Explicit Serre Weight Conjectures in Dimension Four

by

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by **Whitney Berard**, titled ***Explicit Serre Weight Conjectures in Dimension Four*** and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Abstract

A generalization of the weight part of Serre's conjecture asks for which Serre weights a given mod p representation of the absolute Galois group of $\mathbb{Q}$ is modular. This set is expected to depend only on the restriction of the representation to the Galois group of $\mathbb{Q}_p$. Let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{F}_p)$ be a continuous representation of the absolute Galois group of $\mathbb{Q}_p$ that is moreover semisimple. In [GHS16], a certain set $W_{\mathrm{expl}}(\overline{\rho})$ of Serre weights (which is defined in a very explicit way) is conjectured to be the correct set of Serre weights as long as $\overline{\rho}$ is sufficiently generic.

However, in the non-generic cases that occur in dimension $n \geq 4$ it is not known whether this set behaves in the way it should under certain functorial operations, like tensor products. This thesis shows that in dimension four, the set of explicit Serre weights $W_{\mathrm{expl}}(\overline{\rho})$ of $\overline{\rho}$ defined in [GHS16] is closed under taking tensor products of two two-dimensional representations.

Chapter 1

Introduction

Let p be a prime, and $\mathbb{Q}_p$ the field of p-adic numbers. For a field K, we write G_K for the absolute Galois group of K.

In [Ser87], J.P. Serre conjectured that every odd, irreducible, continuous representation $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ arises from a modular eigenform. He also gave a recipe, which depends only on the restriction of $\overline{r}$ to $G_{\mathbb{Q}_p}$, for the minimal weight of a modular eigenform of prime-to-p level giving rise to $\overline{r}$. This conjecture is now a theorem due to Khare and Wintenberger [KW09].

Recent work [BDJ10, GHS16, Her09, Sch08] has been done to generalize this conjecture to higher dimensions, and to fields other than $\mathbb{Q}$. For n-dimensional representations of $G_{\mathbb{Q}_p}$, the correct generalization of "weight" is an irreducible $\overline{\mathbb{F}}_p$-representation of the finite group $\mathrm{GL}_n(\mathbb{F}_p)$. Because of this connection, we call such irreducible representations *Serre weights*. A motivating question in this area is to determine the set of Serre weights for which a given $\bmod p$ Galois representation is (conjecturally) modular.

This thesis is concerned with the comparison between two versions of the weight part of Serre's conjecture for four-dimensional representations of $G_{\mathbb{Q}}$. Since both versions are formulated in a purely local way, we will be concerned only with representations of $G_{\mathbb{Q}_p}$.

By a theorem of Steinberg, irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\mathbb{F}_p)$ can be parametrized by tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ that are dominant (i.e. $\lambda_1 \geq \cdots \geq \lambda_n$), and p-restricted ($\lambda_i - \lambda_{i+1} \leq p-1$). More precisely, for λ as above, the Serre weight $F(\lambda)$ is the socle of the Weyl module $W(\lambda)$ of highest weight λ, and $F(\lambda) \cong F(\lambda')$ if and only if $\lambda - \lambda' \in (p-1, \ldots, p-1)\mathbb{Z}$.

The first set of Serre weights associated to $\overline{\rho}$ that we will consider is formulated by considering the crystalline lifts of $\overline{\rho}$. If $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ is a crystalline representation, then associated to ρ are two invariants:

- An n-tuple of integers called Hodge-Tate weights, and
- A $\bmod p$ representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$.

There may be many different crystalline representations that all reduce to the same $\overline{\rho}$. We call these the *lifts* of $\overline{\rho}$. Each of these lifts has an associated list of Hodge-Tate weights.

If ρ has Hodge-Tate weights $r_1 > \cdots > r_n$ that are regular (i.e. distinct) and p-restricted (successive differences are all less than or equal to p), then we say that ρ has *Hodge type* $(r_1, \ldots, r_n) - (n-1, n-2, \ldots, 1, 0)$. Notice that if the set of Hodge-Tate weights is dominant and p-restricted, then the Hodge type $(\lambda_1, \ldots, \lambda_n) = (r_1, \ldots, r_n) - (n-1, n-2, \ldots, 1, 0)$ has $0 \leq \lambda_i - \lambda_{i+1} \leq p-1$, which are exactly the conditions on Serre weights.

Given a $\bmod p$ representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ we define the *crystalline weights for* $\overline{\rho}$, $W_{\mathrm{cris}}(\overline{\rho})$, to be the set of Serre weights $F(\lambda)$ such that $\overline{\rho}$ has a crystalline lift that is regular of Hodge type λ.

Question 1.0.1. *For a* $\bmod p$ *representation* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$*, what is* $W_{\mathrm{cris}}(\overline{\rho})$*?*

The second set of Serre weights associated to $\overline{\rho}$ that we will consider is defined in [GHS16] by a much more explicit process. Specifically, a set of *obvious weights*, which are read off of representations that are easy to write down. Then additional *shadow weights* are added, which are found by applying certain reflections to the obvious weights. And finally, *obscure weights* are added to form all together the set $W_{\mathrm{expl}}(\overline{\rho})$ of *explicit Serre weights.* These sets are as follows:

$$W_{\mathrm{obv}}(\rho) \subseteq \mathcal{C}(W_{\mathrm{obv}}(\overline{\rho})) \subseteq W_{\mathrm{expl}}(\overline{\rho}). \tag{1.1}$$

For sufficiently generic $\overline{\rho}$, it is conjectured in [GHS16] that $W_{\mathrm{expl}}(\overline{\rho}) = W_{\mathrm{cris}}(\overline{\rho})$. In the non-generic case, there does not seem to be a good reason to expect that this equality still holds. However, at this time we do not know of a counterexample.

We may ask whether the set $W_{\mathrm{expl}}(\overline{\rho})$ behaves in ways that we know $W_{\mathrm{cris}}(\overline{\rho})$ must behave. For instance, is should be closed under different functorial operations on $\bmod p$ representations, like tensor products.

Suppose that the $\bmod p$ representations $\overline{\rho}'$ and $\overline{\rho}''$ have crystalline lifts ρ' and ρ'' with Hodge-Tate weights $\{r'_1, \ldots, r'_m\}$ and $\{r''_1, \ldots, r''_n\}$ respectively. Then $\overline{\rho}' \otimes \overline{\rho}''$ has a crystalline lift with Hodge-Tate weights $\{r'_i + r''_j\}$. Write $(r_1, \ldots, r_{mn})$ for this list of integers arranged in decreasing order. If the r_i's are distinct, and have successive differences at most p, and if $W_{\mathrm{cris}}(\overline{\rho}) = W_{\mathrm{expl}}(\overline{\rho})$, then we should find that $F((r_i - (mn - i)))$ lies in $W_{\mathrm{expl}}(\overline{\rho})$.

In this thesis, we prove that this is indeed the case when $m = n = 2$.

Theorem A. *If $\overline{\rho}'$ and $\overline{\rho}''$ are two-dimensional representations with crystalline lifts of Hodge-Tate weights $(s,0)$ and $(t,0)$ such that $(s+t,t,s,0)-\eta$ is dominant and p-restricted, then $F((s+t,t,s,0)-\eta)$ is in $W_{\mathrm{expl}}(\overline{\rho}'\otimes\overline{\rho}'')$.*

In order to calculate explicit weights, we will want to know the Jordan-Hölder factors of Weyl modules $W(\lambda)$ for various λ. These Jordan-Hölder factors are computed in Chapter 2. This chapter begins with some background in the representation theory of algebraic groups. Then in Sections 2.2 and 2.3 we apply this background to compute the required Jordan-Hölder factors.

Chapter 3 follows [GHS16, §7] and defines the sets of Serre weights in (1.1).

Chapter 4 contains a proof of Theorem A. Roughly speaking, the strategy is as follows. Because of our assumptions on s and t, there are only eight forms that $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ may have, four of them only occurring if $t>p$. In the first four cases, it will be easy to see that $\lambda=(s+t,t,s,0)-\eta$ is an obvious weight. In the final four cases (which occur only if $t>p$), we will have a different result depending on where (in which alcove/boundary of an alcove) the weight lies. In these cases, λ is either a shadow weight or an obscure weight. These are treated in Sections 4.3 and 4.4, respectively. Section 4.5 compiles the results of the previous sections in order to prove Theorem A.

Finally, Chapter 5 contains various results that make the arguments in Chapter 4 more complete, but which are not strictly necessary to understand the main ideas and the proof of the main theorem. When $p=2$ or $p=3$, Theorem A requires a different proof than that given in Chapter 4. This argument is more explicit, not as enlightening, and is given in Section 5.1. The proof of Theorem A that is given in Chapter 4 may seem a bit intricate at times - especially when we get to the explicit/obscure weight arguments in Section 4.4. Sections 5.2 and 5.3 explain why these intricacies are truly necessary.

