TEACHER VIEWS OF MATHEMATICAL MODELING

by

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A Thesis Submitted to the Faculty of the

DEPARTMENT OF MATHEMATICS

In Partial Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

2016

STATEMENT BY AUTHOR

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Acknowledgements

I would like to thank Dr. Marta Civil for her guidance and thorough reviews of my writing. This study was made possible by the Mathematical Modeling in the Middle Grades (M^3) professional development (PD) project[1]. I owe a huge thank you to Dr. Erin Turner and Dr. Cynthia Anhalt for their support of my research and for including me in their project. The knowledge I take from the PD and from my committee members (Marta, Erin, and Cynthia) after this process is invaluable. Additional thanks goes to William, Jamie, Griff, Donna, and Arturo for answering questions and participating in my study.

[1]The research is supported by the Arizona Board of Regents under the Improving Teacher Quality (ITQ) grant no. ITQ015-08UA, *Mathematical Modeling in the Middle Grades (M^3).*

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Abstract

As mathematical modeling gains popularity in K-12 classrooms, it is important to define what this entails for both students and teachers. The following study reviews various definitions of mathematical modeling and how these definitions are relevant for middle grades (5-9) teachers. Following a professional development workshop on mathematical modeling, four middle school teachers expressed their views about teaching mathematics through modeling tasks. This study documents the teachers' perceptions of what it means to model with mathematics, which tasks are most appropriate for their students, and why this is important in each of their classrooms. Although the teachers varied in their views depending on the context and circumstances surrounding each modeling task, they agreed that mathematical modeling helps students build critical thinking skills and provides an opportunity to align mathematics concepts with engaging, realistic phenomena.

Chapter 1: Introduction

The Mathematical Modeling in the Middle Grades professional development (PD) project was created with a goal of teaching teachers about mathematical modeling and helping them to implement modeling tasks in their middle-school classrooms. When the PD sessions began, the concept of modeling with mathematics at a middle grades level was new to the teachers. They did not know what constituted a model nor what the modeling process entailed. During one of the first days of the PD, I overheard one of the teachers say to another, “That’s not a model! You don’t have any numbers!” This made me wonder what teachers thought that modeling with mathematics really was. A couple of the teachers were familiar with modeling at the undergraduate level or thought of modeling as something that a computer does. These notions are possibly what inspired their number-centered perceptions of modeling. Past and present literature provide several outlines and diagrams for describing the modeling process, but the act of “mathematizing” a real-world situation is not a clear-cut algorithm, nor is it a purely numerical process. The goal of this study is to understand what mathematical modeling means to four middle school teachers in light of these existing definitions of modeling in the literature.

Mathematical modeling is well-researched in many countries such as Germany, Turkey, and Australia, and is gradually making its way to the foreground of mathematics education research in the United States. Modeling with mathematics is a Mathematical Practice for all grade levels in the Common Core State Standards in Mathematics. With this new emphasis on modeling, mathematics K-12 classrooms are striving to incorporate modeling into their curricula. Teachers are looking for opportunities to implement modeling, but many are finding that the teaching techniques and practices for facilitating modeling tasks are quite different from the practices to which they are accustomed. The concept of an open-ended task solved by collaborative learning is often seen as counter-cultural to the unfortunate, but common, procedure-based mathematics classroom found in many K-12 schools. Modeling tasks are meant to challenge students and encourage them to use critical thinking skills. Due to the unfamiliar nature of modeling, it is sometimes challenging for teachers to determine exactly how to teach mathematics via modeling.

I created this study in an attempt to answer the following two research questions: (1) *What are middle school teachers' views of teaching mathematical modeling?* and (2) *What are middle school teachers' perceptions of their role during the planning and implementation of mathematical modeling tasks?* I began by observing all the teachers during the PD sessions and collecting student work samples from them. During individual interviews with four of the teachers, I asked them about their personal views of mathematical modeling and how they facilitate modeling tasks in their classrooms. Due to a large distance between the site for the project and the corresponding schools, I was unable to observe the teachers as they taught modeling lessons, which is one shortcoming of this study. However, I was able to get a glimpse of how each teacher would interact with students in the context of modeling by asking them to analyze and assess student sample work. This allowed me to see the connection (if any) between their views on modeling and their teaching practices. In the next chapter I will present literature that is relevant to my study. First, I discuss literature related to mathematical modeling and its place in the K-12 classroom. Then, I transition to topics about teacher roles and teacher views in the context of modeling.

Chapter 2: Review of Literature

Why Modeling?

The National Council of Teachers in Mathematics emphasizes the importance of representing physical, social, and mathematical phenomena through modeling (NCTM, 2000). The Common Core State Standards in Mathematics (CCSSM) includes a Standard for Mathematical Practice entitled Model with Mathematics which defines mathematical modeling via a cycle with elements such as analyzing, developing, formulating, computing, interpreting, validating, and reporting (CCSSI, 2010). This mathematical practice is designed for use at any grade level. The CCSSM also lists modeling as a high school standard and states that modeling "links classroom mathematics and statistics to everyday life, work, and decision-making" (CCSSI, 2010). This quote implies that modeling is a process or tool that can be used in any domain of mathematics (e.g. algebra, geometry, statistics); however, mathematical modeling can also be thought of as its own domain or branch of mathematics. So what *is* mathematical modeling? Although the CCSSM gives its own definition of mathematical modeling, many other definitions and notions of modeling have existed for decades. Here, I present some of these prior definitions, give an overview of current perceptions of mathematical modeling, and discuss the role that modeling can play for teachers and students.

About Mathematical Modeling

Through my review of the literature, I have encountered three different perspectives of mathematical modeling: applied problem solving, real-world mathematics, and a mathematical process. These three definitions shape the designs of studies surrounding mathematical modeling, whether they be at the elementary, secondary, or undergraduate level. Of course, not all authors choose only one of these definitions; many combine two or even all three views of modeling to give a more complete explanation of this complex, mathematical concept.

Modeling as applied problem solving. Blum and Niss (1991) described mathematical modeling as an "applied problem solving process" (p. 38) and use the terms "applications" and "modeling" synonymously to refer to any represented relationship between mathematics and the real-world. "Mathematical problem solving" (Mousoulides, Christou, & Sriraman, 2008, p. 298) differs from traditional word problems in its ability to connect multiple mathematics concepts and to elicit generalized solutions. These two articles focused on applying mathematical knowledge but also addressed problem solving as a process. Using familiar wording such as identifying, interpreting, and validating, these studies provided diagrams (see Figures 1 and 2) which acted as precursors to what we now call the modeling cycle in standards (see Figure 3).

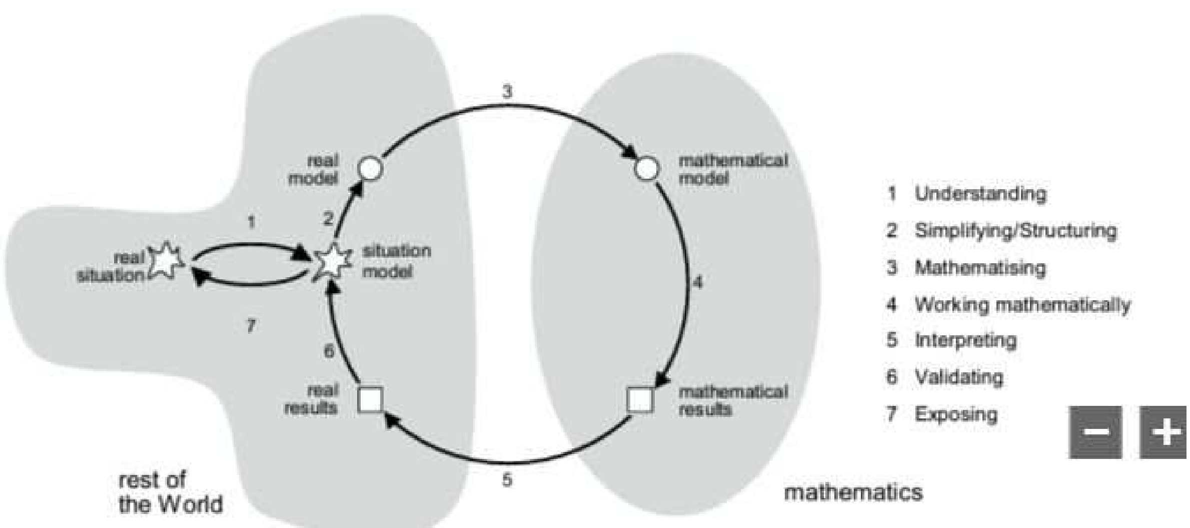

Figure 1. Modeling process from Blum & Leiss (2007, p. 223). This cycle demonstrates how real events are mathematized, solved, and interpreted again in their real-world sense.

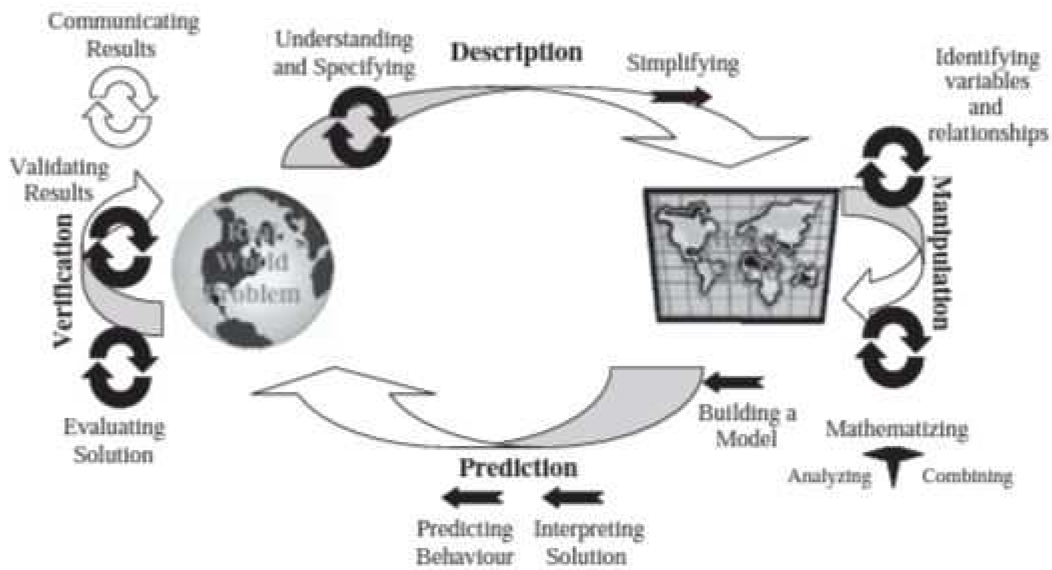

Figure 2. Modeling Process in Mousoulides, et al. (2008, p. 298). This diagram illustrates the many sub-processes that can occur, such as understanding, communicating, and validating, while takes place as a whole.

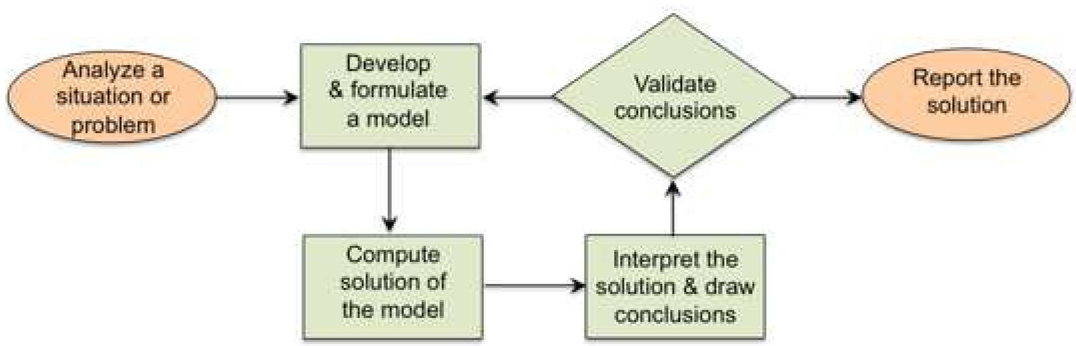

Figure 3. Modeling cycle from the CCSSM, adapted by Anhalt and Cortez (2015, p. 4). Anhalt and Cortez elaborated on the terms given in the CCSSM, producing a clearer meaning of each modeling element.

Viewing mathematical modeling as applied problem solving can also refer to the application of students' prior knowledge to the current task. Tekin, Kula, Hidiroglu, Bukova-Guzel, and Ugurel (2011) found in a study with pre-service teachers (PSTs) that the most common description of mathematical modeling was “concretization,” or applying abstract mathematical concepts to the more concrete situations in modeling tasks. Another way the PSTs expressed their views of modeling was as a means of reiteration, transfer, and retention of mathematical knowledge that had already been taught in a lesson. Modeling was perceived as a way to make certain mathematical topics more applicable to students, and the tasks allowed the students to apply this new topic while utilizing their problem-solving skills.

Schoenfeld (2013) discussed the ability of modeling problems to elicit a better conceptual understanding of mathematics in students. Traditional problem solving should be seen as a subset of “applied problem solving,” a phrase which he used synonymously with modeling. However, Schoenfeld argued that traditional problem solving and applied problem solving are overlapping subcategories of a larger category called “Sense Making.” Students are able to foster this sense-making process while struggling with challenging tasks, such as modeling tasks, during which they discover and make sense of the content on their own rather than being told about it. Stender & Kaiser (2015) agreed with this notion of modeling as giving students the autonomy to solve problems and make sense of the mathematics in their own way and stated that tasks should be seen as “independent activities in which students solve a problem on their own” (p. 1255). They emphasized that the element of independence is important so that the tasks resemble real-world situations, where students are expected to make choices about the most appropriate way to approach a problem.

Modeling as real-world mathematics. Stender and Kaiser (2015) brought up the idea that modeling connects mathematics to real-world situations, and, in fact, other researchers agree. Kaiser and Sriraman (2006) combined two views of modeling: students are solving problems on their own, but these problems are always rooted in real-world contexts. In their global overview of mathematical modeling, Kaiser and Sriraman admit that there is an ongoing debate about what it really means to model with mathematics, because definitions depend on the discipline in which they were founded. For example, they claimed that a scientist will have a different view of modeling than a pedagogy expert, because for each of these people, modeling serves a different purpose and leads to a different goal.

Turker, Saglam, and Umay (2010) referred to mathematical modeling as the "transformation of real life problems into mathematical ones" (p. 4622). Indeed, the tasks in their study all involved common, real-world experiences, such as delivering mail, mowing lawns, and packaging cans at a factory. Particularly noteworthy of the problems in Turker et al. is that they seem to closely resemble word problems with real-world contexts. So then what is the distinction between word problems and modeling tasks? To answer this question, it is helpful to examine the third perception of modeling.

Modeling as a mathematical process. The lines dividing these views of modeling can often be blurred. Schukajlow, Kolter, and Blum (2015) claimed that modeling includes "the ability to solve problems related to the real world" (p. 1241), but they also referenced the multi-step modeling cycle shown in Figure 1 (Blum & Leiss, 2007). Similarly, Blum & Borromeo Ferri (2009) included the diagram in Figure 1 and defined modeling as "the process of translating between the real-world and mathematics in both directions" (p. 45). Again, the authors mentioned both the aspect of real-world situations and described modeling as a complex cycle with several steps and potential iterations.

Sriraman and Lesh (2006) assert that mathematical modeling "is primarily about purposeful description, explanation, or conceptualization (quantification, dimensionalization, coordinatization, or in general mathematization) - even though computation and deduction processes also are involved" (p. 247). Their overview of modeling conceptions stressed that *any* student can engage in the process of making assumptions and solving model-eliciting activities.

Anhalt and Cortez (2015) focused on the modeling cycle that is given in the CCSSM (CCSSI, 2010), but acknowledged that mathematical modeling had historically been accepted as a means to solve problems from everyday situations. During an instructional module with PSTs, Anhalt and Cortez emphasized the elements of the modeling process with their students and found that the PSTs were better able to make sense of the problem and solution and attend to precision as a result of iterating the modeling cycle multiple times.

If mathematical modeling is ill-defined in the literature, then it also vaguely defined in educational settings. This is exactly what Bautista, Wilkerson-Jerde, Tobin, & Brizuela (2014) found in a study of middle school mathematics teachers' perceptions of modeling.

> We understand modeling to be a rather messy process in which a modeler mathematizes key elements of a real-world or theoretical situation, establishing connections among quantities that reflect the system based on his or her mathematical and non-mathematical knowledge. Mathematical models can be externalized by means of multiple kinds of representations (p. 3).

Modeling can be a process to mathematize real-life situations and relate the mathematics to concrete applications (Eraslan & Kant, 2015). Regardless of the definition(s) of modeling that a researcher or educator chooses to accept, mathematical modeling can serve various purposes; thus, the objectives for using modeling tasks in the classroom need to be established.

Need for Mathematical Modeling in Middle Grades

Turker et al. (2010) claimed that mathematics must be used in students' and teachers' daily lives or else it may be forgotten or even "leave negative impressions" on them. They view modeling as a process that allows students and teachers to develop mathematical representations "in order to find a solution to the problems we come across with in real life" (p. 4622).

Another key feature of modeling is its ability to highlight the relevance of certain mathematical topics. Oftentimes, modeling tasks are designed around a subject or event that interests the students; this can make *doing the math* more appealing and not simply meaningless calculations. "Anytime that modeling is used to explain an everyday situation or problem, the goal(s) of the modeling problem should be considered and made explicit; that is, there should be a purpose for creating models" (Anhalt & Cortez, 2015, p. 3).

Over the past couple decades, numerous articles have been written about the importance of mathematical modeling in classrooms, particularly at the secondary level and above (Anhalt & Cortez, 2015; Doerr, 2006; Stender & Kaiser, 2015). As modeling is often viewed as a complex process that attempts to illustrate the natural (and oftentimes messy) world with mathematics, educators tend to give these tasks to older students (grades 9-12), particularly those tasks which involve functions and statistics. Recently, researchers have been exploring the effects of mathematical modeling tasks in the elementary grades (K-5). English and Watters (2005) conducted a longitudinal study with children from third through fifth grade as they learned about mathematical modeling and solved tasks that were appropriate for their age group. English and Watters concluded that modeling tasks "foster and reveal children's mathematical thinking thus enabling teachers to capitalise on the insights gained into their children's mathematical developments" (p. 302). English (2012) later studied even younger students – first graders – and their ability to select relevant data, analyze it, and represent their analysis with drawings, numerical symbols, and other inscriptions. She argued for a stronger emphasis on statistical reasoning in the early grades and claimed that "[d]ata modelling is a powerful means of illuminating young children's learning potential" (p. 27).

Few studies explore mathematical modeling at the middle school levels (grades 6-8). An exception would be Eraslan and Kant (2015), who studied the modeling processes of Turkish fourth-year middle school students. The students in this study had a difficult time understanding the problem, formulating a model, and mathematizing the modeling process. The authors speculated that these issues could have been a result of the students not finding the task interesting, being unaware of how much time they were supposed to allot to this task, and/or not understanding the overall process or goal of modeling tasks. Because of these unforeseen factors, Eraslan and Kant advocated for future research about "how modeling knowledge develops and changes over time" (p. 822), particularly for middle school students. In order for students to engage in these tasks, their teachers must also understand the modeling process and be well-instructed on how to convey this process to their students.

Although Sriraman & Lesh (2006) referred to modeling problems as "important 'pieces of knowledge' that should be emphasized in teaching and learning…from very early grades" (p. 247-248), a gap appears in the middle school research on mathematical modeling. Schukajlow, Kolter, & Blum (2015) stated that previous studies show students have an insufficient level of modeling competency by the end of ninth or tenth grade and argued for an increased emphasis on modeling prior to high school. "Thus, research on mathematical modelling should be focused on instructional methods that can support the acquisition of mathematical modelling competency from primary through to secondary school" (p. 1241).

