REAL GROUPS AND SYLOW 2-SUBGROUPS

GABRIEL NAVARRO AND PHAM HUU TIEP

Abstract. If G is a finite real group and $P \in \mathrm{Syl}_2(G)$, then P/P' is elementary abelian. This confirms a conjecture of Roderick Gow. In fact, we prove a much stronger result that implies Gow's conjecture.

1. Introduction

Reality questions on finite groups are almost as old as finite group theory itself, and they trace back to the work of Frobenius, Schur and Burnside.

An element g of a group G is **real** if g is conjugate to its inverse, and a character χ of G is **real-valued** if χ only takes real values. It is well-known that the number of real-valued irreducible complex characters of a finite group G is the number of conjugacy classes of G consisting of real elements, and a group G is **real** if all of its elements are real.

R. Gow's conjecture asserts that if G is a finite real group, then P/P' is an elementary abelian 2-group, where P is a Sylow 2-subgroup of G, and P' is the derived subgroup of P. This conjecture follows the main philosophy in finite group representation theory: global implies local or vice-versa.

In this paper we prove Gow's conjecture. The key is that we have found a more general statement with good inductive conditions which holds true (see Theorem 2.11 below). This statement has interest on its own, and in its simplest case already implies a much stronger version of Gow's conjecture.

2010 *Mathematics Subject Classification.* Primary 20C15; Secondary 20C33, 20D06, 20D20.

Key words and phrases. Real groups, Real characters, Sylow 2-subgroups.

The research of the first author is supported by the Prometeo/Generalitat Valenciana, Proyectos MTM2013-40464-P. The second author gratefully acknowledges the support of the NSF (grant DMS-1201374) and the Simons Foundation Fellowship 305247.

The authors are grateful to the Mathematisches Forschungsinstitut Oberwolfach, where this project was started. Parts of the paper were written while the second author visited the Departments of Mathematics, University of Chicago. It is a pleasure to thank Ngo Bao Chau and the University of Chicago for generous hospitality and stimulating environment.

The authors are grateful to Roderick Gow and Martin Liebeck for helpful discussions, and to the referees for careful reading and helpful comments on the paper.

Theorem A. *Let G be a finite group with a Sylow 2-subgroup P. If all the odd-degree irreducible characters of G are real-valued, then P/P' is elementary abelian.*

Our proof of Theorem A uses the Classification of Finite Simple Groups; in fact, we are able to reduce it to a problem on almost simple groups. The converse of Theorem A is not true, however, even for solvable groups.

There is no block version of Theorem A, except for the principal block. It is simply not true that if all the height zero characters in a 2-block B with defect group P are real-valued then P/P' is elementary abelian: D_{24}, for instance, is already a counterexample. On the other hand, we have the following non-trivial extension of Theorem A.

Theorem B. *Let G be a finite group with a Sylow 2-subgroup P. If all the odd-degree irreducible characters in the principal 2-block of G are real-valued, then P/P' is elementary abelian.*

It is not true that if G is a real group, then $P \in \mathrm{Syl}_2(G)$ is real (a statement that would imply Gow's conjecture). The smallest counterexample is the group $\mathsf{A}_4 : Q_8$.

Theorems A and B lead to an interesting problem: is it possible to give a group characterization of when all the odd-degree irreducible characters of G are real-valued? For solvable groups, we can show that this is the case if and only if $P = \mathbf{N}_G(P)$ and $P' = \Phi(P)$, the Frattini subgroup of P (i.e. the intersection of all maximal subgroups of P), see Theorem 2.13 below. But this is not true outside solvable groups, as $G = \mathsf{A}_5$ or $G = \mathsf{A}_9$ show us (in both directions). We also note that the converse of Theorem B is false, as shown by the example of $SL_3(2)$.

Finally, we have some evidence that there might be a version of Gow's conjecture for odd primes, but we will leave this problem for another place [NT4].

2. Theorem B

In this section, we prove Theorem B (which trivially implies Theorem A) assuming that Theorem 2.1 (below) on almost-simple groups is true.

Our notation for characters follows [Is], while the notation for blocks follows [N]. If G is a finite group, then $\mathrm{Irr}(G)$ is the set of the irreducible complex characters of G. If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of irreducible characters $\chi \in \mathrm{Irr}(G)$ such that $[\chi_N, \theta] \neq 0$. (In other words, this is the set of the irreducible constituents of the induced character θ^G.) If p is a prime, then $\mathrm{Irr}_{p'}(G)$ is the set of $\chi \in \mathrm{Irr}(G)$ of degree not divisible by p, and $\mathrm{Irr}_{p'}(G|\theta) = \mathrm{Irr}(G|\theta) \cap \mathrm{Irr}_{p'}(G)$. In this paper, we denote by $B_0(G)$ the set of irreducible characters of G contained in the principal 2-block of G. For brevity, we will also say that $\chi \in \mathrm{Irr}(G)$ is **real** if χ is real-valued.

If P is a p-group, recall that the **Frattini subgroup** $\Phi(P)$ is the smallest normal subgroup such that $P/\Phi(P)$ is elementary abelian (equivalently, $\Phi(P)$ is the intersection of all maximal subgroups of P).

Theorem 2.1. *Let $S \triangleleft G$, where S is a non-abelian finite simple group, G/S is a 2-group, and $\mathbf{C}_G(S) = 1$. Let $P \in \mathrm{Syl}_2(G)$ and $Q = P \cap S$. Suppose that all the P-invariant complex irreducible characters of odd degree in the principal 2-block of S are real-valued. Then every linear P-invariant character of Q is real.*

We need several preliminary results. The first one is well known and follows e.g. from [Gow2, Corollary 1.3]:

Proposition 2.2. *Let G be a finite group, and let $\chi \in \mathrm{Irr}(G)$ be real of odd degree. Then χ belongs to the principal 2-block of G.* □

Suppose that $N \triangleleft G$ and that $\theta \in \mathrm{Irr}(N)$ extends to G. If θ is in the principal block of N, it is not true that there is an extension of θ in the principal block of G. The easiest type of counterexamples is of the following form. Set $p = 3$, $N = (C_3 \times C_3) : C_2$, and let $Q = Q_8$ act on N with $Z = \mathbf{Z}(Q) = \mathbf{C}_Q(N)$. Then the unique non-trivial linear character of N extends to G but does not have an extension containing $Z = \mathbf{O}_{3'}(G)$ in its kernel. However, we have the following.

Theorem 2.3. *Suppose that $\theta \in \mathrm{Irr}(N)$ is real-valued of odd degree and extends to G. Then there exists an extension $\eta \in \mathrm{Irr}(G)$ of θ that lies in the principal 2-block of G.*

Proof. Let $\eta \in \mathrm{Irr}(G)$ be an extension of θ. Then $\bar{\eta} \in \mathrm{Irr}(G)$ is an extension of $\bar{\theta} = \theta$. By Gallagher's Corollary (6.17) of [Is], there exists a linear $\lambda \in \mathrm{Irr}(G/K)$ such that $\bar{\eta} = \lambda\eta$. Now, let $K = \ker(\lambda)$, so that G/K is cyclic. Notice that $\gamma = \bar{\eta}_K = \eta_K$. Therefore $\gamma \in \mathrm{Irr}(K)$ is real-valued of odd degree, and G-invariant. Let $P/K \in \mathrm{Syl}_2(G/K)$, and let L/K be a 2-complement. Then γ has a canonical extension $\hat{\gamma} \in \mathrm{Irr}(L)$ by Lemma 2.1 of [NT1]. By uniqueness, we have that $\hat{\gamma}$ is G-invariant. By Corollary (4.2) of [I], we have that restriction defines a bijection

$$\mathrm{Irr}(G|\hat{\gamma}) \to \mathrm{Irr}(P|\gamma)\,.$$

Let $\tau = \eta_P$ which is an extension of γ to P. Then there exists $\rho \in \mathrm{Irr}(G|\hat{\gamma})$ such that $\rho_P = \tau$. Now, ρ is an extension of γ to G, and we claim that ρ lies in the principal block of G. Notice that $\rho_L = \hat{\gamma}$ is real of odd degree, and therefore ρ_L lies in the principal 2-block of L by Proposition 2.2. Now G/L is a 2-group, and there is only one block of G covering the block of $\hat{\gamma}$ (Corollary (9.6) of [N]), which is the principal block of G. Therefore ρ lies in the principal block of G. □

We will also use the following in some parts in the proof of our main results.

Lemma 2.4. *Let H be a subgroup of G, b a p-block of H, and let $\xi \in \mathrm{Irr}(b)$. Assume that $b^G = B$ is defined. Then the p-part of $|G : H|\xi(1)$ is the p-part of $\sum_{\chi \in \mathrm{Irr}(B)} [\xi^G, \chi]\chi(1)$. In particular, if $p \nmid |G : H|\xi(1)$, then there exists $\chi \in \mathrm{Irr}(B)$ of p'-degree such that $p \nmid [\xi^G, \chi]$.*

Proof. This is Corollary (6.4) of [N]. □

The following lemma is elementary.

Lemma 2.5. (a) *Suppose that $R \le H \le G$. Let $\chi \in \mathrm{Irr}(G)$ be real-valued. Suppose that there exists a real-valued $\nu \in \mathrm{Irr}(R)$ such that $[\chi_R, \nu]$ is odd. Then there exists $\gamma \in \mathrm{Irr}(H)$ real-valued such that $[\chi_H, \gamma][\gamma_R, \nu]$ is odd.*

(b) *Suppose that $R \le K \le G$, $|K : R|$ is odd, P is a 2-group of G normalizing R and K. If $\nu \in \mathrm{Irr}(R)$ is P-invariant, has odd degree, and is real-valued, then there exist a P-invariant real-valued odd-degree $\gamma \in \mathrm{Irr}(K)$ such that $[\gamma_R, \nu]$ is odd.*

Proof. (a) Let $\mathcal{A}$ be the set of real-valued irreducible characters of H, and let $\mathcal{B}$ be a complete set of representatives of $\mathrm{Irr}(H) - \mathcal{A}$ under complex-conjugation. We can write

$$\chi_H = \sum_{\gamma \in \mathcal{A},\ 2 \nmid [\chi_H, \gamma]} [\chi_H, \gamma]\gamma + 2 \sum_{\gamma \in \mathcal{A},\ 2 | [\chi_H, \gamma]} \frac{[\chi_H, \gamma]}{2}\gamma + \sum_{\gamma \in \mathcal{B}} [\chi_H, \gamma](\gamma + \bar{\gamma})\,.$$

Since ν is real, $[\gamma_R, \nu] = [\bar{\gamma}_R, \nu]$ for all $\gamma \in \mathcal{B}$, and we conclude that

$$1 \equiv [\chi_R, \nu] \equiv \sum_{\gamma \in \mathcal{A},\ 2 \nmid [\chi_H, \gamma]} [\chi_H, \gamma][\gamma_R, \nu] \pmod 2\,.$$

Now, part (a) easily follows.

(b) We have that P acts on the irreducible constituents of ν^K and $[\nu^K, \eta] = [\nu^K, \eta^x]$ for $x \in P$ and $\eta \in \mathrm{Irr}(K)$. Also, $[\nu^K, \eta] = [\nu^K, \bar{\eta}]$. Now, use that $\nu^K(1)$ is odd and a similar argument as in part (a). □

Before proceeding to the proof of our main result, we record some results on principal blocks in the following lemma. The first two parts are elementary, but the third part is deeper and is due to M. Murai.

Lemma 2.6. *Let G be a finite group, and let $N \triangleleft G$.*

(a) *We have that $B_0(G/N) \subseteq B_0(G)$.*

(b) *If H_i are finite groups, and $\gamma_i \in B_0(H_i)$, then $\gamma_1 \times \cdots \times \gamma_t \in B_0(H_1 \times \cdots \times H_t)$.*

(c) *Suppose that $\theta \in B_0(N)$ has p'-degree and extends to NP, where $P \in \mathrm{Syl}_p(G)$. Then there exists $\chi \in B_0(G)$ of p'-degree over θ.*

Proof. Recall that every block $\bar{B}$ of G/N is contained in a unique block B of G (see e.g. the discussion before Theorem (7.6) of [N].) This easily proves (a).

Suppose now that $\gamma_i \in B_0(H_i)$ for $i = 1, 2$. Let $K_i = x_i^{H_i}$ be a conjugacy class of $x_i \in H_i$. Now, using the notation in Chapter 3 of [N], we have that

$$\lambda_{\gamma_1\times\gamma_2}(\widehat{K_1 \times K_2}) = \left(\frac{|K_1||K_2|\gamma_1(x_1)\gamma_2(x_2))}{\gamma_1(1)\gamma_2(1)}\right)^* = \left(\frac{|K_1|\gamma_1(x_1)}{\gamma_1(1)}\right)^* \left(\frac{|K_2|\gamma_1(x_2)}{\gamma_2(1)}\right)^*$$
$$= |K_1|^*|K_2|^* = |K_1 \times K_2|^* .$$

This proves (b).

Part (c) directly follows from Lemma 4.3 of [Mu]. □

We will need the following results in different parts of the paper.

Lemma 2.7. *Let p be a prime and let G a finite group.*

(i) *Suppose that $\mathbf{N}_G(P) = P \times A$ for $P \in \mathrm{Syl}_p(G)$ and some subgroup A. Then the number of p-blocks of G of maximal defect is $|\mathrm{Irr}(A)|$.*

(ii) *Suppose that $S = G/N$ for some p'-subgroup $N \lhd G$. Let $\alpha, \beta \in \mathrm{Irr}(S)$ and view α, β as characters of G. Then α, β belong to the same p-block of S if and only if α, β belong to the same p-block of G. Furthermore, $\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(B_0(S))$.*

(iii) *Suppose that G has only one conjugacy class of p-central subgroups of order p, $P \in \mathrm{Syl}_p(G)$ is non-abelian, and $C \leq \mathbf{Z}(P)$ has order p. Then $C \leq P'$.*

(iv) *Suppose that $A \leq B$ are normal subgroups of G such that A and G/B are p-groups. If B/A has a self-normalizing Sylow p-subgroup, then so does G.*

Proof. (i) Certainly, $A = \mathbf{O}_{p'}(\mathbf{N}_G(P))$ and $\mathbf{C}_G(P) = \mathbf{Z}(P) \times A$. Now we can apply [NT2, Lemma 3.1] (which is essentially Brauer's First Main Theorem).

(ii) is well known and easily follows from [N, Theorem (9.9(c))].