Chapter 2

Weyl Module Decompositions

Fix a prime $p > 3$ for the remainder of this chapter. We won't require the results in this chapter when $p = 2, 3$, and imposing this restriction will simplify some of the statements. In this chapter, we compute the Jordan-Hölder factors of certain Weyl modules that will be needed in subsequent chapters. We begin with general background on the representation theory of the algebraic group GL_n over $\mathbb{F}_p$. Then we apply this background to some particular Weyl modules in the cases $n = 2$ and $n = 3$.

2.1 Representation Theory of GL_n over $\mathbb{F}_p$

In this section, we review some representation theory of the algebraic group $G = \mathrm{GL}_n$. The main reference for this section is [Jan], Sections II.1 - II.3 and II.5 - II.8.

Let $T \subset \mathrm{GL}_n$ be the diagonal matrices, and let $B \subset \mathrm{GL}_n$ be the upper triangular matrices. Let $X(T)$ be the character group for GL_n, and let $\epsilon_i \in X(T)$ be the weight

$$\epsilon_i : \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto t_i.$$

Let $R = \{\epsilon_i - \epsilon_j\}$ be the root system of GL_n corresponding to the choice of B and T, $R^+ = \{\epsilon_i - \epsilon_j : i > j\}$ be the positive roots, and $S = \{\epsilon_i - \epsilon_{i+1}\}$ the simple roots. This character group $X(T)$ can be naturally identified with $\mathbb{Z}^n$ by writing $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ for the weight

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}.$$

Write $R^\vee$ for the dual root system, and recall that we have a natural pairing $\langle \cdot, \cdot \rangle : X(T) \times R^\vee \to \mathbb{Z}$ given by $(\lambda \circ \alpha^\vee)(t) = t^{\langle \lambda, \alpha^\vee \rangle}$. For $\lambda = (\lambda_1, \ldots, \lambda_n)$, and $\epsilon_i^\vee - \epsilon_j^\vee \in R^\vee$, this pairing is $\langle \lambda, \epsilon_i^\vee - \epsilon_j^\vee \rangle = \lambda_i - \lambda_j$. For $\alpha \in R$, define the reflection s_α on $X(T)$ by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

The Weyl group W of GL_n is the group generated by the reflections s_α for $\alpha \in R$. For $\alpha \in R$ and $r \in \mathbb{Z}$, define the *affine* reflection $s_{\alpha,r}$ on $X(T)$ by

$$s_{\alpha,r}(\lambda) = s_\alpha(\lambda) + r\alpha.$$

The affine Weyl group W_p is the group generated by $s_{\alpha,np}$ for $\alpha \in R, n \in \mathbb{Z}$.

Let $\eta = (n-1, n-2, \ldots, 1, 0) \in \frac{1}{2}\sum_{\alpha \in R^+} \alpha + (X(T) \otimes_{\mathbb{Z}} \mathbb{Q})^W$. For $w \in W$ or W_p, we define a "dot" action on $X(T)$ by

$$w \bullet \lambda = w(\lambda + \eta) - \eta. \tag{2.1}$$

This action stabilizes the following family of hyperplanes:

$$\{\lambda \in X(T) \otimes \mathbb{R} : \langle \lambda + \eta, \alpha^\vee \rangle = np\}_{n \in \mathbb{Z}, \alpha \in R}. \tag{2.2}$$

Definition 2.1.1. A *facet* is a subset F of $X(T) \otimes \mathbb{R}$ defined by

$$F = \{\lambda \in X(T) \otimes \mathbb{R} \mid \langle \lambda + \eta, \alpha^\vee \rangle = n_\alpha p \quad \text{for } \alpha \in R_0,$$
$$n_\alpha p < \langle \lambda + \eta, \alpha^\vee \rangle < (n_\alpha + 1)p \quad \text{for } \alpha \in R_1\}.$$

for some integers n_α, and some disjoint subsets R_0, R_1 with $R_0 \sqcup R_1 = R^+$.

An *alcove* is a subset C of $X(T) \otimes \mathbb{R}$ defined by

$$C = \{\lambda \in X(T) \otimes \mathbb{R} \mid n_\alpha p < \langle \lambda + \eta, \alpha^\vee \rangle < (n_\alpha + 1)p \quad \text{for } \alpha \in R^+\}.$$

Note that an alcove is a facet with $R_0 = \varnothing$. The alcove with all $n_\alpha = 0$ is called the *standard alcove*, and will be denoted in this thesis by C_0. We will write $\overline{C}$ for the closure of the alcove C, and $\widehat{C}$ for the upper closure of the alcove C, which are defined as follows:

$$\overline{C} = \{\lambda \in X(T) \otimes \mathbb{R} \mid n_\alpha p \leq \langle \lambda + \eta, \alpha^\vee \rangle \leq (n_\alpha + 1)p \text{ for all } \alpha \in R^+\},$$
$$\widehat{C} = \{\lambda \in X(T) \otimes \mathbb{R} \mid n_\alpha p < \langle \lambda + \eta, \alpha^\vee \rangle \leq (n_\alpha + 1)p \text{ for all } \alpha \in R^+\}.$$

We note that alcoves are the connected components of the complement of the union of the hyperplanes defined in (2.2). For $w \in W_p$, the dot action defined in (2.1) is a reflection about one of the hyperplanes in (2.2). By Theorem 2 in Ch. V, §3 of [Bou81], the closure of the standard alcove $\overline{C}_0$ is a fundamental domain for the action of W_p on $X(T) \otimes \mathbb{R}$.

Recall that we have a partial order $\leq$ on $X(T)$ given by $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a sum of positive roots, that is, if $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbb{N}\alpha$. We have a partial order $\uparrow$ on $X(T)$ given by the following: $\lambda \uparrow \mu$ if and only if there exist $w_1, \ldots, w_r \in W_p$ such that

$$\lambda \leq w_1 \bullet \lambda \leq w_2 w_1 \bullet \lambda \leq \cdots \leq w_r \cdots w_1 \bullet \lambda = \mu.$$

If $\lambda \uparrow \mu$, then $\lambda \leq \mu$, and $\lambda \in W_p \bullet \mu$. This ordering on $X(T)$ extends to give us an ordering on the alcoves. Suppose C_1, C_2 are two alcoves in $X(T) \otimes \mathbb{R}$, and suppose $\lambda_1 \in C_1 \cap X(T)$. Since $\overline{C}_1$ is a fundamental domain for the action of W_p, there is a unique $\lambda_2 \in C_2 \cap X(T)$ such that $\lambda_2 \in W_p \bullet \lambda_1$. We say that $C_1 \uparrow C_2$ if and only if $\lambda_1 \uparrow \lambda_2$.

Definition 2.1.2. The *p-restricted region* $X_1(T)$ is:

$$X_1(T) = \{(\lambda_1, \ldots, \lambda_n) \in X(T) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq p-1 \text{ for } i = 1, \ldots, n-1\}$$

The set of *dominant weights* $X(T)_+$ is:

$$X(T)_+ = \{(\lambda_1, \ldots, \lambda_n) \in X(T) : 0 \leq \lambda_i - \lambda_{i+1} \text{ for } i = 1, \ldots, n-1\}.$$

Let w_0 be the longest element in the Weyl group W for our choice of positive roots R^+. Let $W'(\lambda) = \mathrm{Ind}_B^G(w_0\lambda)$ be the "dual" Weyl module. By [Jan, Prop II.2.6], $W'(\lambda) = 0$ for $\lambda \notin X(T)_+$. Let $L(\lambda)$ be the socle of $W'(\lambda)$ (i.e. the maximal semisimple subrepresentation of $W'(\lambda)$). If $\lambda \in X(T)_+$, then $L(\lambda)$ is irreducible:

Theorem 2.1.3 (Steinberg)**.** *The simple* GL_n*-modules are* $\{L(\lambda) \mid \lambda \in X(T)_+\}$.