Teacher's Role in Mathematical Modeling

The work of researchers in the area of collaborative learning can be applied to mathematical modeling due to the collaborative nature of modeling tasks. In the recent edition of their book “Designing Groupwork,” Cohen and Lotan (2014) described the change that must occur in a teacher’s role in order to promote effective collaborative learning: “Groupwork changes a teacher’s role dramatically. No longer are you a direct supervisor of students, responsible for ensuring that they do their work exactly as you direct. … Instead, authority is delegated to students and to groups of students” (p. 130). The authors highlighted delegation of authority as a key feature that is essential to effective group work. Because of this shared authority, students are held more responsible for their actions and learning in the classroom. Cohen and Lotan emphasized that giving students more freedom is not the same as letting them do all of the work while the teaching remains idle:

> After teachers discover that they do not appear to be needed because everything is running without them, they often say, “I feel like I’ve been done out of my job; it all works without me. What am I supposed to be doing?” … Delegating authority does not mean that you are abdicating your role and your responsibilities. You are now free for a more demanding and ambitious teacher role. You now have a chance to observe students carefully and to listen to the discussion from a discreet distance (p. 135).

Here, Cohen and Lotan (2014) speak directly to teachers as they stress the importance of their new roles as facilitators of group work. Indeed, there is a delicate balance that needs to be maintained between the work the teacher does and the work students are held accountable for.

Simply telling teachers to avoid teaching via direct instruction “suggests that teachers are not supposed to act, regardless of what is going on in the classroom” (Chazan & Ball, 1999, p. 2). Avoiding telling, as Chazan and Ball refer to the common proposal at the time, “contributes nothing toward examining what teachers should or could do" (p. 2). They continued to stress the importance of clarifying a teacher's role in the classroom, especially regarding what they should or should not do:

> While it is intended to allow students a larger role in classroom discussions, it oversimplifies the teacher's role, leaving educators with no framework for the kinds of specific, constructive pedagogical moves that teachers might make (p. 2).

Teacher intervention has its proper time and place in a mathematics classroom (Cohen & Lotan, 2014). If students are arguing, they could be having a productive debate about relevant content. Alternatively, there could be personality issues, and the argument could be impeding the students' ability to be successful. Teachers can make clarifying moves that help students get on the right track without telling them too much, but it is often difficult to decide when these moves are necessary. To address this lack of guidance, Chazan and Ball (1999) recommended employing teaching techniques that create a sense of disequilibrium and disagreement that students have to grapple with and resolve. Pushing students to question *why* at every step of a problem helps them to become familiar with justifying their work and ensuring that all students understand how a solution is found. An important piece of the disequilibrium method comes at the end when students must reflect and finally reach a consensus. Without this agreement, students can become frustrated with each other and with the task, causing them to give up or harbor negative feelings toward that mathematics topic. Chazan and Ball recommended that teachers help to shape the discussion by "inserting substantive mathematical comments" (p. 9) and bringing up new mathematics by asking challenging questions.

The teacher practices described above can be applied to classrooms in which mathematical modeling is taught. The moves that a teacher makes are often a reflection of their understanding of the material. In their study with secondary mathematics pre-service teachers, Anhalt and Cortez (2015) found that "[t]eaching mathematical modeling requires a full understanding of the practice of modeling as a process in which new perspectives about solving mathematical problems must occur for the students" (p. 2). Anhalt and Cortez insisted that pre- and in-service teachers develop their mathematical knowledge in order to teach modeling and foster an interest in modeling among their students.

In four case studies, Doerr (2006) examined the roles of four secondary teachers as they engaged their students in various modeling tasks and lessons. She found that each teacher created her own way of interpreting her students' thinking processes and developed schema for how to systematically listen and respond to her students. While two teachers mastered the art of listening to students and not telling them the next step, a third provided her students with advanced tools for investigation (such as graphing calculators) when it was not possible to let them spend a lot of time on individual work. Doerr concluded that considerable variation exists in approaches that teachers take when implementing modeling tasks; however, teachers should always make sure their actions promote learning: "When teaching mathematics through modeling, an essential task confronting the teacher as she listens to students' thinking is to respond in ways that enable students to further develop their emerging models" (p. 267).

Doerr (2006) noted that engaging students in modeling tasks requires a different way of teaching that oftentimes can be more demanding of the instructor. As she writes, "this shifts the focus of teaching from determining what students need to be taught (as prerequisite knowledge and skills) to seeing and interpreting more deeply the knowledge that students already have" (p. 266). Blum and Borromeo Ferri (2009) echoed this idea when they posed a thought-provoking question:

> Why do we find only so few modelling in everyday classrooms, why is there this gap between the educational debate (and even official curricula), on the one hand, and classroom practice, on the other hand? The main reason is that modelling is difficult also for teachers, for real world knowledge is needed, and teaching becomes more open and less predictable (p. 47).

In a study with eight PSTs in an undergraduate mathematical modeling course, Doerr (2007) spoke of the new role that the teacher will play when implementing modeling tasks into her classroom, referring to some responsibilities as “substantial pedagogical knowledge demands” (p. 77). After analyzing an episode between the teacher and a student explaining his work, she summed up these demands on a teachers’ knowledge base:

> This case illustrates four characteristics of the teachers' knowledge: (1) to be able to listen for anticipated ambiguities, (2) to offer useful representations of student ideas, (3) to hear unexpected approaches, and (4) to support students in making connections to other representations (p. 77).

Certainly, a dramatic transformation of the role of the teacher is being described here. The teacher no longer is seen solely as an evaluator of student work after the students have passively absorbed knowledge, but rather, the teacher must put students in a position to be their own evaluators, which includes making decisions on the accuracy and reasonableness of their models.

> A modelling approach to teaching mathematics calls for a major reversal in the usual roles of teachers and students. Students need to do more evaluating of their own ideas and teachers need to create opportunities where this evaluation can productively occur (p. 78).

Teachers’ Beliefs

The effects of teacher beliefs in the classroom have been well documented (Philipps, 2007; Thompson, 1992; Schoenfeld, 1992). For instance, Schoenfeld (1992) stated that

> Belief structures are important not only for students, but for teachers as well. Simply put, a teacher's sense of the mathematical enterprise determines the nature of the classroom environment that the teacher creates. That environment, in turn, shapes students' beliefs about the nature of mathematics (p.359).

The aforementioned authors define beliefs and discuss how they are manifested through teachers' practices and techniques in the classroom. Using this knowledge, it is possible to examine teachers' motives for preferring one view or teaching style over another.

Thompson (1992) differentiated between teachers' beliefs and teachers' knowledge and argued that the two should be studied together, as one term that she called *conceptions.* She also used the term *orientation* to refer to teachers' views on mathematics and how it is taught. There are two subsets of orientation, namely *conceptual orientation* and *calculational* orientation, which refer to teachers' views about particular pedagogical tasks and their purposes. Teachers with different mathematical orientations may have the same goal for their students but differ in their approaches to reach that goal. A teacher who has a conceptual orientation may be "driven by an image of a system of ideas and ways of thinking she intends her students to develop" (Philipps, 2007, p. 303) and design tasks around this view; meanwhile, a teacher with a calculational orientation may view mathematical tasks as "the application of calculations and procedures for deriving numerical results" (p. 304).

Philipps (2007) gives an example of teacher who values students' development of mathematical proficiency but who is constrained by school curricula and standardized test content that must be taught. Her beliefs about what is most important in mathematics education may not match how she teaches and what her students are learning. "Simply observing this teacher's performance is unlikely to provide one with an accurate assessment of her values or beliefs" (p. 266). Raymond (1997) gives an example of a teacher whose non-traditional beliefs about mathematics teaching and learning did not match her traditional methods in the classroom. Raymond concluded that "the teacher's beliefs and practice were not wholly consistent and that her practice was more closely related to her beliefs about mathematics content than to her beliefs about mathematics learning and teaching" (Philipps, 2007, p. 272). Oftentimes, teachers feel the need to take on multiple roles in order to balance mathematical learning, personal growth (in areas such as self-confidence and autonomy), and classroom management.

Teachers' resistance to change beliefs. Teachers often experience a resistance to change their teaching practices (Philipps, 2007), especially if these practices are tied to deeply-held beliefs or have been successfully used by the teacher for many years. Research has shown that teachers sometimes internalize one view of teaching mathematics and even profess that view, but then enact a completely different view in the classroom (Cooney, 1985; Grant, Hiebert, and Wearne, 1998). Benbow (1995) found that some pre-service teachers' self-efficacy was more influential on their approach to teaching mathematics than their early field experience. Philipps (2007) claimed that professional development opportunities are necessary if teachers are expected to change teaching practices that are rooted in their beliefs:

> While teachers are learning to see differently, they challenge their existing beliefs, leading to associated beliefs change. Change in teachers' beliefs may not lead to change in their practices, or vice versa, but I conjecture that the most lasting change will result from professional development experiences that provide teachers with opportunities to coordinate incremental change in beliefs with corresponding change in practice. Teacher

> educators and professional developers must better understand not only what beliefs teachers hold but also how they hold them, because the ways that teachers hold their beliefs affect the extent to which existing beliefs can be challenged (p. 281).

In Cooney (1985), a new high school teacher, Fred, views authentic, “fun” problems as motivators or time-fillers for students but does not connect them to the content he is teaching. He wants to use these problems to teach heuristics, but argues that there is no time to teach anything other than content in a high school classroom. Fred wants to incorporate genuine tasks that will engage his students in the mathematics, but complains that creating new tasks and lessons is difficult and time-consuming. Fred accepts that although teaching mathematics as routine, textbook exercises conflicts with his pedagogical beliefs, he will teach in this traditional style because it is easier. “The notion that a teacher could be a strong and forceful classroom leader and yet use a problem-solving approach did not seem to fit Fred’s conception of teaching mathematics” (p. 335). I present this case from past literature because it is a classic scenario of when professed beliefs do not match actions. As teachers learn about mathematical modeling and are asked to implement their new ideas into their classrooms, how much of what they *want* to do or *plan* to do will be carried out?

Teachers’ Ideas about Mathematical Modeling

It is not only beliefs that influence a teacher’s views of a mathematical concept. Bautista et al. (2014) conducted a study with middle grades (5-9) mathematics teachers and found that their educational backgrounds affected their ideas of mathematical models of real-world phenomena. Few studies have focused on the relationship between mathematics teachers’ educations and their views of mathematical modeling, but similar results have been found in science education literature. Due to varying teacher certification rules depending on state, middle school mathematics teachers may have a range of different educational backgrounds, which may affect how they interpret and teach mathematical modeling. Particularly regarding models that were created based on rich empirical data, Bautista et al. (2014) found that teachers with a degree in a field other than mathematics – usually natural sciences, technology, or social sciences – tended to be more disparate in their choice of the best model for a set of data. While the teachers with backgrounds in mathematics and mathematics education almost exclusively preferred a line of best fit or equation to represent the data, the science and humanities teachers were more inclined to value discrete data points. Also noteworthy was that less than half of the teachers with social sciences backgrounds believed that the model conveyed more information that the data, whereas almost all of the mathematics teachers believed that the model conveyed more information.

The results of this study showed that teachers' educational backgrounds can influence their views of mathematical modeling. Bautista et al. concluded that teachers interpret the term mathematical model differently and describe models in various ways:

> Teachers with degrees in mathematics education extensively referred to the advantages of having a model (e.g., to visualize patterns, estimate unknown data, see trends in the data). However, they did not describe in detail the specific conceptual information of the mathematical model at hand, as teachers from the sub-group Natural Sciences and Technology did. … Moreover, it was primarily the Natural Sciences and Technology sub-group who tended to see the model presented as just one of many possible models, and who discussed other possibilities (i.e., an exponential model) (p. 23).

Teachers' perspectives of modeling – what it means and why it is important – also affect how modeling tasks are taught (Lesh & Lehrer, 2003). Depending on how a teacher views modeling, she may implement modeling tasks in her classroom with the goal that her students will develop critical thinking skills and "powerful constructs and conceptual systems" (p. 124). These systems may relate to real-world phenomena and allow students to mathematize their everyday experiences. Contrarily, the models may exist for the sake of "pure" mathematics in an attempt to emphasize important mathematical skills. The commonality between Bautista et al. (2014) and Lesh and Lehrer (2003) is that they both underscore the importance of the teachers' views of the modeling process. Before the students complete a modeling task, a teacher must select the task and situate it in a certain way in her classroom. These choices are determined by her perspectives of mathematical modeling.

In this review of the literature, I have discussed different views of modeling that apply to many grade levels and contexts. The literature I explored supplied abundant examples of modeling in high school and undergraduate work, but very few in the middle grades. The research presented in this study explores modeling tasks at the middle grades level (5-9) in an attempt to address this gap in the literature. Research on teacher views in the context of mathematical modeling is scarce. Although there is substantial literature about teacher views framed in the context of beliefs, this study is interested in teachers' beliefs only regarding how these beliefs may or may not influence a teacher's view of modeling.

My definition of mathematical modeling aligns closely with that of Blum and Borromeo Ferri (2009) in that I see modeling as a multi-step process used to understand and represent real-world phenomena through mathematics. Due to my focus on teacher views and mathematical modeling, I adapted pre-existing frameworks to create the framework for this study. In the next section, I discuss this framework.

Chapter 3: Theoretical Framework

The theoretical framework of this study was created in an attempt to connect the idea of "mathematical thinking styles" (Borromeo Ferri, 2006) to Kuhs and Ball's (1986) framework for "mathematics teaching styles" and Thompson's (1992) notion of "conceptions" regarding conceptual and calculational orientations. These three frameworks are related in their objective to describe teachers' views and how these views affect their teaching practices.

Borromeo Ferri (2006) defined "mathematical thinking styles" of students and teachers in the context of mathematical modeling. Her work is an extension of past research of "thinking styles" (Sternberg & Zhang, 2005; Zhang, 2002; Zhang & Sternberg, 2005), but specifically aligned with mathematical modeling. In her framework, she focused on the "complete modelling process" (p. 2), paying special attention to the role of the teacher. When students do not understand what their teacher is trying to convey, it may not be because the teacher's explanation is "bad" but rather because the student and teacher use different mathematical thinking styles. In this way, a teacher's beliefs and views on mathematical modeling will affect her thinking style, and this in turn will affect the way that her students learn – or are expected to learn – mathematical modeling. Borromeo Ferri found that teachers are likely to emphasize different aspects of the modeling process or even avoid certain modeling elements depending on their preferences toward a certain mathematical thinking style.

In two case studies, Borromeo Ferri (2006) found that one teacher, who was an analytical thinker, tended to emphasize the interpretation and validation steps; meanwhile, a second teacher, who was a visual thinker, provided elaborate descriptions for her students that were based in reality. She created three categories of mathematical thinking styles, namely, analytical, visual, and a combination of the first two and generated the following hypothesis based on her study: "Teachers, who differ in their mathematical thinking styles, have the preferences of focusing on different parts of the modeling-cycle while discussing the solutions of the problems" (p. 7). Borromeo Ferri also noted that a teacher does not have to fall into one category or another; a teacher can have multiple thinking styles or even a combination of them.

I designed my framework for this study in a similar way to Borromeo Ferri's technique, taking an older framework and adapting it to my specific research questions. Similar to Borromeo Ferri, I will consider which elements of the modeling process are emphasized or avoided by these teachers and connect those elements to their views of learning and teaching mathematics. Borromeo Ferri adapted the notion of "thinking styles" (Sternberg & Zhang, 2005) to align with the elements of mathematical modeling; here, I make a similar adaptation to Kuhs and Ball's (1986) and Thompson's (1992) frameworks by adjusting the terminology to match teacher views and practices in the context of mathematical modeling. Thompson (1992) defined a mathematical orientation as a combination of a teacher's beliefs and knowledge and categorized all orientations as two distinct types: calculational or conceptual.

Thompson (1992) reviewed multiple structures for analyzing beliefs and practices for teaching mathematics. One of the frameworks discussed in Thompson was created by Kuhs and Ball (1986) and outlines "four dominant and distinctive views of how mathematics should be taught" (Kuhs & Ball, 1986, in Thompson, 1992, p. 136). The four views are given as follows:

- Learner-focused: mathematics teaching that focuses on the learner's personal construction of mathematical knowledge;
- Content-focused with an emphasis on conceptual understanding: mathematics teaching that is driven by the content itself but emphasizes conceptual understanding;

- Content-focused with an emphasis on performance: mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures;
- Classroom-focused: mathematics teaching based on knowledge about effective classrooms (Thompson, 1992, p. 136).

Clearly, the second view, with its emphasis on conceptual understanding, aligns with Thompson's conceptual orientation, while the third view, focused on procedures and performance, aligns with her calculational orientation. I chose to adapt this framework as the basis of my own because I appreciate the dichotomy of procedural fluency versus conceptual understanding presented in Thompson. I think these two orientations articulate the difference between the two "content-focused" views in Kuhs and Ball (1986) more thoroughly.

The learner-focused and classroom-focused views in Kuhs and Ball are not as clearly aligned with one orientation over another. It seems that more evidence about the teacher's practices would need to be provided in order to classify her teaching view as a specific orientation toward mathematics. For instance, a teacher may be classroom-focused, always employing effective group work strategies and opportunities for students to teach one another; however, she may use these strategies to stress mastery of algorithms and procedures rather than conceptual understanding, or vice versa. In this way, teachers' practices and views of mathematics, in this case specifically mathematical modeling, can be analyzed through the framework of mathematical orientations (Thompson, 1992) and teaching views (Kuhs & Ball, 1986).

In my framework, I expanded the Content-focused views in Kuhs and Ball (1986) to include curriculum components that teachers weigh into their decisions about teaching. I maintained the two versions of this view from Kuhs and Ball, but adapted them slightly to align with the two orientations in Thompson (1992). One of my content-centered views is based on conceptual understanding of the mathematics, and the second view manifests itself by an emphasis on student calculations and procedural fluency. The other two views are student-centered and real world-centered. In short, the framework for this study consists of the following four views of teaching mathematical modeling:

Content-centered (conceptual): mathematics teaching and curriculum design that is centered on conceptual understanding of content and development of critical thinking skills;

Content-centered (calculational): mathematics teaching and curriculum design that is centered on procedural knowledge of content and fluency with mathematics calculations;

Student-centered: mathematics teaching and curriculum design that is centered on the background knowledge and interests of the students;

Real world-centered: mathematics teaching and curriculum design that is centered on the incorporation of real-world events and issues into the classroom.

I chose to include the real world-centered view because modeling tasks are designed from real-world situations and phenomena. I realized that it is very probable that a teacher would have a view of mathematical modeling tasks being a bridge from real life to the mathematics in the classroom. This bridge connects not only mathematical content and skills, but also the dispositions and practices of the teacher and learners. Teachers who have a real world-centered view of mathematical modeling see modeling tasks as being very different activities from those that are typically done in a mathematics classroom; in fact, they are more *realistic*. I also changed the focus of the third view from “learner” to “student” to emphasize that my framework’s student-centered view focuses on the student’s interests and personal experiences. This is a subtle difference from Kuhs and Ball’s (1986) learner-focused view that looks at how learners construct their mathematical knowledge. These elements of constructivism did not seem relevant to this study.

Chapter 4: Methods

The goal of this study is to determine the views of four middle school teachers regarding mathematical modeling and analyze how these views affect their teaching. The research questions that motivated the study and analysis that was conducted were:

RQ1: What are middle school teachers' views of teaching mathematical modeling?

RQ2: What are middle school teachers' perceptions of their role during the planning and implementation of mathematical modeling tasks?

