\chi))(S_{19}^2) \]
\[ = \text{COVERRESFCN}(p(P_{19}^1)). \]

The outputs are \( \xi_{19}^1(\chi) = \chi(S_{19}^1) \), \( \xi_{19}^2(\chi) = (S_{19}^2) \).

Transpiration Distribution

The transpiration component determines the amount of water transpired by the plants. It is necessary, however, to distribute the soil water loss resulting from transpiration over the soil profile. The method for doing this is based on work by Gardner (1964) on relation of root distribution to water uptake.

Gardner (1964) divides the soil profile into \( n \) layers of equal thickness \( h \). He then defines an integrated soil suction \( \tau \) as

\[ \tau = \frac{\sum_{i=1}^{n} K_i L_i (\tau_i + z_i)}{\sum_{i=1}^{n} K_i L_i} \quad (5.68) \]

where \( K_i \) is the conductivity of the \( i \)th soil layer, \( L_i \) is the length of roots per unit volume of soil on the \( i \)th layer, and \( z_i \) is the depth from the soil surface to the center of the \( i \)th layer. He also defines the integrated conductance as

\[ K = \sum_{i=1}^{n} K_i L_i \quad (5.69) \]

Equation 5.68 weights the suction at each depth by the number of roots at that depth as well as the conductivity at that depth. The suction, conductivity and root activity are important factors in determining the availability of soil water to plants.
Symbol Definitions-- Define the following notation:

(TRANSFRAC(d))(t) is the fraction of total transpiration coming from the layer of soil at depth d at time t, d ∈ DEPTHS, where TRANSFRAC ∈ FUNCTIONS(DEPTHS{$\cup$0,SOILDEPTH}, REALS[0,1]);

(ROOTFACTOR(d))(t) is the contribution to ROOTGEOMFACT(t) at depth d at time t where ROOTFACTOR ∈ FUNCTIONS(DEPTHS, REALS).

The relationship between ROOTFACTOR(d,t) and ROOTGEOMFACT(t) is not clear; however, it intuitively seems that there should be a relationship dependent on root size, density and activity. Such a relationship would come from an analysis of the plant component.

To determine the transpiration loss from the ith layer at time t, it is assumed that the fraction of the total transpiration at time t, coming from the ith layer, is directly proportional to 1 minus the fractional contribution of that layer to the integrated potential. Hence fraction of total transpiration loss may be written in terms of previously defined system notation as

\[
(TRANSFRAC(d))(t) = 1 - \left[ (KAP(d,M(d)))(t-1) \times (ROOTFACTOR(d))(t-1) \times \left[ (PSI(d,M(d)))(t-1) + d \right] / \sum \left[ (KAP(Z,M(Z)))(t-1) \times (ROOTFACTOR(Z))(t-1) \times [(PSI(Z,M(Z))(t-1) + Z] : Z \in (DEPTHS\cup{0, SOILDEPTH})) \right],
\]

where d ∈ (DEPTHS\cup{0, SOILDEPTH}) AND M = MOISTUREDIST(t-1).  (5.70)

Note that equation 5.70 assumes that no transpiration losses occur from the surface and lower boundary half-layers.

Then the actual transpiration loss from the layer at depth d at time t is
\[ (\text{TRANSDIST}(d))(t) = (\text{TRANSFRACT}(d))(t-1) \times \text{TRANS}(t-1) \]

where \( d \in \text{DEPTHS}\{0, \text{SOILDEPTH}\} \).

(5.71)

The transpiration distribution function will be modeled as a single system in the following definition 5.20.

The Transpiration Distribution System, \( Z_{20} \) (Definition 5.20)

This system models equation 5.70 and 5.71 discretely to determine the transpiration distribution function.

Define the discrete system \( Z_{20} = (S_{20}, P_{20}, F_{20}, N_{20}, T_{20}, \sigma_{20}) \) where:

\[ S_{20} = \text{FUNCTIONS}(\text{DEPTHS}\{0, \text{SOILDEPTH}\}, \text{TRANSLOSS}); \]

\[ P_{20} = X \{ p_{20}^1, p_{20}^2, p_{20}^3 \} \text{ where } p_{20}^1 = \text{TRANS}, \]

\[ p_{20}^2 = \text{FUNCTIONS}(\text{DEPTHS}, \text{CAPCOND} \times \text{POTNTLS}), \]

\[ p_{20}^3 = \text{FUNCTIONS}(\text{DEPTHS}, \text{REALS}); \]

and for all \( p \in P_{20}, t \in T_{20}, x \in S_{20}, \) and

\[ \text{SOILPROPERTY} = p(P_{20}^2), \]

\[ \text{ROOTFACTOR} = p(P_{20}^3): \]

\[ (\sigma_{20}(\text{CONSTANT}(\text{REALS}, p),1))(x) \]

\[ = \text{TRANSDIST} = \{(d, \text{TRANSDIST}(d)): d \in \text{DEPTHS}\{0, \text{SOILDEPTH}\}, \]

\[ \text{TRANSDIST}(d) = (1 - \text{PROJECTION(SOILPROPERTY}(d))(\text{CAPCOND}) \times \text{ROOTFACTOR}(d) \times \text{PROJECTION(SOILPROPERTY}(d))(\text{POTNTLS}) \]

\[ /\{\text{PROJECTION(SOILPROPERTY}(Z))(\text{CAPCOND}) \times \text{ROOTFACTOR}(Z) \]
An analysis of the water balance of an ecosystem has been presented. The ecosystem water balance was broken up into the canopy water balance, mulch layer water balance, soil surface water balance, and plant water balance. Each of these major components is made up of smaller components. An attempt was made to model interactions. However, in order to do a complete job in modeling interactions it is necessary to consider plant growth. The result of the analysis is 20 component systems which must now be coupled into an overall ecosystem water balance system.
CHAPTER 6

SYNTHESIS OF THE WATER BALANCE

The objective of this chapter is to couple the systems defined in Chapter 5 into a single system. The coupling is illustrated in Figure 6.1. The coupling must necessarily be done within the system theoretic framework given in Chapter 3. After giving the coupling recipe the resultant of the couple is deduced. The behavior of the resultant for a given input function and an initial state may be determined from a computational table of the form of Table 6.1.

The Input Ports

In order to minimize possible confusion in the following discussions the sets in the set of input ports for each system are listed and the definitions repeated. Note that the definitions were made using notation of the form $P^j_i$ for the $i$th input port of the $j$th system.

INPUTPORTS ($Z_1$) = \{$P^1_1, P^2_1, P^3_1, P^4_1$\} where $P^1_1$ = PRECIPRATE, $P^2_1$ = COVERDENSITY, $P^3_1$ = IMAX, $P^4_1$ = ISTOR.

INPUTPORTS ($Z_2$) = \{$P^1_2, P^2_2, P^3_2, P^4_2, P^5_2$\} where $P^1_2$ = COVERDENSITY, $P^2_2$ = PRECIPRATE, $P^3_2$ = ISTOR, $P^4_2$ = IMAX, $P^5_2$ = KSF.

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Figure 6.1. The Ecosystem Water Balance System, Z.
INPUTPORTS \((Z_3) = \{P_3^1, P_3^2, P_3^3\}\) where \(P_3^1 = \text{EWET}, P_3^2 = \text{ISTOR}, P_3^3 = \text{IMAX}\).

\[\text{INPUTPORTS \((Z_4) = \{P_4^1, P_4^2, P_4^3, P_4^4\}\) where \(P_4^1 = \text{INTR}, P_4^2 = \text{STMFLW}, P_4^3 = \text{IEVAP}, P_4^4 = \text{IMAX}.}\]

INPUTPORTS \((Z_5) = \{P_5^1, P_5^2, P_5^3\}\) where \(P_5^1 = \text{EWET}, P_5^2 = \text{MSTOR}, P_5^3 = \text{MMAX}\).

INPUTPORTS \((Z_6) = \{P_6^1, P_6^2, P_6^3, P_6^4\}\) where \(P_6^1 = \text{EFFPRECIPRATE}, P_6^2 = \text{MEVAP}, P_6^3 = \text{MCOVR}\).

INPUTPORTS \((Z_7) = \{P_7^1, P_7^2\}\) where \(P_7^1 = \text{MSTOR}, P_7^2 = \text{EFFPRECIPRATE}\).

INPUTPORTS \((Z_8) = \{P_8^1, P_8^2, P_8^3, P_8^4, P_8^5, P_8^6\}\) where \(P_8^1 = \text{NETPRECIPRATE}, P_8^2 = \text{SRFCFLOWIN}, P_8^3 = \text{STMFLW}, P_8^4 = \text{SRFCFLOWOUT}, P_8^5 = \text{INFIL}, P_8^6 = \text{EWET}\).

INPUTPORTS \((Z_9) = \{P_9^1, P_9^2, P_9^3\}\) where \(P_9^1 = \text{FUNCTIONS (DEPTHS, SOILH20)}, P_9^2 = \text{FUNCTIONS (REALS [0, SOILDEPTH] } \times \text{SOILH20, CAPCOND)}, P_9^3 = \text{FUNCTIONS (REALS [0, SOILDEPTH] } \times \text{SOILH20, POTNL5)}\).