It is a special case of [Jan, Prop II.6.16] that if $\lambda \in X(T)_+$, then $L(\lambda)$ is a composition factor of $W'(\lambda)$ with multiplicity one. We will also make use of the following:

Proposition 2.1.4 (Strong linkage principle). *If $L(\mu)$ is a composition factor of $W'(\lambda)$, then $\mu \uparrow \lambda$.*

Let $\{e(\lambda)\}_{\lambda\in X(T)}$ be the canonical basis of the group ring $\mathbb{Z}[X(T)]$, with $e(\mu)e(\lambda) = e(\mu+\lambda)$. There is a formal character $\mathrm{ch} : \{\text{finite-dimensional } \mathrm{GL}_n-\text{modules}\} \to \mathbb{Z}[X(T)]^W$ defined by $\mathrm{ch}\, M = \sum_{\lambda\in X(T)} \dim M_\lambda\, e(\lambda)$ for any finite-dimensional GL_n-module (where $M = \bigoplus_{\lambda\in X(T)} M_\lambda$ is the weight space decomposition of M). For λ dominant, define $\chi(\lambda) := \mathrm{ch}\, W'(\lambda)$. Recall (see [Jan, Prop. II.5.10]) that by Weyl's character formula, we have

$$\chi(\lambda) = \frac{A(\lambda+\eta)}{A(\eta)}, \tag{2.3}$$

where

$$A(\lambda) = \sum_{w\in W} \det(w) e(w\lambda). \tag{2.4}$$

We use this formula to extend the definition of $\chi(\lambda)$ to all of $X(T)$ by defining $\chi(\lambda) = A(\lambda+\eta)/A(\eta)$ for all $\lambda \in X(T)$. From (2.4), notice that $A(w\lambda) = \det w\; A(\lambda)$. Then, by using (2.3) and (2.1), we get $\chi(w\bullet\lambda) = \det w\; \chi(\lambda)$. Since $w^2 = id$, we can write $\chi(\lambda) = \det w\; \chi(w\bullet\lambda)$. In particular, if $w\bullet\lambda$ is dominant, then

$$\chi(\lambda) = \det w\;\; \mathrm{ch}\, W'(w\bullet\lambda). \tag{2.5}$$

From the definition of ch, if M is a GL_n-module with $\mathrm{ch}\, M = \sum a_\lambda e(\lambda)$, then $\dim M = \sum a_\lambda$. We recall that the dimensions of the $W'(\lambda)$'s are given by Weyl's dimension formula:

$$\dim W'(\lambda) = \frac{\prod_{\alpha\in R^+}\langle \lambda+\eta, \alpha^\vee\rangle}{\prod_{\alpha\in R^+}\langle \eta, \alpha^\vee\rangle} \tag{2.6}$$

For $\lambda, \mu \in \overline{C}_0 \cap X(T)$, there exists a functor $T_\lambda^\mu : \{G\text{-modules}\} \to \{G\text{-modules}\}$ called the *translation functor from* λ *to* μ. We refer the reader to [Jan, II.7.6] for the definition, which we won't require. Instead, we mention the properties needed for the proofs in sections 2.2 and 2.3.

Proposition 2.1.5. *Let $\lambda, \mu \in \overline{C}_0 \cap X(T)$ such that μ belongs to the closure of the facet containing λ.*

1. *The functor T_λ^μ is exact.*

2. *Let $w \in W_p$ with $w \centerdot \lambda \in X(T)_+$ and let F be the facet with $w \centerdot \lambda \in F$. Then*

$$T_\lambda^\mu W'(w \centerdot \lambda) \cong W'(w \centerdot \mu), \tag{2.7}$$

$$T_\lambda^\mu L(w \centerdot \lambda) \cong \begin{cases} L(w \centerdot \mu) & \text{if } w \centerdot \mu \in \widehat{F}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

For a proof, see II.7.11 and II.7.15 in [Jan].

The affine Weyl group W_1 is the group generated by $s_{\alpha,k}$ for $\alpha \in R, k \in \mathbb{Z}$. There is an isomorphism of affine Weyl groups $W_1 \to W_p$ defined by $s_{\alpha,n} \mapsto s_{\alpha,pn}$, which we denote by $w \mapsto w[p]$. Let W_p^+ be the subset of W_p

$$W_p^+ = \{w \in W_p \mid \langle w(x+\eta), \alpha^\vee\rangle > 0 \text{ for all } \alpha \in R^+, x \in C_0 \cap X(T)\}$$

that maps an element of the standard alcove to $X(T)_+$. For λ in the standard alcove and $w \in W_p^+$, we have

$$\operatorname{ch} L(w \centerdot \lambda) = \sum_{w' \in W_p^+} a_{w,w'} \chi(w' \centerdot \lambda)$$

for some unique integers $a_{w,w'}$. By relabeling the elements of the affine Weyl group, we use the following theorem to compute the coefficients $a_{w,w'}$.

Theorem 2.1.6 (Lusztig's Conjecture)**.** *Let $w \in W_1^+$. For $p \gg 0$, we have*

$$a_{w[p],w'[p]} = \varepsilon(w)\varepsilon(w')P_{w_0w',w_0w}(1) \tag{2.9}$$

for all $w, w' \in W_1^+$, where $P_{x,y}$ is the Kazhdan-Lusztig polynomial for W_1.

For a definition of Kazhdan-Lusztig polynomials, see e.g. [Hum90, §7.9]. Theorem 2.1.6 was first conjectured by Lusztig in [Lus80]. It is now known to be false for small values of p (see [Wil15]). However, it is a theorem for p large enough. In [Fie12], Peter Fiebig proved this for an explicit (very large) lower bound on p.

Now, we turn our attention to the $\mathbb{F}_p$-points of $W'(\lambda)$ and $L(\lambda)$. Let $W(\lambda)$ be the representation of $\mathrm{GL}_n(\mathbb{F}_p)$ obtained by evaluating $W'(\lambda)$ on $\mathbb{F}_p$, and let $F(\lambda)$ be the representation of $\mathrm{GL}_n(\mathbb{F}_p)$ obtained by evaluating $L(\lambda)$ on $\mathbb{F}_p$.

Theorem 2.1.7 (Steinberg)**.** *The irreducible representations of $\mathrm{GL}_n(\mathbb{F}_p)$ are $F(\lambda)$, where $\lambda \in X_1(T)$. Moreover, $F(\lambda) \cong F(\lambda')$ if and only if $\lambda - \lambda' \in \mathbb{Z}(p-1, \ldots, p-1)$.*

For a proof, see Sections 2.11 and 2.12 of [Hum06].

Notice in particular that if $L(\lambda)$ is irreducible, $F(\lambda)$ may not be. Steinberg's tensor product theorem givens us a way to write any $F(\lambda)$ as a tensor produt of irreducible $F(\lambda_i)$.

Theorem 2.1.8 (Steinberg's tensor product theorem). *Let* $\lambda = \sum_{i=1}^{m} p^i \lambda_i$ *with* $\lambda_i \in X_1(T)$. *Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_m)^{[m]},$$

where $L(\lambda_i)^{[i]}$ *is the composition of* $L(\lambda_i)$ *with the* i*th power of Frobenius* $\mathrm{Fr} : \mathrm{GL}_n \to \mathrm{GL}_n$.

For a proof, see [Jan, II.3.16-II.3.17].

Proposition 2.1.9 (Brauer's formula). *If* M, V *are finite-dimensional* GL_n*-modules, and* $\mathrm{ch}\, V = \sum_{\mu \in X(T)} a_\mu \chi(\mu)$, *then*

$$\mathrm{ch}(M \otimes V) = \sum_{\mu \in X(T)} \sum_{\nu \in X(T)} a_\mu \dim(M_\nu) \chi(\mu + \nu).$$

For a proof, see [Jan, II.7.4].

2.2 Weyl Module Decompositions for $G = \mathrm{GL}_2$

2.2.1 Root System and Alcoves for $G = \mathrm{GL}_2$

Let $G = \mathrm{GL}_2$, so that $R^+ = \{\epsilon_1 - \epsilon_2\}$ is the standard system of positive roots. We write $\alpha = \epsilon_1 - \epsilon_2$. Then $s_\alpha(\lambda)$ is the reflection that swaps the two entries of $\lambda \in X(T) \cong \mathbb{Z}^2$. In Chapter 4, the GL_2 cases we will need are those when $\lambda = (a, b)$ has $0 \le a - b \le 3p - 1$. So, we will consider the (upper) closures of the following alcoves:

$$C_0: \qquad \{(a,b) \in \mathbb{R}^2 : -1 < a-b < p-1\},$$

$$C_1: \qquad \{(a,b) \in \mathbb{R}^2 : p-1 < a-b < 2p-1\},$$

$$C_2: \qquad \{(a,b) \in \mathbb{R}^2 : 2p-1 < a-b < 3p-1\}.$$

With this notation, the dot action of the affine Weyl group W_p on $X(T)$ given by $s_{\alpha,np} \cdot (a,b) = (b+pn-1, a-pn+1)$ reflects a weight about the lower boundary of the alcove C_n.

2.2.2 Jordan-Hölder Factors of $W(a,b)$

In this section, we calculate the irreducible constituents of $W(a,b)$ for all $0 \leq a-b < 3p-1$. In the language of alcoves, we find $JH(W(a,b))$ for $(a,b) \in \widehat{C}_0, \widehat{C}_1$, and $\widehat{C}_2$.