Context for the Study

Professional development (PD) sessions were held for five full days (six hours each) during the summer of 2015. Teachers were first asked to individually brainstorm what mathematical modeling might entail and then to draw a modeling process with their small group. Teachers were then shown various modeling cycles that have appeared in recent literature and asked to reference these as they completed modeling tasks throughout the week. Teachers were given ample time to design lesson plans around some of the tasks they completed; they were also able to adapt problems from their schools' textbooks to include elements of modeling. Participants frequently created posters to display their group's solution to a modeling task and then engaged in a gallery walk to view other posters.

Additional two-hour PD sessions were given once during September, October, November, January, February, and April, so a total of six sessions during the school year. The general format of the school-year PD sessions was as follows: teachers spent about 30 minutes discussing how they had implemented a modeling task with their students and shared student work samples in small groups; then for about an hour, teachers brainstormed what they knew about the topic of a new modeling task first as a whole group, and then the teachers solved this task in small groups, sometimes organized by grade level; teachers and facilitators spent the final 30 minutes of the session debriefing and planning a lesson to incorporate the new modeling task into a classroom setting. Each session had one or two different focuses in addition to the new modeling task. These focuses included launching the task, scaffolding the task to be appropriate for lower (or higher) grade levels, extending the task to align it with relevant content in the CCSSM, creating task cards to promote equitable group work, finding a generalized model with unknown parameters, creating or modifying rubrics to assess student solutions, and adapting tasks to incorporate relevant issues in the teachers' communities.

The PD focused on mathematics content, not on pedagogy. While the facilitators modeled appropriate ways to implement modeling tasks, they did not explicitly stress one teaching approach over another. However, the facilitators have backgrounds in teacher education and the methods of Complex Instruction (CI) (Cohen & Lotan, 2014). CI is a teaching approach that uses collaborative learning strategies to address issues in equity. CI functions properly when students work in groups on a task that is open-ended, encourages multiple abilities and solutions methods, and has a high cognitive demand. By recognizing and treating status issues in the classroom, CI claims that students are able to bring their individual abilities to the group and enable the group to be more successful than any member acting alone. Because the PD instructors have studied this approach, techniques from CI often were used during facilitation of modeling tasks. Many of the modeling tasks given to the teachers are what Cohen and Lotan (2014) would call "groupworthy," and require the teacher to take on a new role in the classroom.

The Participants

Five teachers consented to participate in this study: William, Jamie, Griff, Donna, and Arturo. It is important to note that findings will only span the responses of four of the five teachers in my study. Arturo chose to answer the interview questions through typed responses via email. In reading his responses, I realized that it was difficult to gauge his meaning when he expressed his opinions. To illustrate this, here is an example of an interview question that I asked and his non-committal response:

> Question: Do you think that your personal views and opinions about modeling affect how you teach it? If so, how?

> Arturo: I believe that personal views have a big impact on the way you teach modeling. If someone does not like to make models for regular lessons as you teach (such as graphs, equations, etc.), then modeling is going to be a hassle all the time for them, because that's the base of modeling, which is application.

Notice how Arturo responded to a question directed at him by using the pronoun "you," instead of using "I" to speak about himself and his views. His response is also hypothetical, "If someone," rather than an assertion of what he thinks. His other interview responses followed a similar pattern; therefore, I decided to exclude his data from the discussion.

Short profiles of each teacher. William is a seventh grade mathematics teacher. He received a Bachelor's Degree in Education about 10 years ago from a nearby university and earned a Master's Degree in Education from an online university. William teaches five sections of seventh graders every day. Mathematics is a tracked subject at his school, and William has all "regular" classes, as opposed to "honors" classes. A few of his colleagues are participants in the PD workshop, so he frequently collaborates with them in designing and implementing modeling tasks.

Jamie is an eighth grade mathematics teacher at the same public middle school at which William teaches. She has been a teacher for 14 years, but, like most of the PD participants, had never used modeling tasks in her classroom. She teaches six classes of eighth graders every day: two honors classes, three regular classes, and one enrichment class. The enrichment class is mandatory for all students for one semester, and so, is a heterogeneous group of students. During this semester, students will be enrolled in two mathematics classes: their typical eighth-grade mathematics class and an enrichment class. In Jamie's section of the enrichment class, the focus was solely on mathematical modeling. Recently, she requested to teach more sections of the enrichment class next school year with the curriculum specifically designed around mathematical modeling.

Griff is a seventh and eighth grade mathematics teacher at a different public middle school. He has been teaching mathematics for 17 years. Although he teaches multiple sections of eighth graders, he has only done modeling tasks with his REACH students. REACH is an enrichment class for higher-ability or struggling students that meets for an hour every day. This class is not mandatory, but is comprised of a heterogeneous group of students.

Donna is the district coach and instructional specialist for a small district. She was formally the mathematics district specialist for a larger district, but now she oversees all disciplines. She is also the Professional Development facilitator for all teachers and a coach to new teachers. She does not teach her own students but occasionally co-teaches or model-teaches with the new teachers in order to share strategies. There are three mathematics teachers in her district: one for upper elementary (grades 4-5), one for middle school (grades 6-8), and one for high school (grades 9-12). The upper elementary and middle school teachers are also participants in the M^3 PD workshop.

Sources of Data

The table below summarizes the sources of data that were collected for this study and the timeline in which they were collected. Not every participant contributed equally to each type of data. Following the table, I describe the data sources.

	June 2015 (Summer Institute for 5 days)	**Sep 2015**	**Oct 2015**	**Nov 2015**	**Jan 2016**	**Feb 2016**	**March 2016**
Field Notes	**William, Jamie, Griff, Donna**	**Griff**	**Griff, William**	**William, Jamie, Donna**	**William, Jamie**	**William, Griff, Jamie**	
Interview *(date, length of interview)*				**Griff** *(12th, 38')*, **William** *(19th,44')*		**Jamie** *(11th, 54')* **Donna** *(22nd,44')*	
Collect Student Work Samples		**Griff, Jamie**	**Griff, Jamie**	**William, Jamie**		**Jamie**	
Email Correspondence				**Griff, William**	**William**	**Donna, Jamie, William**	**Jamie**

Table 2. Sources of data collected per participant, organized by type and date.

Descriptions of Data Sources

Field notes. During the summer and the school-year sessions of the PD workshop, I took notes based on my interactions with the teachers and my observations of their participation. These notes sometimes included direct quotations or summaries of episodes that occurred between teachers. I later compared my field notes to the interview transcriptions from the teachers in the study. This allowed me to corroborate some of their claims regarding how they view mathematical modeling and implement it in their classrooms. It is important to note that during the summer sessions, I did not target specific teachers, but rather focused on the entire group. This is why field notes were gathered for all four teachers in the study during June 2015. Since the focus of the sessions were on PD and not research, my time and attention were dedicated to facilitating the modeling material rather than meticulously collecting field notes.

During the school year, my notes were focused on specific participants of the study each session depending on who was most vocal and my proximity to them. Data in the form of field notes were not collected equally for all participants in the study.

Collect student work samples. At the end of each PD session, many teachers would bring forward student work samples from modeling tasks that they had given their students during the previous month. The co-facilitators and I would collect these samples and take de-identified pictures for future analysis. Many of the samples we collected from various teachers came from the same modeling tasks, which allowed us to compare how the task was implemented in different classrooms and how different groups of students approached the task. During the interviews with participants in my study, I referenced some of the samples that they provided so that we could discuss what mathematical and modeling processes the student used.

Interviews. The main source of data for this study came from interviews with each of four teachers. During each interview, I asked the teachers what modeling tasks they had given their students and how they incorporated modeling into their classroom. I also specifically asked about their views of mathematical modeling and what elements they felt were necessary to define a “modeling task.” Teachers were asked to look at student work samples—samples that they provided from their own students or samples that I created to simulate a student’s solution—in order to compare them and decide which elements of the modeling process were apparent in the students’ work. Interviews lasted between 38 and 54 minutes and were conducted in-person or via phone call. The interview questions are given in Appendix A. As I went over their answers to the interview questions, I sometimes had follow-up questions that I pursued through email correspondence as described below.

Email correspondence. Because most of the teachers did not live near me, I corresponded with them via email outside of PD sessions. Although I conducted an in-person interview with William, I followed his interview with additional, clarifying questions via email. Jamie sent (de-identified) student work samples to me via email, because she had some more recent samples (than what I had collected from PD sessions) that she wanted to discuss in her interview. I also emailed Jamie clarifying questions directed toward her view of modeling. My only email communication with Griff and Donna regarded logistics of scheduling an interview.

Data Coding and Analysis

The data were collected in order to conduct qualitative analysis. I used the codling and analysis procedures outlined in Saldaña (2015). Following the collection of data, I initially read through all interviews and field notes to get a feel for the themes that seemed to emerge from the text. I then created codes based on specific quotes by the participants and grouped them based on categories related to each research question. The *in vivo* codes (Saldaña, 2015) are included with descriptions in the Codebook provided (Table 2). For example, to address the first research question, I noticed initially that three views of modeling emerged from the participants' interviews, so the *in vivo* codes included specific words and phrases that participants said in reference to their view of modeling. The descriptions of these codes list what phrases were coded as such and which were not. For instance, the word "process" is coded as a view of mathematical modeling only when it is clearly in reference to the modeling process and not when it refers to other generic thought processes. Other codes were not *in vivo* and required multiple passes through the data to ensure consistency in the coding of certain phrases and inferences. For example, one participant stated that "So I mean it's not, it's the same word 'modeling' but, and it's not the same as math modeling…" which seems to allude to a view of modeling, assuming that we understand the antecedent of the word “it.” In fact, this excerpt is referencing “modeling” in the sense of showing someone what to do so that they can imitate the actions, rather than mathematical modeling. Thus, this datum is excluded from coding for modeling views.

Code & Examples	Description
DEFINITIONS OF MATH MODELING	
Modeling as Real-world Mathematics “real-world example”, “Something they would actually do…”, “This is reality.”	Teacher sees modeling as the mathematics that is embedded in everyday phenomena. Mathematical solutions are realistic and make sense in real-world context.
Modeling as a Mathematics Application “…students apply prior knowledge…,”	Teacher sees modeling as opportunity for students to practice mathematics skills and concepts that they have already learned.
Modeling as a Process “They went through the entire process…”	Teacher refers specifically to the modeling process or thinking about how to model; not other “thought processes.”
What counts or does not count as modeling “I thought this was a good example of a modeling task.” “I think the assumptions piece is what’s different from the traditional math that teachers have been doing.”	Teacher refers to specific elements of modeling or specific tasks and why they do or do not count as modeling
Elements of Modeling Emphasized “If it doesn’t have purpose, conclusion, reflection, it’s lacking.”	Teacher lists specific modeling elements and explains why they are important.
TEACHING MODELING	
Role of Teacher “I had to bite my tongue, because, you know, then if I started talking, I’m going back to the old style [of teaching].”	Teacher mentions role in the classroom while students engaged in mathematical modeling.
Intervention “I guided the students…” “I didn’t say anything.”	Teacher specifically states practices used to intervene (or not) while students worked on modeling tasks.
Change in teaching approach/methods “…move away from direct instruction.” “I’ve had to really let go of a lot of my methods that I’ve used for the last 14 years.”	Teacher adds new approach or alters current methods of teaching due to components of modeling.
Resources used “I allowed them to use their phones in the classroom…” “If they asked to use a computer, the answer was yes.”	Teacher explicitly states resources that students used or were allowed to use during modeling tasks.
Resistance to changing practices	Teacher expresses some difficulty or resistance to using modeling tasks due to new teaching practices required or specific traits of modeling.

"This has been *really* hard for me." "You do something for a certain way for so long, it's hard to change that thought process…"	
Issues with curriculum/standards "…like a waste of time or taking away from this list of standards that he's got to get through." "…you're pulling several standards together and even the math practices…"	Teacher expresses the need to align modeling tasks with state/national standards. Or, teacher struggles to use modeling due to strict teaching schedule or constraints with standards.
Past experiences affect teaching of modeling "…a lot of us took how we would have done it personally and how we would have done it in our own families." "They're assuming three people in the family, which wouldn't make sense in my family…"	Teacher recounts past experience or personal background that influenced approach to teaching a certain modeling element or task. Or, teacher is influenced by personality or disposition.

Table 2. Codes with examples and descriptions used during analysis of data.

The findings are presented in a case study format in order to get an in-depth view of the perceptions of four teachers with regard to mathematical modeling and their individual teaching roles. I first address the research questions with respect to each teacher. Then, I later discuss the findings across all four teachers and provide implications for teaching and future research.

Chapter 5: Findings

The goal of this study was to address two research questions:

1. What are middle school teachers' views of teaching mathematical modeling?

2. What are middle school teachers' perceptions of their role during the planning and implementation of mathematical modeling tasks?

This study focused on four teachers' perceptions of what it means to model with mathematics at the middle school level and how they teach modeling tasks to their students. The teachers' professed views revealed that they see their own role as a teacher as being altered from more traditional roles of teachers in mathematics classrooms. They also see modeling having a specific place within curriculum; sometimes that placement is in enrichment classes, which are separate from regular mathematics classes. Here, findings are presented in the form of case studies per participant. The cases are organized into two main sections, each addressing one of the research questions.

Views of Mathematical Modeling

Although all four teachers attended the PD workshop on mathematical modeling, there were noteworthy variations in their views of this concept. While discussing the teachers' views of mathematical modeling, I present specific episodes that address what does or does not count as modeling to the teachers as well as examples of how teachers' past experiences and personalities affected their reported teaching of modeling.

William. William stated that he does not use the word 'modeling' with his students, but rather refers to each task as a "real-world example." He emphasizes that mathematics is inherently tied to the world around his students, and the modeling tasks (or real-world examples) illustrate the connection between mathematics and real life: "It's saying, ok, here's reality. Here's something that's actually going to happen."

He enjoys the familiarity of the concepts that appear in the seventh grade mathematics curriculum: “What I always explain seventh grade math is, [it] is the math you will use in your lifetime.” When choosing or designing mathematical modeling tasks for his classes, he pays special attention to the relevance of the task to his students’ lives. During the fall semester, he designed a task called the Recipe task [see Appendix B] in which the students needed to decide how much of each ingredient to buy to make desserts for a party. The statement of the task is given here for reference: *A group has requested this recipe [for Mississippi Mud Pie] for the caterer to prepare for their party of ___ people. Adjust the amounts to know how much the caterer needs to buy of each ingredient.*

During our interview, William explained that he chose the varying number of guests for the party to be numbers that do not share common factors with 12, which is the serving yield for the given recipe:

> So those were decisions they had to make on their own. But mostly it was to let them realize that, you know, the math that we do is real and here’s an example, something that you actually *would do* for a party.

Here, his statement backs up his earlier view of modeling tasks as “real-world examples.” The decisions that students had to make related to the remaining number of brownies that the students would end up with, depending on which solution method they chose. If the students used unit rates, they could make the exact number of brownies they needed; but, if they found the multiple of 12 closest to their needed amount, they could have many brownies (up to 11) left over. William assessed some of his students’ solutions from work samples that he brought to our interview. When evaluating the mathematical work, he focused on the reasonableness of the solution as it related to the context of the problem. One of the students [Sample E1] found a solution by overestimating how much of each ingredient she needed to buy [see Figure 4 and Appendix C]. William was critical of this answer but acknowledged that it was a valid approach:

> So, very different approach, very effective – granted, in the real world they'd have a lot of waste cost there and they'd probably be, you know, fired [laughs] but it was an excellent approach to do, because if I was going to make it at home, that's probably the method that I would use.

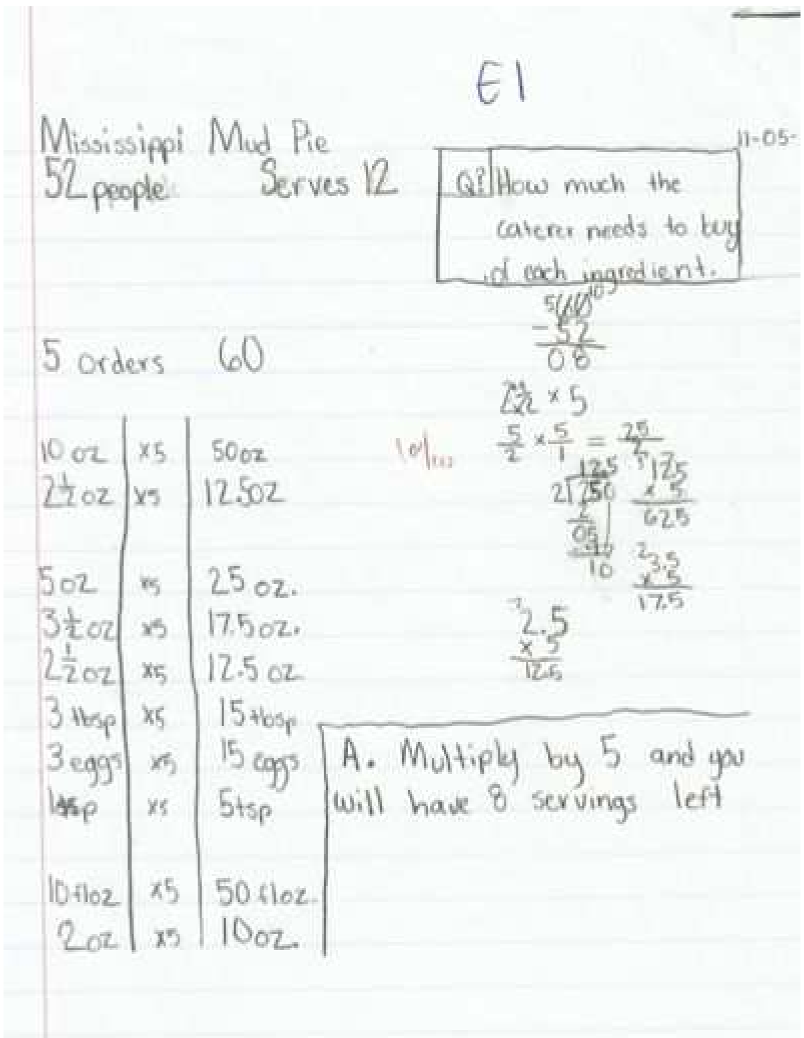

Figure 4. Student work sample [E1] from Recipe task. William's student solved the task using multiples of 12, leaving 8 servings as a remainder.

William recalled a task that he labeled as modeling which he completed when he was in the fourth grade. In this task, he was asked to calculate the number of skittles contained in a large jar. Even though he admitted that this task is not as relevant as others, such as the Recipe task, he decided to give it to his students in order to let them grapple with the idea of volume. He jokingly said that his students probably would not calculate the number of skittles in a jar before eating them, but, as a fourth grader, he was very intrigued by this question:

> But that [the Skittles task] stuck with me, which I think is the most important thing about the modeling in the first place. That's why I say a real-world example, because some day in their life, they're gonna eat food and they're going to have to prepare it for somebody else. Unless they're some hermit out in the middle of nowhere, they're going to be preparing it for a party [laughs] at some time in their life…and it's not going to be a nice number to work with, and so that's why [the Recipe task] was designed that way.

When William gave his students the Recipe task, they had just learned about proportional reasoning and unit rates. William expected the students to use this knowledge to help them complete the task. He views modeling tasks as a way for students to apply what they have already learned to a real-life situation: "What I'm trying to do is pull that knowledge out of them that's already been taught so they're actually making a practical use of it [the knowledge]." For William, mathematical modeling is all about relating mathematics concepts in the classroom back to the real world.