(iii) Recall $x \in G$ is said to be p-central if x centralizes a Sylow p-subgroup of G. Pick $x \in \mathbf{Z}(P) \cap P'$ of order p. By hypothesis, we may assume that x is conjugate to some generator y of C. By Burnside's fusion control lemma, $y = x^g$ for some $g \in \mathbf{N}_G(P) \leq \mathbf{N}_G(P')$. It follows that $y \in P'$.

(iv) Note that $G = BP$ for some $P \in \mathrm{Syl}_p(G)$, and $A \leq Q := P \cap B \in \mathrm{Syl}_p(B)$. By hypothesis, $\mathbf{N}_{B/A}(Q/A) = Q/A$. Hence, $\mathbf{N}_B(Q) = Q$ and $\mathbf{N}_G(P) = P$. □

Lemma 2.8. *Let G be a finite group, $N \lhd G$ and $P \in \mathrm{Syl}_2(G)$. Let $Q = P \cap N$. Suppose that P/QP' and Q/Q' are elementary abelian. If $2 \nmid |N/N'|$, then P/P' is elementary abelian.*

Proof. We may assume that $G = NP$. Let $\lambda \in \mathrm{Irr}(P)$ be linear. Then $\nu = \lambda_Q$ is P-invariant, and real as $Q' = \Phi(Q)$. By Lemma 2.5(b), there exists $\eta \in \mathrm{Irr}(N)$ which is P-invariant, of odd degree, real-valued, with $[\eta_Q, \nu]$ being odd. As $2 \nmid |N/N'|$, the determinantal order $o(\eta)$ is odd. Hence, by [Is, Corollary (8.16)], η has a unique extension ψ to G with $o(\psi) = o(\eta)$; in particular, $\psi = \bar{\psi}$ and $[\psi_Q, \nu] = [\eta_Q, \nu]$ is odd. By Lemma 2.5(a), ψ_P contains a real-valued character $\xi \in \mathrm{Irr}(P)$ such that $[\xi_P, \nu]$ is odd. Since ν is P-invariant and P is a 2-group, it follows that $\xi_Q = \nu = \lambda_Q$, whence

$\lambda = \epsilon\xi$ for some linear $\epsilon \in \mathrm{Irr}(P/Q)$ by Gallagher's Corollary (6.17) of [Is]. Since P/QP' is elementary abelian, we see that ϵ is real, and so λ is real. □

The following result will be useful in constructing irreducible characters belonging to the principal p-block of finite simple groups of Lie type. We refer the reader to [C], [DM] for basics of the Deligne-Lusztig theory.

Proposition 2.9. *Let p be a prime and let $\mathcal{G}$ be a simple algebraic group over a field of characteristic $\ell \neq p$ of adjoint type. Let $F : \mathcal{G} \to \mathcal{G}$ be a Frobenius endomorphism, $G := \mathcal{G}^F$, $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$, and let $G^* := (\mathcal{G}^*)^{F^*}$, $S := [G, G]$. Let $s \in G^*$ be a semisimple element. Then the following statements hold.*

(i) *Suppose that s is a p-element. Then the semisimple character χ_s corresponding to s belongs to the principal p-block $B_0(G)$ of G. Furthermore, every irreducible constituent of $(\chi_s)_S$ belongs to the principal p-block of S.*

(ii) *Suppose that s is not G^*-conjugate to sz whenever $1 \neq z \in \mathbf{Z}(G^*)$. Then $\theta := (\chi_s)_S \in \mathrm{Irr}(S)$. More generally, any $\varphi \in \mathcal{E}(G, (s))$ is irreducible over S. Moreover, if $k \in \mathbb{Z}$ is such that s^k is not G^*-conjugate to sz whenever $1 \neq z \in \mathbf{Z}(G^*)$, then $(\chi_s)_S = (\chi_{s^k})_S$ if and only if s and s^k are conjugate in G^*. In particular, if s^{-1} is not G^*-conjugate to any sz with $1 \neq z \in \mathbf{Z}(G^*)$, then θ is real if and only if s is real in G^*.*

(iii) *Suppose that $\gcd(|s|, |\mathbf{Z}(G^*)|) = 1$. Then $\theta := (\chi_s)_S \in \mathrm{Irr}(S)$. More generally, any $\varphi \in \mathcal{E}(G, (s))$ is irreducible over S. Moreover, if $k \in \mathbb{Z}$, then $(\chi_s)_S = (\chi_{s^k})_S$ if and only if s and s^k are conjugate in G^*. In particular, θ is real if and only if s is real in G^*.*

Proof. Since $\mathcal{G}$ is of adjoint type, $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, and so χ_s is well defined and has degree $[G^* : \mathbf{C}_{G^*}(s)]_{\ell'}$, cf. [DM, §14]. Now, (i) follows from [BNOT, Lemma 3.1] (which relies on the main results of [H]).

For (ii), we consider $z, t \in \mathbf{Z}(G^*)$. If $st = g(sz)g^{-1}$ for some $g \in G^*$, then $gsg^{-1} = s(tz^{-1})$, and so $z = t$ by hypothesis. Thus the elements sz with $z \in \mathbf{Z}(G^*)$ are pairwise non-conjugate. It follows that the rational series $\mathcal{E}(G, (sz))$, with z running over $\mathbf{Z}(G^*)$, are disjoint. In particular, the characters χ_{sz} are well defined and pairwise distinct for $z \in \mathbf{Z}(G^*)$. More generally, as shown in the proof of [DM, Proposition 13.30], if $\varphi \in \mathcal{E}(G, (s))$, then the $|G/S|$ characters $\varphi\lambda$, $\lambda \in \mathrm{Irr}(G/S)$, belong to these $|\mathbf{Z}(G^*)|$ disjoint series (note that $|G/S| = |\mathbf{Z}(G^*)|$), and so are pairwise distinct. Arguing as in the proof of [MT, Proposition 4.3] and using [MT, Lemma 4.2], we arrive at the first two statements of (ii), as well as

$$\mathrm{Irr}(G|\theta) = \{\chi_{sz} \mid z \in \mathbf{Z}(G^*)\}. \tag{2.1}$$

Now, if s and s^k are G^*-conjugate, then $\chi_s = \chi_{s^k}$. Conversely, assume that $(\chi_s)_S = (\chi_{s^k})_S$. Then $\chi_{s^k} \in \mathrm{Irr}(G|\theta)$, and so by (2.1) we have that $\chi_{s^k} = \chi_{st}$ for some $t \in \mathbf{Z}(L)$. Thus s^k and st are conjugate in G^*, whence $t = 1$ by hypothesis, proving the third statement. For the fourth statement, just note that $\overline{\chi_s} = \chi_{s^{-1}}$ and apply the third statement to $k = -1$.

For (iii), first we observe that the condition $\gcd(|s|, |\mathbf{Z}(G^*)|) = 1$ implies that s and sz are not conjugate in G^* whenever $1 \neq z \in \mathbf{Z}(G^*)$. Next, suppose that s^k is G^*-conjugate to st for some $k \in \mathbb{Z}$ and $t \in \mathbf{Z}(G^*)$. Then

$$|s^k| = |st| = |s| \cdot |t|.$$

As $|s^k|$ divides $|s|$, we conclude that $t = 1$. Now all the statements in (iii) follow from (ii). □

Remark 2.10. Note that the case P is abelian of Theorem B is quite straightforward, as the structure of finite groups with abelian Sylow 2-subgroups is known. Moreover, the case P is abelian of Gow's conjecture is elementary. More generally, if G is real, then it is easy to check that $\mathbf{Z}(P)$ is elementary abelian. (Every $z \in \mathbf{Z}(P)$ is conjugate to z^{-1} in G and so in $\mathbf{N}_G(P)$ by Burnside's fusion control lemma, whence $z^2 = 1$.) One more trivial remark is that if $P \in \mathrm{Syl}_2(G)$ is real, then every linear character of P is real, so $P' = \Phi(P)$.

We are ready to prove our main result. Recall that a simple group S is a **section** of a finite group G if there exists $Y \triangleleft X \leq G$ such that X/Y is isomorphic to S.

Theorem 2.11. *Suppose that $N \triangleleft G$. Set $p = 2$. Let $P/N \in \mathrm{Syl}_p(G/N)$. Let $\theta \in \mathrm{Irr}(N)$ be real P-invariant of odd degree. Suppose that the set $\mathrm{Irr}_{p'}(G|\theta) \cap B_0(G)$ is not empty and consists of real-valued characters. Also, suppose that Theorem* 2.1 *holds whenever the simple group S in Theorem* 2.1 *is a section of G/N. Then $P/P'N$ is elementary abelian.*

Proof. We argue by induction first on $|G|$, and then on $|G : N|$. By elementary group theory, recall that $(P/N)' = P'N/N$. Thus $(P/N)/(P/N)'$ is isomorphic to $P/P'N$. Hence our statement is equivalent to proving that the linear characters of P/N are real-valued.

Let $P \leq T$ be the stabilizer of θ in G. By the Clifford correspondence (Theorem (6.11) of [Is]), we know that induction defines a bijection $\mathrm{Irr}(T|\theta) \to \mathrm{Irr}(G|\theta)$. Since $|G : T|$ is odd, then induction defines a bijection $\mathrm{Irr}_{p'}(T|\theta) \to \mathrm{Irr}_{p'}(G|\theta)$. Since θ is real-valued, then we have that $\psi \in \mathrm{Irr}(T|\theta)$ is real-valued if and only if ψ^G is real-valued, by the uniqueness in the Clifford correspondence. Now, using Corollary (6.2) of [N], and the Third Main Theorem (6.7) of [N], if $T < G$, then we are done by induction. Therefore, we may assume that θ is G-invariant.

Now, let $\chi \in \mathrm{Irr}_{p'}(G|\theta) \cap B_0(G)$. Hence, χ_P contains an odd-degree irreducible constituent $\gamma \in \mathrm{Irr}_{p'}(P|\theta)$. Then $\gamma_P = \theta$ by Corollary (11.29) of [Is]. If Q/N is a Sylow q-subgroup of G/N for some odd prime q, since θ is real, then θ extends to Q/N by Lemma 2.1 of [NT1]. Hence, we have that θ extends to G by Corollary (11.31) of [Is]. By Theorem 2.3, let $\beta \in \mathrm{Irr}(G|\theta)$ be an extension of θ in the principal block of G, which by hypothesis is real-valued. Now, let $\gamma \in \mathrm{Irr}_{p'}(G/N)$ be in the principal block of G/N. In particular, γ lies in the principal block of G. By Lemma (3.5) of [NT2] and Gallagher's Corollary (6.17) of [Is], we have that $\beta\gamma$ lies in the principal block of G. Since $\beta\gamma$ has p'-degree, then we have that $\beta\gamma$ is real valued by hypothesis. Thus $\beta\bar{\gamma} = \beta\gamma$, and by the uniqueness in (6.17) of [Is], we have that $\bar{\gamma} = \gamma$. Hence, we have that $\mathrm{Irr}_{p'}(G/N) \cap B_0(G/N)$ are all real-valued (and certainly is not empty, because it contains the trivial character). Suppose that $|G/N| < |G|$. Applying the inductive hypothesis in the group G/N with respect to the trivial subgroup, we conclude that $(P/N)/(P/N)'$ is elementary abelian. Then $P/P'N$ is elementary abelian, and we are done.

Hence, we may assume that $N = 1$. Our current hypothesis is that every $\chi \in \mathrm{Irr}_{p'}(G) \cap B_0(G)$ is real-valued and we wish to prove that P/P' is elementary abelian, where $P \in \mathrm{Syl}_p(G)$. Or in other words, that the linear characters of P are real-valued.

If $1 < L$ is a normal subgroup of G, then we have that every $\chi \in \mathrm{Irr}_{p'}(G/L) \cap B_0(G/L)$ is real-valued, and by induction (and using Lemma 2.6 (a)) we conclude by induction that every linear character of PL/L is real-valued. In particular, we may assume that $\mathbf{O}_{2'}(G) = 1$ (that is, G has no odd order normal subgroups).

Now, let K be a minimal normal subgroup of G. Assume first that K is abelian. Therefore K is an elementary abelian 2-group, and $K \leq P$. Let $\lambda \in \mathrm{Irr}(P)$ be linear. Write $\nu = \lambda_K$ and let $P \leq I$ be the stabilizer of ν in G. If $\beta \in \mathrm{Irr}(I|\nu)$ has odd degree and lies in the principal block of I, then β^G has odd degree, lies in the principal block of G, and therefore is real-valued by hypothesis. Hence, notice that β is real (using that ν is real and the uniqueness in the Clifford correspondence). Now, notice that $\mathbf{C}_G(P) \leq \mathbf{C}_G(K) \leq I$, and that we may write $P\mathbf{C}_G(P) = P \times R$, by the Schur-Zassenhaus theorem. Let $\hat{\lambda} = \lambda \times 1_R \in \mathrm{Irr}(P\mathbf{C}_G(P))$, and let b be the principal block of $H := P\mathbf{C}_G(P)$ to which $\hat{\lambda}$ belongs. Now, $\hat{\lambda}^I$ has odd degree. Next, $P\mathbf{C}_I(P) \leq P\mathbf{C}_G(P) = H \leq I$ and so $H \leq \mathbf{N}_I(P)$, whence b^I is defined (by Theorem (4.14) of [N]). Therefore, by Lemma 2.4 there is some $\beta \in \mathrm{Irr}(I)$ of odd degree over $\hat{\lambda}$ in the principal block of I. In particular, β lies over ν, and we know that β is real. By Lemma 2.5(a) (applied when $R = 1$), there is some irreducible constituent ϵ of β_P which is linear and real, and over ν. By Gallagher, $\lambda = \epsilon\rho$, for some linear $\rho \in \mathrm{Irr}(P/K)$. By induction, we have that ρ is real-valued, so λ is real-valued. Hence we may assume that K is not abelian.