Lemma 2.2.1. *The following identities hold in the Grothendieck group of* $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_p)]$*:*

$$\begin{aligned}
F(a,b) &= W(a,b) && 0 \leq a-b \leq p-1 \\
F(a,b) &= W(b+1,b) && a-b = p \\
F(a,b) &= W(a-p+1,b) + W(a-p,b+1) && p < a-b \leq 2p-1 \\
F(a,b) &= W(b+2,b) && a-b = 2p \\
F(a,b) &= W(b+3,b) + W(b+2,b+1) && a-b = 2p+1 \\
F(a,b) &= W(a-2p+2,b) + W(a-2p+1,b+1) && 2p+1 < a-b \leq 3p-1 \\
&\qquad + W(a-2p,b+2)
\end{aligned}$$

Proof. First, if $0 \le a-b \le p-1$, then Theorem 2.1.7 says that $F(a,b)$ is irreducible so that $F(a,b) = W(a,b)$. For (a,b) with $p-1 < a-b < 2p-1$, we may write $(a,b) = (a-p,b) + p(1,0)$ with $(a-p,b) \in X_1(T)$. So, by Steinberg's tensor product theorem $L(a,b) \cong L(a-p,b) \otimes L(1,0)^{[1]}$. In fact, since Frobenius is trivial on $\mathbb{F}_p$, we have

$$F(a,b) \cong F(a-p,b) \otimes F(1,0). \tag{2.10}$$

We investigate the right hand side of (2.10) at the level of GL_n-modules by applying Brauer's formula. It follows from the Weyl character formula that $\operatorname{ch} L(1,0) = \operatorname{ch} W'(1,0) = e(1,0) + e(0,1)$. So, we have $\dim(F(1,0)_\nu) = 1$ for $\nu = (1,0),(0,1)$, and $\dim(F(1,0)_\nu) = 0$ for all other $\nu \in X(T)$. By Brauer's formula,

$$\begin{aligned}
\operatorname{ch}(L(a-p,b) \otimes L(1,0)) &= \operatorname{ch}(W'(a-p,b) \otimes L(1,0)) \\
&= \sum_{\nu \in X(T)} \dim(L(1,0)_\nu)\chi((a-p,b)+\nu) \\
&= \chi(a-p+1,b) + \chi(a-p,b+1). \qquad (2.11)
\end{aligned}$$

If $p = a-b$, then the second term is $\chi(a-p,b+1) = \chi(b,b+1)$. When applying Weyl's character formula to this term, the numerator is $e(b+1,b+1) - e(b+1,b+1) = 0$. So $\chi(b,b+1) = 0$ and $\operatorname{ch} L(a-p,b) \otimes L(1,0) = W'(a-p+1,b)$ in the Grothendieck group of $\overline{\mathbb{F}}_p[\mathrm{GL}_2]$. Evaluating on $\mathbb{F}_p$-points and using (2.10) gives us $F(b+p,b) = W(b+1,b)$ in the Grothendieck group $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_p)]$.

If $p < a-b < 2p-1$, then $(a-p+1,b)$ and $(a-p,b+1)$ are both dominant. So $L(a-p,b) \otimes L(1,0) = W'(a-p+1,b) + W'(a-p,b+1)$, and evaluating on $\mathbb{F}_p$-points gives us

$$F(a,b) = W(a-p+1,b) + W(a-p,b+1)$$

in the Grothendieck group $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_p)]$.

For (a,b) with $2p-1 < a-b \leq 3p-1$, we may write $(a,b) = (a-2p,b) + p(2,0)$ where $(a-2p,b) \in X_1(T)$. Steinberg's tensor product theorem, and the fact that Frobenius is trivial on $\mathbb{F}_p$ gives us

$$F(a,b) \cong F(a-2p,b) \otimes F(2,0). \tag{2.12}$$

Since $\mathrm{ch}\, L(2,0) = \mathrm{ch}\, W'(2,0) = e(2,0)+e(1,1)+e(0,2)$, we have $\dim(F(2,0)_\nu) = 1$ for $\nu = (2,0),(1,1)$, or $(0,2)$, and $\dim(F(2,0)_\nu) = 0$ otherwise. Then by Brauer's formula,

$$\begin{aligned}
&\mathrm{ch}(L(a-2p,b)\otimes L(2,0)) \\
&\qquad = \mathrm{ch}(W'(a-2p,b) \otimes L(2,0)) \\
&\qquad = \sum_{\nu \in X(T)} \dim(L(2,0)_\nu)\chi(\mu+\nu) \\
&\qquad = \chi(a-2p+2,b) + \chi(a-2p+1,b+1) + \chi(a-2p,b+2).
\end{aligned} \tag{2.13}$$

If $a-b = 2p$ or $2p+1$, then $(a-2p,b+2) \notin X(T)_+$. Consider first the case $a-b=2p$. Using the formula in (2.5) for computing the character $\chi(\lambda)$ with λ

non-dominant, we can show that $\chi(b, b+2) = -\chi(b+1, b+1)$. Then (2.13) is

$$\begin{aligned} \mathrm{ch}(L(b,b) \otimes L(2,0)) &= \chi(b+2,b) + \chi(b+1,b+1) + \chi(b,b+2) \\ &= \chi(b+2,b) + \chi(b+1,b+1) - \chi(b+1,b+1) \\ &= \chi(b+2,b). \end{aligned}$$

So $L(b,b) \otimes L(2,0) = W'(b+2,b)$, and evaluating on $\mathbb{F}_p$-points gives us $F(a,b) = W(b+2,b)$.

Next, consider the case $a-b = 2p+1$. Then $\chi(a-2p, b+2) = \chi(b+1, b+2)$, and when applying Weyl's character formula, the numerator is $e(b+2) - e(b+2) = 0$. So $\chi(a-2p, b+2) = 0$ in (2.13) implies $L(a-2p, b) \otimes L(2,0) = W'(a-2p+2, b) + W'(a-2p+1, b+1)$. Evaluating on $\mathbb{F}_p$-points as before, we get $F(a,b) = W(a-2p+2, b) + W(a-2p+1, b+1)$ in the Grothendieck group $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_p)]$.

If $2p+1 < a-b \leq 3p-1$, then $(a-2p+2, b), (a-2p+1, b+1)$, and $(a-2p, b+2)$ are all dominant. So

$$F(a,b) = W(a-2p+2,b) + W(a-2p+1,b+1) + W(a-2p,b+2)$$

in the Grothendieck group $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_p)]$. This provides the final row of the result, and completes the proof.

$\square$

Lemma 2.2.2. *Suppose* (a,b) *has* $0 \le a-b \le 3p-1$. *Then*

$$\begin{aligned}
W'(a,b) &= \operatorname{ch} L(a,b) && 0 \le a-b \le p-1,\\
W'(a,b) &= \operatorname{ch} L(a,b) + \operatorname{ch} L(b+p-1, a-p+1) && p-1 < a-b < 2p-1,\\
W'(a,b) &= \operatorname{ch} L(a,b) && a-b = 2p-1,\\
W'(a,b) &= \operatorname{ch} L(a,b) + \operatorname{ch} L(b+2p-1, a-2p+1) && 2p-1 < a-b < 3p-1,\\
W'(a,b) &= \operatorname{ch} L(a,b) && a-b = 3p-1.
\end{aligned}$$

Proof. It is an immediate consequence of the strong linkage principle that if $\lambda = (a,b) \in \widehat{C}_0$ or if $a-b = 2p-1$, then $JH(W'(\lambda)) = \{L(\lambda)\}$. Suppose $\lambda = (a,b) \in C_1$. Let $\lambda_0 = (b+p-1, a-p+1) \in C_0$ be the unique W_p-translate of (a,b) in alcove C_0. Then $s_{\alpha,p} \cdot \lambda_0 = \lambda$, so $\lambda_0 \uparrow \lambda$, and λ_0 is the only dominant weight strictly less than λ in the $\uparrow$ ordering. By the strong linkage principle, the only possible constituents of $W'(\lambda)$ are $L(\lambda)$ and $L(\lambda_0)$. Since $L(\lambda)$ has multiplicity 1 in $W'(\lambda)$, we have an identity of formal characters

$$\operatorname{ch} W'(\lambda) = \operatorname{ch} L(\lambda) + m \operatorname{ch} L(\lambda_0). \tag{2.14}$$

Next, we use the translation functor and Proposition 2.1.5 to find m. Choose $\mu = (p-1, 0) \in \widehat{C}_0$, and apply the translation functor $T_{\lambda_0}^{\mu}$ to (2.14). Since $T_{\lambda_0}^{\mu}$ is exact,

$$\operatorname{ch} T_{\lambda_0}^{\mu} W'(\lambda) = \operatorname{ch} T_{\lambda_0}^{\mu} L(\lambda) + m \operatorname{ch} T_{\lambda_0}^{\mu} L(\lambda_0). \tag{2.15}$$

We use Proposition 2.1.5 to calculate the three translated terms separately. By (2.7) we have

$$T_{\lambda_0}^{\mu} W'(\lambda) = T_{\lambda_0}^{\mu} W'(s_{\alpha,p} \centerdot \lambda_0) \cong W'(s_{\alpha,p} \centerdot \mu) = W'(\mu).$$