In response to the student work samples of the T-Shirt and Shampoo Bottle tasks [see Appendix C], William was very critical of the work and solutions. The statements of the T-Shirt and Shampoo Bottle tasks are given here for reference: *The school is looking buy T-shirts for a volleyball team. Tees-R-Us has the sale "buy one, get one half off" and Sport-Mart has the sale* *"buy two, get one free." Which is the better choice if the school needs 11 shirts?* (T-Shirt) and *A family shares a bottle of shampoo when they wash their hair in the shower. (They usually buy a large bottle that holds 1 liter or 33.8 fluid ounces.) How often do you think the family should buy a new bottle of shampoo?* (Shampoo Bottle).

In William's opinion, something was missing from the students' models to make them count as "true modeling." While evaluating student work from the T-Shirt task, William criticized the lack of explanation and reflection in the group's poster [see Figure 5]. Although the students show their calculations and give a solution, William decided their work is not satisfactory:

> So the modeling, they're using proportionality to find what they're doing; they're using it as a tool, but they're not really modeling the project, as far as representing, in my mind. Yes, they solved the problem, and they did some basic arithmetic in Sample A, but once again, there's no explanation as to why anything was done, which is I think what's lacking in true modeling.

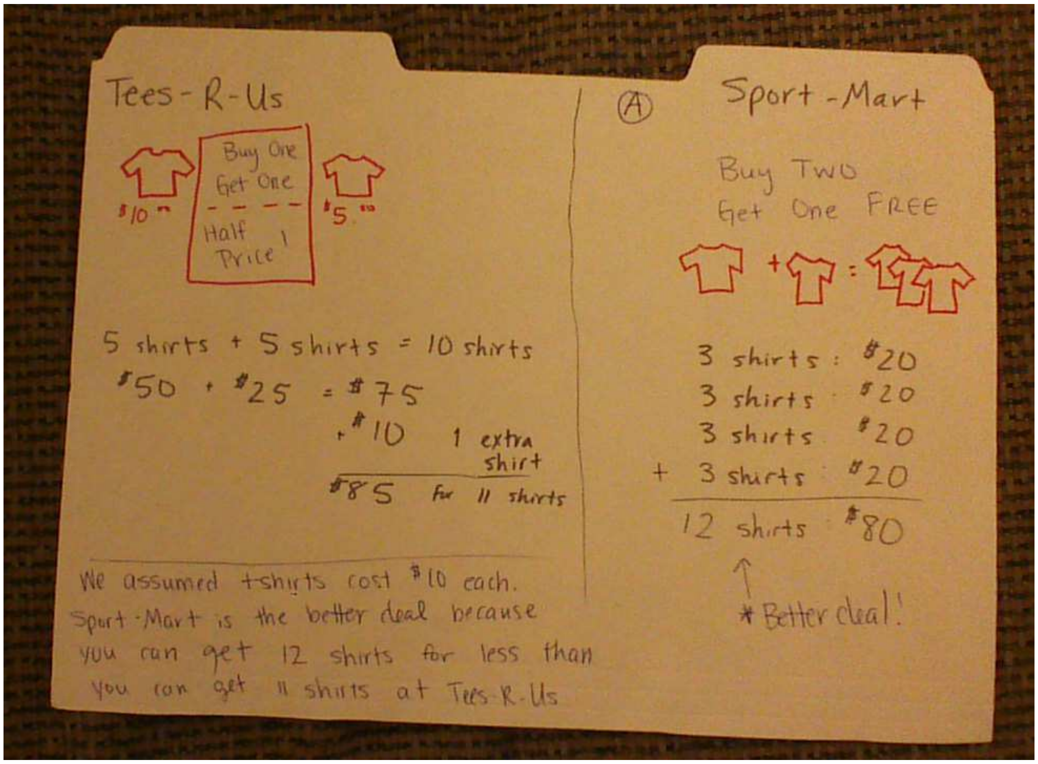

Figure 5. Student work of T-Shirt task, Sample A. This student showed arithmetic work but did not justify assumptions or reflect on solution.

William seems to distinguish between work that is "true modeling" and work that is not. Later in our interview, I asked William which modeling tasks he would choose to best represent the concept of mathematical modeling. He did not agree that such a hierarchy of modeling tasks existed, but still used the phrase "full modeling project" in his response:

> So, which ones are more modeling? That's opinion, quite frankly, I would say. However, I would suggest that if they are going to be a full modeling project, they have to have related math concepts to what we're doing [in class]. They can't just be out of the blue. I would rate it on importance as it would be relative to what we're doing.

William believes that modeling tasks should be related to the mathematics concepts that are being taught in class. In fact, this is what constitutes a "full modeling project" in his mind. Above, he mentioned that representing and explaining are important and now he states that connecting concepts is a vital component in mathematical modeling tasks. But it is not only these elements that outline what counts as modeling for him. He also considers familiar mathematical objects and tools, such as algorithms and formulas, as examples of models: "Because, math algorithms themselves are modeling, because we're organizing things into structure. So even a formula itself is a form of modeling, just so that you can get around the abstract – at least in the way I see it."

A noteworthy feature of the Skittles task is that the model students must create to estimate the number of skittles is essentially the formula for the volume of a cylinder. This may be what William means by a formula being "a form of modeling." It appears that William has a clear view in his mind as to what counts as modeling, whether it be symbolic equations, hand-drawn diagrams, or even generated formulas. The key feature of a modeling task and solution is its connection to the real world.

Jamie. In my interview with Jamie, she stated that she described mathematical modeling as "application math." When I first heard this, I thought she meant that modeling tasks help students apply certain mathematical knowledge they have learned to a particular task. Jamie also stated that when teachers implement modeling tasks, they are "giving the students an opportunity to be creative and develop critical thinking and apply." Again, using the word "apply" made me think that the primary function of modeling tasks in her classroom was to let students practice a mathematics topic in the context of a modeling task. I emailed Jamie to gain clarification on exactly what she meant by her description of "application math." She replied, "In both cases the word "Apply" simply means using math in the real world so that students have an opportunity to make that connection." So, even though her view of mathematical modeling is very similar to William's view, the language she used to describe it was different and a little ambiguous.

To help Jamie elaborate, I presented her with the three views of modeling that I have encountered in the literature, gave a brief description of each of these views, and asked her which one she agreed with the most. This is the prompt I sent her:

> I have come across three main views of mathematical modeling in my research: real-world math, applied problem solving, and a formal process for critical thinking. To elaborate on the first view, I would think about modeling as anything to relate the mathematics of the classroom back to a real-life situation or real-world phenomenon. The second view is more about applying what has been taught in the classroom to a problem (real-world or not) in order for the students to practice that particular mathematics concept. The third view of modeling as a process is a more holistic view where the mathematical concepts that the students use aren't as important as the overall understanding, decision-making, and justifying that they do during the entire task. One way to think about this is that using the modeling cycle is more of the end-goal than a tool used to reach some goal or solution. Which of these views (or maybe a combination) do you agree with most? Please explain why. [email sent 3/24/16]

She replied, "I agree with all views and I see all three being used to some degree on every task." Indeed, this answer may seem inconsistent with her prior response since she specifically referred to modeling as "application math", but Jamie also spoke about modeling as a "process" prior to my clarification questions:

> I think most of [the students] are learning something new from the task. I think the process, the modeling, you know, that has to be taught, but I think every task is teaching them something new. And I think, because it's all application, so it's building background knowledge that they don't already have.

Jamie sees modeling as a holistic process, but also as an opportunity to solve real-world problems by applying mathematics knowledge learned in the classroom.

Griff. Griff views modeling as a process for doing real-world mathematics. He emphasized the formal process or cycle of modeling as very important for students to keep in mind while working on a task:

> Probably one of the things I'm going to do personally is dig out the copy of the math modeling cycle as you've given to us [PD participants] and maybe next time before I even start the lesson, go through the cycle and, you know, just discuss with them what the cycle entails. I think it was taking place pretty much probably naturally as part of the process, but I wasn't specific enough with that.

Griff has a general idea of a diagram of the modeling cycle (see Figure 6) in his mind and he claims to use it as an aid to remind him of the process for mathematizing a real-world event and creating a model. While Griff thought that some of the elements of the cycle were occurring naturally as his students worked toward a solution without having to be continually prompted, he wished that he would have emphasized the benefit of completing the entire cycle instead of just certain aspects of it. In particular, he focuses on the understanding of the task, which leads to the mathematical work:

> And so, you know, when we do this mathematical modeling, not only is it good for math, the math portion of it, the understanding of it, but also for just, you know, a lot of those other things that are important as you become adults, and go into adulthood, and go into the real world.

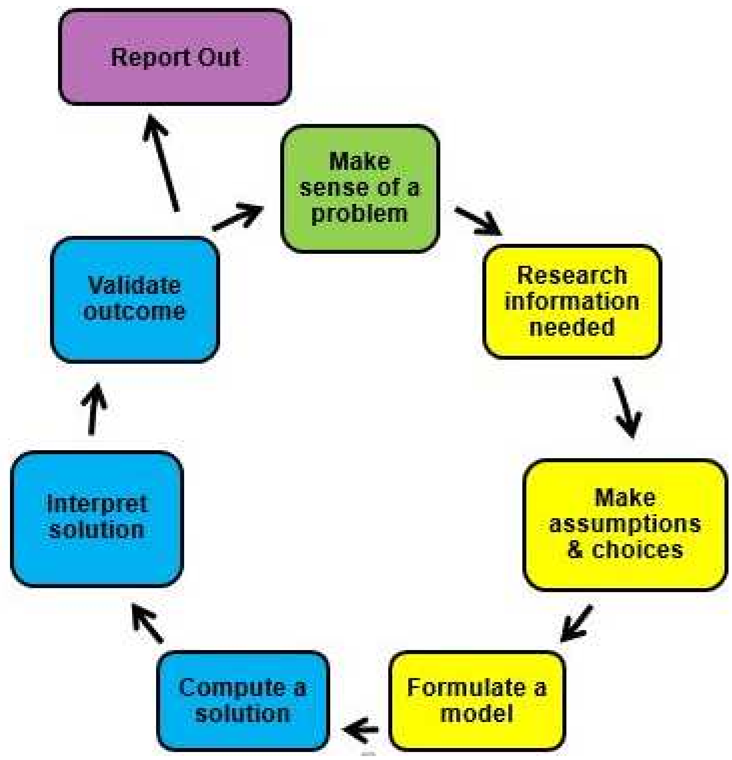

Figure 6. A version of the modeling cycle as given to PD participants. Griff used this version as a reference when teaching his students the elements of mathematical modeling.

While the mathematical work is important to Griff, the extension of the calculations to real-world situations is just as important. This is why he said that he emphasizes the reflection step after students have come to a conclusion. Griff spoke specifically about a group's solution to the Shampoo Bottle task [see Appendix B], in which they claimed that a family should buy a new bottle of shampoo every week:

> To say that it's going to take one [bottle] per week, I just think some of the numbers that they came up with really kinda confused their realistic thought process, you know. They got too tied up in the numbers and didn't think of it from real world.

Griff thought the claim that a new shampoo bottle is needed every seven days was not realistic and that the students who found that solution did not reflect on their answer to see if it made sense. Although the Shampoo Bottle task is open-ended and does not have one correct answer, Griff was concerned the process that some students used did not lead them to a realistic conclusion. To him, the modeling process, including justification and reflection, is just as important as the final answer. Griff analyzes modeling tasks and solutions holistically, taking into consideration the work from start to finish, assumptions to reflection.

Griff had a very positive opinion about all of the modeling tasks that he was introduced to during the PD workshop. He stated that all of the tasks were good examples of mathematical modeling because they all included some aspects of the modeling cycle. A few of the tasks stood out to him due to specific characteristics. For example, Griff especially liked the Car Wash task [see Appendix B], because of the many "assumptions you had to make" and the "re-thinking you had to do." The statement of the Car Wash task is given here for reference: *A locally owned automated car wash advertises that it serves millions of satisfied customers each year. Is this a reasonable claim?*

Griff also thought that bringing personal background knowledge into a mathematical task is one of the defining characteristics of a modeling task, as shown in the Papi's Birthday Party task [see Appendix B]:

> So I was just thinking, you know, I remembered with that one [Papi's Birthday Party task] that a lot of us took how we would have done it [planned the party] personally and how we would have done it in our own families. And I guess when you think about it, it is part of the modeling task. You take…your own background knowledge and you apply it to a situation that's a little different. That's not to say that background information or background knowledge isn't important in some of the other [tasks] that are done, but I

think in [these particular modeling tasks], they have a big effect on how you approach that problem and how you solve that, you know, how you look at it.

The statement of Papi's Birthday Party task is given here for reference:

It was Señor Aguirre's 70th Birthday. His three children wanted to throw him a big party to celebrate. The hall rental, mariachi, food, and decorations will cost a total of $4,500. The brother, a special medical doctor (anesthesiologist) who makes about $20,000 per month, suggested that the three children split the cost equally.

One of the sisters, a university professor who makes about $6,000 per month, said that would not be fair. She suggested the following: the brother pays $3150. She would pay $900, and the other sister, a partner in the family business and single mom with 2 boys who makes about $3000 per month, should pay $450.

Write a position statement using mathematical evidence (e.g. proportions, ratios, percent) to support your conclusion to the following questions:

•Which person do you agree with and why?

•What is fair in this situation?

•Can you think of an alternative financial arrangement that might be better (more fair)?

Although Griff seems to have a broader view of what counts as modeling, there are still specific elements of tasks that he emphasized, such as assumptions and the ability to bring personal knowledge into the task.

Donna. Through her work with teachers and students, Donna has developed the view of modeling as an opportunity to bring real-world issues and decision-making into the mathematics classroom:

> I would say it [mathematical modeling] is working on problems that are more realistic that don't have everything defined for them. … And the real world has lots of different situations going on. They have to make decisions in order to define where they're going.

Similar to William, she likes the real-world aspect of modeling tasks, because they give a purpose to the mathematical work that students must carry out. During our interview, I asked Donna if the realness of the tasks was a defining characteristic of modeling. She replied:

> It's realistic, yes, but it's also, there's also a purpose. It also gives them a life skill that they're going to use. … But what's more purposeful is having to make a decision and that a lot of those decisions you can't go back and change. So it's more important to have everything lined up to make sure you have the right one, because you're probably gonna have to live with it.

Donna goes a step beyond viewing modeling as "math in the real world;" she adds the element of decision-making and talks about how the consequences of choices make the task more meaningful. She elaborated by giving an account of a conversation between herself and a friend while on a road trip. Her friend was trying to decide which car to buy by ranking certain features of each type of car and then determining which had the highest overall rating:

> So she's sitting there and picking out what five characteristics she's looking at and giving them a rating of one to five in each of the categories and using that to help her decide. And I [laughs], I just looked at her and said, 'That's a modeling problem.' So that's something that I think is so real for what we want these students of ours to be able to do when you're buying a house or trying to decide on a job. That is so – that's so relevant."

During our interview, Donna spoke extensively about what types of tasks she liked best and which tasks she did not think illustrated mathematical modeling. Currently in her school, the high school teacher she coaches is using word problems about patterns and sequences to encourage students to use multiple solution methods. Donna stated that these word problems are not the same as modeling problems, because the students do not have to make assumptions. She hopes to move toward more open-ended problems, but for now, she feels the need to scaffold tasks and progressively reach modeling-type problems when the students get used to them.

Donna sees modeling tasks as open-ended problems that challenge students to make assumptions and think about multiple solutions, but they are beyond typical word problems: "I think the assumptions piece is what's different from the traditional math that teachers have been doing. I mean the math is still there, the concepts are still there, but [it is] a problem that's not so clearly defined […]." She differentiates between traditional mathematics problems and modeling tasks and hopes that teachers will incorporate more of the latter into their classrooms.

It was when I asked Donna which modeling tasks from the PD workshop were her favorites that she told me the funny story (mentioned above) about her friend who was trying to decide on a car to buy:

> And I [laughs] I just looked at her and said, 'That's a modeling problem.'…And I suppose I also like anything that will make the world a better place…like recognizing the toothbrush and the water use [Water Conservation task]. Anything that has to do with conservation or ecology.

The statement of the Water Conservation task is given here for reference:

> *A newspaper says that each family can save almost 7,000 gallons of water every year if they turn off the water while brushing their teeth.*
>
> *Is this reasonable? How can this be possible? List all assumptions and show all mathematical work that led to your solution.*

Clearly, making decisions is an important aspect of modeling tasks, as well as the topic and purpose of the task. Donna emphasized that some tasks require students to "live with" the decisions they have made for their model, and these decisions are related to non-trivial issues: "The ones [tasks] that are successful are the ones that have those types of issues that are either connected to the area that the kids live in or critical issues in the world." She talked about how some modeling tasks come from real-world situations, but are so mundane that they are not considered *good* tasks. For Donna, a good modeling task has to be related to a critical issue and require critical thinking and decision making. The Softball task [see Appendix B], in which the student must decide which players make the final team roster, was a great example of this:

> There are some modeling tasks that just ask, 'How long will it take to get there?' or 'How much water will be used?' or 'How much shampoo?' Those things I don't think are *as good* as ones where it's more of a decision making, which would point to the ones that I picked [as favorites]. You know, having to decide who's going to be on the team and why.

The statement of the Softball task is given here for reference:

> *The softball league president is looking for a fair way to select players for the All-Star softball team. He wants to use information from tryouts and scrimmages to come up with a system for comparing the performances of each player. He also wants to consider comments from the coaches about each player.*

While the question "How much water will be used?" can be seen as relating to the critical issue of water conservation, Donna focused on the decisions that need to be made and how lasting their effects are. If a person calculates that she uses more water during her shower than during a bath, she may still decide to take shower. This decision is not as critical because she will take another bath or shower regardless of the outcome. But, in the Softball Task, if certain players are selected for a softball team, then the team is set; there is no going back.

In sum, many mathematical tasks can be created from real-world phenomena, but not all of these tasks will require students to make decisions based off of a model and then live with those decisions. For Donna, modeling tasks are unique in their ability to help students develop the life skill of making difficult decisions.

Teaching Modeling

This section presents the teaching views with respect to mathematical modeling of the four teachers in this study. It is important to note that teachers *reported* their teaching practices as I was not able to observe them in the classroom. All four teachers felt that they needed to alter their primary role in the classroom to align with the teaching and learning practices of modeling. The teachers in this study were eager to talk about their new role as a mathematical modeling facilitator, but they were also open about the challenges they faced while trying to adapt their teaching styles and methods to this new type of task. Three of the teachers seemed to easily adjust to the new role, but the fourth teacher, Griff, admitted that the transition is a constant struggle.

The teachers also commented on the place of mathematical modeling tasks in their curricula, particularly in connection to the type of classroom in which modeling tasks can be done. Jamie and Griff only give modeling tasks to their enrichment students, while William and Donna try to implement modeling in standard mathematics classes. The teachers shared examples of times when their past experiences or personalities affected their facilitation and evaluation of modeling tasks.

William. Although William primarily uses a lecture-style format to teach mathematics, when it comes to modeling tasks, William sees his role altered to that of a facilitator who is seldom needed in order for the students to be successful. He gives his students very little instructions or guidance and lets them decide how to approach each modeling task, using any resources available. In the following quote, William discusses how this approach worked fine during the T-Shirt task, but not so during other modeling tasks:

> So, the very first time I did it [gave students a modeling task]…I refused to answer any mathematical questions and I refused to tell them what was available either. So if they asked to use a computer, the answer was 'yes.' If they asked to use a calculator, the answer was 'yes.' If they asked for paper, the answer was 'yes,' but I didn't tell them what was available. They had to come up with everything, which worked for that project, but in order to pull off the others [tasks], I wasn't able to do that; I had to actually coach them. … If I'd have actually had two or three days, I'd have let them suffer, so to speak.