Now, let $\gamma \in \mathrm{Irr}_{p'}(KP) \cap B_0(KP)$. We claim that γ is real-valued. We have that $\gamma_K \in \mathrm{Irr}(K)$ by degrees (Corollary (11.29) of [Is]), and also γ_K belongs to the principal block of K (Theorem (9.2) of [N]). By Lemma 2.6 (c), there exists $\chi \in B_0(G)$ of odd degree over γ_K. We have that χ is real-valued, by hypothesis. Now, by Lemma 2.5(a) (applied in the case $R = 1$), we have that χ_{KP} contains an odd-degree real constituent $\tau \in \mathrm{Irr}(KP)$, and then $\tau_K \in \mathrm{Irr}(K)$ is real and lies under χ. By Clifford's theorem, γ_K and τ_K are conjugate, and therefore $\xi = \gamma_K$ is real too. Since K is not abelian, notice that the determinantal order of ξ is 1. Hence, ξ is P-invariant and $o(\xi)\xi(1)$ is odd, so ξ has a canonical extension $\hat{\xi} \in \mathrm{Irr}(KP)$ (by Corollary (6.28) of [Is]), which is real because it is canonical and uniquely determined by ξ. By Gallagher, $\gamma = \hat{\xi}\lambda$, for some $\lambda \in \mathrm{Irr}(PK/K)$ linear, which we know is real-valued by induction. Therefore, we conclude that γ is real. This proves the claim. By the inductive hypothesis, we may assume that $KP = G$.

Let $S \lhd K$ be a non-abelian simple group. Let $H = \mathbf{N}_G(S)$. Thus $G = HP$ and $Q = P \cap H \in \mathrm{Syl}_p(H)$. Let $R = K \cap P = K \cap Q$, and let $R_1 = R \cap S = P \cap S = Q \cap S \in \mathrm{Syl}_p(S)$. We can write $K = S^{x_1} \times \cdots \times S^{x_t}$, where $P = \bigcup_{j=1}^{t} Qx_j$ with $x_1 = 1$. Notice that

$$R = R_1^{x_1} \times \cdots \times R_1^{x_t}\,.$$

Furthermore, we claim that $Q = \mathbf{N}_P(R_1)$. Since $R_1 = P \cap S$ and $Q = \mathbf{N}_P(S)$, it follows that $Q \le \mathbf{N}_P(R_1)$. Conversely, suppose that $z \in \mathbf{N}_P(R_1)$. Let $1 \ne v \in R_1$. Then $v^z \in R_1 \le S$. On the other hand $v^z \in S^z = S^{x_j}$ for some j and $v^z \in S \cap S^{x_j}$. Necessarily $S^{x_j} = S = S^z$ and $z \in \mathbf{N}_P(S) = Q$.

Now, let $C = \mathbf{C}_G(S)$. Thus $S \le K \le SC \le H$, and H/C is almost simple with H/SC a 2-group. Also $QC/C \in \mathrm{Syl}_p(H/C)$. By hypothesis, we know that Theorem 2.1 is true for $S \cong SC/C$. We wish to apply Theorem 2.1 to H/C. Let $\gamma \in \mathrm{Irr}_{p'}(SC/C)$ be H-invariant in the principal 2-block of SC/C. Since $C \cap S = 1$, we have that $\gamma_S = \tau \in \mathrm{Irr}(S)$ is H-invariant of odd degree, and lies in the principal block of S. By Lemma 4.1 of [NTT1], we have that $\rho = \tau^{x_1} \times \cdots \times \tau^{x_t} \in \mathrm{Irr}(K)$ is G-invariant of odd degree. Also, it lies in the principal block of K by Lemma 2.6 (b). Now, ρ has a canonical extension $\hat{\rho}$ to G (by Corollary (6.28) of [Is]), which necessarily lies in the principal block of G (because G/K is a 2-group), and therefore is real by hypothesis. Since ρ_S is a multiple of τ, we conclude that τ (and therefore γ) are real-valued. By Theorem 2.1, we have that all the Q-invariant linear characters of R_1 are real-valued.

Now, let $\lambda \in \mathrm{Irr}(P)$ be linear. Then $\nu_1 = \lambda_{R_1}$ is Q-invariant, and therefore is real-valued. Also $\lambda_{R_1^{x_i}} = (\nu_1)^{x_i}$, and we conclude that $\nu = \lambda_R = \nu_1^{x_1} \times \cdots \times \nu_1^{x_t}$ is real-valued and P-invariant. Note that ν^K is a character of K which is real-valued, of odd-degree, and P-invariant. Hence, by Lemma 2.5 (b), ν^K contains an irreducible constituent η which is P-invariant, of odd degree, and real-valued with $[\nu^K, \eta] = [\eta_R, \nu]$ being odd. Let $\psi = \hat{\eta}$ be the canonical extension of η to G. Then $[\psi_R, \nu]$ is odd. Notice that ψ is real-valued of odd degree. Hence, ψ_P contains a real-valued character $\xi \in \mathrm{Irr}(P)$ such that $[\xi_P, \nu]$ is odd, by Lemma 2.5 (a). Since ν is P-invariant and P is a 2-group, it follows that $\xi_R = \nu$ and $\lambda = \epsilon\xi$, by Gallagher, for some linear $\epsilon \in \mathrm{Irr}(P/R)$. Since the linear characters of P/R are the linear characters of G/K, they are all real-valued, and the proof is complete. □

Now we can prove Theorem A (modulo Theorem 2.1 which we will prove in the next two sections).

Corollary 2.12. *Let G be a finite group with a Sylow 2-subgroup P. If all the odd-degree irreducible characters of G are real-valued, then P/P' is elementary abelian.*

Proof. Set $N = 1$ in the previous theorem. □

In the Introduction, we mentioned the following not so well-known fact on solvable groups. We take this opportunity to properly record it.

Theorem 2.13. *Let G be a finite solvable group, let $P \in \mathrm{Syl}_2(G)$, and let $N = \mathbf{N}_G(P)$. Then the number of real-valued odd-degree irreducible characters of G coincides with the number of real-valued odd-degree irreducible characters of N. In particular, all the odd-degree irreducible characters of G are real-valued if and only if $P' = \Phi(P)$ and $\mathbf{N}_G(P) = P$.*

Proof. By Theorem (10.9) of [I], there is a natural (choice-free) bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N)$. This bijection commutes with complex-conjugation, and therefore the first part of the theorem follows. Now, it suffices to show that all the odd-degree irreducible characters of N are real-valued if and only if $P = N$ and $P' = \Phi(P)$. One direction is obvious. For the other direction, notice that N/P is a group of odd order, and by Burnside's theorem does not have non-trivial real irreducible characters. Now the proof of the theorem easily follows. □

In fact, as one of the referees pointed out, Theorem 2.13 also follows from Theorems 5 and 6 of [Gow1].

Outside solvable groups, reality questions and McKay correspondences do not mix well. (In some sense, this makes Gow's conjecture even more interesting.) Already in A_5, the number of odd-degree real-valued characters of A_5 does not coincide with the number of odd-degree real-valued characters in the 2-Sylow normalizer, and while every odd-degree irreducible character is real-valued, we do not have that $P = \mathbf{N}_G(P)$. In the other direction, $G = \mathsf{A}_8$ has a self-normalizing Sylow 2-subgroup P with $P' = \Phi(P)$, but not every odd-degree irreducible character is real-valued.

3. Almost Simple Groups. I

In this and the next sections we prove Theorem 2.1.

Definition 3.1. Let p be a prime and a be any positive integer. A finite p-group P is called *p^a-good*, if $\exp(P/P') \leq p^a$. A finite group G is called *p^a-good* if $P \in \mathrm{Syl}_p(G)$ is p^a-good.

In what follows, N_p denotes the p-part of any positive integer N. We begin with some elementary observations.

Lemma 3.2. *Let p be a prime and a any positive integer, and let P, Q, R be finite p-groups.*

(i) *The set* $P[a] := \{x \in P \mid x^{p^a} \in P'\}$ *is a normal subgroup of* P. *Furthermore,* P *is* p^a*-good if and only* $P[a] = P$. *Also,* P *is* p*-good if and only if* $P' = \Phi(P)$.

(ii) *Suppose that* Q *and* R *are* p^a*-good subgroups of* P. *Then* $\langle Q, R\rangle$ *as well as any quotient of* Q *are* p^a*-good.*

(iii) *Suppose that* $P = \langle P', x_1, \ldots, x_m\rangle$ *with* $x_i^{p^a} \in P'$. *Then* P *is* p^a*-good.*

Proof. (i) is obvious. For (ii), note that $Q[a] = Q$ by (i), and so any quotient of Q is p^a-good. Similarly, $R[a] = R$. Now if $A := \langle Q, R\rangle$, then $A[a] \geq Q[a] = Q$ and $A[a] \geq R[a] = R$. It follows that $A[a] = A$ and so A is p^a-good.

For (iii), note that $P[a] \geq P'$ and $P[a] \ni x_i$ for all i, whence $P[a] = P$. □

The following statement is well known, but we give a proof for the reader's convenience.

Lemma 3.3. *Let* p *be any prime and* $n \in \mathbb{N}$ *any positive integer. Then* S_n *and* A_n *are* p*-good.*

Proof. (a) First we consider the case $G = \mathsf{S}_n$ and let $P \in \mathrm{Syl}_p(G)$. We prove by induction on n that P is p-good, with the induction base $1 \leq n \leq p$ being obvious. For the inductive step, first assume that $n = p^k > p$. Then $P \cong Q \wr C_p \cong Q^p \rtimes C_p$ with $Q \in \mathrm{Syl}_p(\mathsf{S}_{p^{k-1}})$ and $C \cong C_p$. By induction, Q and C_p are p-good, when P is p-good by Lemma 3.2(ii). Now if $n = ap^k$ with $k \geq 1$ and $1 < a < p$, then $P \cong R^a$ and $R \in \mathrm{Syl}_p(\mathsf{S}_{p^k})$ is p-good, whence P is p-good. Finally, suppose that $n = ap^k + b$ with $k \geq 1$, $1 \leq a < p$, and $1 \leq b < p^k$. Then $P \cong Q \times R$ with $Q \in \mathrm{Syl}_p(\mathsf{S}_{ap^k})$ and $R \in \mathrm{Syl}_p(\mathsf{S}_b)$. Since Q and R are p-good by induction, we conclude again by Lemma 3.2(ii) that P is p-good.

(b) Now we consider $H = \mathsf{A}_n < G = \mathsf{S}_n$ and let $Q \in \mathrm{Syl}_p(H)$. If $p > 2$, then $Q \in \mathrm{Syl}_p(G)$ and so Q is p-good by (a). So we may assume $p = 2$ and we will prove by induction on n that Q is 2-good. The induction base $n \leq 2$ is trivial. For the inductive step, if n is odd, then we can choose $Q \in \mathrm{Syl}_2(\mathsf{A}_{n-1})$ and apply the induction hypothesis. Suppose $n = 2m$ is even, and consider the imprimitive subgroup $A \cong C_2 \wr \mathsf{S}_m$ of G, which preserves the decomposition of $\{1, 2, \ldots, 2m\}$ into m pairs $\{i, i+m\}$, $1 \leq i \leq m$. Note that $A = E \rtimes B$, where $E < G$ is generated by the m transpositions $t_i = (i, i+m)$, $1 \leq i \leq m$, and $B \cong \mathsf{S}_m$ is generated by $s_1, \ldots, s_{m-1}$, where $s_i = (i, i+1)(i+m, i+m+1)$. In particular, $B < H$, and $E_1 := E \cap H \cong C_2^{m-1}$. By (a), $R \in \mathrm{Syl}_2(B)$ is 2-good, of order $(m!)_2$. Since $|Q| = 2^{m-1} \cdot (m!)_2$, we can take $Q = E_1 \rtimes R$ and conclude by Lemma 3.2(ii) that Q is 2-good. □

Lemma 3.4. *All sporadic simple groups are* p*-good for all primes* p.

Proof. The statement is verified in [W] (see the penultimate paragraph of [W, §1]). □

We will now start proving Theorem 2.1 for simple classical groups in odd characteristic.

Proposition 3.5. *Let q be any odd prime power. Then $L = Sp_{2n}(q)$ is 2-good.*

Proof. (a) First we consider the case $n = 1$ and choose $\varepsilon = \pm 1$ such that $4|(q-\varepsilon)$. Then for $Q \in \mathrm{Syl}_2(L)$ we have

$$Q = \langle x, y \mid |x| = (q-\varepsilon)_2,\ |y| = 4,\ y^2 = x^{(q-\varepsilon)_2/2}, yxy^{-1} = x^{-1}\rangle.$$

In particular, $Q' = \langle x^2 \rangle$ and $Q/Q' \cong C_2^2$, whence Q is 2-good.

(b) In the general case, note that $|P| = |Q|^n \cdot (n!)_2$, if $P \in \mathrm{Syl}_2(G)$ and $Q \in \mathrm{Syl}_2(Sp_2(q))$. Decompose the natural G-module $V = \mathbb{F}_q^{2n}$ into an orthogonal sum of n non-degenerate 2-dimensional subspaces $V_i = \langle e_i, f_i \rangle_{\mathbb{F}_q}$, with $(e_i, f_i) = 1$, $1 \leq i \leq n$ (and $(*, *)$ denotes the symplectic form on V). The stabilizer of this decomposition in $L = Sp_{2n}(q)$ is of the form $Sp_2(q) \wr \mathsf{S}_n$ (where S_n is generated by

$$s_{ij} : e_i \leftrightarrow e_j,\ \ f_i \leftrightarrow f_j, e_k \mapsto e_k, f_k \mapsto f_k, \forall k \neq i, j,$$

$1 \leq i \neq j \leq n$). Hence we can choose $P = Q^n \rtimes R$, with $R \in \mathrm{Syl}_2(\mathsf{S}_n)$. Since Q is 2-good by (a) and R is 2-good by Lemma 3.3, we are done. $\square$

Proposition 3.6. *Let q be any odd prime power and $m \in \mathbb{N}$. Then $L = \Omega^+_{4m}(q)$ is 2-good.*

Proof. Consider the natural L-module $V = \mathbb{F}_q^{4m} = V_1 \oplus V_2 \oplus \ldots \oplus V_m$ as an orthogonal sum of m isometric copies V_i, $1 \leq i \leq m$, of a quadratic 4-dimensional space V_1 of type $+$. Let F denote the quadratic form on V.

(a) First we define some "flip" τ_{12} between V_1 and V_2 that belongs to $L = \Omega(V)$. To do this, we fix an isometry $\sigma : V_1 \to V_2$ and an orthogonal basis (v_1, v_2, v_3, v_4) of V_1, such that $F(v_1) = F(v_3)$ and $F(v_2) = F(v_4)$. Then we define

(3.1)
$$\begin{aligned}
\tau_1 &: v_1 \leftrightarrow \sigma(v_1),\ v_2 \leftrightarrow \sigma(v_2),\ v_3 \mapsto v_3,\ v_4 \mapsto v_4,\ \sigma(v_3) \mapsto \sigma(v_3),\ \sigma(v_4) \mapsto \sigma(v_4),\\
\tau_2 &: v_3 \leftrightarrow \sigma(v_3),\ v_4 \leftrightarrow \sigma(v_4),\ v_1 \mapsto v_1,\ v_2 \mapsto v_2,\ \sigma(v_1) \mapsto \sigma(v_1),\ \sigma(v_2) \mapsto \sigma(v_2),\\
\tau_{12} &:= \tau_1\tau_2 : v_i \leftrightarrow \sigma(v_i),\ 1 \leq i \leq 4
\end{aligned}$$

(and with $\tau_{1,2}$ acting trivially on $V_3 \oplus \ldots \oplus V_m$). Then τ_1 and τ_2 both belong to $SO(V)$ and have the same spinor norm. It follows that $\tau_{12} \in \Omega(V)$.