INPUTPORTS \((Z_{10}) = \{P_{10}^1, P_{10}^2, P_{10}^3, P_{10}^4, P_{10}^5, P_{10}^6\}\) where \(P_{10}^1 = \text{SURFACEFLUX}, P_{10}^2 = \text{LOWERH2OCON}, P_{10}^3 = \text{FUNCTIONS (DEPTHS } \times \{0, \text{SOILDEPTH}\})\).
TRANSLOSS), \( p^{4}_{10} = \text{SURFACE H2ODEPTH} \), \( p^{5}_{10} = \text{FUNCTIONS (REALS [0, SOILDEPTH])} \times \text{SOILH2O, CAPCOND)} \), \( p^{6}_{10} = \text{FUNCTIONS (REALS [0, SOILDEPTH])} \times \text{SOILH2O, POINTLS)} \).

INPUTPORTS \((Z_{11}) = \{p^{1}_{11}, p^{2}_{11}, p^{3}_{11}\} \) where \( p^{1}_{11} = \text{SOILMOISTURE} \), \( p^{2}_{11} = \text{SOILEVAP} \), \( p^{3}_{11} = \text{TRANS} \).

INPUTPORTS \((Z_{12}) = \{p^{1}_{12}, p^{2}_{12}, p^{3}_{12}\} \) where \( p^{1}_{12} = \text{NETPRECIPRATE} \), \( p^{2}_{12} = \text{EWET} \), \( p^{3}_{12} = \text{SOILEVAP} \).

INPUTPORTS \((Z_{13}) = \{p^{1}_{13}, p^{2}_{13}, p^{3}_{13}, p^{4}_{13}, p^{5}_{13}, p^{6}_{13}\} \) where \( p^{1}_{13} = \text{DELTA} \), \( p^{2}_{13} = \text{NETRAD} \), \( p^{3}_{13} = \text{PSYCHCON} \), \( p^{4}_{13} = \text{WIND} \), \( p^{5}_{13} = \text{VAPPRESDEF} \), \( p^{6}_{13} = \text{ROUGHNESS} \).

INPUTPORTS \((Z_{14}) = \{p^{1}_{14}, p^{2}_{14}, p^{3}_{14}, p^{4}_{14}, p^{5}_{14}, p^{6}_{14}, p^{7}_{14}, p^{8}_{14}\} \) where \( p^{1}_{14} = \text{DELTA} \), \( p^{2}_{14} = \text{PSYCHCON} \), \( p^{3}_{14} = \text{EWET} \), \( p^{4}_{14} = \text{IEVAP} \), \( p^{5}_{14} = \text{ROUGHNESS} \), \( p^{6}_{14} = \text{WIND} \), \( p^{7}_{14} = \text{CROPDIFFRES} \), \( p^{8}_{14} = \text{PARTLCROPROS} \).

INPUTPORTS \((Z_{15}) = \{p^{15}\} \) where \( p^{15} = \text{ROUGHNESSLENGTH} \).

INPUTPORTS \((Z_{16}) = \{p^{1}_{16}, p^{2}_{16}, p^{3}_{16}\} \) where \( p^{1}_{16} = \text{COVERRES} \), \( p^{2}_{16} = \text{SOILPLANTRES} \), \( p^{3}_{16} = \text{LIGHTRES} \).

INPUTPORTS \((Z_{17}) = \{p^{1}_{17}, p^{2}_{17}, p^{3}_{17}, p^{4}_{17}, p^{5}_{17}, p^{6}_{17}\} \) where
\( p^1_{17} = \text{POTNTLTRANS}, p^2_{17} = \text{PLANTRES}, p^3_{17} = \text{ROOTGEOMFACT}, p^4_{17} = \text{DSOILMOISTDT}, p^5_{17} = \text{MEANCAP}, p^6_{17} = \text{MEANPOT}. \)

INPUTPORTS \( (Z_{18}) = \{p^1_{18}, p^2_{18}\} \text{ where } p^1_{18} = \text{LEAFPOTNTL, } p^2_{18} = \text{FUNCTIONS (LEAFPOTNTL, SOILPLANTRES).} \)

INPUTPORTS \( (Z_{19}) = \{p^1_{19}, p^2_{19}, p^3_{19}, p^4_{19}\} \text{ where } p^1_{19} = \text{COVERDENSITY}, p^2_{19} = \text{MEANNETRAD, } p^3_{19} = \text{FUNCTIONS (MEANNETRAD, LIGHTRES), } p^4_{19} = \text{FUNCTIONS (COVERDENSITY, COVERRES).} \)

INPUTPORTS \( (Z_{20}) = \{p^1_{20}, p^2_{20}, p^3_{20}\} \text{ where } p^1_{20} = \text{TRANS, } p^2_{20} = \text{FUNCTIONS (DEPTHS, CAPCOND X POTNTLS), } p^3_{20} = \text{FUNCTIONS (DEPTHS, REALS).} \)

**The Coupling Recipe**

Having gotten a notational handle on the input ports of each system, it is now possible to couple the systems. This is done in the following coupling recipe, which defines mathematically the diagram of the coupling given in Figure 6.1.

Let \( C = (\xi, a, o) \) be a couple where:

\[ \xi = \{Z_i : i \in \text{INTEGERS } [1, 20]\}; \]

and for all \( i \in \text{INTEGERS } [1, 20]\):

the input port assignments of \( C \) are

\[ a(Z_1, Z_i) = \{p^4_i\} \text{ if } i = 4, \]

\[ = \emptyset \text{ otherwise}; \]
\[ a(Z_2, Z_1) = \{P_2^3\} \text{ if } i = 4, \]
\[ = \emptyset \text{ otherwise}; \]
\[ a(Z_3, Z_1) = \{P_3^2\} \text{ if } i = 4, \]
\[ = \{P_3^1\} \text{ if } i = 13, \]
\[ = \emptyset \text{ otherwise}; \]
\[ a(Z_4, Z_1) = \{P_4^1\} \text{ if } i = 1, \]
\[ = \{P_4^2\} \text{ if } i = 2, \]
\[ = \{P_4^3\} \text{ if } i = 3, \]
\[ = \emptyset \text{ otherwise}; \]
\[ a(Z_5, Z_1) = \{P_5^1\} \text{ if } i = 13, \]
\[ = \{P_5^2\} \text{ if } i = 6, \]
\[ = \emptyset \text{ otherwise}; \]
\[ a(Z_6, Z_1) = \{P_6^3\} \text{ if } i = 5, \]
\[ = \{P_6^2\} \text{ if } i = 1, \]
\[ = \emptyset \text{ otherwise}; \]
\[ a(Z_7, Z_1) = \{P_7^1\} \text{ if } i = 6, \]
\[ = \{P_7^2\} \text{ if } i = 1, \]
\[ a(Z_8, Z_i) = \begin{cases} P_8^5 & \text{if } i = 11, \\ P_8^1 & \text{if } i = 7, \\ P_8^3 & \text{if } i = 2, \\ P_8^6 & \text{if } i = 13, \\ \phi & \text{otherwise}; \end{cases} \]

\[ a(Z_9, Z_i) = \begin{cases} P_9^1 & \text{if } i = 10, \\ \phi & \text{otherwise}; \end{cases} \]

\[ a(Z_{10}, Z_i) = \begin{cases} P_{10}^4 & \text{if } i = 8, \\ P_{10}^3 & \text{if } i = 20, \\ P_{10}^1 & \text{if } i = 12, \\ \phi & \text{otherwise}; \end{cases} \]

\[ a(Z_{11}, Z_i) = \begin{cases} P_{11}^1 & \text{if } i = 10, \\ P_{11}^3 & \text{if } i = 14, \\ \phi & \text{otherwise}; \end{cases} \]

\[ a(Z_{12}, Z_i) = \begin{cases} P_{12}^2 & \text{if } i = 13, \\ P_{12}^1 & \text{if } i = 7, \\ \phi & \text{otherwise}; \end{cases} \]
a(Z_{13}, Z_i) = \{P_{13}^6\} \text{ if } i = 15,
\quad = \emptyset \text{ otherwise};

a(Z_{14}, Z_i) = \{P_{14}^5\} \text{ if } i = 15,
\quad = \{P_{14}^3\} \text{ if } i = 13,
\quad = \{P_{14}^7, P_{14}^8\} \text{ if } i = 16,
\quad = \{P_{14}^4\} \text{ if } i = 3,
\quad = \emptyset \text{ otherwise};

a(Z_{15}, Z_i) = \emptyset \text{ for all } i;

a(Z_{16}, Z_i) = \{P_{16}^1, P_{16}^3\} \text{ if } i = 19,
\quad = \{P_{16}^2\} \text{ if } i = 18,
\quad = \emptyset \text{ otherwise};

a(Z_{17}, Z_i) = \{P_{17}^1\} \text{ if } i = 14,
= \{P^4_{17}\} \text{ if } i = 11,
= \{P^5_{17}, P^6_{17}\} \text{ if } i = 9,
= \phi \text{ otherwise; }

a(Z_{18}, Z_i) = \begin{cases} 1 & \text{if } i = 17, \\ \phi & \text{otherwise; } \end{cases}

a(Z_{19}, Z_i) = \phi \text{ for all } i;

a(Z_{20}, Z_i) = \begin{cases} 1 & \text{if } i = 14, \\ 2 & \text{if } i \in \{6, 7\}, \\ \phi & \text{otherwise; } \end{cases}

and the output function assignments of C are

o(Z_1, Z_i) = \begin{cases} 1 & \text{if } i = 4, \\ 2 & \text{if } i \in \{6, 7\}, \\ \phi & \text{otherwise; } \end{cases}

o(Z_2, Z_i) = \text{IDENTITY } (S_2) \text{ if } i \in \{4, 8\},
= \phi \text{ otherwise; }

o(Z_3, Z_i) = \text{IDENTITY } (S_3) \text{ if } i \in \{4, 14\},
= \phi \text{ otherwise; }
\( o(Z_4, Z_i) = \text{IDENTITY } (S_4) \) if \( i \in \{1, 2, 3\}, \)
\[ = \phi \text{ otherwise; } \]