William claimed that time limitations were a large factor in his classroom; namely, that he would have given his students more time to "suffer" or struggle with the mathematics before he gave them further guidance. Due to time constraints, he coached the students through the task a little more than he would have liked. The first modeling task that William used was the T-Shirt task, and he was able to give his students a lot of freedom in terms of resources and strategies with that task. The reason he stated it "worked for that project," but not others, may be that the T-Shirt task is a slightly less open-ended task, because there are some given constraints and (arguably) one correct answer. However, the students still need to research relevant information about t-shirt sales in order to make assumptions to justify their mathematics. It was during this research phase that William allowed his students to use computers to look up information.

Months later, after William had implemented three or four more modeling tasks, he tried a similar implementation strategy with a different modeling task. He gave his students the Shower vs. Bath task and the autonomy to research what they needed to know to solve the problem. However, this time he had different results: "I gave my students the task and a computer and 30 minutes later, they hadn't done anything. They were just sitting there wondering what to do. … I didn't tell them anything!" The statement of the Shower vs. Bath task is given here for reference: *In order to be part of the water conservation effort, some people say that one uses less water when taking a bath than when taking a shower. Others say this is not true! Who is right?* With the more open-ended Shower vs. Bath task, William's unlimited-freedom method seemed to be ineffective. The students struggled to begin the task, but William was persistent in his efforts to not give them too much information or guidance.

His strategy of minimal instruction goes beyond the initial steps of the modeling task; William also lets his students make decisions about which mathematics concepts and tools to use to solve each task:

> I didn't tell them to use unit rates or proportions, but most groups did, because that was the natural tool to use. What I'm trying to do is pull that knowledge out of them that's already been taught so they're actually making a practical use of it [the knowledge].

His rationale for giving the student mathematical autonomy is that by allowing students to choose unit rates, proportions, or something else, they are seeing opportunities to use classroom knowledge in everyday life. This particular quote referred to the Recipe task, which William created with multiple objectives in mind. He designed it so the students would connect classroom mathematics to real-world situations, as he explained in his views of modeling, and also specifically so that students could apply their recently learned conceptions of proportional reasoning:

> The goal [of the Recipe task] was to use the skills that they had already learned in class. Just to recognize that they should use the unit rate and proportionality to double, triple,

> [or] quadruple whatever recipe, and to think outside the box to realize that these numbers [for how many party guests] were not your usual multiples that you find in the times tables. I did that on purpose so that they would not have a perfect multiple from the times table; they had to round or decide, ok, do I have an extra brownie or do I lack a brownie? [laughs] That sort of thing.

As a part of his role as facilitator in the classroom, William does much of his work outside of class, designing tasks and preparing lessons during which the students do most of the labor. They are making decisions and coming to mathematical conclusions on their own while William listens and anticipates follow-up questions he can ask them. From William's own accounts, his facilitation appears to be more like observation as he shies from intervening even when students are confused and struggling.

William did not express a change in beliefs or practices regarding mathematical modeling. From the first day of our PD workshop, he seemed very open to modeling tasks and has since then continued to use them in his classroom. A precursory glance of the tasks he has used and designed suggests that he picks specific tasks that align well with his current curriculum. In this way, he has not needed to adjust his teaching practices in terms of content; instead, he has altered his method of presentation: giving his students a task with minimal direction or guidance.

William had a preconceived notion of mathematical modeling before beginning the PD workshop. He believed that modeling tasks were only accessible to older students, but later, changed his mind: "I mean, if there's anything that I learned in the seminar [the PD project] at all, was [that] modeling includes all sorts of different levels." After a few months of working with fellow teachers in the PD workshop, William decided that modeling tasks can be used at any grade level.

William is always very focused on whether or not his students reflect on their solutions to check for sense-making. While we discussed the Shampoo Bottle task, William talked about his family size and the importance of his own background knowledge for this problem. He stated that his immediate and extended family are very large, so when his students assumed a family size of three to complete the Shampoo Bottle task, he thought this was unreasonable. He did not tell the students that they were wrong, but he viewed their solution to be not as good as other students due to his personal experience.

Jamie. Jamie frequently arranges her students in groups in her enrichment classes. At the beginning of the semester, she established group norms and formed the groups in which they would be working. She sees herself as a guide who is constantly probing her students to ensure they understand each task and are justifying their mathematical work. She repeated multiple times during our interview that “the assumptions have to justify the mathematical work, and the math has to justify the conclusion.” As an instructor, she tries to guide her students through the critical thinking process so that their answer makes sense realistically and mathematically, but this is not accomplished immediately: “For me, teaching the modeling is kind of, let the first [modeling task solution] just be what it is, and then use it to try to build on that critical thinking.” She believes that this critical thinking process begins at the assumption-making stage. At first, students may not know what assumptions are necessary to construct a model, so Jamie’s role is to ask questions and guide them toward reasonable decisions:

> But you have to let them put those [assumptions] in in the beginning, and then, this is something I learned after the first semester, the silly assumptions have to stay in at first,

> and then you get them to step back and say, 'Ok, which one of those assumptions is *really* not necessary to help you solve the problem?'. … For this particular age group, for middle school kids, you just have to let them use their silly imaginations and then look at their own silly imaginations critically, and say, "Oh yeah, that doesn't [work]; that's silly.'

When groups report their solutions to the class, Jamie always asks the class which assumptions are relevant, and the students presenting get a chance to explain why they made particular assumptions. If the class decides that some assumptions on a group's poster are not necessary, then together they cross off those assumptions. Constantly redirecting and focusing the students on the justification of assumptions made is one of Jamie's key roles in her classroom. Jamie views each modeling task as an opportunity for a class discussion of the mathematical concepts, rather than a teacher-centered, lecture-style instruction:

> Sometimes when I'm listening to a group, I might guide the conversation a little bit or maybe, just a tiny bit – really, just a tiny bit. The other thing is that sometimes a group will ask me a very poignant question and so then we have a 'freeze:' everyone has to stop talking. I sort of answer the question to the whole class. I'll be like "Oh, group five brings up a good point," but that's it! That's all I do. I love it!

Jamie emphasized that her role in the classroom has changed with the implementation of modeling tasks. She feels like she "doesn't do anything," but instead, walks around listening to and facilitating student conversations: "But what I was amazed at is the part where I anticipated, I thought I would have to do a lot more work. I really don't do anything. I just walk around and watch…don't tell my principal! [laughs]." Although the prospect of having "less work" to do during class is appealing, her new role has challenged her to "let go" of some of her control and authority in the classroom while the students work on their own: "[Y]ou cannot solve problems until you develop critical thinkers. And if I just give them the answer every time, I'm not developing critical thinkers. So it's hard. This has been *really* hard for me."

Again, she reiterated that her role is a guide during the process of modeling and not a lecturer who dictates the instructions and mathematical information from the front of the classroom:

> But I felt that, if I tell them at the beginning of every task, you must do this first, then I don't think I'm really allowing for authentic learning of modeling to take place. I have to let them just *go.*

While acting as a guide, sometimes Jamie has to let her students take the lead. Her role as a modeling task facilitator, which is different from a traditional mathematics lecturer, is new and challenging, but has great potential to promote critical thinking in her students.

Jamie's beliefs about teaching align well with her new role as a mathematical modeling facilitator; however, some of her practices have changed. She stated that she has had to "let go" of a lot of the teaching methods that she has used for the past 14 years as an educator. Student assessment is one of the areas in which her methods have changed:

> I think that's another important part; if I want to continue to encourage them to [do] that whole thought process of math modeling, I have to let them make mistakes and not penalize them so much that they don't want to do it anymore.

Jamie realized that the freedom she gives to her students during the solving of the task must be extended to the evaluation of the solution. Students are not willing to be creative and explore multiple solution methods if they think they are going to be penalized for not completing the task a certain, predetermined way. Sometimes, this means that the student will make mistakes, but this is a small price to pay if they are truly engaging in the modeling process and thinking critically about the various mathematical tools they can use. Oftentimes, the students are not as much making mistakes as they are simply focusing on the wrong things. Because her students are 12-14 year-old kids, Jamie acknowledged that they can get distracted by opportunities to express their creativity. She discovered this when her students spent more time making elaborate posters for the Milk Cow task [see Appendix B] than they did solving the task:

> [T]he posters have to start to become less about pictures of cows – which they love – but they have to become also more precise mathematically. And so I'm now learning that if I want to see some really good math, I almost have to have them create two separate posters [one that is more artsy and creative and one that is more mathematical]. And then I also discovered as I was going about this, that some of the tasks were really straightforward, like there's just one right answer. So [for those tasks] I required them only to make a poster that was mathematical, absolutely no puffy paint…

The statement of the Milk Cow task is given here for reference:

> *It is recommended that children aged between 2 and 12 drink 2 glasses of milk a day. How many cows are needed to supply every child in your family with the amount of milk recommended each day? What about for each child in your class?*

Jamie said that the key to consistently incorporating modeling tasks into mathematics lessons is to be flexible. When she saw that her students were missing the point of the tasks, which was to learn and apply mathematical knowledge, she adapted her format so that the students were better able to focus on the lesson's objectives. She did not completely throw out the element of creating posters since she knew that her students enjoyed this feature and she also liked having them report out their solutions. This is a practice that Jamie added to her teaching routine when she adapted her enrichment curriculum to include modeling tasks. She described the effect that the modeling PD workshop has had on her as an educator as “life-changing.” And although she readily admitted that changing her teaching practices from the past 14 years has been really hard, she thinks that teaching mathematical modeling is good for the future of K-12 education:

> Here’s how math modeling changed the way I teach, to this extreme. I have one enrichment class and my other five classes are regular math classes, two honor and three regular eighth grade. So based on this, I created…a curriculum that would be an entire semester long. Some of the stuff you gave us [PD participants] and some of the stuff I’m writing. And then I sent a formal letter to my principal requesting that I have four enrichment classes next year and two regular math classes. So I requested…to become the enrichment math teacher for my school based on mathematical modeling. And [he] said yes.

Not only does Jamie solely teach modeling in her current enrichment class, but she now will also be teaching multiple enrichment sections next year!

Jamie is on board with modeling tasks in general, but there are other ideas that she still needs to ponder. Usually, Jamie employs a modeling task to drive home a mathematical concept that has already been taught, but during our interview, she contemplated using a task to introduce a concept, such as linear functions and the slope-intercept equation for a line: “Could I use a task in our group to introduce y=mx+b? Never thought about it. So, I could try that.” Her perceptions of the utility of modeling tasks are still evolving; she has already changed many of her practices to adapt her curriculum to incorporate mathematical modeling, and still more practices may change.

During one of the very first PD sessions, Jamie shared that she bought her purse at the local car wash. This seemed like an insignificant piece of information at the time, but it came up later when she let her own students work on the Car Wash task. She mentioned in our interview that she was disappointed that none of the students talked about the definition of a customer in their assumptions. When she bought her purse at the car wash, she considered herself to be a "customer" even though she did not wash her car. When her students completed the task, they considered only people who got their cars washed to be "customers." Jamie shared this oversight with her students after they had reported their solutions to the entire class, asking them to think about what other kinds of products one can buy at a car wash. She later told me that when she uses this task again, she wants to first have a discussion with her class about how to define a "customer." This seemingly off-topic discussion is important to Jamie because of her personal experience as a customer of various products at her local car wash.

Griff. Griff sees his role as a mathematical modeling instructor as a balancing act. On one end is the more traditional approach to teaching problem solving that he has used successfully for years, and on the opposite end is the new role of teacher as guide who only intervenes when necessary. He enjoys facilitating modeling tasks and is working hard to give students more freedom as they solve the tasks, but he acknowledges that it is difficult to be so hands-off, especially when students approach the task differently than he would have:

> Some looked at [the task] differently and it was hard. That was hard for me as I walked around and I saw that. There was part of me that wanted to say 'uhh…' and I had to bite my tongue, because, you know, then if I started talking, I'm going back to the old style [of teaching]. … I definitely had to bite my tongue at times, wanting to say or do more and just kinda sit back and let them work their way through it.

Griff guided students through the launch of the T-Shirt task, so they could invest themselves in the context and get “warmed up to the idea or concept of buying something.” Then he “cut them loose” with the basic information needed and “let them go with it” while he walked around and engaged the students in conversations about their mathematical work. Even though he eventually left the students to their own devices during the task, he still felt a need to initially motivate students to engage in the task by presenting a formal introduction and warmup. After the setup, he feels more comfortable letting the students go in whatever direction they choose.

He emphasized that his job as a teacher is to let the students discover the mathematics, rather than be taught through direct instruction: “I did very little. I just walked around and maybe asked a question here or there, but that was it.” He views modeling facilitation as doing little work outside of observing and asking follow-up questions, even though it does not come easily to him. He also acknowledged that teachers have so much pressure to teach for the standardized tests that they often do not have the time to do many in-depth tasks.

> I sometimes feel like I have to push, push, push, push, push, when I really wish I could slow down and let kids do more of this [modeling], ‘cause I think this has such a benefit and it’s actually going to make mathematical thinkers out of them, instead of mathematical doers. It’s that internal struggle and actually using [modeling tasks] and implementing [them] and feeling comfortable enough knowing that it will be ok.

Griff distinguished between “mathematical thinkers” and “mathematical doers” because he strives to develop critical thinking skills in his students, rather than create automatons who are fluent at rote mathematical calculations with no meaning. And he claims that modeling tasks are the means to get students to this level of critical thinking, even though he is often uncomfortable spending so much time on longer tasks when he needs to prepare students for state-mandated exams.

Griff's first attempt to change his teaching practices in the context of a modeling task came during the Shampoo Bottle task. Although it was contrary to his typical policies, he allowed his students to use their cell phones during class "so they could dig up some data about how much [shampoo] is normally used and this type of thing." It was very challenging for him to give this freedom to his students, but it was even more shocking for them:

> I have a reputation as being the guy who nails everybody for having cell phones out so I think when I said that [students could use their phones], it probably floored them. I mean, they were shocked that I was gonna let them.

Griff mentioned this incident at the PD session and said that deciding to allow the students to research information using their phones was a tough decision because he usually has such a strict cell phone policy in his classroom. He also said that he is not opposed to cell phones in general or to technology in the classroom, but students often abuse their right to use these resources. In the end, he admitted that it was not as hard as he thought it would be to relinquish his firm policy on phones:

> I think sitting in the [PD sessions] with you all and doing that [using phones during modeling tasks], and realizing how helpful and important that was made it easier for me to do it in the classroom. Now, I'm trying to embrace this whole idea [of using technology in the classroom] – even though I've only been teaching for 17 years, you know, I've been around for a while and I come from a different time and a different era, you know, so I'm grasping *slowly* and I'm moving into the technology and the good of it.

Griff does not have issues with technology in general; his hesitancies appear when he has to incorporate technology into his teaching routine. It was not until he personally experienced the positive effects of using cell phones during a mathematical task that he considered doing it himself. While Griff showed the most resistance to changing his teaching practices, he hinted at a possible opening when he mentioned that he saw "the good in it": "You know, you do something for a certain way for so long, it's hard to change that thought process, even though I see the good in [using modeling tasks]. I absolutely see the good in it."

For Griff, the problem is not that he cannot see benefits to mathematical modeling; his resistance comes from many other logistical and circumstantial issues at the local and state levels:

> [B]asically the problem I have, I think this [modeling] stuff is awesome. I absolutely believe that this is good stuff and certainly is very beneficial. It's implementing it to the point where I'm comfortable knowing that by doing this…my kids are still going to, at the end of the year, be in good shape and know what they need to know to move on to the next grade. Also, to – and I hate to say it – but to pass the…state tests we happen to have that year. There's that problem right there and it's been an ongoing issue.

Griff describes himself as someone who is slow to change his beliefs and practices, even though he has been trying out new ideas and approaches in his classroom. He is struggling to align mathematical modeling with the state standards, and claims this is the biggest challenge for him in his new role of modeling facilitator.

During one of the first PD sessions, Griff worked on the Papi's Birthday task with a few colleagues. When his group shared their solution, Griff emphasized that his personal family life influenced how he solved the task, leading him to weigh in non-quantifiable parameters with mathematical work while creating the model. The purpose of the task is to equitably distribute the costs and labor of organizing a birthday party. The specific context of this task is that the party for a 60-year-old father and the subjects of the scenario are his three grown children, trying to coordinate with one another long distance. Griff instantly felt connected to this task because he has siblings and has previously found himself in a similar situation. He naturally related to the educator (in this case, a university professor) in the scenario, and expressed that he knows most teachers do not have a lot of extra money lying around to spend on DJs or food for parties. He also felt great sympathy for the single mother who had limited time and money to give. In the end, his group decided to create a model illustrating which sibling would contribute which resources based completely on personal experience and empathy rather than on the numbers given in the task. Students completing this task are expected to use proportional reasoning and/or percentages to find a fair arrangement. Griff and his group created a model that did not include the single mother in the monetary contributions, even though they could not justify this with mathematical work. They decided that the other two siblings should split the cost for the party, and the mother can help out with the logistics and decorations since she is (assumed to be) located in the same place as the party. For this particular task, his personal views dramatically influenced how he used mathematical modeling to find a solution.

Donna. Donna is a unique case because she does not teach and interact with her own students on a daily basis. Her view of the role of a mathematical modeling instructor is projected onto how she coaches the mathematics teachers at her school. In particular, she coaches one high school mathematics teacher at their school who opted to not participate in the PD workshop on modeling. She focuses on teaching strategies that are easily incorporated into a modeling lesson, such as group work, multiple solution paths, and increased student engagement: “I’m not so much focused on the modeling but on the engagement piece of getting more students to talk about what they're doing and thinking, and move away from direct instruction."

One major area of concern for the high school teacher Donna coaches is that modeling tasks do not always address one specific content area, which then creates challenges when it comes to checking off state content standards that were covered in class:

> [O]ne of the pushbacks I get from [the high school teacher] is he wants to know, 'Well, what standard are you actually working on?' We have so many standards in high school and it's just impossible to get them all covered. So he thinks it's kind of like a waste of time or taking away from this list of standards that he's got to get through. So another factor I've looked at is showing that when you do a rich task, that you're pulling several standards together and even the math practices, and it's not just about one particular thing that's going to be on [the state standardized] test.

Donna disagrees with her colleague, claiming that richer tasks provide for more cognitively demanding learning opportunities than the problems that are typically employed to address state standards. Her roles as a teacher coach and curriculum specialist are to encourage teachers to promote higher-order thinking, which she claims can be achieved through modeling tasks. She is "hoping to build a culture of students" that are willing to engage in more open-ended tasks, because it allows for teachers to cover multiple state content standards with the same task.

> I'm seeing that…teachers are starting to recognize it's not just about preparing the students for the test and…I think they're also starting to see that when they give their kids the challenging tasks, they're…putting them in the situation where they have to think. And you may miss one of the standards in your teaching, but if you've taught them how

> to be good thinkers, critical thinkers, then they have a better chance of working through something, even if they haven't mastered a particular skill.

Donna believes that teaching thinking skills is more important than teaching specific calculation-based skills that geared toward exam preparation. But increased critical thinking is not the only benefit of modeling tasks. Donna focuses on multiple attributes of modeling in the classroom, including student engagement. When Donna sees how excited students get as they engage in a modeling task, she reciprocates this emotion and high energy:

> [T]he energy that happens when you're doing one of these tasks, the engagement is so much higher. And, it's like the students are more engaged, so I am more interested and engaged and…it all feeds in a positive way. I have a pretty low affect or calm way of doing things. … But when I get into a problem, and [talk] about, 'Oh cool!' how that works, and it just builds and feeds and the time passes in class and people feel good and I feel good. So that energy is like palpable, and I try to talk about that afterwards.

As an instructional coach, her role as a modeling facilitator is slightly different than the other teachers, but Donna has still witnessed newly emphasized elements in her teaching that were not apparent when she coached traditional methods of instruction.