(b) Next we study $Q_1 \in \mathrm{Syl}_2(GO(V_1))$ in more detail. First, we choose $\alpha = \pm 1$ such that $4|(q+\alpha)$ and note that $\alpha \in \mathbb{F}_q^{\times 2}$. Then we fix an orthogonal basis (e_1, f_1, e_2, f_2) of V_1 such that $F(e_i) = 1$ and $F(f_i) = \alpha$ for $i = 1, 2$. It follows that the transformation

$$\gamma_1 : e_1 \leftrightarrow e_2,\ \ f_1 \leftrightarrow f_2$$

belongs to $GO(V_1)$. In fact, if ρ_v denotes the reflection through the hyperplane $v^\perp$ for any anisotropic $v \in V$, then $\gamma_1 = \rho_{e_1-e_2} \cdot \rho_{f_1-f_2}$. Hence, $\det(\gamma_1) = 1$ and γ_1 has spinor norm $F(e_1 - e_2)F(f_1 - f_2) = 4\alpha \in \mathbb{F}_q^{\times 2}$, i.e. $\gamma_1 \in \Omega(V_1)$. Denoting $U_i = \langle e_i, f_i \rangle_{\mathbb{F}_q}$, we see that $V_1 = U_1 \oplus U_2$ and both U_1, U_2 are quadratic spaces of type $-\alpha$. In particular, $D_1 \in \mathrm{Syl}_2(GO(U_1))$ is dihedral of order $2(q+\alpha)_2$, and we can write

$$D_1 = \langle x_1, y_1 \mid |x_1| = (q+\alpha)_2, |y_1| = 2, y_1 x_1 y_1^{-1} = x_1^{-1} \rangle \in \mathrm{Syl}_2(GO(U_1)),$$

with $x_1 \in SO(U_1)$, and $y_1 \in GO(U_1) \smallsetminus SO(U_1)$. Then we choose $x_2 = \gamma_1 x_1 \gamma_1$ and $y_2 = \gamma_1 y_1 \gamma_1$. This choice of x_2 and y_2 ensures that $x_1 x_2, y_1 y_2 \in \Omega(V_1)$; furthermore,

$$D_2 = \langle x_2, y_2 \rangle \in \mathrm{Syl}_2(GO(U_2)).$$

Since γ_1 flips D_1 and D_2, it is easy to see that

$$Q_1 = \langle D_1, D_2, \gamma_1 \rangle = \langle x_1, x_2, y_1, y_2, \gamma_1 \rangle,\ Q^\circ := Q_1 \cap \Omega(V_1) = \langle x_1^2, x_2^2, x_1 x_2, y_1 y_2, \gamma_1 \rangle.$$

Furthermore, $D_1, D_2 \cong D_{2(q+\alpha)_2}$ are 2-good and $|\gamma_1| = 2$, so $Q_1 \cong (D_1 \times D_2) \rtimes C_2$ is 2-good by Lemma 3.2(ii). Also note that

$$[y_1 y_2, x_1^2] = x_1^4,\ [\gamma_1, x_1^2] = x_2^2 x_1^{-2},$$

i.e. $x_i^4, x_1^2 x_2^2 \in (Q^\circ)'$. It follows by Lemma 3.2(iii) that Q° is 2-good; in particular, we are done if $m = 1$. (This also follows from Proposition 3.5 as $\Omega_4^+(q) \cong SL_2(q) \circ SL_2(q)$ is a quotient of $Sp_2(q)^2$.)

(c) Assume now that $m \geq 2$. For each $2 \leq i \leq m$ we fix an isometry $\sigma_i : V_1 \to V_i$ and use (3.1) to define the flip τ_{1i}. Next, we define

$$Q_i := \langle x_{2i-1}, x_{2i}, y_{2i-1}, y_{2i}, \gamma_i \rangle \in \mathrm{Syl}_2(GO(V_i)),$$

with

$$x_{2i-1} = \tau_{1i} x_1 \tau_{1i},\ x_{2i} = \tau_{1i} x_2 \tau_{1i},\ y_{2i-1} = \tau_{1i} y_1 \tau_{1i},\ y_{2i} = \tau_{1i} y_2 \tau_{1i},\ \gamma_i = \tau_{1i} \gamma_1 \tau_{1i}.$$

Note that

$$T := \langle \tau_{12}, \tau_{13}, \ldots, \tau_{1m} \rangle \cong \mathsf{S}_m$$

is contained in $\Omega(V)$. The stabilizer A in $GO(V)$ of the decomposition $V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$ is then of the form

$$GO(V_1) \wr \mathsf{S}_m = (GO(V_1) \times GO(V_2) \times \ldots \times GO(V_m)) \rtimes T.$$

One can check that A has *odd* index in $GO(V)$. It follows that

$$\tilde{P} := (Q_1 \times Q_2 \times \ldots \times Q_m) \rtimes R \in \mathrm{Syl}_2(GO(V)),$$

if we choose $R \in \mathrm{Syl}_2(T)$. As $R < \Omega(V)$, we then have

$$P := Q \rtimes R \in \mathrm{Syl}_2(\Omega(V)) \tag{3.2}$$

for $Q := (Q_1 \times Q_2 \times \ldots \times Q_m) \cap \Omega(V)$. Our construction ensures that $x_i x_j, y_i y_j \in \Omega(V)$ for all i, j. Consequently,

$$Q = \langle x_i^2, \gamma_k, x_i x_j, y_i y_j \mid 1 \leq i, j \leq 2m, 1 \leq k \leq m \rangle. \tag{3.3}$$

Since $m \geq 2$, we can observe that $(y_1 y_2) x_1 x_3 (y_1 y_2)^{-1} = x_1^{-1} x_3$, and so $x_1^2 \in Q'$. Similarly, $x_i^2 \in Q'$ for all i. Now we have that $(x_i x_j)^2 \in Q'$, $(y_i y_j)^2 = \gamma_k^2 = 1$, and so (3.3) implies by Lemma 3.2(iii) that Q is 2-good. The latter conclusion, together with (3.2), then implies by Lemma 3.2(ii) that P is 2-good. □

Proposition 3.6 will play a key role in the following treatment of all orthogonal groups:

Proposition 3.7. *Let q be an odd prime power, $\epsilon = \pm$, and $n \geq 7$ be an integer. Then $L = \Omega^{\epsilon}_n(q)$ is 2-good.*

Proof. We will keep the notation $F, V, P, \tilde{P}$ of the proof of Proposition 3.6. In particular, $V = \mathbb{F}_q^{4m}$ is endowed with a quadratic form F of type $+$, $\tilde{P} \in \mathrm{Syl}_2(GO(V))$, and $P \lhd \tilde{P}$.

(a) Here we show that if $W = \mathbb{F}_q^d$ is a non-degenerate quadratic space over $\mathbb{F}_q$ of dimension $1 \leq d \leq 4$, then $GO(W)$ is 2-good. If $d = 1$, then $GO(W) \cong C_2$ and so we are done. If $d = 2$, then $GO(W)$ is dihedral of order $2(q \mp 1)$ and the claim follows. Suppose $d = 3$. Then, in the notation of part (b) of the proof of Proposition 3.6, we may assume that

$$W = \langle e_1, f_1, g\rangle_{\mathbb{F}_q} = U_1 \oplus \langle g\rangle_{\mathbb{F}_q}$$

with $g \perp U_1$ and $F(g) = 1$. Choosing

$$D := \langle D_1, \rho_g\rangle \cong D_{2(q+\alpha)_2} \times C_2$$

we see that $D \in \mathrm{Syl}_2(GO(W))$ and D is 2-good.

Assume now that $d = 4$. The case where W is of type $+$ is already treated in part (b) of the proof of Proposition 3.6. Hence we may assume that W is of type $-$ and write $W = W_+ \oplus W_-$ with $W_+ \perp W_-$, and W_β is of type β and dimension 2 for $\beta = \pm$. Picking $D_\beta \in \mathrm{Syl}_2(GO(W_\beta))$, we then have

$$D := D_+ \times D_- \cong D_{2(q-1)_2} \times D_{2(q+1)_2} \in \mathrm{Syl}_2(GO(W)),$$

and D is 2-good.

(b) Now we write $n = 4m + d$ with $1 \leq d \leq 4$. By Proposition 3.6, we may assume $\epsilon = -$ if $d = 4$. We can write $L = \Omega(V^*)$ with $V^* = V \oplus W$, with W as in (a), of type ϵ, and $V \perp W$. We will assume in addition that $m \geq 2$ if $(d, \epsilon) = (4, -)$. Our assumptions guarantee that V contains a subspace W_0 isometric to W. We will embed $GO(W_0)$ in $GO(V)$ by letting each $f \in GO(W_0)$ act trivially on the orthogonal complement to W_0 in V. If $D_0 \in \mathrm{Syl}_2(GO(W_0))$, then D_0 is contained in a conjugate $h\tilde{P}h^{-1}$ of $\tilde{P} \in \mathrm{Syl}_2(GO(V))$, with $h \in GO(V)$. Replacing W_0 by $h^{-1}(W_0)$, we then have that W_0 is isometric to W and $D_0 \leq \tilde{P}$. The latter implies that D_0 normalizes $P \in \mathrm{Syl}_2(\Omega(V))$, whereas the former implies that $D_0 \cong D$ where $D \in \mathrm{Syl}_2(GO(W))$; fix an isomorphism $\sigma : D_0 \to D$. Now we define

$$D^* := \{t\sigma(t) \mid t \in D_0\},\ \ P^* = PD^*.$$

Since $[D, \tilde{P}] = 1$ and $D_0 \leq \tilde{P}$, we see that D^* is a group isomorphic to D. Furthermore, for any $t \in D_0$, t normalizes P and $\sigma(t) \in D$ centralizes P, whence D^* normalizes P and P^* is a group. As $\tilde{P} \cap D = 1$, we also have $P \cap D^* = 1$. Thus $P^* = P \rtimes D^*$. Since P is 2-good by Proposition 3.6 and $D^* \cong D$ is 2-good by (a), we conclude by Lemma 3.2(ii) that P^* is 2-good. It is straightforward to check that $P^* \in \mathrm{Syl}_2(\Omega(V^*))$.

(c) It remains to consider the case $(n, \epsilon) = (8, -)$. In this case, we can write $L = \Omega(V^*)$ with $V^* = V \oplus W$, where, changing the notation, now we have that $V = \mathbb{F}_q^6$ is a quadratic space of type $+$ and $W = \mathbb{F}_q^2$ is a quadratic space of type $-$ orthogonal to V. According to (b), $P \in \mathrm{Syl}_2(\Omega(V))$ is 2-good. Furthermore, $D \in \mathrm{Syl}_2(GO(W))$ is 2-good as mentioned in (a). Now we can argue as in (b) to get a Sylow 2-subgroup $P^* = P \rtimes D^*$ of $\Omega(V^*)$, with $D^* \cong D$, and conclude that P^* is 2-good. $\square$

In what follows, we use the notation SL^ϵ to denote SL if $\epsilon = +$ and SU if $\epsilon = -$, and similarly for GL^ϵ. Also, abusing the notation, we will treat ϵ with $\epsilon = \pm$ as $\epsilon 1$ in expressions like $q - \epsilon$, etc.

Proposition 3.8. *Let q be an odd prime power and $\epsilon = \pm$. Then $L = SL^\epsilon_{2m}(q)$ is 2-good if $4|(q - \epsilon)$ and m is a 2-power, or if $4|(q + \epsilon)$.*

Proof. (a) We define $q^* = q$ if $\epsilon = +$ and $q^* = q^2$ if $\epsilon = -$. Since the case $m = 1$ follows from Proposition 3.5, we may assume that $m \geq 2$. Embed L in $G = GL^\epsilon_{2m}(q)$ and consider the natural $\mathbb{F}_{q^*}G$-module $V = \mathbb{F}_{q^*}^{2m}$. Then we can decompose V into a direct, orthogonal if $\epsilon = -$, sum of m 2-dimensional subspaces over $\mathbb{F}_{q^*}$:

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_m. \tag{3.4}$$

Fixing an isomorphism $\sigma_i : V_1 \cong V_i$, isometric if $\epsilon = -$, for each $2 \leq i \leq m$, we can define a flip $\tau_{1i} : V_1 \leftrightarrow V_i$ by

$$\tau_{1i} : u \leftrightarrow \sigma_i(u), \ \forall u \in V_1$$

(and with τ_{1i} acting trivially on all V_j with $j \neq 1, i$). Since $\dim V_i = 2$, we see that $\tau_{1i} \in L$, and $A := \langle \tau_{1i} \mid 2 \leq i \leq m \rangle \cong \mathsf{S}_m$ permutes $V_1, \ldots, V_m$ naturally. Then the stabilizer B of the decomposition (3.4) in G is

$$B = (X_1 \times X_2 \times \ldots \times X_m) \rtimes A,$$

where $GL(V_i) \geq X_i \cong GL^\epsilon_2(q)$. We also set $r := (q^2 - 1)_2/2$.