The facet containing $\lambda = s_{\alpha,p} \centerdot \lambda_0$ is the alcove $C_1 = \{(a,b) \mid p-1 < a-b < 2p-1\}$. Since μ is on the upper boundary of C_0, the reflection $s_{\alpha,p}$ fixes μ. In particular, $s_{\alpha,p} \centerdot \mu \notin \widehat{C}_1$, so by (2.8) in Proposition 2.1.5,

$$T_{\lambda_0}^{\mu} L(\lambda) = T_{\lambda_0}^{\mu} L(s_{\alpha,p} \centerdot \lambda_0) = 0.$$

Next, the facet containing λ_0 is the alcove C_0, and notice that $\mu \in \widehat{C}_0$. So by (2.8),

$$T_{\lambda_0}^{\mu} L(\lambda_0) = T_{\lambda_0}^{\mu} L(id \centerdot \lambda_0) = L(id \centerdot \mu) = L(\mu).$$

Thus the translated identity in (2.15) is $\operatorname{ch} W'(\mu) = \operatorname{ch} 0 + m \operatorname{ch} L(\mu)$. Since $L(\mu)$ has multiplicity 1 in $W'(\mu)$, $m = 1$ and

$$\operatorname{ch} W'(a,b) = \operatorname{ch} L(a,b) + \operatorname{ch} L(b+p-1, a-p+1). \tag{2.16}$$

Suppose $\lambda = (a,b) \in C_2$. Let $\lambda_1 = (b+2p-1, a-2p+1)$ and $\lambda_0 = (a-p, b+p)$ be the unique W_p-translates of λ lying in alcoves C_1 and C_0, respectively. By the strong linkage principle, the possible constituents of $W'(\lambda)$ are $L(\lambda)$, $L(\lambda_1)$, and $L(\lambda_0)$. Since $L(\lambda)$ has multiplicity 1 in $W'(\lambda)$, we have an identity of formal characters

$$\operatorname{ch} W'(\lambda) = \operatorname{ch} L(\lambda) + m \operatorname{ch} L(\lambda_1) + n \operatorname{ch} L(\lambda_0). \tag{2.17}$$

As before, we use the translation functor and Proposition 2.2 to find m. Choose $\mu = (2p-1, 0)$ on the upper boundary of C_1, and apply the translation functor $T_{\lambda_1}^{\mu}$

to (2.17). Since $T_{\lambda_1}^{\mu}$ is exact,

$$\operatorname{ch} T_{\lambda_1}^{\mu} W'(\lambda) = \operatorname{ch} T_{\lambda_1}^{\mu} L(\lambda) + m \operatorname{ch} T_{\lambda_1}^{\mu} L(\lambda_1) + n \operatorname{ch} T_{\lambda_1}^{\mu} L(\lambda_0). \tag{2.18}$$

We now use Proposition 2.1.5 to calculate the four translated terms separately. By (2.7) we have

$$T_{\lambda_1}^{\mu} W'(\lambda) = T_{\lambda_1}^{\mu} W'(s_{\alpha,2p} \cdot \lambda_1) \cong W'(s_{\alpha,2p} \cdot \mu) = W'(2p-1,0).$$

The facet containing $\lambda = s_{\alpha,2p} \cdot \lambda_1$ is C_2, and since μ is on the lower boundary of C_2, the reflection $s_{\alpha,2p}$ fixes μ. In particular, $s_{\alpha,2p} \notin \widehat{C}_2$, so by (2.8) in Proposition 2.1.5,

$$T_{\lambda_1}^{\mu} L(\lambda) = T_{\lambda_1}^{\mu} L(s_{\alpha,2p} \cdot \lambda_1) = 0.$$

The facet containing λ_1 is C_1, and since $\mu \in \widehat{C}_1$, we have

$$T_{\lambda_1}^{\mu} L(\lambda_1) = T_{\lambda_1}^{\mu} L(id \cdot \lambda_0) = L(id \cdot \mu) = L(\mu).$$

The facet containing $\lambda_0 = s_{\alpha,p} \cdot \lambda_1$ is C_0, but $s_{\alpha,p} \cdot \mu = (p-1,p)$ is on the lower boundary of C_0. So by (2.8),

$$T_{\lambda_1}^{\mu} L(\lambda_0) = T_{\lambda_1}^{\mu} L(s_{\alpha,p} \cdot \lambda_1) = 0.$$

Thus the translated identity in 2.18 is $\operatorname{ch} W'(\mu) = \operatorname{ch} 0 + m \operatorname{ch} L(2p-1,0) + \operatorname{ch} 0$. Since $L(\mu)$ has multiplicity 1 in $W'(\mu)$, $m = 1$ and

$$\operatorname{ch} W'(\lambda) = \operatorname{ch} L(\lambda) + \operatorname{ch} L(\lambda_1) + n \operatorname{ch} L(\lambda_0) \tag{2.19}$$

To calculate the value of n in (2.19), we let $\mu = (p-1,0)$, and apply the translation functor $T_{\lambda_1}^{\mu}$ to (2.17). By arguing in the same way as above, we obtain the identity

$$\operatorname{ch} W'(3p-1,0) = \operatorname{ch} L(3p-1,0) + n \operatorname{ch} L(2p-1,p). \tag{2.20}$$

Now we compare dimensions on either side of this identity. In two dimensions, Weyl's dimension formula is $\dim W'(x,y) = x - y + 1$. On the left hand side of (2.20), $\dim W'(3p-1,0) = 3p$. To calculate the dimensions on the right hand side, we use the fact that if $\lambda \in \widehat{C}_0$, then $W'(\lambda) = L(\lambda)$. By Steinberg's tensor product theorem, we can write $L(3p-1,0) = L(p-1,0) \otimes L(2,0)^{[1]}$. On the right hand side, $\dim L(p-1,0) = p$ and $\dim L(2,0) = 3$. So $\dim L(3p-1,0) = 3p$ and we must have $n = 0$ in (2.20). Then equation (2.20) gives us the result for $a-b = 3p-1$. And since $n = 0$ in (2.20), then $n = 0$ in (2.19). This gives us the final line in the lemma. $\square$

Theorem 2.2.3. *The decompositions of the Weyl modules $W(a,b)$ with $0 \leq a-b \leq 3p-1$ into irreducible modules are given by the following table. The right hand side of* *the table gives the decomposition of* $W(a,b)$ *in the Grothendieck group of* $\overline{\mathbb{F}}_p[\mathrm{GL}_n(\mathbb{F}_p)]$.

condition	$W(a,b)$
$0 \le a-b < p$	$F(a,b)$
$a-b=p$	$F(b+1,b)+F(b+p-1,b+1)$
$p < a-b < 2p-1$	$F(a-p+1,b)+F(a-p,b+1)+F(b+p-1,a-p+1)$
$a-b=2p-1$	$F(b+1,b)+2F(a-p,b+1)$
$a-b=2p$	$F(b+2,b)+F(b+p,b+1)+F(b+p-1,b+2)$
$a-b=2p+1$	$F(b+3,b)+F(b+2,b+1)$ $+F(b+p,b+2)+F(b+p-1,b+3)$
$2p+1 < a-b < 3p-2$	$F(a-2p+2,b)+F(a-2p+1,b+1)$ $+F(a-2p,b+2)+F(b+p,a-2p+1)$ $+F(b+p-1,a-2p+2)$
$a-b=3p-2$	$2F(b+1,b)+2F(b+p-1,b+1)$ $+F(b+p-2,b+2)$
$a-b=3p-1$	$F(b+2,b)+F(b+1,b+1)$ $+2F(b+p-1,b+2)+F(b+p,b+1)$

In particular, all the $F(\lambda)$ *listed in this table have* $\lambda \in X_1(T)$*, so that all the* $F(\lambda)$ *are irreducible* $\mathrm{GL}_n(\mathbb{F}_p)$*-modules.*

Proof. To prove this theorem, we use Lemmas 2.2.2 and 2.2.1 to rewrite each $W(\lambda)$ as a sum of $W(\lambda_i)$, where the λ_i's lie in lower alcoves than λ. This procedure is then repeated if necessary until all the λ_i's lie in the restricted region $X_1(T)$, so that $W(\lambda_i) = F(\lambda_i)$ and by Theorem 2.1.7, each of these $F(\lambda_i)$ are irreducible.

To illustrate, we work out lines eight and nine of the table. The proofs of the rest of the cases follow similarly.