\( o(Z_5, Z_i) = \text{IDENTITY } (S_5) \) if \( i = 6, \)
\[ = \phi \text{ otherwise; } \]

\( o(Z_6, Z_i) = \text{IDENTITY } (S_6) \) if \( i \in \{5, 7\}, \)
\[ = \phi \text{ otherwise; } \]

\( o(Z_7, Z_i) = \tau_7 \) if \( i \in \{8, 12\}, \)
\[ = \phi \text{ otherwise; } \]

\( o(Z_8, Z_i) = \text{IDENTITY } (S_8) \) if \( i = 10, \)
\[ = \phi \text{ otherwise; } \]

\( o(Z_9, Z_i) = \text{IDENTITY } (S_9) \) if \( i = 20, \)
\[ = ((x, \{(\mathcal{P}^5_{17}, \zeta_9(x)), (\mathcal{P}^6_{17}, \zeta_9(x))\}) : x \in S_9) \]
\[ = \phi \text{ otherwise; } \]

\( o(Z_{10}, Z_i) = \tau_{10}^1 \) if \( i = 11, \)
\[ = \text{IDENTITY } (S_{10}) \) if \( i = 9, \)
\[ = \phi \text{ otherwise; } \]
\( o(Z_{11}, z_i) = \begin{cases} \zeta_{11}^1 & \text{if } i = 8, \\ \zeta_{11}^2 & \text{if } i = 17, \\ \phi & \text{otherwise}; \end{cases} \)

\( o(Z_{12}, z_i) = \text{IDENTITY } (S_{12}) \text{ if } i = 10, \)

\( = \phi \text{ otherwise}; \)

\( o(Z_{13}, z_i) = \text{IDENTITY } (S_{13}) \text{ if } i \in \{3, 5, 8, 12, 14\}, \)

\( = \phi \text{ otherwise}; \)

\( o(Z_{14}, z_i) = \begin{cases} \zeta_{14}^1 & \text{if } i \in \{11, 20\}, \\ \zeta_{14}^2 & \text{if } i = 17, \\ \phi & \text{otherwise}; \end{cases} \)

\( o(Z_{15}, z_i) = \text{IDENTITY } (S_{15}) \text{ if } i \in \{13, 14\}, \)

\( = \phi \text{ otherwise}; \)

\( o(Z_{16}, z_i) = \begin{cases} (x, \{(p_{14}^7, \zeta_{16}^1(x)), (p_{14}^8, \zeta_{16}^2(x))\}) : \\ x \in S_{16} \} \text{ if } i = 14, \\ \phi \text{ otherwise}; \end{cases} \)

\( o(Z_{17}, z_i) = \text{IDENTITY } (S_{17}) \text{ if } i = 18, \)

\( = \phi \text{ otherwise}; \)
\( o(Z_{18}, Z_i) = \text{IDENTITY } (S_{18}) \) if \( i = 16 \),

\( = \emptyset \) otherwise;

\( o(Z_{19}, Z_i) = \{(x, \{(P_{16}, \zeta_{19}(x)), (P_{16}, \zeta_{19}(x))\}) : x \in S_{19}\} \) if \( i = 19 \),

\( = \emptyset \) otherwise;

\( o(Z_{20}, Z_i) = \text{IDENTITY } (S_{20}) \) if \( i = 10 \),

\( = \emptyset \) otherwise.

The set of input ports of \( Z_i \) \( \in \xi \) designated as occupied by \( C \) is \( \text{OCCUPIEDPORTS } (Z_i, C) \), where \( \text{OCCUPIED PORTS } (Z_i, C) \)

\( = \{P_1^4\} \) if \( i = 1 \),

\( = \{P_2^3\} \) if \( i = 2 \),

\( = \{P_3^1, P_3^2\} \) if \( i = 3 \),

\( = \{P_4^1, P_4^2, P_4^3\} \) if \( i = 4 \),

\( = \{P_5^1, P_5^2\} \) if \( i = 5 \),

\( = \{P_6^2, P_6^3\} \) if \( i = 6 \),

\( = \{P_7^1, P_7^2\} \) if \( i = 7 \),

\( = \{P_8^1, P_8^3, P_8^5, P_8^6\} \) if \( i = 8 \),
= \{P^1_9\} \text{ if } i = 9,
= \{P^1_{10}, P^3_{10}, P^4_{10}\} \text{ if } i = 10,
= \{P^1_{11}, P^3_{11}\} \text{ if } i = 11,
= \{P^1_{12}, P^2_{12}\} \text{ if } i = 12,
= \{P^6_{13}\} \text{ if } i = 13,
= \{P^3_{14}, P^4_{14}, P^5_{14}, P^7_{14}, P^8_{14}\} \text{ if } i = 14,
= \emptyset \text{ if } i = 15,
= \{P^1_{16}, P^2_{16}, P^3_{16}\} \text{ if } i = 16,
= \{P^1_{17}, P^4_{17}, P^5_{17}, P^6_{17}\} \text{ if } i = 17,
= \{P^1_{18}\} \text{ if } i = 18,
= \emptyset \text{ if } i = 19,
= \{P^1_{20}, P^2_{20}\} \text{ if } i = 20.

The set of input ports of \(Z_i \in \xi \) left unoccupied by \(C\) is

\text{UNOCCUPIEDPORTS} (Z_i, C) \text{ where } \text{UNOCCUPIEDPORTS} (Z_1, C)

= \{P^1_1, P^2_1, P^3_1\} \text{ if } i = 1,
= \{P^1_2, P^2_2, P^4_2, P^5_2\} \text{ if } i = 2,
= \{P^3_3\} \text{ if } i = 3,
= \{P^4_4\} \text{ if } i = 4,
= \{p^3_5\} \text{ if } i = 5, \\
= \{p^1_6, p^4_6\} \text{ if } i = 6, \\
= \emptyset \text{ if } i = 7, \\
= \{p^2_8, p^4_8\} \text{ if } i = 8, \\
= \{p^2_9, p^3_9\} \text{ if } i = 9, \\
= \{p^2_{10}, p^5_{10}, p^6_{10}\} \text{ if } i = 10, \\
= \{p^2_{11}\} \text{ if } i = 11, \\
= \{p^3_{12}\} \text{ if } i = 12, \\
= \{p^1_{13}, p^2_{13}, p^3_{13}, p^4_{13}, p^5_{13}\} \text{ if } i = 13, \\
= \{p^1_{14}, p^2_{14}, p^6_{14}\} \text{ if } i = 14, \\
= \{p^1_{15}\} \text{ if } i = 15, \\
= \emptyset \text{ if } i = 16, \\
= \{p^2_{17}, p^3_{17}\} \text{ if } i = 17, \\
= \{p^2_{18}\} \text{ if } i = 18, \\
= \{p^1_{19}, p^2_{19}, p^3_{19}, p^4_{19}\} \text{ if } i = 19, \\
= \{p^3_{20}\} \text{ if } i = 20.
Therefore the set of all input ports occupied by C is

\[ \text{TOTALOCCUPIEDPORTS}(C) = \{P_4^1, P_3^3, P_3^2, P_4^1, P_4^2, P_3^3, P_5^1, P_5^2, P_6^2, P_6^3, P_7^1, P_7^2, \]
\[ P_8^1, P_8^3, P_8^5, P_9^1, P_9^3, P_9^4, P_{10}^1, P_{10}^3, P_{10}^4, P_{11}^1, P_{11}^3, P_{11}^4, P_{12}^2, P_{12}^3, P_{13}^4, P_{14}^2, \]
\[ P_{14}^5, P_{14}^7, P_{14}^8, P_{16}^1, P_{16}^2, P_{16}^3, P_{17}^1, P_{17}^4, P_{17}^5, P_{17}^6, P_{18}^1, P_{20}^1, P_{20}^2}. \]

The set of all input ports left unoccupied by C is \( \text{TOTALUNOCCUPIEDPORTS}(C) \)

\[ = \{P_1^1, P_1^2, P_1^3, P_1^2, P_2^2, P_2^4, P_2^5, P_2^3, P_2^4, P_2^5, P_2^6, P_2^4, P_2^2, P_8^1, P_8^2, P_8^3, P_8^5, P_9^1, P_9^2, P_9^3, P_9^4, P_{10}^1, P_{10}^2, P_{10}^3, P_{10}^4, P_{10}^5, P_{10}^1, P_{10}^3, P_{10}^4, P_{10}^5, P_{11}^2, P_{11}^3, P_{11}^4, P_{11}^1, P_{12}^1, P_{12}^3, P_{12}^4, P_{12}^5, P_{12}^1, P_{13}^2, P_{13}^3, P_{13}^4, P_{13}^5, P_{13}^1, P_{14}^2, P_{14}^3, P_{14}^4, P_{14}^5, P_{14}^6, P_{14}^7, P_{14}^8, P_{17}^1, P_{17}^2, P_{17}^3, P_{17}^4, P_{17}^5, P_{17}^6, P_{18}^1, P_{18}^2, P_{18}^3, P_{18}^4, P_{18}^5, P_{18}^6, P_{18}^7, P_{18}^8, P_{18}^9, P_{18}^10, P_{18}^11, P_{18}^12, P_{18}^13, P_{18}^14, P_{18}^15, P_{18}^16, P_{18}^17, P_{18}^18, P_{18}^19, P_{18}^20}. \]

The Resultant

Since the systems in the couple C are discrete, then from definition 3.9 the resultant determined by C is a discrete system. This discussion will explore the nature of the resultant system \( Z = \text{RESULTANT}(C) \) which may be determined from definition 3.9.