Donna did not express any resistance to the concepts of mathematical modeling or to using this type of task in the classroom. As an instructional coach, the biggest resistance to change that she has witnessed comes from the high school teacher whom she coaches. She mentioned that he thinks mathematical modeling is a "waste of time" because it does not address a particular state standard. Donna believes that modeling tasks can address multiple standards and has adjusted her teaching style to incorporate more tasks that involve various content domains and promote multiple solution methods. As a mentor and coach, she has had to constantly emphasize the benefits of modeling, either through her example as an instructor or by relaying the information she has learned in the PD sessions:

> I'm not as successful with the high school teacher because he's an old-fashion: get up there, and [the students] take notes and he does his thing. When I model for him, he says, 'Well, that was interesting, *but*…' [Donna laughs]. And the hard thing is – as hard as I try – they're not my students, so it's really hard to, to get them engaged… So I'm just going to keep trying and sharing.

When we first began the PD sessions, I remember that Donna would ask a lot of questions about the objectives for modeling with mathematics. She was very focused on what mathematics would be covered in the tasks and how those content areas aligned with the CCSSM standards. As the year progressed, she became less critical of tasks and more willing to use them in conjunction with different content domains. Her views of the benefits of modeling gradually changed throughout the year and she slowly incorporated modeling tasks into her curriculum. As the curriculum specialist for her school district, this is a big change not only for her, but also for the teachers she coaches.

Donna expressed that she is not a "super bubbly" person, but modeling tasks make her more excited about teaching certain mathematical concepts. She described herself as having "low affect," meaning she is not excitedly jumping around the classroom and pumping up her students while teaching. She has a very calm demeanor, but when she spoke about her friend's real-world modeling opportunity when choosing a new car, she became very animated. She said that her demeanor changes when she teaches modeling tasks, and her students become more engaged as well. Because she is able to "get in to a problem," her excitement for the task and the energy level in the classroom is "palpable." Although disposition toward teaching mathematics in general has not changed, Donna's views about the "coolness" of modeling influence her demeanor while facilitating modeling tasks.

Teacher	**Grade, Type of Class (number of sections) Taught**	**View of Teaching Mathematical Modeling**	**Role while Teaching Mathematical Modeling**
William	7th, regular (5)	Real world-centered	Facilitator who gives students complete autonomy and few directions
Jamie	8th, regular (3), honors (2), enrichment (1)	Content-centered (conceptual)	Guide who constantly redirects and refocuses students on the mathematics
Griff	8th, regular (4), enrichment (1)	Student-centered	Guide who infrequently intervenes but lets students use technology as resources
Donna	District instructional coach	Content-centered (conceptual) and real world-centered	Coach who motivates cooperating teachers to use modeling tasks

Table 3: Summary of teachers' duties, view of modeling, and teaching roles.

The findings presented here describe the mathematical modeling views of four middle school teachers and how these teachers implement modeling tasks in their classrooms. All four teachers have different views of what modeling is, what constitutes a model, and how to best facilitate this method of teaching mathematics. In the next chapter, I will discuss the teachers' views through a theoretical framework created specifically for this study and look for similarities and differences across the four case studies.

Chapter 6: Discussion and Implications

As seen in chapter three, I developed a theoretical framework for this study building on several relevant frameworks. The framework for this study consists of four teaching views in the context of mathematical modeling. These views span three foci – content, student, and real-world – with the content view divided into conceptual and calculational subcategories. The four mathematical modeling teaching views are listed here:

Content-centered (conceptual): mathematics teaching and curriculum design centered on conceptual understanding of content and development of critical thinking skills;

Content-centered (calculational): mathematics teaching and curriculum design centered on procedural knowledge of content and fluency with mathematics calculations;

Student-centered: mathematics teaching and curriculum design centered on the background knowledge and interests of the students;

Real world-centered: mathematics teaching and curriculum design centered on the incorporation of real-world events and issues into the classroom.

The conceptual content-centered view emphasizes a deeper understanding of the mathematics involved in a modeling task. A teacher with this view may urge her students to think critically about their assumptions and conclusions and how they support or are supported by the mathematics. The modeling objective for a teacher with the calculational content-centered view is for her students to perform the necessary computations to correctly find a solution to the task, which aligns with the "calculational orientation" in Thompson (1992). A teacher with this view stresses accurate computations and chooses modeling tasks that allow students to practice particular mathematical algorithms and operations. A teacher with a student-centered view is concerned with what engages students during class or allows them to use their background and non-mathematical knowledge that they bring into the classroom. Such a teacher would be constantly adapting his lesson to address the needs and interests of the students. In the case of modeling, this view is held by a teacher who selects tasks based on students' interests and presents them in a way that helps the students to form a connection with the topic of a task. A teacher who is real world-centered will see her classroom as a way to connect the mathematics of the classroom to issues in the students' everyday lives. This teacher will choose modeling tasks that are rooted in realistic and critical events.

The lens of mathematical modeling teaching views allows me to analyze the instructional choices and methods of the four teachers in this study in order to discuss their views of modeling. The following discussion sections are organized according to the research question they address. These sections analyze the views and teaching practices of all four teachers in the study, taking data from the case studies presented in the previous chapter. After addressing both research questions, I propose possible teaching and research implications of this study.

Views of Teaching Mathematical Modeling

This section addresses the first research question, which asks about the teachers' views of mathematical modeling. First, the teachers' views were examined using the theoretical framework of this study. Then, these views were analyzed to determine how they are manifested as the teachers implement mathematical modeling. All four teachers mentioned specific elements of modeling that they emphasize as they teach and why those elements are important. As Borromeo Ferri (2006) found, these elements can give indications to the teachers' views of mathematical modeling.

The teaching views of William, Jamie, Griff, and Donna in the context of mathematical modeling can be categorized based on framework of this study. Note that the categorization of views is not exclusive; a teacher may have multiple views or a combination of them. William has a real world-centered view of teaching modeling. His actions are motivated by his desire to create a classroom experience that is similar to “the real world.” Jamie has a content-centered view with an emphasis on conceptual understanding. She stresses the importance of critical thinking to her students as they complete a modeling task. Griff has a student-centered view when teaching modeling tasks. He is cognizant of his students’ interests and works hard to launch each task in a creative, engaging way. Donna has a combination of views; she pays considerable attention to how each task aligns with the curriculum while also choosing modeling tasks that are rooted in realistic phenomena. In this way, her view is both content-centered and real world-centered. None of the teachers reported teaching practices that indicated a content-centered view with an emphasis on calculations. Again, the reader should keep in mind that all teaching practices of the participants were self-reported.

Teachers’ definitions of modeling and their views of teaching. The four teachers illustrated a subtle distinction between *views* and *definitions* of mathematical modeling. Three of the teachers, William, Jamie, and Donna, had very clear views of modeling, and they were able to articulate their views in a way that aligned with a particular definition of modeling. For William, this definition was that modeling is a “real-world example.” His other descriptions of modeling and his views about how modeling should be taught and assessed also aligned with this definition. Jamie shared a concise definition for mathematical modeling in her own words: “application math.” Notice that, as detailed in the findings, William and Jamie mean essentially the same thing by their definitions, but their choice of words reflect that they may have different orientations towards modeling. William bases his definition on the ability of modeling to bring real-world events into the mathematics classroom as an educational activity. To him, modeling reveals an "example" of the world to which students are then able to apply their mathematics knowledge.

Donna defined mathematical modeling as "working on problems that are more realistic that don't have everything defined for them." Her definition is consistent with her view that modeling tasks should involve decision-making in the context of critical issues. Donna has a content-centered view of teaching modeling that is apparent from her efforts to align all word problems and modeling tasks with the curriculum. She often uses non-model-eliciting word problems instead of modeling tasks because they encourage group work and multiple solution methods, but are easier to adapt to cover particular standards. It appears that she is more concerned with the mathematical content, which can be covered using traditional word problems, than with implementing model-eliciting tasks. This could be partly due to the resistance she received from the high school teacher with whom she worked closely. She is unable to implement modeling tasks as much as she would like, so her compromise is to use group work and problem solving. Her position as an instructional coach and curriculum specialist may also influence what she does in the classroom. When Donna does teach with modeling tasks, she selects tasks that focus on critical issues and require important decisions to be made. This is where her real world-centered view comes into play. Even though some word problems can encourage group work and problem solving, one of the major advantages of modeling tasks is their ability to force students to make a decision and justify their calculations based on those decisions.

Meanwhile, Griff did not express as clear of a definition of mathematical modeling as the other three teachers did. There are many features of modeling tasks that he can list; he especially likes the complete process of the modeling cycle, but he struggled to state a concise definition of modeling. But even though Griff did not have a clearly-stated definition of mathematical modeling, his view of teaching modeling can determined to be a student-centered view. Griff also exhibited elements of a content-centered view that emphasizes the conceptual understanding of the mathematics. He frequently spoke about the importance of critical thinking skills and the entire modeling process. When evaluating student work, he commented on the students' thought processes rather than the procedures and calculations they used to find a solution.

Griff especially appreciated the personal connection that he made with the topics of some of the modeling tasks. For example, Griff repeatedly mentioned Papi's Birthday task and how this task created an excellent opportunity to incorporate non-mathematical decisions into a mathematical task. When Griff launches modeling tasks to his students, he claims to present the tasks in a way that piques his students' interests and helps them to place themselves into the situation of the task. He argues that this makes the tasks engaging and students are more willing to perform the mathematical duties when they feel personally invested. Analyzing his descriptions of these classroom practices, it seems that Griff strives to create a student-centered classroom experience, and that mathematical modeling tasks are helping him to achieve this. In his classroom, the students have taken ownership of the mathematical content via the modeling tasks and are better suited to acquire new mathematics knowledge using their personal experiences and resources. Some students used familiar objects such as tablespoons to estimate the amount of shampoo and later converted the tablespoons to ounces or milliliters. When Griff's students were allowed to use their cell phones to research how much shampoo the average person uses during a shower, they were able to use their own resources and draw upon past experiences to connect to the task.

It is worthwhile to compare the four teachers in terms of their teaching views. They used many of the same modeling tasks and taught essentially the same mathematical content, but their methods depended upon preference, style, and goals. While William and Jamie advocated critical thinking at the end of modeling tasks, namely reflecting on solutions to check for reasonableness, Griff pushed his students to think critically during the entire process of completing the modeling tasks. This subtle difference is likely due to the teachers' purposes for using a modeling task.

It could be argued that William and Jamie have a student-centered view because they also allowed their students to research information and use multiple resources; however, they did not mention spending time on the launch so that students felt personally invested in the task before beginning. The reason for this could be explained by the purpose of the tasks and the circumstances surrounding them. Griff and Jamie use modeling tasks in their enrichment classes, which have arguably more freedom in terms of material covered. William implements modeling tasks in his standard math classes where the schedule is stricter and the content of tasks must fit into the curriculum. Griff may have the ability to spend more time on the launch of the task in his enrichment class, while William has a limited amount of time with his students every day and a list of standards to cover. The restriction of having to teach specific content for the standardized exams was one of the biggest reasons that Griff only used modeling tasks in his enrichment class. Jamie also teaches modeling only in her enrichment class, but she reportedly spends a lot of time at the end of the task on the poster design and reporting stages. It was not apparent that she spent a similar amount of time at the beginning of the task, ensuring that all students feel a personal connection to the content. This does not mean that William and Jamie do not have a student-centered view of teaching mathematics; on the contrary, they seem to be very aware of what their students prefer or resist doing. However, specifically in the context of mathematical modeling, William and Jamie do not focus on the personal investment that their students stake in a given task as much as Griff does. William and Jamie focus on the content and real-world components of the task, while Griff takes care that his enrichment students are engaged.

Elements of mathematical modeling most emphasized while teaching. Mathematical modeling is a complex process containing multiple steps and iterations. In the PD sessions, the participants discussed and explored various versions of the modeling cycle extensively. Their views and teaching preferences determined not only how they taught mathematical modeling as a whole, but also which elements of modeling they emphasized. The first element that the teachers in this study mentioned occurs even before students see the task. Naturally, the teachers were concerned with the design and purpose for using the task. William and Donna stated this most clearly. William boldly claimed that "if [the task] doesn't have a purpose, conclusion, reflection, it's lacking." Donna did not like tasks that did not address a critical issue in society, so the purpose – here, solving one of the world's problems – was very important to her. Once a modeling task has been selected, the next step is to present it to the students so they can understand the objectives. This step is often referred to as the "launch" because it determines how students will "take off" with the task. Griff places emphasis on the launch; he gives his students time to think about the task and process what their goal is. Griff and Jamie repeatedly mentioned the importance of making valid assumptions at the beginning of each modeling task. Both teachers claimed to evaluate their students' mathematical solutions on how well they were justified by the assumptions. Griff and Jamie were also able to check for conceptual understanding by assessing the reasonableness of assumptions. Recall that Jamie admonished her students for not considering the nuances of being a "customer" at a car wash. She was focused on how her students were thinking critically, but not on what personal background information they brought to the task.

It is interesting that William did not stress the assumption-making step of the modeling process to his students even though he has a real world-centered view. Assumptions, both implicit and explicit, are pervasive in everyday situations and are necessary for making important life decisions. As he said, he evaluates every task in terms of its purpose, conclusion, and reflection. A follow-up question for William would be, on what should the students be reflecting? It may be that William intends for his students to reflect on their assumptions once they reach a conclusion, but this was not explicitly stated. The purpose of a task conveys its goal, and the conclusion is a claim that this goal has been achieved. The reflection step for William's students is the crucial moment when they look back on the purpose of the task and decide whether or not their conclusion has answered the original question and completed the task. The modeling process as William describes it does not seem to emphasize assumptions, but they may be implicitly included in the reflection phase. This does not mean that William ignores valid assumptions; on the contrary, his view of modeling as rooted in real-world problems mandates that all assumptions are realistic. However, his view on the importance of assumptions is not necessarily expressed to his students during the implementation of the modeling tasks. He mentioned that his plan for the next modeling task was to focus on the just the assumptions of a task, even if the students did not get to completely solve it.

Donna's unique position as instructional coach, curriculum specialist, and modeling PD participant reveals a unique perspective of teaching mathematical modeling that is not given from the other three teachers in this study. In addition to the purpose of each modeling task, Donna also emphasizes the decision-making aspect and the collaboration of students while working on a task. These two features of modeling align well with her view of teaching modeling as a combination of content-centered and real world-centered. Because Donna coaches other teaches at her school in terms of pedagogy, she focuses on encouraging group work and multiple solution methods from students. Similar to William, she appreciates when tasks are based on realistic events, especially when they require students to make critical decisions.

Toward the end of a modeling task, Griff said that he pushes his students to create a generalized model to illustrate their solution. This usually takes the form of an equation with one or several parameters. None of the other teachers stressed this element of the modeling process to their students, although at the time of her interview, Jamie was just beginning to help her students create generalized models. Jamie said she spent a large of amount of time after each task making sure her students validated their conclusions and reported their solutions to the class. The class acted as peer reviewers, offering constructive feedback to the presenters about the reasonableness of their solution, the organization of their poster, and even the accuracy of their mathematical steps. While Griff also typically let his students create posters for each task, Jamie insisted that all work be displayed and justified on a poster. It was almost as though she felt designing and sharing posters were integral parts of the modeling process. This will be discussed at greater length later. I now address the second research question, which leads to a discussion about a teacher's role during the facilitation of mathematical modeling tasks and how this role aligns with the teaching views discussed above.

Role of Teacher During Modeling Tasks

This section addresses the second research question regarding the role of the teacher during planning and implementation of modeling tasks. First, I discuss what the new roles of teaching mathematical modeling look like and how these new roles align with teaching views.

As roles change, a theme among the teachers was that they felt they were doing nothing. I discuss the difference between when a teacher "does nothing" and when a teacher is facilitating an engaging class activity, such as a modeling task.

Role as modeling facilitator aligns with teaching views. Do the teachers' professed views of modeling align with the role that they assume in class? All four teachers spoke about their roles changing to guides or facilitators who have to relinquish some of their authority in the classroom. This was a common theme across all four participants, but the degree to which they reported a change varied from teacher to teacher. William gave his students complete freedom when working on modeling tasks to the extent that they seemed to be sometimes unsure of what to do next. He tried to not intervene even when students asked specific questions about how they should proceed. Although he let the students make choices, they seemed uncomfortable and even ill-prepared to do so. He stated that he wanted the students to choose which mathematical concepts and tools to employ so that they could "make practical use" of the knowledge they have already acquired. William's practices in these episodes align with his real world-view of teaching mathematical modeling. In everyday life, people make difficult decisions and apply their knowledge from prior experiences to situations where the next step is uncertain. William is striving to create an environment where this *everyday* reality is also a *classroom* reality.

Jamie did not give her students as much free range as William did. She reportedly guided and redirected her students during the modeling process to ensure that they were on the right track. But, as she acknowledged, her enrichment students still had more freedom than her students in the standard mathematics classes do. The reason she constantly redirected her class was to emphasize certain content-related and conceptual nuances that she assumed they missed while working on a task. Her view was very content-centered as she guided her students. Griff tried to find a balance between telling the students too much and trying to let them work on their own. This was often a difficult balance to maintain, but Griff erred on the side of “biting his tongue” rather than telling students what to do. Aligning with his student-centered view, he chose to “let the students go” rather than directing their actions and decisions. This was only possible after he helped his students become personally invested in the objective of each modeling task. Donna is trying to get “more students to talk about what they’re doing and thinking,” while she moves away from direct instruction and toward collaborative learning strategies. Because she teaches as a guest in other teachers’ classrooms, she does not get to implement modeling tasks as much as she would like, but she finds these tasks to be a great way to model effective teaching practices to her mentees.

“Doing nothing” versus facilitating. Jamie mentioned in almost every PD session that she loves using modeling tasks in her classroom because she “does not have to do anything.” This idea of a teacher “doing nothing” reminds me of the words of Cohen and Lotan (2014) about teachers who use groupworthy tasks in their classrooms. They claimed that teachers often feel like they do nothing during these collaborative tasks, because the students are the ones doing the mathematics (as it should be) and the teachers are now facilitating instead of teaching via direct instruction. The key here is to note the difference between “doing nothing” and facilitating. Facilitating is about observing, listening, anticipating students’ next moves, and thinking about questions to ask to follow up. As Cohen and Lotan mentioned, much of a teacher’s work for group tasks is done outside of class, but this does not mean that they remain idle during class.

William stated that he “didn’t tell his students anything” when they were confused how to proceed with a task, which might have been a slight exaggeration on his part. There is a difference between “doing nothing” and not intervening, the question is which one William actually meant. Griff also said that he “did very little” while his students worked on modeling tasks and downplayed his role as facilitator: “I just walked around and maybe asked a question here or there, but that was it.”

Three of the teachers, Jamie, Griff and William seemed to share this idea of “not doing much” when teaching modeling. It may be worthwhile studying further how teachers view their role when teaching modeling versus teaching “regular” mathematics. In this section, teacher roles in the context of modeling are clearly enacted and articulated by the participants. What is not clear are their motives for these actions. In the next section, I revisit the discussion of teachers’ views of teaching modeling by addressing an emergent question.

What Is Inherent to Teaching Modeling?

The findings chapter presented data illustrating *what* teachers do during modeling tasks; here, I discuss *why.* It is apparent that the four teachers in this study use different teaching practices when they teach via modeling tasks than when they teach a typical mathematics lesson. The new techniques include student group work, increased student autonomy and use of technology, limited instructor intervention, and poster design with student presentations. These techniques were not only new, but oftentimes challenging for the teachers to incorporate into their routine. Some of the teachers even reported resistance toward these practices, such as Griff’s reluctance to let his students use their cell phones in the classroom. Additionally, the students also were resistant to some of their teacher’s new methods. William’s students outright refused to work in groups, which is why he designed the Recipe task as pair-work.