Suppose that (a,b) has $2p+1 < a-b < 3p-1$. By Lemma 2.2.2,

$$\operatorname{ch} W'(a,b) = \operatorname{ch} L(a,b) + \operatorname{ch} L(b+2p-1, a-2p+1).$$

Evaluating on $\mathbb{F}_p$, we have

$$W(a,b) = F(a,b) + F(b+2p-1, a-2p+1).$$

Now, use Lemma 2.2.1 to rewrite each term on the right hand side in terms of lower alcoves.

$$F(a,b) = W(a-2p+2, b) + W(a-2p+1, b+1) + W(a-2p, b+2), \tag{2.21}$$

$$F(b+2p-1, a-2p+1) = \begin{cases} W(b+1,b) & \text{if } a-b = 3p-2, \\ W(b+p-1, a-2p+2) + W(b+p, a-2p+1) & \text{if } 2p+1 < a-b < 3p-2. \end{cases} \tag{2.22}$$

There are two cases. If $2p+1 < a-b < 3p-2$, then

$$\begin{aligned} W(a,b) = W(a-2p+2, b) + W(a-2p+1, b+1) + W(a-2p, b+2) \\ + W(b+p, a-2p+1) + W(b+p-1, a-2p+2). \end{aligned}$$

We can check that each of the pairs on the right side are p-restricted, so that

$$\begin{aligned} W(a,b) = F(a-2p+2, b) + F(a-2p+1, b+1) + F(a-2p, b+2) \\ + F(b+p, a-2p+1) + F(b+p-1, a-2p+2), \end{aligned}$$

and each of the $F(\lambda_i)$ on the right is irreducible. If $a-b=3p-2$, then

$$W(a,b) = W(b+p,b) + W(b+p-1,b+1) + W(b+p-2,b+2) + W(b+1,b). \quad (2.23)$$

The final three pairs in (2.23) are in C_0, and by the first line of the theorem, we have $W(b+p,b) = F(b+1,b) + F(b+p-1,b+1)$. Putting these facts together with (2.23) gives us the remaining line of the table:

$$W(a,b) = 2F(b+1,b) + 2F(b+p-1,b+1) + F(b+p-2,b+2).$$

□

2.3 Weyl Module Decompositions for $G = \mathrm{GL}_3$

In this section, we calculate the irreducible constituents of $W(a, b, c)$ for $(a, b, c) \in C_2$.

2.3.1 Root System and Alcoves for GL_3

Let $G = \mathrm{GL}_3$, so that $R^+ = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_3\}$ is the system of positive roots. We will write $\alpha_{12} = \epsilon_1 - \epsilon_2, \alpha_{23} = \epsilon_2 - \epsilon_3$, and $\alpha_{13} = \epsilon_1 - \epsilon_3$, so that the reflection $s_{\alpha_{ij}}(\lambda)$ swaps the ith and jth entries of λ. In Chapter 4, the GL_3 cases we will need are those when $\lambda = (a, b, c)$ has entries where $a - b$ and $b - c$ are at most $2p - 1$. So we will consider the following alcoves:

$$
\begin{aligned}
C_0 : &\quad \{(a,b,c) \in \mathbb{R}^3 : -1 < a-b, b-c, \quad a-c < p-2\} \\
C_{0'} : &\quad \{(a,b,c) \in \mathbb{R}^3 : a-b, b-c < p-1, \quad p-2 < a-c\} \\
C_1 : &\quad \{(a,b,c) \in \mathbb{R}^3 : \ -1 < a-b, \quad p-1 < b-c, \quad a-c < 2p-2\} \\
C_{1'} : &\quad \{(a,b,c) \in \mathbb{R}^3 : a-b < p-1, \quad b-c < 2p-1, \quad 2p-2 < a-c\} \\
C_2 : &\quad \{(a,b,c) \in \mathbb{R}^3 : p-1 < a-b, \quad -1 < b-c, \quad a-c < 2p-2\} \\
C_{2'} : &\quad \{(a,b,c) \in \mathbb{R}^3 : a-b < 2p-1, \quad b-c < p-1, \quad 2p-2 < a-c\},
\end{aligned}
\tag{2.24}
$$

where $C_0, C_{0'}$ are the lower and upper alcoves in [Her06, §4]. Each pair $(C_i, C_{i'})$ is a translation of the alcoves $C_0, C_{0'}$ that lie in the restricted region $X_1(T)$. The partial order on these alcoves can be visualized as in Figure 2.1.

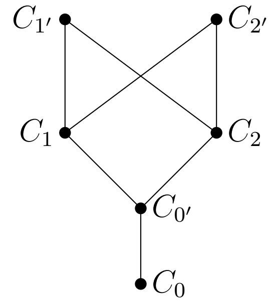

FIGURE 2.1. Partial order on GL_3 alcoves.

2.3.2 Jordan-Hölder Factors of $W(a, b, c)$.

Lemma 2.3.1. *Suppose $(a, b, c) \in C_2$. Then the following identities hold in the Grothendieck group of $\overline{\mathbb{F}}_p[\mathrm{GL}_3(\mathbb{F}_p)]$:*

$$F(a,b,c) = W(a-p+1,b,c) + W(a-p,b+1,c) + W(a-p,b,c+1) \qquad a-b \neq p, b-c \neq 0$$

$$F(a,b,c) = W(a-p+1,b,c) + W(a-p,b,c+1) \qquad a-b = p, b-c \neq 0$$

$$F(a,b,c) = W(a-p+1,b,c) + W(a-p,b+1,c) \qquad a-b \neq p, b-c = 0$$

$$F(a,b,c) = W(a-p+1,b,c) \qquad a-b = p, b-c = 0$$

Proof. For $(a, b, c) \in C_2$, we can write $(a, b, c) = (a - p, b, c) + p(1, 0, 0)$ with $(a - p, b, c) \in C_0$. Steinberg's tensor product theorem gives us $L(a, b, c) \cong L(a - p, b, c) \otimes L(1, 0, 0)^{[1]}$. Since Frobenius is trivial on $\mathbb{F}_p$, evaluating both sides of this isomorphism on $\mathbb{F}_p$ gives us

$$F(a, b, c) \cong F(a - p, b, c) \otimes F(1, 0, 0). \tag{2.25}$$

Now, we consider the right hand side of (2.25) at the level of GL_3-modules. Using

$\mathrm{ch}\, L(1,0,0) = e(1,0,0) + e(0,1,0) + e(0,0,1)$ and Brauer's formula, we have

$$\mathrm{ch}(L(a-p,b,c)\otimes L(1,0,0)) = \chi(a-p+1,b,c)+\chi(a-p,b+1,c)+\chi(a-p,b,c+1) \quad (2.26)$$

If $a-b=p$, then $\chi(a-p,b+1,c) = \chi(b,b+1,c)$. When applying Weyl's character formula to this, the numerator is $A(b+2,b+2,c) = 0$, so $\chi(a-p,b+1,c) = 0$. If $a-b \neq p$, then $(a-p,b+1,c) \in X(T)_+$, so $\chi(a-p,b+1,c) = \mathrm{ch}\, W'(a-p,b+1,c)$.

If $b-c=0$, then $\chi(a-p,b,c+1) = \chi(a-p,b,b+1)$. Applying Weyl's character formula as before, the numerator is $A(a-p+2,b+1,b+1) = 0$, so $\chi(a-p,b,c+1) = 0$. If $b-c \neq 0$, then $(a-p,b,c+1) \in X(T)_+$, so $\chi(a-p,b,c+1) = \mathrm{ch}\, W'(a-p,b,c+1)$.

The first term on the right hand side of (2.26) has $(a-p+1,b,c) \in X(T)_+$ for all $(a,b,c) \in C_2$. Evaluating (2.26) on $\mathbb{F}_p$-points, and then using (2.25) we have

$$F(a,b,c) = W(a-p+1,b,c) + W(a-p,b+1,c) + W(a-p,b,c+1),$$

where the term $W(a-p,b+1,c)$ occurs only if $a-p \neq p$, and the term $W(a-p,b,c+1)$ occurs only if $b-c \neq 0$. □

Next, we prove an analog of Lemma 2.2.2. The argument in the proof of that lemma used translation functors. In order to make the argument in this case less tedious, we will argue using Theorem 2.1.6 (Lusztig's conjecture) instead. While this theorem only holds for large p, the result listed can also be obtained using translation functors so that Lemma 2.3.2 does indeed hold for all $p > 3$.

Lemma 2.3.2. *If $\lambda_2 \in C_2$, then*

$$\operatorname{ch} W'(\lambda_2) = \operatorname{ch} L(\lambda_2) + \operatorname{ch} L(\lambda_{0'}), \tag{2.27}$$

where $\lambda_{0'}$ is the unique W_p-translate of λ_2 lying in $C_{0'}$.

Proof for $p \gg 0$. Let $\lambda_2 \in C_2$, and let $\lambda_{0'}, \lambda_0$ be the unique W_p-translates of λ_2 that lie in $C_{0'}, C_0$ respectively. Then we can write

$$\operatorname{ch} L(\lambda_i) = a_{i,0}\chi(\lambda_0) + a_{i,0'}\chi(\lambda_{0'}) + a_{i,2}\chi(\lambda_2)$$

for some integers $a_{i,j}$. We now use Lusztig's conjecture to show that the matrix of these a_{ij} is

$$A = (a_{ij})_{i,j=0,0',2} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}. \tag{2.28}$$

Let $w_1 = s_{\alpha_{13},1}$ so that $w_1[p] = s_{\alpha_{13},p}$, and $w_2 = s_{\alpha_{12},1}$ so that $w_2[p] = s_{\alpha_{12},p}$. First, $a_{0,0'} = a_{\mathrm{id},w_1[p]} = 0$ since $\mathrm{id} < w_1$. Similarly, $a_{0,2} = a_{\mathrm{id},w_2w_1[p]} = 0$ since $\mathrm{id} < w_2w_1$ and $a_{0',2} = a_{w_1[p],w_2w_1[p]} = 0$ since $w_1 < w_2w_1$.