The set of inputs to Z is the set of unoccupied input ports of \( \text{RESULTANT}(C) \). That is, \( \text{INPUTS}(Z) \)

\[ = \text{RESULTANTINPUTS}(C) = X \text{TOTALUNOCCUPIEDPORTS}(C). \]

The set of states of Z is \( \text{STATES}(Z) \)

\[ = \text{RESULTANTSTATESET}(C) = X \{\text{STATES}(Z_i) : Z_i \epsilon \xi}. \]
The set of input functions of Z is INPUTFUNCTIONS (\(Z\))
\[= \text{RESULTANTINPUTFUNCTIONS} (C)\]
\[= \{ f : f \in \text{FUNCTIONS} (\text{REALS}, \text{RESULTANTINPUTS} (C)) \},\]
for all \(Z_i \in \mathcal{S}\), there exists \(g \in \text{INPUTFUNCTIONS}(Z_i)\) such that
\[
\text{PROJECTION} (\text{UNOCCUPIEDPORTS} (Z_i, C)) f
\]
\[= \text{PROJECTION} (\text{UNOCCUPIEDPORTS} (Z_i, C)) g\}.

The behavior of Z is BEHAVIOR(Z)
\[= \text{RANGE} (\text{MOTION} (Z))\]

The timescale of Z is TIMESCALE(Z)
\[= \text{NONNEGATIVEINTEGERS}.

For all \(f \in \text{INPUTFUNCTIONS}(Z), t \in \text{TIMESCALE}(Z), x \in \text{STATES}(Z), Z_i \in \mathcal{S}, \text{MOTION}(Z) = \sigma\) where \((\sigma (f, t) (x)) (S_i)\)
\[= \sigma_i (\text{PROJECTION} (\text{INPUTS}(Z_i))) \kappa (f, x), t) (x(S_i)), \text{where } \kappa \in \text{FUNCTIONS} (\text{INPUTFUNCTIONS}(Z) \times \text{STATES}(Z), \text{TOTALINPUTFUNCTIONS}(C));\]
and where, for all \(f \in \text{INPUTFUNCTIONS}(Z), x \in \text{STATES}(Z), \ t \in \text{TIMESCALE}(Z), \ V \in \text{TOTALINPUTPORTS}(C);\)
\((\kappa (f, x)) (t)) (V)
\[= (f(t)) (V) \text{ if } V \in \text{TOTALUNOCCUPIEDPORTS}(C),\]
\[= (\text{IDENTITY} \ (\sigma_4 (\text{PROJECTION} (P_4) \kappa (f, x), t) (x(S_4)))) (V)\]
if \(V \in \{P_1^4, P_2^3, P_3^2\},\)
\[= (\text{IDENTITY} \ (\sigma_{13} (\text{PROJECTION} (P_{13}) \kappa (f, x), t) (x(S_{13})))) (V)\]
if \( V \in \{P_3^1, P_5^1, P_8^6, P_{12}^2, P_{14}^3\}, \)

\[
(\xi_1^1 (\sigma_1 (\text{PROJECTION } (P_1) \times (f, x), t) (x(S_1)))) (P_4^3)
\]

if \( V = P_4^1, \)

\[
(\text{IDENTITY } (\sigma_2 (\text{PROJECTION } (P_2) \times (f, x), t) (x(S_2)))) (V)
\]

if \( V = P_4^2, P_8^3, \)

\[
(\text{IDENTITY } (\sigma_3 (\text{PROJECTION } (P_3) \times (f, x), t) (x(S_3)))) (V)
\]

if \( V \in \{P_4^3, P_{14}^3\}, \)

\[
(\text{IDENTITY } (\sigma_6 (\text{PROJECTION } (P_6) \times (f, x), t) (x(S_6)))) (V)
\]

if \( V \in \{P_5^2, P_7^2\}, \)

\[
(\xi_1^2 (\sigma_1 (\text{PROJECTION } (P_1) \times (f, x), t) (x(S_1)))) (V)
\]

if \( V \in \{P_6^2, P_7^2\}, \)

\[
(\text{IDENTITY } (\sigma_5 (\text{PROJECTION } (P_5) \times (f, x), t) (x(S_5))))
\]

if \( V = P_6^3, \)

\[
(\xi_7 (\sigma_7 (\text{PROJECTION } (P_7) \times (f, x), t) (x(S_7)))) (V)
\]

if \( V \in \{P_8^1, P_{12}^1\}, \)

\[
(\xi_1^1 (\sigma_1 (\text{PROJECTION } (P_{11}) \times (f, x), t) (x(S_{11}))))
\]

if \( V = P_8^5, \)

\[
(\text{IDENTITY } (\sigma_{10} (\text{PROJECTION } (P_{10}) \times (f, x), t) (x(S_{10})))) (P_9^1)
\]
if $V = P^1_9$,

$$= (\text{IDENTITY } (\sigma_{12}(\text{PROJECTION } (P^1_{12}) \times (f, x), t) (x(S_{12}))))(P^1_{10})$$

if $V = P^1_{10}$,

$$= (\text{IDENTITY } (\sigma_{20}(\text{PROJECTION } (P^3_{20}) \times (f, x), t) (x(S_{20}))))(P^3_{10})$$

if $V = P^3_{10}$,

$$= (\text{IDENTITY } (\sigma_{8}(\text{PROJECTION } (P^4_{8}) \times (f, x), t) (x(S_{8}))))(P^4_{10})$$

if $V = P^4_{10}$,

$$= (\sigma^1_{10}(\text{PROJECTION } (P^1_{10}) \times (f, x), t) (x(S_{10}))))(P^1_{11})$$

if $V = P^1_{11}$,

$$= (\sigma^1_{14}(\text{PROJECTION } (P^1_{14}) \times (f, x), t) (x(S_{14}))))(V)$$

if $V \in \{P^3_{11}, P^4_{20}\}$,

$$= (\text{IDENTITY } (\sigma_{15}(\text{PROJECTION } (P^5_{15}) \times (f, x), t) (x(S_{15}))))(V)$$

if $V \in \{P^6_{13}, P^5_{14}\}$,

$$= (\sigma^1_{16}(\text{PROJECTION } (P^7_{16}) \times (f, x), t) (x(S_{16}))))(P^7_{14})$$

if $V = P^7_{14}$,

$$= (\sigma^2_{16}(\text{PROJECTION } (P^8_{16}) \times (f, x), t) (x(S_{16}))))(P^8_{14})$$

if $V = P^8_{14}$,

$$= (\sigma^2_{19}(\text{PROJECTION } (P^1_{19}) \times (f, x), t) (x(S_{19}))))(P^1_{16})$$
if \( V = p^1_{16} \),

\[
= (\tau^1_9 (\sigma_9 (\text{PROJECTION} (P_9) \times (f, x), t), (x(S_9)))) (P^5_{17})
\]

if \( V = p^2_{16} \)

\[
= (\text{IDENTITY} (\sigma_9 (\text{PROJECTION} (P_9) \times (f, x), t), (x(S_9)))) (P^1_{18})
\]

if \( V = p^3_{16} \)

\[
= (\text{IDENTITY} (\sigma_9 (\text{PROJECTION} (P_9) \times (f, x), t), (x(S_9)))) (P^2_{20})
\]

if \( V = p^4_{17} \)

\[
= (\tau^2_9 (\sigma_9 (\text{PROJECTION} (P_9) \times (f, x), t), (x(S_9)))) (P^5_{17})
\]

if \( V = p^5_{17} \)

\[
= (\text{IDENTITY} (\sigma_9 (\text{PROJECTION} (P_9) \times (f, x), t), (x(S_9)))) (P^1_{18})
\]

if \( V = p^6_{17} \)

\[
= (\text{IDENTITY} (\sigma_9 (\text{PROJECTION} (P_9) \times (f, x), t), (x(S_9)))) (P^2_{20})
\]

if \( V = p^7_{17} \).
Rather than solve the previous set of equations in a recursive way explicitly for the $\kappa$ function, it is easier to set up a table. This table is given as Table 6.1. For each component $Z_i \in \mathcal{E}$ the table shows how to determine the input to each input port of $\text{TOTALINPUTPORTS}(C)$ given an input function $f \in \text{INPUTFUNCTIONS}(Z)$, $t \in \text{TIMESCALE}(Z)$, and the initial state $\chi \in \text{STATES}(Z)$. The table also shows how to compute the next state of each component. Note that the state transitions are recursive, a property of discrete systems. Hence the behavior of the system may be determined in a rather straightforward manner on a digital computer, if the state transition function is known.