(b) Here we consider the case $4|(q - \epsilon)$, so that $r = (q - \epsilon)_2$, and assume that m is a 2-power. Then a Sylow 2-subgroup $Y_1 \cong C_r^2 \rtimes C_2$ of X_1 can be generated by x_1, x_2, t_1, where

$$|x_1| = |x_2| = r, \ t_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in L, \ x_2 = t_1 x_1 t_1^{-1}, [x_1, x_2] = 1, (x_1 x_2^{-1})^{r/2} = t_1^2. \tag{3.5}$$

Next, for each $2 \le i \le m$, we set $Y_i := \langle x_{2i-1}, x_{2i}, t_i \rangle$, where

$$x_{2i-1} = \tau_{1i} x_1 \tau_{1i},\ x_{2i} = \tau_{1i} x_2 \tau_{1i},\ t_i = \tau_{1i} t_1 \tau_{1i}.$$

Since $[G : B]$ is odd, we can find $\tilde{P} := \tilde{Q} \rtimes T \in \mathrm{Syl}_2(G)$ with

$$\tilde{Q} := Y_1 \times Y_2 \times \ldots \times Y_m,\ T \in \mathrm{Syl}_2(A).$$

It follows that $P := Q \rtimes T \in \mathrm{Syl}_2(L)$, where

$$Q := \tilde{Q} \cap L = \langle t_k, x_i x_j^{-1} \mid 1 \le k \le m, 1 \le i \ne j \le 2m \rangle.$$

Note that Q is A-invariant, so we can choose any Sylow 2-subgroup of A to play the role of T. Since $m \ge 2$ is a 2-power, we can identify $\{1, 2, \ldots, m\}$ with a vector space W over $\mathbb{F}_2$ and then choose T to contain the regular subgroup T_1 of all translations $t_v : u \mapsto u + v$ on W. In particular, given any two indices $1 \le i \ne j \le m$, we can find a translation $\tau'_{ij} \in T_1$ that interchanges i and j. Now,

$$(x_1 x_2^{-1})^2 = [x_1 x_2^{-1}, t_1],\ (x_1 x_3^{-1})^2 = [x_1 x_3^{-1}, \tau'_{12}],$$

and similarly $(x_i x_j^{-1})^2 \in P'$ for all $i \ne j$. As $4|r$, (3.5) then implies that

$$t_1^2 = ((x_1 x_2^{-1})^2)^{r/4} \in P',$$

and similarly $t_i^2 \in P'$ for all i. We have shown that $P[1] \ni t_i, x_i x_j^{-1}$ for all $i \ne j$, and so $P[1] \ge Q$. Furthermore, T is 2-good by Lemma 3.3, whence $P[1] \ge T$. Consequently, $P[1] = P$, i.e. P is 2-good.

(c) Now we may assume $4|(q + \epsilon)$, so that $r := (q + \epsilon)_2$. Then a Sylow 2-subgroup $Y_1 \cong C_{2r} \cdot C_2$ of X_1 can be generated by x_1, t_1, where

$$\text{(3.6)} \qquad |x_1| = 2r,\ |t_1| = 4,\ t_1^2 = x_1^r,\ t_1 x_1 t_1^{-1} = x_1^{r-1},\ \det(x_1) = -1,\ \det(t_1) = 1.$$

(Indeed, we can find $\alpha \in \mathbb{F}_{q^2}^\times$ of order $2r = (q^2 - 1)_2$. Now if $\epsilon = +$, then we can identify V_1 with $\mathbb{F}_{q^2}$ and take x_1 to be the multiplication by α, z_1 to be the Frobenius map $v \mapsto v^q$, and set $t_1 := x_1 z_1$. If $\epsilon = -$, then we can choose a Witt basis (e_1, f_1) for $V_1 = \mathbb{F}_{q^2}^2$, and take

$$x_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-q} \end{pmatrix},\ z_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in this basis, and again set $t_1 := x_1 z_1$.) Next, for each $2 \le i \le m$, we set $Y_i := \langle x_i, t_i \rangle$, where

$$x_i = \tau_{1i} x_1 \tau_{1i},\ t_i = \tau_{1i} t_1 \tau_{1i}.$$

Since $[G : B]$ is odd, we can find $\tilde{P} := \tilde{Q} \rtimes T \in \mathrm{Syl}_2(G)$ with

$$\tilde{Q} := Y_1 \times Y_2 \times \ldots \times Y_m,\ T \in \mathrm{Syl}_2(A).$$

It follows that $P := Q \rtimes T \in \mathrm{Syl}_2(L)$, where

$$Q := \tilde{Q} \cap L = \langle t_i, x_i^2, x_i x_j^{-1} \mid 1 \le i \ne j \le m \rangle.$$

Recall that $m \geq 2$. Now,

$$x_1^{-2} = (x_1^{r-2})^{r/2+1} = [x_1^{-1}x_2, t_1]^{r/2+1},$$

and so $x_i^2 \in Q'$ for all i. Next, (3.6) implies that $t_1^2 = (x_1^2)^{r/2} \in Q'$, and similarly $t_i^2 \in Q'$ for all i. We have shown that $Q[1] \ni t_i, x_i^2, x_i x_j^{-1}$ for all $i \neq j$, and so $Q[1] \geq Q$. Furthermore, T is 2-good by Lemma 3.3, whence $P[1] \geq T$. Consequently, $P[1] = P$, i.e. P is 2-good. □

Corollary 3.9. *Let q be an odd prime power, $m \in \mathbb{N}$, $\epsilon = \pm$, and $4|(q+\epsilon)$. Then $S = SL^\epsilon_{2m+1}(q)$ is 2-good.*

Proof. We keep the notation of the proof of Proposition 3.8; in particular, $V = \mathbb{F}_{q^*}^{2m}$, $\tilde{P} \in \mathrm{Syl}_2(GL^\epsilon(V))$, and $P \lhd \tilde{P}$. Then we can write $S = SL^\epsilon(V^*)$ with $V^* = V \oplus W$, with $W = \mathbb{F}_{q^*}$ and furthermore $V \perp W$ if $\epsilon = -$. We can pick a subspace $W_0 \subset V$ isometric to W. If $D_0 \in \mathrm{Syl}_2(GL^\epsilon(W_0))$, then $D_0 = \langle d_0 \rangle \cong C_2$ is contained in a conjugate $h\tilde{P}h^{-1}$ of $\tilde{P} \in \mathrm{Syl}_2(GL^\epsilon(V))$, with $h \in GL^\epsilon(V)$. Replacing W_0 by $h^{-1}(W_0)$, we then have that W_0 is isometric to W and $D_0 \leq \tilde{P}$. The latter implies that D_0 normalizes $P \in \mathrm{Syl}_2(SL^\epsilon(V))$, whereas the former implies that $D_0 \cong D$ where $D = \langle d \rangle \in \mathrm{Syl}_2(GL^\epsilon(W))$. Also note that $\det(d_0) = \det(d) = -1$. Now we define

$$D^* := \langle d_0 d \rangle,\ P^* = PD^*.$$

Since $[D, \tilde{P}] = 1$ and $D_0 \leq \tilde{P}$, we see that D^* is a group isomorphic to D and D^* normalizes P, whence P^* is a group. As $\tilde{P} \cap D = 1$, we also have $P \cap D^* = 1$. Thus $P^* = P \rtimes D^* \cong P \rtimes C_2$. Since P is 2-good by Proposition 3.8, we conclude by Lemma 3.2(ii) that P^* is 2-good. Finally, it is easy to check that $P^* \in \mathrm{Syl}_2(SL^\epsilon(V^*))$. □

Now we show that Theorem 2.1 holds for $S = PSL^\epsilon_n(q)$ in the case $4|(q-\epsilon)$ but n is not a 2-power.

Proposition 3.10. *Let $q = p^f$ be a power of an odd prime p, $\epsilon = \pm$, $4|(q-\epsilon)$, and $S = PSL^\epsilon_n(q)$, where $n \geq 3$ is not a 2-power. Let $S \lhd G$, where G is a finite group, G/S is a 2-group, and $\mathbf{C}_G(S) = 1$. Let $R \in \mathrm{Syl}_2(G)$ and $P = R \cap S$. Suppose that all the R-invariant complex irreducible characters of odd degree in the principal 2-block of S are real-valued. Then every linear R-invariant character of P is real.*

Proof. (a) Write $n = 2m + \kappa$ with $\kappa \in \{0, 1\}$. We will view $S = L/\mathbf{Z}(L)$, where $L = SL^\epsilon_n(q)$, and set $r := (q-\epsilon)_2 \geq 4$. Keeping the notation q^* from the proof of Proposition 3.8, consider $V = \mathbb{F}_{q^*}^n$ with a (orthonormal if $\epsilon = -$) basis $(e_1, \ldots, e_n)$. Using this basis, we can define the transpose-inverse automorphism $\tau : Y \mapsto {}^tY^{-1}$ and the Frobenius automorphism $\sigma : Y = (y_{ij}) \mapsto (y_{ij}^p)$ of the groups $GL^\epsilon(V)$, L, and S. Write $f = 2^e f_0$ for some $e \in \mathbb{Z}_{\geq 0}$ and odd f_0, and let $\sigma_0 := \sigma^{f_0}$. (Note that the automorphism σ_0 of L has order 2^e if $\epsilon = +$ and 2^{e+1} if $\epsilon = -$; in the latter case, $\tau = \sigma_0^{2^e}$.) Also, we set

$$M := \{Y \in GL^\epsilon(V) \mid \det(Y)^r = 1\},\ \Gamma := M \rtimes \langle \tau, \sigma_0 \rangle,\ Z := \mathbf{Z}(M).$$

Then Γ/Z induces an odd-index subgroup of $\mathrm{Aut}(S)$. Since $\mathbf{C}_G(S) = 1$ and G/S is a 2-group, after a suitable conjugation in $\mathrm{Aut}(S)$, we may assume that $G \leq \Gamma/Z$.

Fix $\alpha \in \mathbb{F}_{q^*}^\times$ of order r, and define

$$x_i = \mathrm{diag}(I_{i-1}, \alpha, I_{n-i}),\ 1 \leq i \leq n;\ t_j = \mathrm{diag}(I_{2j-2}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_{n-2j}),\ 1 \leq j \leq m$$

in the chosen basis of V. We also consider the "flips"

$$\tau_{1i} : e_1 \leftrightarrow e_{2i-1},\ e_2 \leftrightarrow e_{2i},\ e_j \mapsto e_j,\ \forall j \neq 1, 2, 2i-1, 2i$$

for $1 \leq i \leq m$. Then $A := \langle \tau_{1i} \mid 2 \leq i \leq m \rangle \cong \mathsf{S}_m$, and we fix $T \in \mathrm{Syl}_2(A)$. Now it is easy to check that $\hat{P} := Q \rtimes T \in \mathrm{Syl}_2(L)$, where

$$Q := \langle x_i x_j^{-1}, t_k \mid 1 \leq i \neq j \leq n, 1 \leq k \leq m \rangle. \tag{3.7}$$

Moreover, denoting

$$B := \langle x_1 \rangle \rtimes \langle \tau, \sigma_0 \rangle,$$

we see that $R^* := \hat{P} \rtimes B$ is a Sylow 2-subgroup of Γ. Let $\hat{G}$ denote the full inverse image of G in $\Gamma = L \rtimes B$. Then

$$\hat{G} = L \rtimes (\hat{G} \cap B),\ \hat{R} = \hat{P} \rtimes (\hat{G} \cap B) \in \mathrm{Syl}_2(\hat{G}). \tag{3.8}$$

We can and will identify P with $\hat{P}/(Z \cap \hat{P})$, and R with $\hat{R}/Z$.

(b) Since n is not a 2-power, we can write $n = a + b$, where

$$a = 2^d \geq 2,\ 1 \leq b = 2^{d'} b' \leq 2^d - 1,\ 0 \leq d' < d,\ 2 \nmid b'.$$

As $\gcd(b', r) = 1$, we can find $x, y \in \mathbb{Z}$ such that $xb' = yr - 1$. Setting

$$j = 2^{d-d'} x, \tag{3.9}$$

we then have $a + bj = 2^d yr$ and so we can consider the 2-element

$$s = \mathrm{diag}(\alpha I_a, \alpha^j I_b) \in L. \tag{3.10}$$

Note that the choice (3.9) implies that $2|j$. In what follows, we will work with some 2-powers l, where $r > l \geq 1$. For such an l, $\alpha^l \neq \alpha^{lj}$, and so

$$\mathbf{C}_{GL^\epsilon(V)}(s^l) \cong GL^\epsilon_a(q) \times GL^\epsilon_b(q).$$

Next we observe that if k is any odd integer and s^{kl} is L-conjugate to $s^l z$ for some $z \in \mathbf{Z}(L)$, then

$$z = 1,\ \alpha^{l(k-1)} = 1. \tag{3.11}$$

Indeed, if β is the (unique) eigenvalue of z (acting on V), then the conditions $a > b$, $\alpha^l \neq \alpha^{lj}$, and $\alpha^{kl} \neq \alpha^{klj}$ imply by comparing eigenvalues and their multiplicities that

$$\alpha^{kl} = \alpha^l \beta,\ \alpha^{klj} = \alpha^{lj}\beta.$$

It follows that $\beta = \alpha^{l(k-1)}$; in particular, β is a 2-element; and that $\beta^{j-1} = 1$. Since $2|j$, we conclude that $\beta = 1$ and $z = 1$.

We can also view L as the dual group H^* and S as $[H, H]$, where $H = PGL^{\epsilon}_n(q)$ is of adjoint type. By virtue of (3.11) with $k = 1$, we can apply Proposition 2.9(i), (ii) to the semisimple character χ_{s^l} of H and conclude that $\theta_l := (\chi_{s^l})_S$ is an irreducible character in $B_0(S)$, of degree

$$\theta_l(1) = [L : \mathbf{C}_L(s^l)]_{p'} = \frac{|GL^{\epsilon}_{a+b}(q)|_{p'}}{|GL^{\epsilon}_a(q)|_{p'} \cdot |GL^{\epsilon}_b(q)|_{p'}}$$

which is odd, since $\binom{a+b}{a}$ is odd, see [NT3, Lemma 4.4(i)]. Furthermore, applying (3.11) to $(l, k) = (1, -1)$ and recalling that $|\alpha| = r \geq 4$, we see that $(s^{-1})^L \cap s\mathbf{Z}(L) = \emptyset$. It follows by Proposition 2.9(ii) that θ_1 is non-real. The latter implies by the hypothesis that θ_1 cannot be R-invariant. Since $M/Z \leq H$ and θ_1 is clearly H-invariant, we have therefore shown that $R > P$ and $G(M/Z) > M/Z$, whence $\hat{G}M > M$.