Since w_1, w_2 are each translations of a single reflection, we have $\varepsilon(w_1) = \varepsilon(w_2) = -1$, and $\varepsilon(w_2w_1) = 1$.

We also note that $\ell(w_0w_2w_1) = 5, \ell(w_0w_1) = 4$, and $\ell(w_0) = 3$. To see this, choose $S = \{s_{\alpha_{12}}, s_{\alpha_{23}}, s_{\alpha_{13},1}\}$ as the simple reflections for W_1 and find a reduced expression for each of $w_0w_2w_1$, w_0w_1, and w_0. For instance, the reduced expression for $w_0w_2w_1$ is

$$w_0w_2w_1 = s_{\alpha_{12}}s_{\alpha_{13}}s_{\alpha_{12}}s_{\alpha_{12},1}s_{\alpha_{13},1} = s_{\alpha_{12}}s_{\alpha_{13}}s_{\alpha_{12}}s_{\alpha_{13},1}s_{\alpha_{23}}.$$

In what follows, we will use the fact [Hum90, §7.11] that if $\ell(w) - \ell(x) \leq 2$, then $P_{x,w} = 1$.

- Find $a_{2,0}$. Since $w_0 \leq w_0w_2w_1$, and $\ell(w_0w_2w_1) - \ell(w_0) = 2$, we have $P_{w_0,w_0w_2w_1} = 1$. So

$$a_{2,0} = a_{w_2w_1[p],\mathrm{id}} = \varepsilon(w_2w_1)\varepsilon(\mathrm{id})P_{w_0,w_0w_2w_1}(1) = (1)(1)(1) = 1.$$

- Find $a_{2,0'}$. Since $w_0w_1 \leq w_0w_2w_1$, and $\ell(w_0w_2w_1) - \ell(w_0w_1) = 1 \leq 2$, we have $P_{w_0w_1,w_0w_2w_1} = 1$. So

$$a_{2,0'} = a_{w_2w_1[p],w_1[p]} = \varepsilon(w_2w_1)\varepsilon(w_1)P_{w_0w_1,w_0w_2w_1}(1) = (1)(-1)(1) = -1.$$

- Find $a_{0',0}$. Since $w_0 \leq w_0w_1$, and $\ell(w_0w_1) - \ell(w_0) = 1 \leq 2$, we have $P_{w_0,w_0w_1} = 1$. So

$$a_{0',0} = a_{w_1[p],\mathrm{id}} = \varepsilon(w_1)\varepsilon(\mathrm{id})P_{w_0,w_0w_1}(1) = (-1)(1)(1) = -1.$$

So the matrix of the a_{ij} is indeed the one stated in (2.28). The inverse of A is the matrix of $b_{ij} = [W'(\lambda_i) : L(\lambda_j)]$. Computing the matrix inverse gives us

$$B = (b_{ij})_{i,j=0,0',2} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

In particular,

$$\mathrm{ch}\, W'(\lambda_2) = \mathrm{ch}\, L(\lambda_2) + \mathrm{ch}\, L(\lambda_{0'}).$$

□

Lemma 2.3.3. *For* $(a, b, c) \in C_2$*, the irreducible Jordan-Hölder factors of* $W(a, b, c)$ *are:*

$$JH(W(a,b,c)) = \{F(b+p-1, a-p+1, c),$$
$$F(a-p+1, b, c), F(a-p, b+1, c), F(a-p, b, c+1)\},$$

where the factor $F(a-p, b+1, c)$ *shows up only if* $a - b \neq p$*, and the factor* $F(a-p, b, c+1)$ *shows up only if* $b - c \neq 0$*.*

Proof. By Lemma 2.3.2,

$$\operatorname{ch} W'(a,b,c) = \operatorname{ch} L(a,b,c) + \operatorname{ch} L(b+p-1, a-p+1, c).$$

Evaluating this on $\mathbb{F}_p$, and using Lemma 2.3.1 on the resulting $F(a, b, c)$ term, we have

$$\begin{aligned} W(a,b,c) &= F(a,b,c) + F(b+p-1, a-p+1, c) \\ &= W(a-p+1, b, c) + W(a-p, b+1, c) + W(a-p, b, c+1) \\ &\quad + F(b+p-1, a-p+1, c), \end{aligned}$$

where $W(a-p, b+1, c)$ occurs only if $a - b \neq p$, and $W(a-p, b, c+1)$ occurs only if $b - c \neq 0$. Since each of $(a-p+1, b, c), (a-p, b+1, c), (a-p, b, c+1)$, and $(b+p-1, a-p+1, c)$ are p-restricted, we have

$$\begin{aligned} W(a,b,c) = F(a-p+1, b, c) + F(a-p, b+1, c) + F(a-p, b, c+1) \\ + F(b+p-1, a-p+1, c), \end{aligned}$$

with each term on the right hand side irreducible. □

Chapter 3

Sets of Serre Weights

Let p be a prime, and let $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the absolute Galois group of $\mathbb{Q}_p$. Suppose $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ is a crystalline representation (that is, suppose that $\rho \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$ is crystalline), and let $HT(\rho)$ be the multi-set of Hodge-Tate weights associated to ρ. We will restrict to the representations ρ that are *regular*, i.e. that have *distinct* Hodge-Tate weights.

Recalling the notation η from the second chapter, we will write $\eta_m = (m-1, m-2, \ldots, 1, 0)$.

Definition 3.0.1. If a crystalline representation has distinct Hodge-Tate weights $\{a_1, \ldots, a_n\}$, we will say that it has *Hodge type* $\lambda = \mathrm{dec}(a_1, \ldots, a_n) - \eta_n$, where dec indicates that we put the list $\{a_i\}$ in decreasing order.

We write $\mathbb{Z}^n_+$ for the set of n-tuples $(x_1, \ldots, x_n)$ with $x_1 \geq x_2 \geq \cdots \geq x_n$. Notice that since we restrict to the case of distinct Hodge-Tate weights, a Hodge type is always in $\mathbb{Z}^n_+$.

Definition 3.0.2. Let $\lambda \in \mathbb{Z}_n^+$ be a Hodge type. We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is *p-restricted* if $\lambda_i - \lambda_{i+1} \leq p - 1$ for $i = 1, \ldots, n-1$.

By reducing the entries of $\mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ modulo the maximal ideal of $\overline{\mathbb{Z}}_p$, we obtain a "mod p" representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$. We call ρ a *lift* of $\overline{\rho}$. There may be many different lifts of a mod p representation $\overline{\rho}$, and the regular such lifts may have various Hodge-Tate weights (and so various Hodge types).

Definition 3.0.3. Given a representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ such that $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ is semisimple, define the *crystalline weights for* $\overline{\rho}$, $W_{\mathrm{cris}}(\overline{\rho})$, to be the set of Serre weights $F(\lambda)$ such that $\overline{\rho}$ has a crystalline lift that is regular of Hodge type λ.

We remark that if $F(\lambda) = F(\lambda')$, then $\overline{\rho}$ has a lift of Hodge type λ if and only if it has one of type λ', so the set $W_{\mathrm{cris}}(\overline{\rho})$ is well-defined.

Question 3.0.4. *Given a* mod p *representation* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$*, what is* $W_{\mathrm{cris}}(\overline{\rho})$*?*

This question has been answered in dimensions 1 and 2 (see Example 3.1.5 below), and there is substantial progress towards an answer in dimension 3. In dimensions 4 and greater, the answer to this question is poorly understood. The strategy employed in [GHS16] to study Question 3.0.4 in general is to build up some subsets of Serre weights which are conjecturally contained in $W_{\mathrm{cris}}(\overline{\rho})$. In particular, [GHS16] defines the following subsets:

- $W_{\mathrm{obv}}(\overline{\rho})$: The set of *obvious weights*. These are discussed in Section 3.1 below.
- $\mathcal{C}(W_{\mathrm{obv}}(\overline{\rho}))$: Here $\mathcal{C}$ is a certain "closure" operator, which provides additional *shadow weights*. These are discussed in Section 3.2.
- $W_{\mathrm{expl}}(\overline{\rho})$: The set of explicit predicted weights, which provides additional *obscure weights*. These are discussed in Section 3.3.

If $\overline{\rho}$ is *sufficiently generic* (meaning that the weights $F(\lambda)$ lie far enough away from the boundaries of the alcoves - see [GHS16, §10] for a precise definition of “far enough”), then [GHS16] conjectures that $W_{\text{expl}}(\overline{\rho}) = W_{\text{cris}}(\overline{\rho})$.