It is unclear *why* the teachers employed the practices that they did while teaching mathematics through modeling. Are the teacher moves shown in this study inherent to modeling? Or did they choose these practices based on what they saw the facilitators do during the PD sessions because that is the only way they have seen modeling taught? These questions address what teaching practices are fundamentally tied to the concept of mathematical modeling. If the new roles that the teachers exhibited are not an intrinsic part of modeling and are instead a result of imitation or preference, then these methods could be used to teach any mathematics topic. If this is the case, then the teachers could have still taught their students with this same approach regardless of the content of the PD session. The PD instructors did not specifically talk about pedagogical practices to use with modeling tasks besides group work strategies and some elements of Complex Instruction. Are the teachers changing some of their practices to align with collaborative learning strategies rather than specifically mathematical modeling? Unfortunately, the question of which teaching practices are inherent to modeling and which are a result of preference or imitation cannot be satisfactorily answered. Below I discuss teaching practices of Jamie and Griff in an attempt to address this question further.

Jamie ended every modeling task by having her students create a poster and share their work with the class. She did not have her students in the regular mathematics classes design posters for their work, even though there were probably some projects and units that could be displayed on a poster and shared with the class. At first glance, it appears that Jamie considered poster design as necessary for modeling tasks, but not necessary for other methods of doing mathematics. However, it is also possible that Jamie liked how the facilitators of the PD sessions always had the participants report out their solutions by creating and sharing a poster. Jamie mentioned that her students love creating posters; in fact, sometimes they got a little too focused on the artistic component of the poster and forget about the mathematics. Why did Jamie not let her other students, who are not in the enrichment class, design posters for their work? Was it an issue of time? Designing posters and presenting does take time, and it may be that Jamie could not afford to give up this time in her regular mathematics classes where the schedule is much stricter. Another possible explanation is that Jamie watched the facilitation of poster presentations during the PD workshop and knew no other way to facilitate modeling in her own classroom. Jamie imitated exactly what the instructors did during the PD sessions; she taught her students using the same tasks, lesson structure, and teaching techniques that the instructors used to teach the PD participants.

Griff reported two key practices that distinguish his facilitation of modeling tasks from his mathematics teaching methods in his non-enrichment courses. The first is a lack of intervention while students are working and the second is the use of cell phones during class. Griff said that it was challenging to not intervene when students were solving a task differently than he had anticipated, but he resisted anyway. He referred to intervening as part of “the old style of teaching,” implying that he somehow alters his teaching style when he facilitates modeling tasks. It is possible that Griff now teaches all of his mathematics classes with an interruption-free approach, but it was unclear from his statements. Again, this new teaching practice is a reflection of what the facilitators modeled during the PD sessions. Cohen and Lotan (2014) refer to this move as “no hovering” (p. 133) and claim that teachers must achieve “a delicate balance between avoiding hovering and wisely intervening in a group” (p. 135). The PD facilitators were aware of this balance during the modeling sessions, and so Griff experienced this teaching practice firsthand. His rationale for using this teaching practice – preference, imitation, or belief that “no hovering” is necessary during modeling tasks – is uncertain. Griff’s second key teaching practice allows the use cell phones in class during modeling tasks. The PD facilitators let the teachers use their phones or any other technology available to look up necessary information, and Griff realized that this was very effective. When working on a modeling task, students in his class use their phones to research information about the task in order to make assumptions. Students in Griff's other mathematics classes are not allowed to use their cell phones during class, but would cell phone use aid their learning of the mathematics?

Implications

Here, I make inferences for teachers, teacher educators, and researchers with respect to mathematical modeling, teaching mathematics, and designing research studies focused on PD.

Implications for teaching mathematical modeling. Any teacher – and not just middle school mathematics teachers – can learn from the experiences of the four teachers in this study. They learned a completely new approach to teaching their discipline and recounted their struggles and shortcomings as well as their successes. Even though they all had years of experience teaching middle school mathematics, they grew as educators through their endeavor to teach modeling tasks and excitedly shared their triumphs. This section briefly discusses two prevalent themes in the findings: the influence of past experiences in a teacher's practices and the selection of appropriate modeling tasks.

Past experiences can affect the teaching of modeling tasks. Teachers' past experiences can affect how they view, teach, and assess mathematical modeling. What if a teacher's views and background are very different from her students'? This situation occurred with William when he had his students work on the Shampoo Bottle task. William said that some of the groups of students assumed that families were composed of three or four members, but in his personal experience, family sizes were much larger. Although he did not give the students a lower grade for this assumption, he saw their solution as not as realistic as some of the other groups'. Similarly, Jamie was disappointed that her students did not consider the various definitions of a “customer” in the Car Wash task. Jamie knew from experience that there are many things to buy at a typical car wash: snacks, drinks, souvenirs, and even purses. In other words, she did not restrict her view of a car wash customer to someone who simply washes his or her car. Because her students did not see this possibility, Jamie revisited the concept of assumptions and decided to change the way she would launch the Car Wash task for the following class. If Jamie had not bought her purse at a local car wash, she probably would have been more satisfied with her students’ solutions. While William and Jamie recognized that their students did not do anything *wrong*, they felt their students could have done *better* and seemed disappointed in their students’ solutions. It is interesting that their personal experiences affected how they viewed and assessed their students’ work. In Jamie’s case, her past shopping experiences even affected how she planned to teach the Car Wash task in the future.

Because modeling tasks are usually based on real-world events, they allow for teachers to bring in their background knowledge and past experiences to different elements of the tasks. Teachers may be more likely to choose tasks with topics that personally interest them; it is possible that this interest can be an issue when the students do not solve a task in the way that their teacher anticipated. Jamie was so focused on the fact that her students assumed a different definition for a “customer” than she did that she possibly lost sight of the task’s objectives. She also may have missed other teaching opportunities, such as pointing out what the students did well.

Students may bring background knowledge into their modeling solutions as well; in fact, this is a major benefit of mathematical modeling tasks. When a student has a very different background than her teacher, one may not be able to understand what implicit cultural or contextual assumptions the other is making. The students in Jamie’s class did not seem to understand why someone would go to a car wash except to have their car washed. Likewise, William's students did not see the need to assume a large family for the Shampoo task.

Although Griff did not implement Papi's Birthday Party task in his enrichment class, it can be presumed that his students would not have had the same emotional reaction to the task as he did. Most seventh and eighth graders probably have not had to fund a large party for a parent's birthday, nor do they have a steady income, so they are not likely to form a personal connection with this task. However, I wonder if Griff's students would have come up with a similar model to that which he created – a mathematical model based on many non-mathematical elements, such as sympathy for the hard-working, single mother.

What makes a good modeling task? This chapter would be remiss if it did not discuss the content and structure of the Recipe task that William created. As mentioned above, his motive for the task design was to help his students become more comfortable with working in groups. His real-world-centered view motivated him to choose an authentic recipe and create a task that incorporated the students' current mathematical focus: proportional reasoning. While the task created a great opportunity for students to practice calculating unit rates and scaling ratios, it left out many key features of a modeling task. The Recipe task was not open-ended; the students chose between a couple different solutions methods [See Appendix C], but they amounted to a slight change in operation, rather than a completely new approach to the problem. Another major issue was the lack of assumptions that were required in the task. Students were given the amounts of each ingredient, the serving yield for the recipe, and the total number of people they needed to feed at the party. This task was in fact a word problem in disguise.

As a comparison, I make reference to the famous NAEP (1986) "Bus Problem" that was the impetus for the problem-solving and sense-making movement of the 1990s. In this word problem, students are asked how many buses with capacity of 36 people each it takes to transport 1,128 soldiers to their destination. If the students divide 1,128 by 36, their answer is not a whole number, so they must decide what to do with the remainder. The quotient is in fact 31 1/3, so this is one-third of a 32nd bus that will be filled. Or this can be written as 31 remainder 12, where 12 people need to ride in the 32nd bus. I make this comparison because William claimed that "what to do with the remainder" was the assumption component of the Recipe task. Maybe the "Bus Problem" also involves making assumptions, but I think not. There is a subtle distinction here between assumptions made at the beginning of a task that are necessary in order to solve the problem, and assumptions about how to interpret a solution to a task. There is also the matter of precision. It seemed that for the Recipe task, students could use unit rates to find a precise quantity required for each ingredient, or they could use multiples of 12 and division to find an estimate. The important thing to notice is that students "made the assumption" to estimate their answer after realizing that their answer was not precise, as though it was an afterthought. It was not an initial assumption, but rather a post hoc recognition that their answer was *close enough*.

Why is this important? The issue to keep in mind is what teachers want the focus and purpose of a mathematical activity to be. If it is to practice long division and expression of a remainder, then that is one purpose. But if it is to help the students develop critical thinking skills and expect a logical solution *before* making mindless calculations, then that is a different, more challenging purpose. This purpose requires a special kind of mathematical task. I claim that this task should be model-eliciting and contain elements listed in this study.

While the Recipe task may have contained some elements of modeling, it did not meet the key criteria of being open-ended, allowing for assumptions, and requiring reflection or justification of the solution. It incorporated a real-world event, but for some teachers, like

Donna, this is not enough to be considered a "good" modeling task. What would she think of the Recipe task? She definitely would not select it for her students, because it does not address a critical issue in the world. Does Donna ever see a need for "non-critical" tasks, such as the Shampoo Bottle task or the Recipe task? The purpose of a modeling task is very important to keep in mind. The purpose, and preferences of the teacher, can determine which tasks should be selected for which audience.

Implications for teacher education. A question came to mind as I listened to the four teachers talk about the mathematical modeling work that their students have done in class: Who can model and when can it be done? Jamie and Griff only used modeling tasks in their enrichment classes while William and Donna implemented tasks in standard mathematics classes. Does this mean that modeling tasks are only for students in enrichment classes? Or that the typical mathematics classroom is neither the time nor the place for modeling? This difference of placement and purpose for modeling tasks brings up many important questions, most of which cannot be answered by this study. Where does modeling fit within the middle school curriculum? This is especially an issue regarding state standards and exams. Griff brought up this dilemma during his interview; he worried that modeling tasks took away from the topics he needed to cover in his regular mathematics classes. His concern of not being able to cover all of the standards for students' exams is what drives him to forego modeling tasks in his regular classes. There is much more flexibility in the schedule and curriculum of his enrichment class, so he is able to do modeling tasks with those students.

While his students work on tasks, Griff pushes them to generalize their model. Is this because the students are in an enrichment class and have more time to spend on the task? Does Griff make all students create a generalized model, regardless of their academic level? I worry that because modeling is a complex process, it has a higher status in the eyes of mathematics educators, leading them to only give modeling opportunities to their "best" students. The fact that the PD was designed around mathematical modeling for students in the middle grades should encourage teachers and administrators to allow any student to engage in a modeling task. Mathematical modeling is not an elite program for only the best and brightest, but rather a chance to motivate students to view the world through a mathematical lens and think critically about their surroundings. It is unfortunate that some teachers do not have the opportunity to try a new approach in their own classrooms – particularly in standard mathematics classes – due to restrictions of time, resources, and administration.

Recall that William was surprised to learn that modeling includes every grade level. The tasks that were adapted or created for the PD project inspired William to write his own modeling tasks for his seventh graders. The Skittles task was an adaptation of a task he remembered from his years in elementary school, and while this task is not as open-ended and critical as others, it requires students to generate middle school-level formulas based on their elementary-level knowledge. Modeling tasks have been and continue to be created for the middle grades (and primary grades), but in addition to great tasks, teachers need to be open to implementing them.

Implications for future research. The focus of this study was neither the effectiveness of this PD nor possible structures and suggestions for effective PDs in general. An interesting question for future research regards the transfer of knowledge during PD sessions. What do teachers walk away with and what do they misinterpret or adapt to suit their teaching needs? I first wondered this when William expressed his unique criteria for a modeling task: "If it doesn't have purpose, conclusion, reflection, it's lacking." I wondered how he decided on these three elements if we never mentioned these exact words during the PD sessions and he had no prior experience with mathematical modeling. Future research may explore teachers' knowledge pre- and post-PD and follow up with the teachers at a later date to ascertain their retained knowledge. This proposed longitudinal study could determine if they are in fact using the mathematical skills or pedagogical methods from the PD, or if any of these concepts has been altered.

Since the site of the project was located quite far from the corresponding schools, I was unable to observe the teachers as they taught modeling lessons, which is one shortcoming of this study. In future research, I would like to be able to observe PD participants as they teach what they are learning, in order to see how their views are reflected in their teaching practices.

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Appendix A – Interview Questions*

1. What mathematical modeling tasks have you given to your students so far last fall and this spring?

2. Can you give a brief description of the sequence of instruction you used during your implementation of mathematical modeling? (i.e. you introduced "the modeling cycle" first and then gave students a task, or vice versa)

3. What challenges did you face during the early implementation stages? Were students resistant to engage in modeling tasks?

4. Did you find that your students approached/solved the modeling tasks differently than you had originally solved them? Or different than what you anticipated?

5. Can I ask you to analyze some of your own students' work?

I will send you (de-identified) student work samples from a few of the tasks that you have used this year. For each task, compare the two samples and answer the following questions.

i. Between Sample A and Sample B, which group of students do you think more correctly included elements of modeling? Why?

ii. Did the students of Sample A (or Sample B) complete the modeling cycle? If not, which elements did they include?

6. Have you encouraged your students to create a generalized model (i.e., graph, formula, equation) during the solution stages of every or some tasks?

7. Have you encouraged your students to "report out" in a style similar to our M^3 Workshop, that is, by creating posters and sharing their results with the class in an open discussion setting?

8. If someone asked you to give them a brief definition of mathematical modeling in your own words, what would you tell them?

9. Do you think that your personal views and opinions about modeling affect how you teach it? If so, how?

10. What modeling tasks from this past summer session or fall sessions do you think particularly illustrated mathematical modeling or best gave the opportunity to complete the modeling cycle? Which ones didn't? Why?

11. Do you feel that modeling tasks have added mathematical rigor to your curriculum or taken from it?

What particular aspects of the tasks make you feel this way?

12. Please state whether you agree or disagree with the following statements and then provide a brief explanation for your response.

- In order for a modeling task to add mathematical rigor to a classroom or curricula, it must be specifically designed to align with a certain mathematical concept or standard.
- In order for a modeling task to add mathematical rigor to a classroom or curricula, it must be implemented *after* the corresponding mathematical lesson has been taught.
- In order for a modeling task to add mathematical rigor to a classroom or curricula, it must be completed using all modeling elements in the cycle.
- A mathematical modeling task can be considered "rigorous" even when the students to whom it is given do not yet have the background mathematical knowledge required to complete the task.
- The following two terms have essentially the same meaning: *rigorous* and *thinking outside the box.*
- The following two terms have essentially the same meaning: *critical thinking* and *thinking outside the box.*

*Note that some questions were altered slightly to adapt to Donna's position as teacher coach instead of teacher. However, she could still answer questions about her views of modeling and its place in the curriculum.

Appendix B – Modeling Tasks

Car Wash

Car wash problem:

"A locally owned automated car wash advertises that it serves millions of satisfied customers each year. Is this a reasonable claim?" (NCTM, 2005)

Discuss what is known or unknown, and what assumptions you need to make to come to a conclusion.

T-Shirt

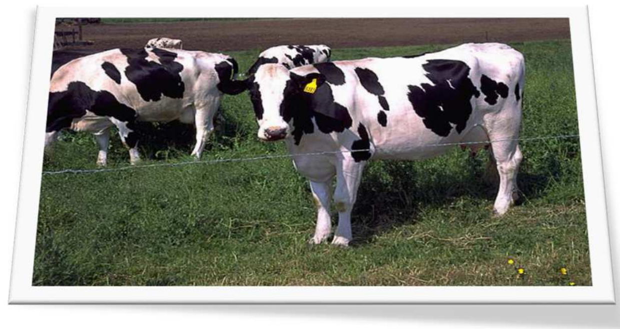
T-Shirt Sales

The school is looking buy T-shirts for a volleyball team. Tees-R-Us has the sale "buy one, get one half off" and Sport-Mart has the sale "buy two, get one free." Which is the better choice if the school needs 11 shirts?

Source: Anhalt, C. and Cortez, R. (2015)

Milk Cow

Milk Cow Task



Milk is a very healthy food and has been used by humans for 5,000 years. Apart from high-quality protein, it contains minerals, in particular calcium and vitamins.

It is recommended that children aged between 2 and 12 drink 2 glasses of milk a day.

How many cows are needed to supply every child in your family with the amount of milk recommended each day? What about for each child in your class?

List all **assumptions** and show all mathematical work that led to your solution.

Relevant AZCCRS Math Standards:
5.OA.1; 5.NBT.5; 5.MD.1; 5.MD.3; 5.MD.4; 5.MD.5

6.G.2; 6.EE.6

(LEMA Project, 2010)

Shampoo Bottle

Family Shampoo Bottle Task

<u>Source</u>: Anhalt, C. (2014)

A family shares a bottle of shampoo when they wash their hair in the shower. (They usually buy a large bottle that holds 1 liter or 33.8 fluid ounces.) How often do you think the family should buy a new bottle of shampoo?

List all **assumptions** and show all mathematical work that led to your solution.

Papi's Birthday Party

Papi's Birthday Task

<u>Source</u>: Aguirre, J. (2010)

It was Señor Aguirre's 70th Birthday. His three children wanted to throw him a big party to celebrate. The hall rental, mariachi, food, and decorations will cost a total of $4,500. The brother, a special medical doctor (anesthesiologist) who makes about $20,000 per month, suggested that the three children split the cost equally.

One of the sisters, a university professor who makes about $6,000 per month, said that would not be fair. She suggested the following: the brother pays $3150. She would pay $900, and the other sister, a partner in the family business and single mom with 2 boys who makes about $3000 per month, should pay $450.

TASK*: Write a position statement using mathematical evidence (e.g. proportions, ratios, percent) to support your conclusion to the following questions:

•Which person do you agree with and why?

•What is *fair* in this situation?

•Can you think of an alternative financial arrangement that might be better (more fair)?

Water Conservation (toothbrush)

Water Conservation Task

A newspaper says that each family can save almost 7,000 gallons of water every year if they turn off the water while brushing their teeth.

Is this reasonable? How can this be possible? List all **assumptions** and show all mathematical work that led to your solution.

You can watch this short video to see possible assumptions and reasons why water conservation is important:

Colgate "Save Water" Video (1:08): https://www.youtube.com/watch?v=mJtEJm4F4nE

Task created by Erin Turner and Cynthia Anhalt, 2015.

Shower vs. Bath

Shower vs. Bath

In order to be part of the water conservation effort, some people say that one uses less water when taking a bath than when taking a shower. Others say this is not true! Who is right?

Keep in mind that older showerheads have a flow rate of up to 3.4 gallons per minute while energy-saving showerheads have a flow rate as low as 1.9 gallons/minute. Bathtubs vary in size from 30-45 gallons when filled to the top.

Task adapted from Anhalt, C. (2014).

Be sure to list all **assumptions** that you make.

Show all mathematical work.

Softball Team

Softball Team Selection Task Card: All-Star Team

Adapted from Case Studies for Kids!
Purdue University (Lesh & English, 2016)

The softball league president is looking for a fair way to select players for the All-Star softball team. He wants to use information from tryouts and scrimmages to come up with a system for comparing the performances of each player. He also wants to consider comments from the coaches about each player.