(c) Assume now that $\tau \in \hat{G}M$. Since $\hat{G} > L$ and $M = L\langle x_1\rangle$, it follows that $\tau = \tau' x_1^{-i_0}$ for some $\tau' \in \hat{G}$ and $i_0 \in \mathbb{Z}$. Thus $\tau' = \tau x_1^{i_0} \in \hat{G} \cap B$, and so $\tau' \in \hat{R}$ by (3.8). Let $\lambda \in \mathrm{Irr}(P/P')$ be R-invariant. Then we can view λ as an $\hat{R}$-invariant character of $\hat{P}/\hat{P}'$. Thus $\lambda = \lambda^{\tau'}$ and so $K := \mathrm{Ker}(\lambda)$ contains $\tau'(x)x^{-1}$ for all $x \in \hat{P}$. Certainly, x_1 centralizes all x_i and τ inverts each x_i. Hence

$$\hat{P}' \leq K \ni \tau'(x_i x_j^{-1})(x_i x_j^{-1})^{-1} = (x_i x_j^{-1})^{-2}$$

for all $1 \leq i \neq j \leq n$. Since $4|r$, this implies that $t_k^2 = (x_{2k-1}x_{2k}^{-1})^{r/2} \in K$. As T is 2-good by Lemma 3.3, we also have that $\hat{P}' \ni y^2$ for all $y \in T$. It follows from (3.7) and Lemma 3.2(iii) that $\hat{P}/K$ is 2-good, whence λ is real.

(d) Note that if $\epsilon = -$, then $e = 0$ and $B = \langle x_1, \tau\rangle$ with $M = L\langle x_1\rangle$. In this case, $\hat{G}M > M$ implies that $\tau \in \hat{G}M$ and so we are done by (c). So it remains to consider the case where $\epsilon = +$, $\tau \notin \hat{G}M$, and $e \geq 1$. Note that $B_1 := \langle\sigma_0\rangle \times \langle\tau\rangle \cong C_{2^e} \times C_2$ and $\hat{G}M = M \rtimes (\hat{G}M \cap B_1)$. Also,

$$\hat{R} \leq \tilde{R} := \langle\hat{P}, x_1\rangle \rtimes (\hat{G}M \cap B_1) \in \mathrm{Syl}_2(\hat{G}M). \tag{3.12}$$

Now the assumption $\tau \notin \hat{G}M > M$ implies that there exist some $1 \leq e_1 \leq e$ and some $j = 0, 1$ such that $\hat{G}M \cap B_1 = \langle\sigma_0^{2^{e-e_1}}\tau^j\rangle \cong C_{2^{e_1}}$. Note that $e_1 > 0$ when $j = 1$, as otherwise $\tau \in \hat{G}M$. Set $q_1 := p^{2^{e-e_1}f_0}$, $\sigma_1 = \sigma_0^{2^{e-e_1}}$, so that $q = q_1^{2^{e_1}}$. Again we can write $\sigma_1\tau^j = \sigma' x_1^{-i_1}$ for some $\sigma' \in \hat{G}$ and $i_1 \in \mathbb{Z}$. Thus $\sigma' = \sigma_1\tau^j x_1^{i_1} \in \hat{G} \cap B$ and so $\sigma' \in \hat{R}$ by (3.8).

Recall that $\alpha \in \mathbb{F}_q^\times$ has order $r = (q-1)_2$. As $e_1 \geq 1$, $q - 1 = q_1^{2^{e_1}} - 1$ is divisible by $q_1 - (-1)^j$. Hence there is a 2-power $1 \leq l < r$ such that α^l has order $(q_1 - (-1)^j)_2 \geq 2$. Our choice of l implies that $\sigma_1\tau^j(\alpha^l) = \alpha^l$ (where we let τ act on $\mathbb{F}_q^\times$ as the inversion), and so the element s^l, with s as defined in (3.10), is $\sigma_1\tau^j$-invariant. It then follows by [NTT2, Corollary 2.5] that the characters χ_{s^l} and θ_l (as defined in (b)) are $\sigma_1\tau^j$-invariant. Since θ_l is invariant under $M/Z \leq H$, we see by (3.12) and Proposition 2.9(i), (ii), that θ_l is an R-invariant irreducible character of odd degree in $B_0(S)$, whence θ_l is real by the hypothesis. The latter implies by (3.11) (with $k = -1$) and Proposition 2.9(ii) that s^l is real in L. Using (3.11) again with $k = -1$, we see that $|\alpha^l| \leq 2$, and so

$$(q_1 - (-1)^j)_2 = 2. \tag{3.13}$$

Consider any R-invariant $\lambda \in \mathrm{Irr}(P/P')$. Then we can view λ as an $\hat{R}$-invariant character of $\hat{P}/\hat{P}'$. As $\sigma' \in \hat{R}$, $\lambda = \lambda^{\sigma'}$ and so $K := \mathrm{Ker}(\lambda)$ contains $\sigma'(x)x^{-1}$ for all $x \in \hat{P}$. Note that $\sigma'(x_i) = \sigma_1\tau^j(x_i) = x_i^{(-1)^j q_1}$, and so $\hat{P}' \leq K \ni (x_i x_j^{-1})^{q_1 - (-1)^j}$ for all $1 \leq i \neq j \leq n$. But $|x_i x_j^{-1}| = r$, so in fact (3.13) implies that $K \ni (x_i x_j^{-1})^2$ for all such i, j. Now we can conclude as in (c) that $\hat{P}/K$ is 2-good, whence λ is real, as desired. □

Combining Propositions 3.5, 3.7, 3.8, 3.10, and Corollary 3.9, we obtain

Corollary 3.11. *Theorem 2.1 holds if S is a simple classical group in odd characteristic.* □

4. Almost Simple Groups. II

In what follows, the notation E_6^ϵ is used for E_6 if $\epsilon = +$ and ${}^2E_6$ if $\epsilon = -$.

Proposition 4.1. *Let q be an odd prime power, and let the simple group S be one of the following exceptional groups: ${}^2G_2(q)$, $G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, $E_7(q)$, $E_8(q)$, or $S = E_6^\epsilon(q)$ with $q \equiv -\epsilon (\mathrm{mod}\, 4)$. Then S is 2-good.*

Proof. Let $P \in \mathrm{Syl}_2(S)$. If $S = {}^2G_2(q)$, then $P \cong C_2^3$, and so we are done. For other types, note that $P' \neq 1$ and S has only one conjugacy class of 2-central involutions (see [GLS, Table 4.5.1] or [Lu]). We will choose an involution $t \in \mathbf{Z}(P)$ and work with $P < C := \mathbf{C}_S(t)$ and have $t \in P'$ by Lemma 2.7(iii).

(a) Suppose $S = G_2(q)$. Then C has a normal subgroup D of index 2 with

$$D = (A \times B)/\langle t \rangle, \tag{4.1}$$

where $A \cong B \cong SL_2(q)$, see [GLS, Table 4.5.1]. By Proposition 3.5 applied to A and B, $P \cap D = Q \in \mathrm{Syl}_2(D)$ is 2-good, i.e. $Q' = \Phi(Q)$. Note that $|D/D'|$ is odd (in fact it is 1 if $q > 3$ and 9 if $q = 3$). Hence we conclude by Lemma 2.8 that $P' = \Phi(P)$.

If $S = {}^3D_4(q)$, then, according to [GLS, Table 4.5.1], C has a normal subgroup D of index 2 as in (4.1), where $A \cong SL_2(q)$ and $B \cong SL_2(q^3)$. So again $|D/D'|$ is odd, and we can finish as above.

(b) Suppose $S = F_4(q)$. According to [GLS, Table 4.5.1], $C \cong Spin_9(q)$. Now $\langle t \rangle = \mathbf{Z}(C)$, and $C/\mathbf{Z}(C) \cong \Omega_9(q)$ is 2-good by Proposition 3.7, whence $\bar{P} := P/\langle t \rangle$ is 2-good. As $t \in P'$, it follows that $P/P' \cong \bar{P}/\bar{P}'$ is elementary abelian.

Now let $S = E_8(q)$. According to [GLS, Table 4.5.1], C has a normal subgroup D of index 2 with $D \cong Spin^+_{16}(q)/C_2$ and $|\mathbf{C}_C(D)| = 2$; in particular, $\langle t \rangle = \mathbf{Z}(D) \cong C_2$ and $D/\mathbf{Z}(D) \cong P\Omega^+_{16}(q)$. By Proposition 3.7, $D/\mathbf{Z}(D)$ is 2-good. Since D is perfect and $C/D \cong C_2$, it follows by Lemma 2.8 that $C/\mathbf{Z}(D)$ is 2-good. Thus $P/\langle t \rangle$ is 2-good, whence P is 2-good as $t \in P'$.

Suppose $S = E_7(q)$. View $S = [H, H]$, where $H = E_7(q)_{\mathrm{ad}}$ (of adjoint type) and embed $P \in \mathrm{Syl}_2(S)$ in $\tilde{P} \in \mathrm{Syl}_2(H)$. As $P' \neq 1$, we can choose an involution $t \in \mathbf{Z}(\tilde{P}) \cap P'$. According to [GLS, Table 4.5.1], $\tilde{C} := \mathbf{C}_H(t)$ has a normal subgroup D of index 2^2 with

$$D = SL_2(q) * Spin^+_{12}(q)/C_2$$

(a quotient of the direct product $E := SL_2(q) \times Spin^+_{12}(q)/C_2$ by a central subgroup of order 2), and $|\mathbf{C}_{\tilde{C}}(D)| = 2$; in particular, $\langle t \rangle = \mathbf{Z}(D) \cong C_2$. Note that the only normal 2-subgroup of order 2^2 of E is $\mathbf{Z}(SL_2(q)) \times \mathbf{Z}(Spin^+_{12}(q)/C_2)$. As the full inverse image of $\mathbf{Z}(D)$ in E has order 2^2, we now see that

$$D/\mathbf{Z}(D) \cong PSL_2(q) \times P\Omega^+_{12}(q).$$

It follows by Propositions 3.5 and 3.7 that $\bar{D} := D/\mathbf{Z}(D)$ is 2-good. Also, $|\bar{D}/\bar{D}'|$ equals 1 if $q > 3$ and 3 if $q = 3$. As $|\tilde{C}/C| \leq 2$, we see that $D \leq C$, and in fact $C/D \cong C_2$ (by comparing the 2-part of the order). Applying Lemma 2.8, we now have that $C/\mathbf{Z}(D)$ is 2-good. Thus $P/\langle t \rangle$ is 2-good, whence P is 2-good as $t \in P'$.

(c) Finallly, we consider the case $S = E^\epsilon_6(q)$ and $q \equiv -\epsilon (\mathrm{mod}\, 4)$. View $S = L/\mathbf{Z}(L)$, where $L = E^\epsilon_6(q)_{\mathrm{sc}}$ (of simply connected type). Since $|\mathbf{Z}(L)|$ is odd, we may view P as a Sylow 2-subgroup of L. According to [GLS, Table 4.5.2], $C := \mathbf{C}_L(t)$ has a normal subgroup D of index $\gcd(4, q-\epsilon) = 2$ with $D = AT$, $A = Spin^\epsilon_{10}(q)$, and $T = \mathbf{C}_C(A)$ has order $q - \epsilon$. As $t \in \mathbf{Z}(C)$, we see that T contains the central subgroup $\langle t \rangle \cong C_2$ as well as $\mathbf{Z}(A) \cong C_2$. But $(q-\epsilon)_2 = 2$, hence $\langle t \rangle = \mathbf{Z}(A) = T \cap A \lhd T$. Thus

$$D/\mathbf{Z}(A) \cong \Omega^\epsilon_{10}(q) \times T/\mathbf{Z}(A),$$

where $|T/\mathbf{Z}(A)| = (q-\epsilon)/2$ is odd. By Proposition 3.7, $\Omega^\epsilon_{10}(q)$ is 2-good, and so is $\bar{D} := D/\mathbf{Z}(A)$. Also, $|\bar{D}/\bar{D}'|$ is odd. It follows by Lemma 2.8 that $C/\mathbf{Z}(A)$ is 2-good. Thus $P/\langle t \rangle$ is 2-good, whence P is 2-good as $t \in P'$. □

Before handling the simple groups $E^\epsilon_6(q)$ with $4|(q-\epsilon)$, we recall some elementary observations:

Lemma 4.2. (i) *Let $A \cong C_m^a$ be a homocyclic abelian group with a homocyclic subgroup $B \cong C_m^b$. Then B is a direct summand in A: $A = B \oplus C$ with $C \cong C_m^{a-b}$.*

(ii) *Let $\mathcal{T}$ be an n-dimensional torus over an algebraically closed field $\mathbb{F}$ of characteristic p. Suppose that $k \in \mathbb{Z}$ and $\varphi : \mathcal{T} \to \mathcal{T}$ is a morphism that induces the map $\lambda \mapsto k\lambda$ on the character group $X(\mathcal{T}) \cong \mathbb{Z}^n$. Then $\varphi(t) = t^k$ for all $t \in \mathcal{T}$.*

Proof. (i) We can find a free abelian group Λ with a basis $(e_1, \dots, e_a)$ such that $A = \Lambda/m\Lambda$. Then $B = \Lambda'/m\Lambda$ for some subgroup $\Lambda' \leq \Lambda$. As $\Lambda' \geq m\Lambda$ and $B \cong C_m^b$, we can find a basis $(f_1, \dots, f_a)$ of Λ such that

$$\Lambda' = \langle f_1, f_2, \dots, f_b, mf_{b+1}, \dots, mf_a \rangle_{\mathbb{Z}}.$$

It remains to note that $A = B \oplus C$, with

$$C := \langle mf_1, mf_2, \dots, mf_b, f_{b+1}, \dots, f_a \rangle_{\mathbb{Z}}/m\Lambda \cong C_m^{a-b}.$$

(ii) By hypothesis, $\alpha(\varphi(t)t^{-k}) = 1$ for all $t \in \mathcal{T}$ and $\alpha \in X(\mathcal{T})$. Representing $\mathcal{T} = \{\mathrm{diag}(t_1, \dots, t_n) \mid t_i \in \mathbb{F}^\times\}$, we see that $X(\mathcal{T})$ is generated by the characters

$$\chi_i : t = \mathrm{diag}(t_1, \dots, t_n) \mapsto t_i, \ 1 \leq i \leq n.$$

Writing $\varphi(t)t^{-k} = \mathrm{diag}(s_1, \dots, s_n)$ for any $t \in \mathcal{T}$ and applying the above identity to $\alpha = \chi_i$, we get $s_i = \chi_i(\varphi(t)t^{-k}) = 1$, i.e. $\varphi(t)t^{-k} = 1$. $\square$

Proposition 4.3. *Let $q = p^e$ be an odd prime power, and let $4|(q - \epsilon)$ for some $\epsilon = \pm$. Then Theorem 2.1 holds for the simple group $S = E_6^\epsilon(q)$.*

Proof. (a) Let $\mathcal{G}$ be a simple, simply connected, algebraic group of type E_6 defined over $\mathbb{F}_p$, and let $F_p : \mathcal{G} \to \mathcal{G}$ be the Frobenius endomorphism of $\mathcal{G}$ defined by this $\mathbb{F}_p$-structure. Let $\mathcal{T}$ be an F_p-stable maximal torus of $\mathcal{G}$ contained in an F_p-stable Borel subgroup. Fix an orthonormal basis $(e_1, \dots, e_8)$ of the Euclidean space $\mathbb{R}^8$ and realize the root system Φ_{E_8} of type E_8 as

$$\{\pm e_i \pm e_j, \ \frac{1}{2}\sum_{l=1}^{8} a_l e_l \mid 1 \leq i < j \leq 8, \ a_l = \pm 1, \ \prod_{l=1}^{8} a_l = 1\}.$$

Then the root system (of type E_6) of $\mathcal{G}$ with respect to $\mathcal{T}$ can be realized as the subset

$$\Phi := \{\alpha \in \Phi_{E_8} \mid \alpha \perp (e_7 + e_8), \ \alpha \perp \frac{1}{2}\sum_{l=1}^{8} e_l\},$$

with a simple root system Δ consisting of

$$\alpha_l := e_l - e_{l+1}, \ 1 \leq l \leq 5, \ \alpha_6 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 - e_8).$$

Consider the involutive symmetry

$$\rho : \alpha_1 \leftrightarrow \alpha_5, \ \alpha_2 \leftrightarrow \alpha_4, \alpha_3 \mapsto \alpha_3, \ \alpha_6 \mapsto \alpha_6$$

of the Dynkin diagram of Φ. By Chevalley's theorem [MTe, Theorem 11.12], ρ gives rise to an involutive graph automorphism γ of $\mathcal{G}$, commuting with F_p, stabilizing $\mathcal{T}$ and inducing the map ρ on Δ.