The state transition function may be readily determined from the definitions of the component systems and the Table 6.1. By definition, for all $f \in \text{INPUTFUNCTIONS}(Z)$, $t \in \text{TIMESCALE}(Z)$, $\chi \in \text{STATES}(Z)$:

$$\text{MOTION}(Z) = \sigma$$

where for all $Z_i \in \mathcal{E}$,

$$\sigma(f, t)(\chi)(S_i) = \sigma_i(\text{PROJECTION}(\text{INPUTS}(Z_i)) \kappa(f, \chi), t)(x(S_i)).$$

Hence it is only necessary to specify $\sigma(f, t)(\chi)(S_i)$ for each $i \in \text{INTEGERS}[1, 20]$. This is readily done from definitions 5.1 to 5.20 inclusive and will be included in the following definition of the $\text{RESULTANT}(C)$. 
### Table 6.1. The State Transition Computational Table for Z

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₁</th>
<th>State of Z₁</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((f(t))(\text{INPUTPORTS}(Z₁)))</td>
<td>(x^t(S₁))</td>
<td>(\chi(S₁))</td>
</tr>
<tr>
<td>(t)</td>
<td></td>
<td>(x^0(S₁))</td>
<td>(\sigma₁(\text{CONSTANT}(\text{REALS},(f(0))(\text{INPUTPORTS}(Z₁))),1)(x°(S₁)))</td>
</tr>
<tr>
<td>0</td>
<td>((f(0))(P₁^1))</td>
<td>((f(0))(P₂^1))</td>
<td>(x^0(S₁))</td>
</tr>
<tr>
<td>1</td>
<td>((f(1))(P₁^1))</td>
<td>((f(1))(P₂^1))</td>
<td>(x^1(S₁))</td>
</tr>
<tr>
<td>2</td>
<td>((f(2))(P₁^1))</td>
<td>((f(2))(P₂^1))</td>
<td>(x^2(S₁))</td>
</tr>
<tr>
<td>3</td>
<td>((f(3))(P₁^1))</td>
<td>((f(3))(P₂^1))</td>
<td>(x^3(S₁))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₂</th>
<th>State of Z₂</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((f(t))(\text{INPUTPORTS}(Z₂)))</td>
<td>(x^t(S₂))</td>
<td>(\chi(S₂))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(x^0(S₂))</td>
<td>(\sigma₂(\text{CONSTANT}(\text{REALS},(f(0))(\text{INPUTPORTS}(Z₂))),1)(x°(S₂)))</td>
</tr>
<tr>
<td>(t)</td>
<td></td>
<td>(x^1(S₂))</td>
<td>(\sigma₁(\text{CONSTANT}(\text{REALS},(f(1))(\text{INPUTPORTS}(Z₁))),1)(x°(S₁)))</td>
</tr>
<tr>
<td>0</td>
<td>((f(0))(P₁^2))</td>
<td>((f(0))(P₂^2))</td>
<td>(x^0(S₂))</td>
</tr>
<tr>
<td>1</td>
<td>((f(1))(P₁^2))</td>
<td>((f(1))(P₂^2))</td>
<td>(x^1(S₂))</td>
</tr>
<tr>
<td>2</td>
<td>((f(2))(P₁^2))</td>
<td>((f(2))(P₂^2))</td>
<td>(x^2(S₂))</td>
</tr>
<tr>
<td>3</td>
<td>((f(3))(P₁^2))</td>
<td>((f(3))(P₂^2))</td>
<td>(x^3(S₂))</td>
</tr>
</tbody>
</table>

\(f_c\) INPUTFUNCTIONS(2), \(t_c\) TIMESCALE(2), \(x_c\) STATES(2)
Table 6.1. The State Transition Computational Table for Z—continued

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₃ ((f(t))(\text{INPUTPORTS}(Z₃)))</th>
<th>State at time (t) (x^t(S₃))</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>((f(t))(P^1₃))</td>
<td>((f(t))(P^2₃))</td>
<td>((f(t))(P^3₃))</td>
</tr>
<tr>
<td>0</td>
<td>(x^0(S₁3))</td>
<td>(x^0(S₄))</td>
<td>((f(0))(P^3₃))</td>
</tr>
<tr>
<td>1</td>
<td>(x^1(S₁3))</td>
<td>(x^1(S₄))</td>
<td>((f(1))(P^3₃))</td>
</tr>
<tr>
<td>2</td>
<td>(x^2(S₁3))</td>
<td>(x^2(S₄))</td>
<td>((f(2))(P^3₃))</td>
</tr>
<tr>
<td>3</td>
<td>(x^3(S₁3))</td>
<td>(x^3(S₄))</td>
<td>((f(3))(P^3₃))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₄ ((f(t))(\text{INPUTPORTS}(Z₄)))</th>
<th>State at time (t) (x^t(S₄))</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>((f(t))(P^1₄))</td>
<td>((f(t))(P^2₄))</td>
<td>((f(t))(P^3₄))</td>
</tr>
<tr>
<td>0</td>
<td>(ζ^0(S₁2))</td>
<td>(x^0(S₃))</td>
<td>(x^0(S₄))</td>
</tr>
<tr>
<td>1</td>
<td>(ζ^1(S₁2))</td>
<td>(x^1(S₃))</td>
<td>(x^1(S₄))</td>
</tr>
<tr>
<td>2</td>
<td>(ζ^2(S₁2))</td>
<td>(x^2(S₃))</td>
<td>(x^2(S₄))</td>
</tr>
<tr>
<td>3</td>
<td>(ζ^3(S₁2))</td>
<td>(x^3(S₃))</td>
<td>(x^3(S₄))</td>
</tr>
</tbody>
</table>
Table 6.1. The State Transition Computational Table for Z--continued

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₅ ((f(t))) (\text{INPUTPORTS}(Z₅))</th>
<th>State at time (t) (x^{t}(S₅))</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>(x^{0}(S₅))</td>
<td>(x^{0}(S₅))</td>
<td>(x^{0}(S₅))</td>
</tr>
<tr>
<td>(1)</td>
<td>(x^{1}(S₅))</td>
<td>(x^{1}(S₅))</td>
<td>(x^{1}(S₅))</td>
</tr>
<tr>
<td>(2)</td>
<td>(x^{2}(S₅))</td>
<td>(x^{2}(S₅))</td>
<td>(x^{2}(S₅))</td>
</tr>
<tr>
<td>(3)</td>
<td>(x^{3}(S₅))</td>
<td>(x^{3}(S₅))</td>
<td>(x^{3}(S₅))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₆ ((f(t))) (\text{INPUTPORTS}(Z₆))</th>
<th>State at time (t) (x^{t}(S₆))</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>((f(0))(P₆))</td>
<td>(x^{0}(S₆))</td>
<td>(x^{0}(S₆))</td>
</tr>
<tr>
<td>(1)</td>
<td>((f(1))(P₆))</td>
<td>(x^{1}(S₆))</td>
<td>(x^{1}(S₆))</td>
</tr>
<tr>
<td>(2)</td>
<td>((f(2))(P₆))</td>
<td>(x^{2}(S₆))</td>
<td>(x^{2}(S₆))</td>
</tr>
<tr>
<td>(3)</td>
<td>((f(3))(P₆))</td>
<td>(x^{3}(S₆))</td>
<td>(x^{3}(S₆))</td>
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</table>
Table 6.1. The State Transition Computational Table for Z—continued

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to $Z_7$ $(f(t))(\text{INPUTPORTS}(Z_7))$</th>
<th>State of $Z_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t$</td>
<td>$(f(t))(P_1^7)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$x^0(S_6)$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$x^1(S_6)$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$x^2(S_6)$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$x^3(S_6)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to $Z_8$ $(f(t))(\text{INPUTPORTS}(Z_8))$</th>
<th>State of $Z_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t$</td>
<td>$(f(t))(P_1^8)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$\zeta_1^8(x^0(S_7))$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\zeta_1^8(x^1(S_7))$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$\zeta_1^8(x^2(S_7))$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$\zeta_1^8(x^3(S_7))$</td>
</tr>
</tbody>
</table>
Table 6.1. The State Transition Computational Table for Z\textsubscript{g} --continued

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z\textsubscript{g} ((f(t))(\text{INPUTPORTS}(Z\textsubscript{g})))</th>
<th>State of Z\textsubscript{g}</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((f(t))(P\textsubscript{1}))</td>
<td>((f(t))(P\textsubscript{2}))</td>
<td>((f(t))(P\textsubscript{3}))</td>
</tr>
<tr>
<td>0</td>
<td>(x^0(S\textsubscript{10}))</td>
<td>((f(0))(P\textsubscript{2}))</td>
<td>((f(0))(P\textsubscript{3}))</td>
</tr>
<tr>
<td>1</td>
<td>(x^1(S\textsubscript{10}))</td>
<td>((f(1))(P\textsubscript{2}))</td>
<td>((f(1))(P\textsubscript{3}))</td>
</tr>
<tr>
<td>2</td>
<td>(x^2(S\textsubscript{10}))</td>
<td>((f(2))(P\textsubscript{2}))</td>
<td>((f(2))(P\textsubscript{3}))</td>
</tr>
<tr>
<td>3</td>
<td>(x^3(S\textsubscript{10}))</td>
<td>((f(3))(P\textsubscript{2}))</td>
<td>((f(3))(P\textsubscript{3}))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z\textsubscript{10} ((f(t))(\text{INPUTPORTS}(Z\textsubscript{10})))</th>
<th>State of Z\textsubscript{10}</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((f(t))(P\textsubscript{1}'))</td>
<td>((f(t))(P\textsubscript{2}'))</td>
<td>((f(t))(P\textsubscript{3}'))</td>
</tr>
<tr>
<td>0</td>
<td>(x^0(S\textsubscript{12}))</td>
<td>((f(0))(P\textsubscript{2}'))</td>
<td>((f(0))(P\textsubscript{3}'))</td>
</tr>
<tr>
<td>1</td>
<td>(x^1(S\textsubscript{12}))</td>
<td>((f(1))(P\textsubscript{2}'))</td>
<td>((f(1))(P\textsubscript{3}'))</td>
</tr>
<tr>
<td>2</td>
<td>(x^2(S\textsubscript{12}))</td>
<td>((f(2))(P\textsubscript{2}'))</td>
<td>((f(2))(P\textsubscript{3}'))</td>
</tr>
<tr>
<td>3</td>
<td>(x^3(S\textsubscript{12}))</td>
<td>((f(3))(P\textsubscript{2}'))</td>
<td>((f(3))(P\textsubscript{3}'))</td>
</tr>
</tbody>
</table>
| Time | Inputs to $Z_{11}$ 
(f(t))(INPUTPORTS(Z$_{11}$)) | State at 
time $t$ | State Transition |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$x^t(S_{11})$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$x^0(S_{10})$</td>
<td>$x^0(S_{11})$</td>
<td>$x(S_{11})$</td>
</tr>
<tr>
<td>1</td>
<td>$x^1(S_{11})$</td>
<td>$x^1(S_{11})$</td>
<td>$s_{11}(\text{CONSTANT(REALS,}(f(0))(INPUTPORTS(Z_{11})))), 1)(x^0(S_{11}))$</td>
</tr>
<tr>
<td>2</td>
<td>$x^2(S_{11})$</td>
<td>$x^2(S_{11})$</td>
<td>$s_{11}(\text{CONSTANT(REALS,}(f(1))(INPUTPORTS(Z_{11})))), 1)(x^1(S_{11}))$</td>
</tr>
<tr>
<td>3</td>
<td>$x^3(S_{11})$</td>
<td>$x^3(S_{11})$</td>
<td>$s_{11}(\text{CONSTANT(REALS,}(f(2))(INPUTPORTS(Z_{11})))), 1)(x^2(S_{11}))$</td>
</tr>
</tbody>
</table>