Task:

- **Individually**, look over the Coaches Comments and the Player Data.
- **Each group member** shares 1 thing that they notice about the data and 1 question that they have about the data. Discuss and clarify all questions.
- **As a group**, generate at least three different ideas for how to use the Coaches Comments and the Player Data to select 9 players for the All-Star team.
- *Discuss and record* each possible method. Make sure everyone in your group understands each method.
- Next, **as a group,** select one method and use the method to evaluate all the players, and to select nine players for the All-Star team.
- Make a mini-poster that represents your group's method, and how you used the method to select nine players for the team.
  - Use tables, words, symbols, and diagrams to clearly represent your method.
  - Explain which nine players you selected, and why.
  - Explain how the softball league president could use your method to select the players for the All-Star team in future seasons.

If time, select a second method and use the method to select nine players for the team. Discuss how the results of your second method compare to the results of your first method.

Softball Team, continued

The Groupwork Expectations you are practicing include:

No talking outside of your group.
- All questions should be group questions.
- Listen and talk equally in your group.

No one is as smart as all of us together

Comments by coaches:

Tisha: She has great fielding skills. She is very easy to work with.

Kim: Very fast athlete. She can get around the bases quickly and to balls quickly. She has gotten along well with teammates in the past.

Sally: No comments available.

Barb: She has a very strong arm and she is capable of playing several positions in a game.

Melissa: Fields exceptionally well. Her batting was improved near the end of the year.

Charlene: She's very mature for her age. When her coordination catches up with her age, she could be quite good.

Cynthia: Does many things well.

Rebecca: She knows the most about the game as anyone I've coached. Her father played college baseball.

Margaret: She misses practice often. Perhaps her parents made her play this sport and she didn't want to play.

Beth: She's a very good offensive weapon and she plays well in the outfield.

Mary Pat: She does many things well.

Kathy: She has a great throwing arm. This was the first year she ever played.

Esther: No Comments available.

Kelly: She helps the other athletes well. She helps the younger athletes.

Softball Team, continued

Player Data from Tryouts and Scrimmages

Person	Throw in ft.		Batting				Run in seconds
Tisha	81	65	Double	Single	Single	Pop-fly	7.2
	79	61	Single	Pop fly	Pop fly	Single	6.9
	69		Pop-fly	Pop- fly			6.9
Kim	67	61	Pop-fly	Pop-fly	Pop-fly	Home run	5.1
	66	59	Pop-fly	Pop-fly	Triple	Double	5.3
	66		Triple	Ground out			5.1
Sally	83	75	Home run	Ground out	Pop-fly	Pop-fly	5.9
	80	82	Single	Double	Ground-out	Double	6.
	80		Pop-fly	Pop-fly			6.2
Barb	97	89	Triple	Ground-out	Pop-fly	Pop-fly	6.4
	98	90	Ground-out		Single	Pop-fly	6.3
	91		Single	Ground-out			6.3
				Pop-fly			
Melissa	98	88	Pop-fly	Home run	Single	Double	6.3
	90	86	Double	Ground-out	Pop-fly	Ground-out	6.5
	84		Pop-fly				6.7
				Pop-fly			
Charlene	86	77	Single	Strike out	Pop-fly	Ground-out	6.5
	83	80	Double	Ground out	Pop-fly	Pop-fly	7.1
	65		Strike-out	Pop-fly			6.5
Cynthia	72	72	Strike out	Pop-fly	Strike-out	Double	6.9
	72	69	Ground-out	Strike-out	Double	Strike-out	6.1
	71		Strike-out	Pop-fly			6.4

Rebecca	49	56	Strike-out	Strike-out	Single	Strike-out	5.5
	52	54	Strike-out	Pop-fly	Double	Single	5.7
	56		Ground-out	Double			5.6
Margaret	88	89	Single	Single	Ground-out	Pop-fly	5.9
	72	70	Strike-out	Pop-fly	Pop-fly	Strike-out	6.1
	75		Ground-out	Strike-out	Double		5.7
Beth	61	54	Single	Pop-fly	Ground-out	Pop-fly	5.9
	61	59	Double	Triple	Single	Home-run	6.0
	57		Strike-out	Ground-out			5.9
Mary Pat	53	50	Strike-out	Strike-out	Pop-fly	Triple	6.5
	57	37	Pop-fly	Double	Single	Single	6.2
	52		Pop-fly	Pop-fly			6.4
Kathy	98	90	Strike-out	Pop-fly	Double	Strike-out	5.6
	97	92	Single	Pop-fly	Pop-fly	Pop-fly	5.9
	99		Single	Pop-fly			5.4
Esther	90	87	Strike-out	Strike-out	Single	Pop-fly	6.1
	90	85	Strike-out	Double	Single	Double	6.4
	86		Pop-fly	Pop-fly			6.2
Kelly	80	80	Home-run	Strike-out	Home-run	Pop-fly	5.2
	79	82	Strike-out	Double	Strike-out	Pop-fly	5.1
	77		Strike-out	Triple			4.9

Cell Phone Plans

Phone Plans Task Card

Grades 7-8

You are concerned that your family is paying too much for their current cell phone plan. You do some research and find a comparison chart for the top four wireless service providers. Which provider and plan is best for your family?

Task:

- **Individually**, look over the Comparison Chart for the four wireless providers.
- **Each group member** shares 1 aspect of a phone plan that they value as important and 1 question that they have about the chart. Discuss and clarify all questions.
- **As a group**, generate at least three different ideas for how to analyze the Comparison Chart and decide what information is relevant for *your family*.
- *Discuss and record* the important features that your phone plan should include. Make sure everyone in your group understands and contributes.

- Next, **as a group,** create ONE model explaining which phone plan is best for your family.
- Then, GENERALIZE the model so that other families can use it.

- Make a mini-poster that represents your group's method, and how you used the method to select the best phone plan for your family.
- Use tables, words, symbols, and diagrams to clearly represent your method.
- List all assumptions and explain why you included certain features in your plan.
- Explain how other families could look at your model and use it to help them select the best phone plan for them.

Are there any features of your phone plan or model that were not included in the chart?

Cell Phone Plans, continued

The Groupwork Expectations you are practicing include:

- No talking outside of your group.
- All questions should be group questions.
- Listen and talk equally in your group.

No one is as smart as all of us together

Phone Plans Comparison Chart

Here's the breakdown by wireless provider (without any early upgrade privileges). The first section shows monthly service pricing for a single phone with unlimited talk and text, with additional charges listed below.

	Verizon	AT&T	Sprint	T-Mobile
< 500 MB	$55 (250 MB)	$60 (300 MB)	–	–
500 MB	$70	–	–	$40
1 GB	$80	$65	$70	$50
2 GB	$90	$80	–	–
3 GB	$100	–	–	$60
4 GB	$110	$110	–	–
5 GB	–	–	–	$70
6 GB	$120	$120	–	–
8 GB	$130	$130	–	–
10 GB	$140	$140	–	–
> 10 GB	$10 per 2 GB	Varies	–	–
Unlimited	–	–	$80	$80
Second Line	$40	$40	$60 to $70	$30 to $50
Third Line	$40	$40	$50 to $60	$10 to $30
More Lines	$40	$40	$40 to $50	$10 to $30
Mobile Hotspot?	Included	Included	$10 (1 GB)	Included

Taken from (TIME Magazine, 2015) http://time.com/7982/which-wireless-plan-is-cheapest/

Task created by Cynthia Anhalt and Amy Been, 2015.

Cell Phone Plans, continued

Phone Plans Task - Guiding Questions:

How many people are in your family? Is everyone getting a phone?

What kind of phone does each person in your family want? Are they smartphones?

What will you use your phone for?

How much data will you family need? Will everyone share it?

Are there any additional features that your family's phone plan should have?

What mathematical operations do you need to calculate how much your family will pay every month?

Recipe

Mississippi Mud Pie Serves 12

Author: F3r5uCrD_bl

Need a chocolate fix? Then indulge yourself with this Mississippi Mud Pie recipe!

Ingredients

For the base

- 300 g (10 oz) Oreo cookies or Bourbon biscuits or chocolate Digestives
- 75 g (2½ oz) butter, melted

For the filling

- 150 g (5 oz) soft brown sugar
- 100 g (3½ oz) plain chocolate
- 75 g (2½ oz) butter
- 3 tbsp Golden syrup or corn syrup or honey
- 3 eggs
- 1 tsp vanilla flavouring

For the topping

- 283 ml (10 fl oz) double cream or whipping cream or heavy cream
- 50 g (2 oz) plain chocolate to make curls

A group has requested this recipe from the caterer to prepare for their party of 152 people. Adjust the amounts to know how much the caterer needs to buy of each ingredient.

Instructions

Make the base

1. Place the cookies in a bag and crush them quite finely with a heavy object.
2. Put the crushed cookies in a bag, add the melted butter and mix well.
3. Pour the mixture into a 23 cm (9") cake tin with a pop-up bottom. Press the mixture firmly with a spoon to create a base and also about 2 cm (1") up the side of the tin.
4. Place in the 'fridge to chill for about 15-20 minutes.

Make the filling

1. Beat together the sugar and the eggs until smooth.
2. Add in the syrup and the vanilla and beat again.
3. Melt the chocolate and butter together in a bowl over a pan of boiling water. Make sure the water is not touching the bottom of the pan.
4. Once the mixture is smooth pour it into the sugar/egg mixture and mix well.
5. Pour this mixture into the chilled cake base and bake at 180°C/356°F for 30-35 minutes until the filling is set.
6. Remove from the oven and allow to cool.
7. Once cooled, remove the cake from the tin and place on a plate.

The topping

1. Make some chocolate curls by warming the chocolate in your hand and cutting thin slivers using a vegetable peeler. Place in the 'fridge to chill.
2. Beat the cream until firm.
3. Spread it evenly over the top of the cake and decorate with the chocolate curls.

Calories: 395 Fat: 24.5 Sugar: 29.8 Fiber: 1.6 Protein: 3.8 Cholesterol: 85.2

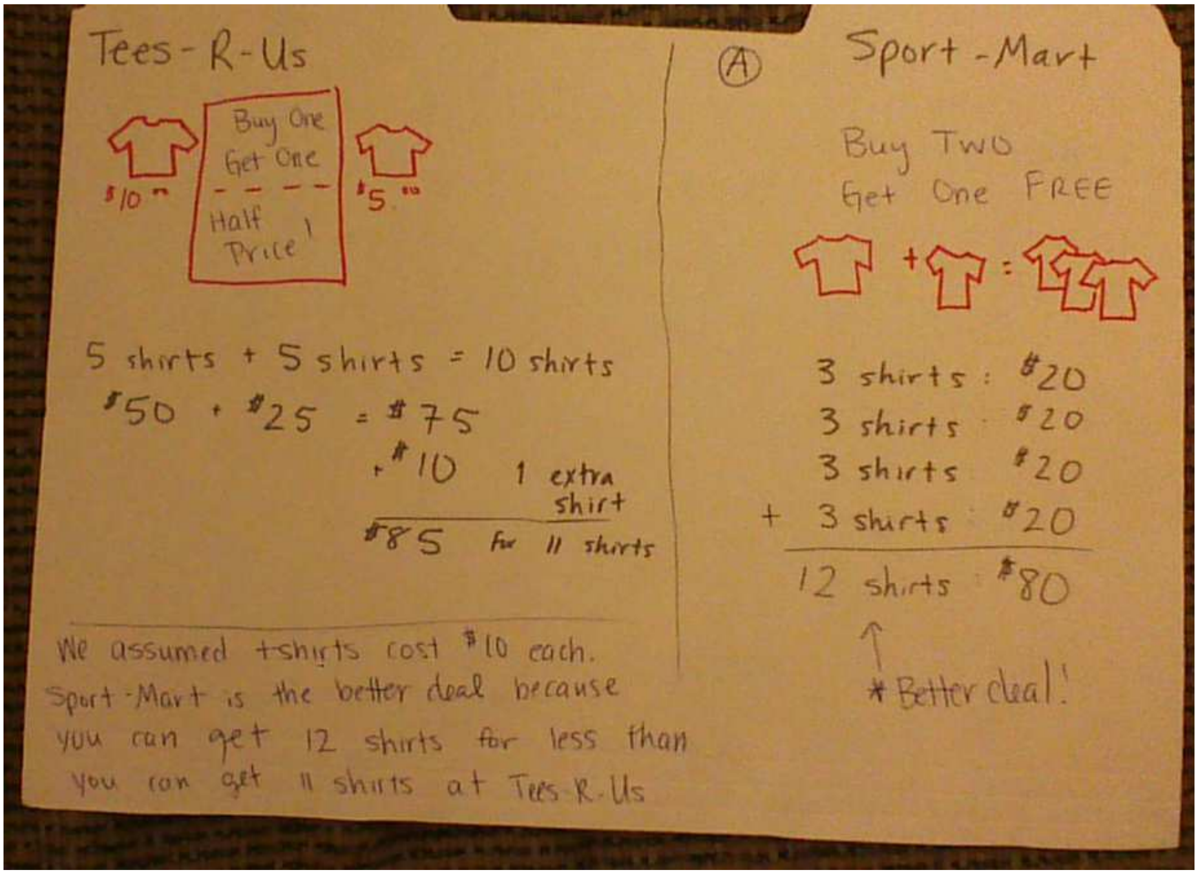
Titli's Tips

To make it even more chocolatey you can either beat a couple of tbsp of cocoa powder into the cream or top the pie with a chocolate custard.

Recipe by **Titli's Busy Kitchen** at http://titlisbusykitchen.com/recipe/mississippi-mud-pie-recipe

Appendix C – Student Work Samples

T-Shirt Task, Sample A



Recipe Task, Sample A

A

11-5-15
P-2

Mississippi Mud Pie

100/100

The caterer needs to buy:

For the base

- 1,075 grams of Oreo Cookies or Bourbon biscuits or chocolate
- 268.75 grams of butter, melted

For the filling

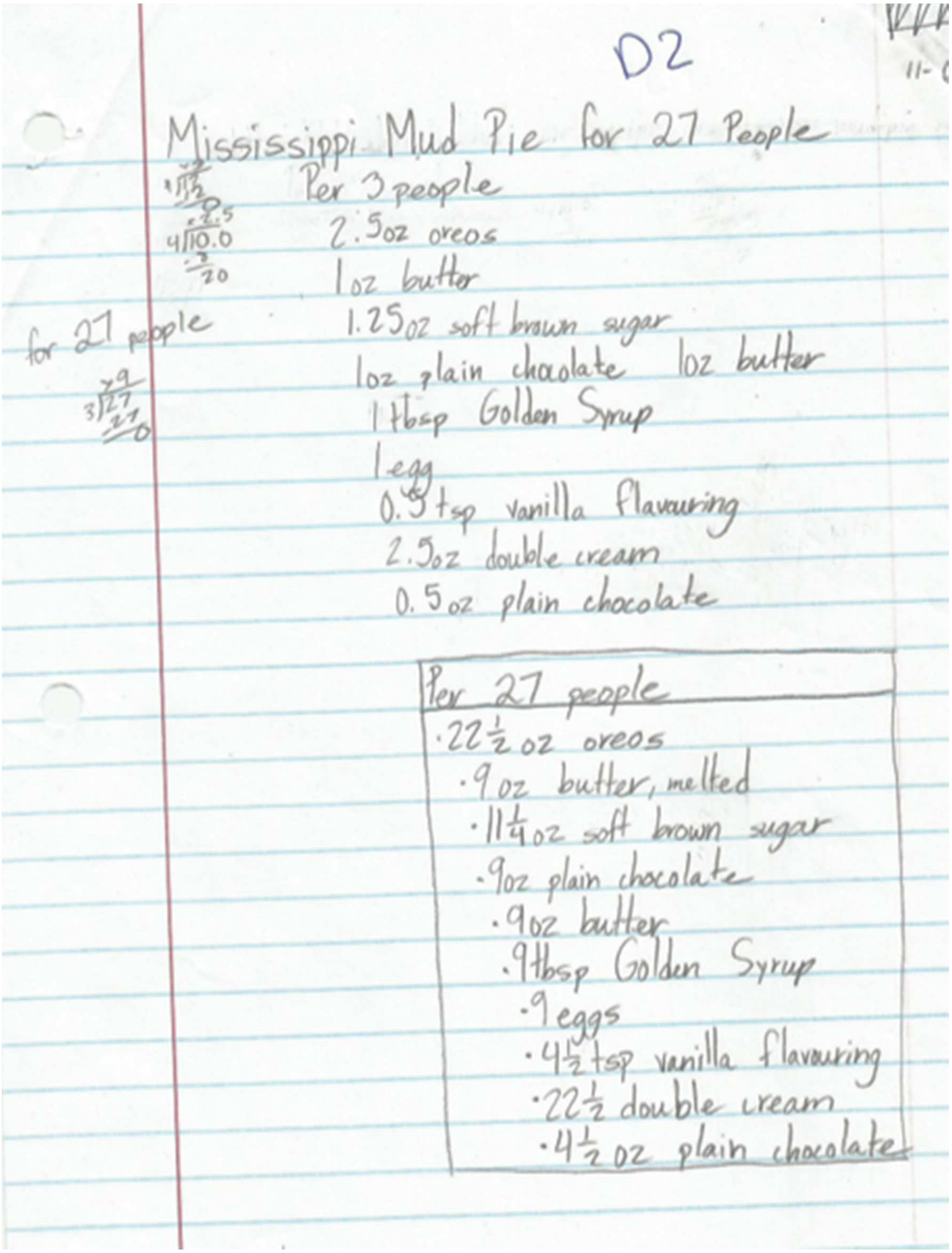
- 537.5g soft brown sugar
- $358.\overline{3}$ g plain chocolate
- 268.75 g butter
- 10.75 tbsp Golden syrup or corn syrup or honey
- 10.75 eggs
- $3.58\overline{3}$ tsp vanilla flavoring.

For the topping

- $1,014.08\overline{3}$ ml double cream or whipping cream or heavy cream
- $179.1\overline{6}$ g - plain chocolate to make curls

We divided each number of ingredients by 12 to get how much it would be for 1 person. Next, we multiplied the unit rate by 43 (number of people) to get the answer.

Recipe Task, Sample D2


D2

Mississippi Mud Pie for 27 People

Per 3 people
2.5oz oreos
1oz butter
1.25oz soft brown sugar
1oz plain chocolate 1oz butter
1tbsp Golden Syrup
1egg
0.5tsp vanilla flavouring
2.5oz double cream
0.5oz plain chocolate

for 27 people
$3\overline{)27}$ = 9

Per 27 people
- $22\frac{1}{2}$ oz oreos
- 9oz butter, melted
- $11\frac{1}{4}$oz soft brown sugar
- 9oz plain chocolate
- 9oz butter
- 9tbsp Golden Syrup
- 9 eggs
- $4\frac{1}{2}$tsp vanilla flavouring
- $22\frac{1}{2}$ double cream
- $4\frac{1}{2}$ oz plain chocolate

Recipe Task, Sample E1

E1

Mississippi Mud Pie 11-05-

52 people Serves 12

Q? How much the caterer needs to buy of each ingredient.

5 orders 60

$60 - 52 = 08$

$2\frac{1}{2} \times 5$

$\frac{5}{2} \times \frac{5}{1} = \frac{25}{2}$

$25.0 \div 2 = 12.5$

$12.5 \times 5 = 62.5$

$3.5 \times 5 = 17.5$

$2.5 \times 5 = 12.5$

10 oz	x5	50oz
$2\frac{1}{2}$ oz	x5	12.5oz
5oz	x5	25 oz.
$3\frac{1}{2}$ oz	x5	17.5oz.
$2\frac{1}{2}$ oz	x5	12.5 oz.
3 tbsp	x5	15 tbsp
3 eggs	x5	15 eggs
1 tsp	x5	5tsp
10 fl oz	x5	50 fl oz.
2oz	x5	10oz.

10/10

A. Multiply by 5 and you will have 8 servings left