Note that $(\alpha_l, 1 \leq l \leq 5)$ is a simple root system of type A_5, with Weyl group isomorphic to S_6 and induced by $\mathrm{Sym}(\{e_1, \dots, e_6\})$. Embedding $SL_6(p)$ in $\mathcal{G}^{F_p}$, we see that $\mathbf{N}_{SL_6(p)}(\mathcal{T})$ contains an element g, such that the conjugation j_g by g induces the permutation $(16)(25)(34) \in \mathsf{S}_6$, which acts as the longest element $w_0 = -\rho$ of the Weyl group W of $\mathcal{G}$ with respect to $\mathcal{T}$, see [KL, p. 192]:

$$w_0 : \alpha_1 \leftrightarrow -\alpha_5,\ \alpha_2 \leftrightarrow -\alpha_4, \alpha_3 \mapsto -\alpha_3,\ \alpha_6 \mapsto -\alpha_6.$$

Define $F_q := F_p^f$ for $q = p^f$, and $d := 1$ if $\epsilon = +$ and $d := 2$ if $\epsilon = -$. We also consider the 1-dimensional subtorus

$$\mathcal{T}_0 := \left(\bigcap_{1 \leq i \leq 6,\ i \neq 5} \mathrm{Ker}(\alpha_i) \right)^{\circ} \tag{4.2}$$

and let $\mathcal{C} := \mathbf{C}_{\mathcal{G}}(\mathcal{T}_0) = \mathcal{T}_0 \mathcal{L}$, where $\mathcal{C}$ is connected reductive by [MTe, Corollary 8.13(a)], and the subgroup $\mathcal{L} = [\mathcal{C}, \mathcal{C}]$ (of type D_5), generated by the root subgroups $X_{\pm\alpha_i}$, $1 \leq i \leq 6$, $i \neq 5$, is simple, simply connected by [MTe, Proposition 12.14].

(b) Here we consider the case $\epsilon = -$. Note that $\mathcal{G}^{\sigma} \cong {}^2E_6(q)_{\mathrm{sc}}$ if $\sigma := \gamma F_q$, and F_q commutes with j_g and γ. It is more convenient for us to work with another endomorphism

$$\sigma' := j_g \sigma,$$

which also commutes with F_q. By the Lang-Steinberg theorem, we can write $g = h^{-1}\sigma(h)$ for some $h \in \mathcal{G}$. Now for any $x \in \mathcal{G}$ we have

$$\sigma'(x) = g\sigma(x)g^{-1} = h^{-1}\sigma(hxh^{-1})h,$$

and so $\sigma'(x) = x$ if and only if $\sigma(hxh^{-1}) = hxh^{-1}$. Hence,

$$\mathcal{G}^{\sigma'} = h^{-1}\mathcal{G}^{\sigma}h \cong \mathcal{G}^{\sigma}.$$

Moreover, $\mathcal{T}$ is σ'-invariant, as it is fixed by j_g, F_p, and γ. Furthermore, γ, respectively j_g, F_q, acts on $X(\mathcal{T})$ via ρ, $-\rho$, $\lambda \mapsto q\lambda$, respectively. Hence, σ' acts on $X(\mathcal{T})$ via $\lambda \mapsto -q\lambda$, whence $\sigma'(t) = t^{-q}$ for all $t \in \mathcal{T}$ by Lemma 4.2(ii). It follows that $|\mathcal{T}^{\sigma'}| = (q+1)^6$, and so $\mathcal{T}$ is a Sylow Φ_d-torus for

$$H := \mathcal{G}^{\sigma'} \cong {}^2E_6(q)_{\mathrm{sc}}$$

and $T := \mathcal{T}^{\sigma'}$ is homocyclic (see the proof of [MTe, Proposition 25.7]). Also note that F_q acts (as an automorphism) on H.

Observe that $\mathcal{T}_0$, $\mathcal{C}$, and $\mathcal{L}$ are all σ'-stable and F_q-stable. It follows that $L := \mathcal{L}^{\sigma'} < H$ and $C := \mathcal{C}^{\sigma'}$ are F_q-stable. Also, $\mathcal{C}$ is a maximal rank subgroup of type D_5T_1 of $\mathcal{G}$, so by [LSS, Table 5.1] we have that $L \cong Spin_{10}^{-}(q)$ and that

$$2 \nmid [H : C]. \tag{4.3}$$

As $\mathcal{T} < \mathcal{C}$ is a Sylow Φ_d-torus of $\mathcal{G}$, it is also a Sylow Φ_d-torus for $\mathcal{C}$. Hence, by [MTe, Theorem 25.11], it contains a (σ'-stable) Sylow Φ_d-torus $\mathcal{T}_1$ of $\mathcal{L}$, which is also a maximal torus of $\mathcal{L}$. Thus $T \cong C_{q+1}^6$ contains the homocyclic subgroup $T_1 := \mathcal{T}_1^{\sigma'} \cong C_{q+1}^5$, whence $T = T_1 \times T_2$ for some $T_2 \cong C_{q+1}$ by Lemma 4.2(i). By Lemma 4.2(ii), F_q acts on $\mathcal{T}$ via $x \mapsto x^q$, so F_q acts on T as $x \mapsto x^{-1}$. As $\mathbf{Z}(\mathcal{L}) \leq \mathbf{C}_{\mathcal{L}}(\mathcal{T}_1) = \mathcal{T}_1 \leq \mathcal{T}$ and $\mathbf{Z}(L) = \mathbf{Z}(\mathcal{L})^{\sigma'}$ (see [C, Proposition 3.6.8]), we see that F_q acts as inversion on $\mathbf{Z}(L) = \langle z \rangle \cong C_4$. In what follows, we will denote by f the 2-part of F_q as an element of $\mathrm{Aut}(H)$; in particular, f acts as inversion on T and $\mathbf{Z}(L)$.

Recall that $4|(q+1)$ in the case under consideration; in particular, $|\mathrm{Out}(S)|_2 = 2$ and f^2 induces an inner automorphism of S. Conjugating within $\mathrm{Aut}(S)$, we may assume that

$$S = H/\mathbf{Z}(H) \lhd G \leq \tilde{H}/\mathbf{Z}(H) = \langle S, f \rangle \cong S \cdot C_2$$

where $\tilde{H} := \langle H, f \rangle$. At the same time, we may view S as $[H^*, H^*]$, where $H^* \cong {}^2E_6(q)_{\mathrm{ad}}$ is dual to H.

(b1) Consider the case $G = S$. Note that the element $z \in \mathbf{Z}(L) < C$ of order 4 centralizes both $\mathcal{L}$ and $\mathcal{T}_0$ and so $z \in \mathbf{Z}(C)$. It follows by (4.3) that $2 \nmid [H : \mathbf{C}_H(z)]$, whence the semisimple character $\chi_z \in \mathrm{Irr}(H^*)$ has odd degree. By Proposition 2.9(i), (iii), $\theta := (\chi_z)_S$ belongs to $B_0(S)$, if we identify S with $[H^*, H^*]$. By hypothesis, θ is real, and so z is real in H by Proposition 2.9(iii). Now we view S as $H/\mathbf{Z}(H)$, and, as $|\mathbf{Z}(H)|$ divides 3, we can assume that $z \in \mathbf{Z}(Q)$, where $Q \in \mathrm{Syl}_2(S)$. Then Burnside's fusion control lemma implies that z and z^{-1} are conjugate in $\mathbf{N}_S(Q)$. But the latter centralizes $\mathbf{Z}(Q)$ by [KM, Theorem 6(c)]. We arrive at the contradiction that $z = z^{-1}$.

(b2) Thus we have shown that $S < G$ and so we may view G as $\tilde{H}/\mathbf{Z}(H) = \langle S, f \rangle$, and $f \notin S$. Recall that $T = T_1 \times T_2$ is f-stable and f is a 2-element. Let $R = \mathbf{O}_2(T) \in \mathrm{Syl}_2(T)$ and write $R = R_1 \times R_2$, where $R_i \in \mathrm{Syl}_2(T_i)$ for $i = 1, 2$ (so that $|R_1| = ((q+1)_2)^5$ and $|R_2| = (q+1)_2$). We extend $\langle R, f \rangle$ to a Sylow 2-subgroup P of $\langle C, f \rangle$ (which contains C properly as $f \notin S$). By (4.3) and since $2 \nmid |\mathbf{Z}(H)|$, we may view $P \in \mathrm{Syl}_2(G)$. Set

$$Q := P \cap C \in \mathrm{Syl}_2(C),\ Q_1 := P \cap L \in \mathrm{Syl}_2(L).$$

Note that $\mathcal{T}_1 \leq \mathcal{T} \cap \mathcal{L} \leq \mathbf{C}_{\mathcal{L}}(\mathcal{T}_1) = \mathcal{T}_1$, and so $\mathcal{T}_1 = \mathcal{T} \cap \mathcal{L}$. It follows that

$$R_2 \cap Q_1 \leq \mathcal{T}^{\sigma'} \cap \mathcal{L}^{\sigma'} = (\mathcal{T} \cap \mathcal{L})^{\sigma'} = \mathcal{T}_1^{\sigma'} = T_1,$$

whence $R_2 \cap Q_1 \leq T_2 \cap T_1 = 1$ and so

$$Q = Q_1 \rtimes R_2. \tag{4.4}$$

Also, note that $f^2 \in P \cap C = Q$ and f normalizes Q.

Now consider any P-invariant linear character λ of Q; in particular, λ is f-invariant. Then $K := \mathrm{Ker}(\lambda)$ contains $f(x)x^{-1}$ for all $x \in Q$. As $Q > Q_1 \in \mathrm{Syl}_2(L)$, Q contains the element $z \in \mathbf{Z}(L)$ of order 4 which is inverted by f. Hence $K \ni z^2$. According to Proposition 3.7, $L/\langle z^2 \rangle \cong \Omega^-_{10}(q)$ is 2-good, whence $Q_1/\langle z^2 \rangle$ is 2-good and so

$$x^2 \in Q_1' \langle z^2 \rangle \leq K \tag{4.5}$$

for all $x \in Q_1$. Next, f acts as inversion on $T > R_2$, whence $K \ni y^2$ for all $y \in R_2$. Together with (4.4) and (4.5), this implies that Q/K is elementary abelian, and so λ is real as desired.

(c) From now on we will assume that $\epsilon = +$. Write $q = p^e$ with $e = e_0 2^a$, where $a \geq 0$ and $2 \nmid e_0$, $q_0 := p^{e_0}$, $F_0 := F_{q_0} = F_p^{e_0}$.

Note that $\tau := j_g \gamma$ normalizes $\mathcal{T}$ and acts as $\lambda \mapsto -\lambda$ on $X(\mathcal{T})$, whence τ acts as inversion on $\mathcal{T}$ by Lemma 4.2(ii). Furthermore, τ commutes with F_p, so τ acts on $H := \mathcal{G}^{F_q} = E_6(q)_{\mathrm{sc}}$. Without loss of generality, we will view τ as an automorphism of H and replace τ by its 2-part. As $|\mathcal{T}^{F_q}| = (q-1)^6$, $\mathcal{T}$ is a Sylow Φ_d-torus for H.

Observe that $\mathcal{T}_0$, $\mathcal{C}$, $\mathcal{L}$, $L := \mathcal{L}^{F_q} < H$, and $C := \mathcal{C}^{F_q}$ are stable under τ and F_p. Also, $\mathcal{C}$ is a maximal rank subgroup of type D_5T_1 of $\mathcal{G}$, so by [LSS, Table 5.1] we have that $L \cong Spin^+_{10}(q)$ and that

$$2 \nmid [H : C]. \tag{4.6}$$

As $\mathcal{T} < \mathcal{C}$ is a Sylow Φ_d-torus of $\mathcal{G}$, it is also a Sylow Φ_d-torus for $\mathcal{C}$. Hence, by [MTe, Theorem 25.11], it contains an F_q-stable Sylow Φ_d-torus $\mathcal{T}_1$ of $\mathcal{L}$, which is also a maximal torus of $\mathcal{L}$. Thus $T := \mathcal{T}^{F_q} \cong C^6_{q-1}$ contains the homocyclic subgroup $T_1 := \mathcal{T}_1^{F_q} \cong C^5_{q-1}$, whence $T = T_1 \times T_2$ for some $T_2 \cong C_{q-1}$ by Lemma 4.2(i).