| Time | Inputs to $Z_{12}$ 
(f(t))(INPUTPORTS(Z$_{12}$)) | State at 
time $t$ | State Transition |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$x^t(S_{12})$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$x^0(S_{11})$</td>
<td>$x^0(S_{12})$</td>
<td>$x(S_{12})$</td>
</tr>
<tr>
<td>1</td>
<td>$x^1(S_{12})$</td>
<td>$x^1(S_{12})$</td>
<td>$s_{12}(\text{CONSTANT(REALS,}(f(0))(INPUTPORTS(Z_{12})))), 1)(x^0(S_{12}))$</td>
</tr>
<tr>
<td>2</td>
<td>$x^2(S_{12})$</td>
<td>$x^2(S_{12})$</td>
<td>$s_{12}(\text{CONSTANT(REALS,}(f(1))(INPUTPORTS(Z_{12})))), 1)(x^1(S_{12}))$</td>
</tr>
<tr>
<td>3</td>
<td>$x^3(S_{12})$</td>
<td>$x^3(S_{12})$</td>
<td>$s_{12}(\text{CONSTANT(REALS,}(f(2))(INPUTPORTS(Z_{12})))), 1)(x^2(S_{12}))$</td>
</tr>
</tbody>
</table>
Table 6.1. The State Transition Computational Table for Z—continued

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to $Z_{13}$ $(f(t))(\text{INPUTPORTS}(Z_{13}))$</th>
<th>State at time $t$ $X^t(S_{13})$</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$(f(t))(P^1_{13})$</td>
<td>$(f(t))(P^2_{13})$</td>
<td>$(f(t))(P^3_{13})$</td>
</tr>
<tr>
<td>0</td>
<td>$(f(0))(P^1_{13})$</td>
<td>$(f(0))(P^2_{13})$</td>
<td>$(f(0))(P^3_{13})$</td>
</tr>
<tr>
<td>1</td>
<td>$(f(1))(P^1_{13})$</td>
<td>$(f(1))(P^2_{13})$</td>
<td>$(f(1))(P^3_{13})$</td>
</tr>
<tr>
<td>2</td>
<td>$(f(2))(P^1_{13})$</td>
<td>$(f(2))(P^2_{13})$</td>
<td>$(f(2))(P^3_{13})$</td>
</tr>
<tr>
<td>3</td>
<td>$(f(3))(P^1_{13})$</td>
<td>$(f(3))(P^2_{13})$</td>
<td>$(f(3))(P^3_{13})$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to $Z_{14}$ $(f(t))(\text{INPUTPORTS}(Z_{14}))$</th>
<th>State at time $t$ $X^t(S_{14})$</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$(f(t))(P^1_{14})$</td>
<td>$(f(t))(P^2_{14})$</td>
<td>$(f(t))(P^3_{14})$</td>
</tr>
<tr>
<td>0</td>
<td>$(f(0))(P^1_{14})$</td>
<td>$(f(0))(P^2_{14})$</td>
<td>$(f(0))(P^3_{14})$</td>
</tr>
<tr>
<td>1</td>
<td>$(f(1))(P^1_{14})$</td>
<td>$(f(1))(P^2_{14})$</td>
<td>$(f(1))(P^3_{14})$</td>
</tr>
<tr>
<td>2</td>
<td>$(f(2))(P^1_{14})$</td>
<td>$(f(2))(P^2_{14})$</td>
<td>$(f(2))(P^3_{14})$</td>
</tr>
<tr>
<td>3</td>
<td>$(f(3))(P^1_{14})$</td>
<td>$(f(3))(P^2_{14})$</td>
<td>$(f(3))(P^3_{14})$</td>
</tr>
</tbody>
</table>
Table 6.1. The State Transition Computational Table for Z—continued

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>Inputs to Z₁₅</th>
<th>State of Z₁₅</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(f(t))(INPUTPORTS(Z₁₅))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>t</td>
<td></td>
<td>Xᵣ(S₁₅)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>(f(0))(P₁₅)</td>
<td>x₀(S₁₅)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(f(1))(P₁₅)</td>
<td>x₁(S₁₅)</td>
<td>σ₁₅(CONSTANT(REALS,(f(0))(P₁₅)),1)(x₀(S₁₅))</td>
</tr>
<tr>
<td>2</td>
<td>(f(2))(P₁₅)</td>
<td>x₂(S₁₅)</td>
<td>σ₁₅(CONSTANT(REALS,(f(1))(P₁₅)),1)(x₁(S₁₅))</td>
</tr>
<tr>
<td>3</td>
<td>(f(3))(P₁₅)</td>
<td>x₃(S₁₅)</td>
<td>σ₁₅(CONSTANT(REALS,(f(2))(P₁₅)),1)(x₂(S₁₅))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>Inputs to Z₁₆</th>
<th>State of Z₁₆</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(f(t))(INPUTPORTS(Z₁₆))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>t</td>
<td></td>
<td>Xᵣ(S₁₆)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>(f(0))(P₁₆)</td>
<td>x₀(S₁₆)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(f(1))(P₁₆)</td>
<td>x₁(S₁₆)</td>
<td>σ₁₆(CONSTANT(REALS,(f(0))(INPUTPORTS(Z₁₆))),1)(x₀(S₁₆))</td>
</tr>
<tr>
<td>2</td>
<td>(f(2))(P₁₆)</td>
<td>x₂(S₁₆)</td>
<td>σ₁₆(CONSTANT(REALS,(f(1))(INPUTPORTS(Z₁₆))),1)(x₁(S₁₆))</td>
</tr>
<tr>
<td>3</td>
<td>(f(3))(P₁₆)</td>
<td>x₃(S₁₆)</td>
<td>σ₁₆(CONSTANT(REALS,(f(2))(INPUTPORTS(Z₁₆))),1)(x₂(S₁₆))</td>
</tr>
</tbody>
</table>
Table 6.1. The State Transition Computational Table for Z--continued

<table>
<thead>
<tr>
<th>Time</th>
<th>( f(t) ) (INPUTPORTS(Z17))</th>
<th>State of Z17</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>State Transition</td>
<td>X(S17)</td>
</tr>
<tr>
<td>0</td>
<td>( x^0(S17) )</td>
<td>( x(S17) )</td>
</tr>
<tr>
<td>1</td>
<td>( x^1(S17) )</td>
<td>( a_{17}(\text{CONSTANT(REALS}, f(0)) \text{(INPUTPORTS(Z17)})) ) ( x^0(S17) )</td>
</tr>
<tr>
<td>2</td>
<td>( x^2(S17) )</td>
<td>( a_{17}(\text{CONSTANT(REALS}, f(1)) \text{(INPUTPORTS(Z17)})) ) ( x^1(S17) )</td>
</tr>
<tr>
<td>3</td>
<td>( x^3(S17) )</td>
<td>( a_{17}(\text{CONSTANT(REALS}, f(2)) \text{(INPUTPORTS(Z17)})) ) ( x^2(S17) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>( f(t) ) (INPUTPORTS(Z18))</th>
<th>State of Z18</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>State Transition</td>
<td>X(S18)</td>
</tr>
<tr>
<td>0</td>
<td>( x^0(S18) )</td>
<td>( x(S18) )</td>
</tr>
<tr>
<td>1</td>
<td>( x^1(S18) )</td>
<td>( a_{18}(\text{CONSTANT(REALS}, f(0)) \text{(INPUTPORTS(Z18)})) ) ( x^0(S18) )</td>
</tr>
<tr>
<td>2</td>
<td>( x^2(S18) )</td>
<td>( a_{18}(\text{CONSTANT(REALS}, f(1)) \text{(INPUTPORTS(Z18)})) ) ( x^1(S18) )</td>
</tr>
<tr>
<td>3</td>
<td>( x^3(S18) )</td>
<td>( a_{18}(\text{CONSTANT(REALS}, f(2)) \text{(INPUTPORTS(Z18)})) ) ( x^2(S18) )</td>
</tr>
</tbody>
</table>
Table 6.1. The State Transition Computational Table for Z—continued

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₁₉</th>
<th>State at time t</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>(f(t))(P₁₉)</td>
<td>Xᵗ(S₁₉)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(f(t))(P²₁₉)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(f(t))(P³₁₉)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>(f(0))(P₁₉)</td>
<td>x⁰(S₁₉)</td>
<td>x(S₁₉)</td>
</tr>
<tr>
<td>1</td>
<td>(f(1))(P₁₉)</td>
<td>x¹(S₁₉)</td>
<td>σ₁₉(CONSTANT(REALS,(f(0))(INPUTPORTS(Z₁₉))),1)(x⁰(S₁₉))</td>
</tr>
<tr>
<td>2</td>
<td>(f(2))(P₁₉)</td>
<td>x²(S₁₉)</td>
<td>σ₁₉(CONSTANT(REALS,(f(1))(INPUTPORTS(Z₁₉))),1)(x¹(S₁₉))</td>
</tr>
<tr>
<td>3</td>
<td>(f(3))(P₁₉)</td>
<td>x³(S₁₉)</td>
<td>σ₁₉(CONSTANT(REALS,(f(2))(INPUTPORTS(Z₁₉))),1)(x²(S₁₉))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Inputs to Z₂₀</th>
<th>State at time t</th>
<th>State Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>(f(t))(P₂₀)</td>
<td>Xᵗ(S₂₀)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(f(t))(P²₂₀)</td>
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<tr>
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<td></td>
</tr>
<tr>
<td>0</td>
<td>(f(0))(P₂₀)</td>
<td>x⁰(S₂₀)</td>
<td>x(S₂₀)</td>
</tr>
<tr>
<td>1</td>
<td>(f(1))(P₂₀)</td>
<td>x¹(S₂₀)</td>
<td>σ₂₀(CONSTANT(REALS,(f(0))(INPUTPORTS(Z₂₀))),1)(x⁰(S₂₀))</td>
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<tr>
<td>2</td>
<td>(f(2))(P₂₀)</td>
<td>x²(S₂₀)</td>
<td>σ₂₀(CONSTANT(REALS,(f(1))(INPUTPORTS(Z₂₀))),1)(x¹(S₂₀))</td>
</tr>
<tr>
<td>3</td>
<td>(f(3))(P₂₀)</td>
<td>x³(S₂₀)</td>
<td>σ₂₀(CONSTANT(REALS,(f(2))(INPUTPORTS(Z₂₀))),1)(x²(S₂₀))</td>
</tr>
</tbody>
</table>
The Ecosystem Water Balance System, $Z$

If

$$Z = \text{RESULTANT}(C),$$

then

$$Z = (S, P, F, M, T, \sigma)$$

is a discrete system where

$$S = X \{S_i: \ i \in \text{INTEGERS}[1, 20]\},$$

$$P = X \{p_1, p_2, p_3, p_4, p_5, p_6,\}
\{p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}, p_{17}, p_{18}, p_{19}, p_{20}\},$$

$$F = \{f: f \in \text{FUNCTIONS} (\text{REALS}, P), \text{for all } Z_i \in \xi, \text{there exists } g \in F_i \text{ such that } \text{projection (UNOCCUPIEDPORTS}(Z_i, C)) f \text{ equals } \text{projection (UNOCCUPIEDPORTS}(Z_i, C)) g\},$$

$$M = \text{RANGE} (\sigma),$$

$$T = \text{NONNEGATIVEINTEGERs},$$

and if $p \in P$, $t \in T$, $x \in S$, then:

$$((\sigma(\text{CONSTANT}(\text{REALS}, p), 1) \ (x)) \ (S_1)) \ (S_1^1) = p \ (P_1^1) \ x \ p \ (P_1^2) \ p \ (P_1^3) \ x \ (x(S_4)) \ / \ p \ (P_1^3),$$

$$((\sigma(\text{CONSTANT}(\text{REALS}, p), 1) \ (x)) \ (S_1)) \ (S_1^2) = p \ (P_1^1) - \ (x(S_1)) \ (S_1^1)$$

(by the definition of $\sigma_1$ in Definition 5.1);
(σ(CONSTANT(REALS, p),1)(x))(S_2)
= p(P_2^2) \times p(P_2^5) \times p(P_2^1) \times x(S_4)/p(P_2^4)
(by the definition of σ_2 in Definition 5.2);

(σ(CONSTANT(REALS, p),1)(x))(S_3)
= x(S_{13}) \times x(S_4)/p(P_3^3)
(by the definition of σ_3 in Definition 5.3);

(σ(CONSTANT(REALS, p),1)(x))(S_4)
= \text{MAX} \{0, \text{MIN} \{p(P_4^1), x(S_4) + (\zeta_1^1(x(S_1)) - x(S_2) - x(S_3)) \times \Delta t\} \}
(by the definition of σ_4 in Definition 5.4);

(σ(CONSTANT(REALS, p),1)(x))(S_5)
= x(S_{13}) \times x(S_6)/p(P_5^3)
(by the definition of σ_5 in Definition 5.5);

(σ(CONSTANT(REALS, p),1)(x))(S_6)
= \text{MAX} \{0, \text{MIN} \{p(P_6^1), x(S_6) + (p(P_6^1) \times (\zeta_1^2(x(S_1)) - x(S_5)) \times \Delta t)\} \},
(by the definition of σ_6 in Definition 5.6);

((σ(CONSTANT(REALS, p),1)(x))(S_7))(S_7^1)
= \zeta_1^2 - (x(S_6) - (x(S_7))(S_7^2))/\Delta t \text{ if } \zeta_1^2(x(S_1))>0,
\[
\begin{align*}
= 0 & \quad \text{if } \zeta_1^2(x(S_1)) \leq 0, \\
\left(\sigma(\text{CONSTANT(REALS, p)}, 1)(x)\right)(S_7) & = \left(\sigma(\text{CONSTANT(REALS, p)}, 1)(x)\right)(S_6)
\end{align*}
\]
(by the definition of \(\sigma_7\) in Definition 5.7);

\[
\left(\sigma(\text{CONSTANT(REALS, p)}, 1)(x)\right)(S_8) = \text{MAX} \left\{0, x(S_8) + (\zeta_7(x(S_7)) + p(P_8^2) + x(S_2) - p(P_8^4) - \zeta_11(x(S_{11})) - x(S_{13}) \times \Delta t}\right\}
\]
(by the definition of \(\sigma_8\) in Definition 5.8);

\[
\left(\sigma(\text{CONSTANT(REALS, p)}, 1)(x)\right)(S_9) = \{(d, ((p(P_9^2))(d, (x(S_{10}))(d)), (p(P_9^3))(d, (x(S_{10}))(d)))) : d \in \text{DEPTH}\}
\]
(by the definition of \(\sigma_9\) in Definition 5.9);

\[
\left(\sigma(\text{CONSTANT(REALS, p)}, 1)(x)\right)(S_{10}) = y \text{ such that for } w = x(S_{10}),
\]

\[
\begin{align*}
\text{PSI} & = p(P_{10}^6), \ KAP = p(P_{10}^5) \\
y(d) & = w(d) + (\Delta t/(\Delta z)^2) \times (KAP(d,w(d)) \times (\text{PSI}(d + \Delta z, w(d + \Delta z))) \\
-2 \times \text{PSI}(d,w(d)) + \text{PSI} (d - \Delta z, w(d - \Delta z)) & + (KAP(d + \Delta z, w(d + \Delta z))) \\
-KAP(d,w(d))) \times (\text{PSI}(d + \Delta z, w(d + \Delta z)) - \text{PSI}(d,w(d)) - \Delta z) \\
x (1/(y(d) - w(d))) \times ((w(d + \Delta z) - w(d)) \times ((KAP(d,y(d))))
\end{align*}
\]
\begin{align*}
-K\text{AP}(d,w(d)) \times (\text{PSI}(d + \Delta z,w(d + \Delta z)) - \text{PSI}(d,w(d)) - \Delta z) \\
+ (K\text{AP}(d + \Delta z, w(d + \Delta z)) - K\text{AP}(d,w(d))) \times (\text{PSI}(d,y(d)) \\
-\text{PSI}(d,w(d))) + 2 \times K\text{AP}(d,w(d)) \times (\text{PSI}(d + \Delta z, y(d + \Delta z)) \\
-\text{PSI}(d,y(d)) - \text{PSI}(d + \Delta z, w(d + \Delta z)) + (\text{PSI}(d, w(d)))) \\
+K\text{AP}(d,w(d)) \times (\text{PSI}(d,y(d)) - \text{PSI}(d,w(d))) \times (w(d + \Delta z) \\
-2 \times w(d) + w(d - \Delta z))) - (x(S_{20}))(d) \\
\text{if } d \in \text{DEPTHS} \sim \{0, \text{SOILDEPTH}\}, \\
y(d) = p(p_{10}^{2}) \text{ if } d = \text{SOILDEPTH}, \text{ and } w \text{ is such that} \\
x(S_{12}) = K\text{AP}(d,w(d)) \times ((\text{PSI}(d + \Delta z,w(d + \Delta z)) - \text{PSI}(d,w(d))) \\
+ (\Delta z \times (\text{PSI}(d,y(d)) - \text{PSI}(d,w(d))) \times (w(d + \Delta z) - w(d)) \\
/(y(d) - w(d))) + \Delta z)/\Delta z \\
\text{if } d = 0 \text{ and } \text{PSI} (0,w(0)) \leq 0, \\
\text{PSI}(0,y(0)) = x(S_{8}) \text{ if } d = 0 \text{ and } \text{PSI} (0,w(0)) > 0 \\
(\text{by the definition of } \sigma_{10} \text{ in Definition 5.10);} \\
((\sigma(\text{CONSTANTREALS}, p),1)(x))(S_{11}))(S_{11}^{1}) \\
= \zeta_{10}^{1}(x(S_{10})) \\
((\sigma(\text{CONSTANTREALS}, p),1)(x))(S_{11}))(S_{11}^{2})
\end{align*}
= 0 if \( p(P^2_{11}) \neq 0\),

\[
= (\tau^1_{10}(x(S_{10})) - (x(S_{11}))(S^1_{11})/\Delta t) + \tau^1_{14}(x(S_{14})) \text{ if } p(P^2_{11}) = 0;
\]

\[
((\sigma(\text{CONSTANT}(\text{REALS}, p), 1)(x))(S^1_{11}))(S^3_{11})
\]

\[
= (\tau^1_{10}(x(S_{10})) - (x(S_{11}))(S^1_{11})/\Delta t
\]