We can view S as either $H/\mathbf{Z}(H)$ or $[H^*, H^*]$, where $H^* \cong E_6(q)_{\mathrm{ad}}$. Note that

$$\mathrm{Out}(S) = H^*/S \rtimes \langle \tau, F_p \rangle.$$

Recall that $T = T_1 \times T_2$ is stable under commuting automorphisms τ and F_0, which are 2-elements of $\mathrm{Aut}(S)$. (Indeed, $F_0^{2^a} = F_q$ acts trivially on H.) Let $R = \mathbf{O}_2(T) \in \mathrm{Syl}_2(T)$ and write $R = R_1 \times R_2$, where $R_i \in \mathrm{Syl}_2(T_i)$ for $i = 1, 2$ (so that $|R_1| = ((q-1)_2)^5$ and $|R_2| = (q-1)_2$). We extend $\langle R, \tau, F_0 \rangle$ to a Sylow 2-subgroup $\tilde{P}$ of $\langle C, \tau, F_0 \rangle$. Note that $\tau^2 \in R$. (Indeed, τ is a graph automorphism of S and so τ^2 acts on S as an inner-diagonal automorphism, i.e. the conjugation j_{h^*} by some $h^* \in H^*$. But we choose τ to be a 2-element, so $h^* \in S$. So τ^2 acts on S as the conjugation j_h for some $h \in H$. But τ^2 centralizes $\mathcal{T}$, so $h \in \mathbf{C}_H(\mathcal{T}) \leq \mathcal{T}^{F_q} = T$. Using the condition τ is a 2-element again, we get $h \in \mathbf{O}_2(T) = R$, yielding $\tau^2 \in R$.) Now (4.6) implies that $\langle C, \tau, F_0 \rangle$ has odd index in $\langle H, \tau, F_p \rangle$. Conjugating inside $\mathrm{Aut}(S)$, and using $2 \nmid |\mathbf{Z}(H)|$, we may assume that $P \leq \tilde{P}$ for $P \in \mathrm{Syl}_2(G)$, and $G \leq \tilde{P}H$. Set

$$Q := P \cap C \in \mathrm{Syl}_2(C),\ Q_1 := P \cap L \in \mathrm{Syl}_2(L).$$

Our discussion implies that $\tilde{P} = Q\langle \tau, F_0 \rangle$ and in fact

$$P/Q \leq \tilde{P}/Q \cong C_2 \times C_{2^a}, \tag{4.7}$$

with C_2 generated by τ and C_{2^a} generated by F_0. As in (b2) we have $\mathcal{T}_1 = \mathcal{T} \cap \mathcal{L}$, and so

$$R_2 \cap Q_1 \leq \mathcal{T}^{F_q} \cap \mathcal{L}^{F_q} = (\mathcal{T} \cap \mathcal{L})^{F_q} = \mathcal{T}_1^{F_q} = T_1,$$

whence $R_2 \cap Q_1 \leq T_2 \cap T_1 = 1$, implying $Q = Q_1 \rtimes R_2$ as in (4.4).

Suppose that $P \ni \tau$. Now consider any P-invariant linear character λ of Q; in particular, λ is τ-invariant. Then $K := \mathrm{Ker}(\lambda)$ contains $\tau(x)x^{-1}$ for all $x \in Q$. As $Q > Q_1 \in \mathrm{Syl}_2(L)$, Q contains a generator z of order 4 of

$$\mathbf{Z}(L) \leq \mathbf{Z}(\mathcal{L}) \leq \mathbf{C}_{\mathcal{L}}(\mathcal{T}_1) = \mathcal{T}_1 < \mathcal{T}.$$

Hence $\tau(z) = z^{-1}$ and $K \ni z^2$. According to Proposition 3.7, $L/\langle z^2 \rangle \cong \Omega_{10}^+(q)$ is 2-good, whence $Q_1/\langle z^2 \rangle$ is 2-good and so $x^2 \in Q_1'\langle z^2 \rangle \leq K$ for all $x \in Q_1$. Next, τ acts as inversion on $T > R_2$, whence $K \ni y^2$ for all $y \in R_2$. Now we can conclude as in (b2) that Q/K is elementary abelian, and so λ is real as desired.

(d) From now on we may assume that $P \not\ni \tau$. Then (4.7) implies that there is some $j \in \{0, 1\}$ and $0 \leq b \leq a$ such that

$$P = \langle Q, F_1\tau^j \rangle,$$

where $F_1 := F_{q_1} = F_0^{2^{a-b}}$ and $q_1 = p^{e_0 \cdot 2^{a-b}}$. Note that $F_1\tau^j(t) = t^{(-1)^j q_1}$ for all $t \in \mathcal{T}$. Next, as F_q acts on $\mathcal{T}$ via $t \mapsto t^q$, $\mathcal{T}_0$ is a Φ_d-torus and so $T_0 := \mathcal{T}_0^{F_q} \cong C_{q-1}$. Furthermore, the condition $P \not\ni \tau$ implies that $(b, j) \neq (0, 1)$ and so $q - 1 = q_1^{2^b} - 1$ is divisible by $q_1 - (-1)^j$. It follows that we can find $s \in T_0 < T$ of order $(q_1 - (-1)^j)_2 \geq 2$. By its choice, $s \in \mathbf{Z}(C)$ and so $2 \nmid [H : \mathbf{C}_H(s)]$. Now Proposition 2.9(i), (iii) implies that $\theta := (\chi_s)_S$ is irreducible, of odd degree, and belongs to $B_0(S)$, if we view S as $[H^*, H^*]$. Also, s is $F_1\tau^j$-invariant, whence θ is P-invariant. By the main hypothesis, θ is real, and so s is real in H by Proposition 2.9(iii). As $s \in \mathbf{O}_2(T) = R < Q$ and $s \in \mathbf{Z}(C)$, we get that $s \in \mathbf{Z}(Q)$. Also, $2 \nmid |\mathbf{Z}(H)$, so, identifying S with $H/\mathbf{Z}(H)$, we have that s is real in S. Now Burnside's fusion control lemma implies that s is real in $\mathbf{N}_S(Q)$. But $\mathbf{N}_S(Q)$ centralizes $\mathbf{Z}(Q)$ by [KM, Theorem 6(c)]. So we conclude that $s = s^{-1}$, i.e.

$$(q_1 - (-1)^j)_2 = 2 \tag{4.8}$$

Let λ be any P-invariant linear character of Q; in particular, λ is $F_1\tau^j$-invariant. Then $K := \mathrm{Ker}(\lambda)$ contains $F_1\tau^j(x)x^{-1}$ for all $x \in Q$. If, in addition, $x \in \mathcal{T}$, then (4.8) implies that

$$F_1\tau^j(x)x^{-1} = x^{(-1)^j(q_1 - (-1)^j)} \tag{4.9}$$

generates $\langle x^2 \rangle$. As $Q > Q_1 \in \mathrm{Syl}_2(L)$, Q contains the generator z of order 4 of $\mathbf{Z}(L) < \mathcal{T}$ (as in (c)). Applying (4.9) to z we get $K \ni z^2$. Again by Proposition 3.7, $L/\langle z^2 \rangle \cong \Omega_{10}^+(q)$ is 2-good, whence $Q_1/\langle z^2 \rangle$ is 2-good and so $x^2 \in Q_1'\langle z^2 \rangle \leq K$ for all $x \in Q_1$. Finally, applying (4.9) to $y \in R_2$ we obtain $K \ni y^2$ for all $y \in R_2$. Using

(4.4) we can conclude as in (b2) that Q/K is elementary abelian, and so $\lambda = \overline{\lambda}$, completing the proof. □

Combining Propositions 4.1 and 4.3 we obtain

Corollary 4.4. *Theorem 2.1 holds if S is an exceptional simple group of Lie type in odd characteristic.* □

Proposition 4.5. *Theorem 2.1 holds if S is a simple group of Lie type in characteristic 2. In fact, S is 2-good unless $S \cong {}^2F_4(2)'$.*

Proof. Note that we do not have to consider the case $S = {}^2F_4(2)'$, since in this case neither S nor $\mathrm{Aut}(S) = {}^2F_4(2)$ can fulfill the hypothesis of Theorem 2.1. In all other cases, we can view $S = H/\mathbf{Z}(H)$, where $H = \mathcal{G}^F$ for some simple, simply connected algebraic group defined over $\mathbb{F}_q$, $q = 2^f$, and $F : \mathcal{G} \to \mathcal{G}$ a Frobenius endomorphism. As $2 \nmid |\mathbf{Z}(H)|$, we may view $Q \in \mathrm{Syl}_2(H)$ as a Sylow 2-subgroup of S.

(a) Suppose first that $\mathcal{G}$ is not of type BC_n or F_4. Then $Q' = \Phi(Q)$ by [GLS, Theorem 3.3.1(b)], and so S is 2-good. Next, suppose that $H = Sp_{2n}(q)$. Embedding Q in a suitable maximal parabolic subgroup of H, we can write $Q = U \rtimes R$, where U is elementary abelian of rank $n(n+1)/2$ and $R \in \mathrm{Syl}_2(SL_n(q))$. By the previous case, R is 2-good, whence so is Q by Lemma 3.2(ii). If $H = {}^2B_2(q)$ with $q \geq 8$, then it is well known (see e.g. [Col, §3]) that $Q' = \Phi(Q)$ has order q, so Q is 2-good.

(b) Assume that $H = F_4(q)$. As shown in [CKS, Proposition 4.5], $Q = U \rtimes R$, where $R \in \mathrm{Syl}_2(Sp_6(q))$ and so is 2-good by (a). Furthermore, $\mathbf{Z}(U)$ is elementary abelian of order q^7, and $U = \mathbf{Z}(U)W$, where $|W| = q^9$ and $W' = \Phi(W)$ (of order q). It follows that U is 2-good and so is Q by Lemma 3.2(ii).

Finally, assume that $H = {}^2F_4(q)$ with $q \geq 8$. Then it is known (see [FS], [LS]) that $Q = U \rtimes R$, where $R \in \mathrm{Syl}_2({}^2B_2(q))$ is 2-good by (a). Furthermore, $U' = \Phi(U)$ and so U is 2-good. Thus Q is 2-good by Lemma 3.2(ii). □

Now Theorem 2.1 follows from Lemmas 3.3, 3.4, Corollaries 3.11, 4.4, and Proposition 4.5. □

References

[BNOT] C. Bessenrodt, G. Navarro, J. Olsson, and Pham Huu Tiep, On the Navarro-Willems conjecture for blocks of finite groups, *J. Pure Appl. Algebra* **208** (2007), 481–484.

[C] R. Carter, '*Finite Groups of Lie type: Conjugacy Classes and Complex Characters*', Wiley, Chichester, 1985.

[Col] M. J. Collins, The characterisation of the Suzuki groups by their Sylow 2-subgroups, *Math. Z.* **123** (1971), 32–48.

[Atlas] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, '*An ATLAS of Finite Groups*', Clarendon Press, Oxford, 1985.

[CKS] C. W. Curtis, W. M. Kantor, and G. M. Seitz, The 2-transitive permutation representations of the finite Chevalley groups, *Trans. Amer. Math. Soc.* **218** (1976), 1–59.

[DM] F. Digne and J. Michel, '*Representations of Finite Groups of Lie Type*', London Mathematical Society Student Texts **21**, Cambridge University Press, 1991.

[FS] P. Fong and G. Seitz, The (B, N)-pairs of rank 2, I, II, *Invent. Math.* **21** (1973), 1–57, **24** (1974), 191–239.

[GAP] The GAP group, 'GAP - *groups, algorithms, and programming*', Version 4.4, 2004, http://www.gap-system.org.

[Gow1] R. Gow, Characters of solvable groups induced by linear characters of Hall subgroups, *Archiv der Math.* **40** (1983), 232–237.

[Gow2] R. Gow, Real 2-blocks of characters of finite groups, *Osaka J. Math.* **25** (1988), 135–147.

[GLS] D. Gorenstein, R. Lyons, and R. Solomon. *The Classification of the Finite Simple Groups*, Number 3. Part I. Chapter A, volume 40 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998.

[H] G. Hiss, Regular and semisimple blocks of finite reductive groups, *J. London Math. Soc.* **41** (1990), 63–68.

[I] I. M. Isaacs, Characters of π-separable groups, *J. Algebra* **86** (1984), 98–128.

[Is] I. M. Isaacs, '*Character Theory of Finite Groups*', AMS-Chelsea, Providence, 2006.

[KL] P. B. Kleidman and M. W. Liebeck, '*The Subgroup Structure of the Finite Classical Groups*', London Math. Soc. Lecture Note Ser. no. **129**, Cambridge University Press, 1990.

[KM] A. S. Kondratiev and V. D. Mazurov, 2-signalizers of finite simple groups, *Algebra and Logic* **42** (2003), 333–348.

[LS] V. Landazuri and G. Seitz, On the minimal degrees of projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418 –443.

[LSS] M. Liebeck, J. Saxl, and G. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, *Proc. London Math. Soc.* **65** (1992), 297–325.

[Lu] F. Lübeck, Elements of order 2 and 3 in exceptional groups of Lie type, `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/23elts.html`

[MTe] G. Malle and D. Testerman, '*Linear Algebraic Groups and Finite Groups of Lie Type*', Cambridge Studies in Advanced Mathematics, **133**, Cambridge University Press, 2011.

[MT] S. Marinelli and Pham Huu Tiep, Zeros of real irreducible characters of finite groups, *Algebra Number Theory* **7** (2013), 567–593.

[Mu] M. Murai, Block induction, normal subgroups and characters of height zero, *Osaka J. Math* **31** (1994), 9–25.

[N] G. Navarro, '*Characters and Blocks of Finite Groups*', Cambridge University Press, 1998.

[NT1] G. Navarro and Pham Huu Tiep, Rational irreducible characters and rational conjugacy classes in finite groups, *Trans. Amer. Math. Soc.* **360** (2008), 2443–2465.

[NT2] G. Navarro and Pham Huu Tiep, Brauer's height zero conjecture for the 2-blocks of maximal defect, *J. Reine Angew. Math.* **669** (2012), 225–247.

[NT3] G. Navarro and Pham Huu Tiep, Irreducible representations of odd degree, *Math. Annalen* (to appear).

[NT4] G. Navarro and Pham Huu Tiep, Sylow subgroups, exponents, and character tables, (in preparation).

[NTT1] G. Navarro, Pham Huu Tiep, and A. Turull, p-rational characters and self-normalizing Sylow p-subgroups, *Represent. Theory* **11** (2007), 84–94.

[NTT2] G. Navarro, Pham Huu Tiep, and A. Turull, Brauer characters with cyclotomic field of values, *J. Pure Appl. Alg.* **212** (2008), 628–635.

[W] R. A. Wilson, The McKay conjecture is true for the sporadic simple groups, *J. Algebra* **207** (1998), 294–305.

Departament d'Àlgebra, Universitat de València, 46100 Burjassot, València, Spain
E-mail address: gabriel.navarro@uv.es

Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA
E-mail address: tiep@math.arizona.edu