(by the definition of \( \sigma_{11} \) in Definition 5.11);

\[
((\sigma(\text{CONSTANT}(\text{REALS}, p), 1)(x))(S^1_{12})
\]

\[
= \text{MAX} \{0, \tau_7(x(S_7)) - x(S_{13})\} \text{ if } \tau_7(x(S_7)) > 0,
\]

\[
= -p(P^3_{12}) \text{ if } \tau_7(x(S_7)) = 0
\]

(by the definition of \( \sigma_{12} \) in Definition 5.12);

\[
((\sigma(\text{CONSTANT}(\text{REALS}, p), 1)(x))(S^1_{13})
\]

\[
= ((p(P^1_{13}) \times p(P^2_{13})/\text{LATENHEAT}) + (p(P^3_{13})
\]

\[
\times p(P^4_{13}) \times p(P^5_{13}) \times x(S_{15}))/p(P^1_{13}) + p(P^3_{13})
\]

(by the definition of \( \sigma_{13} \) in Definition 5.13);

\[
((\sigma(\text{CONSTANT}(\text{REALS}, p), 1)(x))(S^1_{14}))(S^1_{14})
\]

\[
= (p(P^1_{14}) + p(P^2_{14})) \times (x(S_{13}) - x(S^3_3))/(p(P^1_{14})
\]

\[
+ (p(P^2_{14}) \times (1 + (x(S_{15}) \times p(P^6_{14}) \times \tau^1_{16}(x(S_{16}))))))
\]

\[
((\sigma(\text{CONSTANT}(\text{REALS}, p), 1)(x))(S^1_{14}))(S^2_{14})
\]
\[ \begin{align*}
&= (p(P_{14}^1) + p(P_{14}^2)) \times (x(S_{13}) - x(S_3))/(p(P_{14}^1)
+ (p(P_{14}^2) \times (1 + (x(S_{15}) \times p(P_{14}^6) \times \xi_{16}(x^3(S_{16}))))))
\end{align*} \]

(by the definition of \( \sigma_{14} \) in Definition 5.14);

\[ \sigma(\text{CONSTANT(REALS, p),1})(x)(S_{15}) \]

\[ \text{ROUGHNESS} / ((\text{NATURALLOG(MEASELV/p(P_{15}^1)})^2) \]

(by the definition of \( \sigma_{15} \) in Definition 5.15);

\[ ((\sigma(\text{CONSTANT(REALS, p),1})(x))(S_{16}))(S_{16}^1) \]

\[ \xi_{19}^2(x(S_{19})) + x(S_{18}) + \xi_{19}^1(x(S_{19})) \]

\[ ((\sigma(\text{CONSTANT(REALS, p),1})(x))(S_{16}))(S_{16}^2) \]

\[ \xi_{19}^2(x(S_{19})) + \xi_{19}^1(x(S_{19})) \]

(by the definition of \( \sigma_{16} \) in Definition 5.16);

\[ \sigma(\text{CONSTANT(REALS, p),1})(x)(S_{17}) \]

\[ = (\xi_{14}^2(x(S_{14})) \times p(P_{17}^2)) - (p(P_{17}^3) \times \xi_{11}^2(x(S_{11})))
/\xi_{9}^1(x(S_9))) + \xi_{9}^2(x(S_9)) \]

(by the definition of \( \sigma_{17} \) in Definition 5.17);

\[ \sigma(\text{CONSTANT(REALS, p),1})(x)(S_{18}) \]

\[ = (p(P_{18}^2))(x(S_{17})) \]
(by the definition of \( \sigma_{18} \) in Definition 5.18);

\[
((\sigma(\text{CONSTANT(REALS, } p),1)(x))(S_{19}))(S_{19}^1)
\]

\[= (p(p^3_{19}))(p^2_{19}), \]

\[
((\sigma(\text{CONSTANT(REALS, } p),1)(x))(S_{19}))(S_{19}^2)
\]

\[= (p(p^4_{19}))(p(p^1_{19})) \]

(by the definition of \( \sigma_{19} \) in Definition 5.19);

\[
(\sigma(\text{CONSTANTS(REALS, } p),1)(x))(S_{20})
\]

\[= \{(d, \text{TRANSDIST}(d)) : d \in \text{DEPTHS} \cap \{0, \text{SOILDEPTH}\}, \text{TRANSDIST}(d) \]

\[= (1-\text{PROJECTION}((x(S_9))(d))(\text{CAPCOND}) \times (p(p^3_{20}))(d) \]

\[\times \text{PROJECTION}((x(S_9))(d))(\text{POTNTLS})/\sum(p(p^3_{20}))(z) \]

\[\times \text{PROJECTION}((x(S_9))(z))(\text{CAPCOND}) \times \text{PROJECTION}((x(S_9))(z))(\text{POTNTLS}) : \]

\[z \in (\text{DEPTHS} \cap \{0, \text{SOILDEPTH}\})) \times \xi^1_{14}(x(S_{14})) \]

(by the definition of \( \sigma_{20} \) in Definition 5.20).

This definition completely defines the resultant determined by
the couple \( C \) as a discrete system. No attempt was made to implement the
system model of an ecosystem water balance via a digital computer pro-
gram. This work has been done primarily for illustrative purposes.
CHAPTER 7

CONCLUSION

To manage the land in a way that will permit the full development of human potentialities requires a complete understanding of the biosphere. The development of this understanding is giving rise to what is called system ecology. The role of system ecologists is to integrate existing ecological knowledge (Patten 1966, Van Dyne 1969c). An adequate mathematical theory of systems will be required if system ecologists are to integrate ecological information and knowledge into a usable, cohesive theory of ecosystems. The mathematical theory of systems developed by Wymore (1967) provides the foundation for integrating ecological knowledge within existing fundamental concepts of the structure of the biosphere.

The approach to implementing mathematical system theory in integrating ecological knowledge was demonstrated by the analysis and synthesis of the ecosystem water balance. Discrete systems were utilized for purposes of adding some simplicity to the demonstration. Although the intuitive analysis involved only a small part of the ecosystem, it resulted in twenty components. Not all of the components directly involved water movement. Some arose from considering interactions of water movement with the soil, plant, and atmosphere and their effects on water movement.
The individual systems were coupled into one complex system and the results of this couple were determined. The description of the resultant given by Table 6.1 illustrates the way in which system theory can handle complexity. Although the system model was not programmed for a digital computer, Table 6.1 shows how system theory can be used to develop models which can be readily implemented on a computer.

The primary reason for not implementing the model is the great amount of information required concerning the plant effects on the water balance. The straight-forward way to handle this problem would be to expand the model to include the plant component. If this were done sufficiently well, then the basic required inputs would be the climatic inputs which drive the system. A good starting point for modeling the producer component would be the work described by Van Dyne (1969b), Byrne and Tognetti (1969) and particularly that of Paltridge (1970). Paltridge attempts to effectively link the water balance with the plant growth process.

The system models developed here were discrete. However, system theory subsumes the qualitative theory of differential equations and the systems developed could have been continuous. However, this increases the complexity of the problem of determining the coupling function. Coupling functions generally exist when components all consist of discrete systems or all are differentiable systems. If the coupling function exists, the resultant is a well-defined system. If the resultant does not exist, then the system has probably not been modeled or designed correctly. In this case the real world ecological system under study does exist.
The demonstration showed that system theory provides a notational handle for the modeling of extremely complex systems. The structure of the theory provides a common framework for communication between those working on different aspects of a theory of ecosystems.

Stochastic aspects could be included by imposing a probability distribution on the inputs, input functions or possibly even the initial state. The distribution may even be a joint distribution.

This discussion has been primarily directed toward the idea of a theory of ecosystems arising from the work of system ecologists. There is no intent to imply that experimental ecologists should implement mathematical system theory in describing their work. The system ecologist should be able to take the work of the experimental ecologist, incorporate it into the existing theoretical structure and then make suggestions as to where further research is needed. The system ecologist should also be able to guide the experimental ecologist in (1) designing the data network required to perform further research in poorly understood areas, and (2) testing the validity of developed models of the ecosystem.

It would seem that this can only be done within the framework of an adequate theory of ecosystems which integrates ecological knowledge within the existing fundamental concepts of the structure of the biosphere. The fundamental units seem to be biogeocoenose and the ecosystem.
LIST OF REFERENCES


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