3.1 Obvious Weights

The set $W_{\text{cris}}(\overline{\rho})$ conjecturally depends only on the values of $\overline{\rho}$ evaluated on the inertia subgroup. And, if $\overline{\rho}$ is semisimple, then there is a certain class of lifts of $\overline{\rho}$ that are easy to write down.

Fix an embedding $\sigma : \mathbb{F}_{p^n}^\times \hookrightarrow \overline{\mathbb{F}}_p^\times$, and let $\text{Art}^{-1} : I_{\mathbb{Q}_{p^n}} \to \mathbb{Z}_{p^n}^\times$ be the map from class field theory. Let $\omega_n : I_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$ be the composition

$$\omega_n : I_{\mathbb{Q}_p} = I_{\mathbb{Q}_{p^n}} \overset{\text{Art}^{-1}}{\twoheadrightarrow} \mathbb{Z}_{p^n}^\times \to \mathbb{F}_{p^n}^\times \overset{\sigma}{\hookrightarrow} \overline{\mathbb{F}}_p^\times.$$

When we write simply ω, with no subscript, we mean ω_1.

Proposition 3.1.1. *For any n-tuple $\{a_i\}$, there exists a crystalline representation $\Psi_{\{a_i\}} : G_{\mathbb{Q}_p} \to \text{GL}_n(\overline{\mathbb{Z}}_p)$ with Hodge-Tate weights $\{a_i\}$ and*

$$\overline{\Psi}_{\{a_i\}}|_{I_{\mathbb{Q}_p}} = \omega_n^m \oplus \omega_n^{pm} \oplus \cdots \oplus \omega_n^{p^{n-1}m}, \tag{3.1}$$

where $m = a_{n-1}p^{n-1} + \cdots + a_1 p + a_0$.

For a proof, see Corollary 7.1.2 in [GHS16]. Suppose that $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ has the form given on the right hand side of (3.1), and that the set $\{a_i\}$ is p-restricted and regular. Then Proposition 3.1.1 provides us with a Ψ such that $\Psi|_{I_{\mathbb{Q}_p}}$ is a lift of $\overline{\rho}|_{I_{\mathbb{Q}_p}}$. We know the Hodge type of this lift, and since we expect that the set $W_{\text{cris}}(\overline{\rho})$ depends only on inertia, we therefore have a Serre weight that should belong to $W_{\text{cris}}(\overline{\rho})$. This leads us to make the following two definitions (see Definition 7.1.3 in [GHS16]).

Definition 3.1.2. Given a representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ such that $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ is semisimple, we say that $\bigoplus_j \Psi_{\{a_{ij}\}_i}$ is an *obvious lift* of $\overline{\rho}$ if $\bigoplus_j \overline{\Psi}_{\{a_{ij}\}}|_{I_{\mathbb{Q}_p}} = \overline{\rho}|_{I_{\mathbb{Q}_p}}$.

Notice that an obvious lift Ψ of $\overline{\rho}$ is not necessarily an actual lift of $\overline{\rho}$; we only require that the reductions agree on inertia.

Definition 3.1.3. For a representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ such that $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ is semisimple, define the *set of obvious weights for* $\overline{\rho}$, $W_{\text{obv}}(\overline{\rho})$, to be the set of Serre weights $F(\lambda)$ such that $\overline{\rho}$ has an obvious lift of Hodge type λ.

Conjecture 3.1.4. *For $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, we have $W_{\text{obv}}(\overline{\rho}) \subset W_{\text{cris}}(\overline{\rho})$.*

In fact, in dimensions 1 and 2, obvious weights are all we need. That is, if $n = 1$ or 2, then $W_{\text{obv}}(\overline{\rho}) = W_{\text{cris}}(\overline{\rho})$. The argument for $n = 1$ is given in the following example:

Example 3.1.5. Suppose $\overline{\rho} : G_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^{\times}$. Then $\overline{\rho}|_{I_{\mathbb{Q}_p}} = \omega^a$ for some $0 \leq a < p - 1$. By Lemma 5.1.6 of [GHS16], there exists a crystalline character $\psi_{\{a\}} : G_{\mathbb{Q}_p} \to \overline{\mathbb{Z}}_p^{\times}$ with Hodge-Tate weight $\{a\}$, and with $\overline{\psi}_{\{a\}}|_{I_{\mathbb{Q}_p}} = \omega^a$. Moreover, this is unique up to an unramified twist. So $W_{\text{cris}}(\overline{\rho}) = W_{\text{obv}}(\overline{\rho}) = \{F(a)\}$.

When $n = 2$, the fact that $W_{\text{obv}}(\overline{\rho}) = W_{\text{cris}}(\overline{\rho})$ is a consequence of Fontaine-Laffaille theory when $a-b < p-1$, and [BLZ04] for weights $F(a,b)$ with $a-b = p-1$. This equality fails when $n > 2$, as we will see in Example 3.2.6.

3.2 Closure Operator

Let $R(\overline{\rho})$ be the universal lifting $\overline{\mathbb{Z}}_p$-algebra of $\overline{\rho}$, and let $R(\lambda, \overline{\rho})$ be the unique reduced and p-torsion-free quotient (see [Kis08]) of $R(\overline{\rho})$ parametrizing those crystalline lifts with Hodge type λ.

Remark 3.2.1. *For a Serre weight $F(\lambda)$, we have $F(\lambda) \in W_{\text{cris}}(\overline{\rho})$ if and only if $R(\lambda, \overline{\rho}) \neq 0$.*

The semisimplification of a Weyl module $W(\lambda)$ can be written as a direct sum of irreducible representations $F(a)$:

$$W(\lambda)^{ss} = \bigoplus_a F(a)^{n_a(\lambda)}, \tag{3.2}$$

for some integers $n_a(\lambda)$.

Conjecture 3.2.2 (Breuil-Mézard [EG14])**.** *There exist non-negative integers $\mu_a(\overline{\rho})$ for p-restricted $a \in \mathbb{Z}^n_+$ such that for all λ, one has*

$$e(\overline{R(\lambda, \overline{\rho})}) = \sum_a \mu_a(\overline{\rho}) \cdot n_a(\lambda).$$

where the $n_a(\lambda)$ are the integers from (3.2), and e is Hilbert-Samuel multiplicity.

When $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ is semisimple, it is conjectured that $\mu_a(\overline{\rho}) > 0$ if and only if $F(a) \in W_{\text{cris}}(\overline{\rho})$. Suppose that $F(a) \in W_{\text{cris}}(\overline{\rho})$ so that we conjecturally have $\mu_a(\overline{\rho}) > 0$. Suppose also that $F(a)$ shows up as a Jordan-Hölder factor of $W(\lambda)$ so that $n_a(\lambda) > 0$. Then by the Breuil-Mézard conjecture, $e(\overline{R(\lambda, \overline{\rho})}) \neq 0$, which implies that $F(\lambda) \in W_{\text{cris}}(\overline{\rho})$.

To summarize, we will use the following:

$$F(a) \in W_{\text{cris}}(\overline{\rho}) \text{ and } F(a) \in JH(W(\lambda)) \overset{conj.}{\Rightarrow} F(\lambda) \in W_{\text{cris}}(\overline{\rho}). \tag{3.3}$$

Definition 3.2.3. For a set of Serre weights W, define $\mathcal{C}(W)$ to be the smallest set of weights such that

- $W \subset \mathcal{C}(W)$, and
- if $\mathcal{C}(W) \cap JH(W(b)) \neq \varnothing$, then $F(b) \in \mathcal{C}(W)$.

The weights in $\mathcal{C}(W_{\text{obv}}(\overline{\rho})) \setminus W_{\text{obv}}(\overline{\rho})$ are called *shadow weights*.

Example 3.2.4. Suppose $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\mathbb{F}_p)$ has $F(x, y, z) \in W_{\text{obv}}(\overline{\rho})$, where $x - y, y - z < p - 1$ and $x - z < p - 2$. Then by [Her09, Proposition 3.18], there is an exact sequence

$$0 \to F(z + p - 2, y, x - p + 2) \to W(z + p - 2, y, x - p + 2) \to F(x, y, z) \to 0.$$

So $F(z + p - 2, y, x - p + 2) \in \mathcal{C}(W_{\text{obv}}(\overline{\rho}))$.

Conjecture 3.2.5 ([GHS16])**.** *For $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, we have $\mathcal{C}(W_{\text{obv}}(\overline{\rho})) \subset W_{\text{cris}}(\overline{\rho})$*

Example 3.2.6. Let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ be given by $\overline{\rho} = \omega^a \oplus \omega^b \oplus \omega^c$ with $a > b > c$ and $a - c < p - 2$. The obvious and shadow weights of $\overline{\rho}$ are worked out in Example 7.1.8 of [GHS16]. The results are as follows: The obvious weights of $\overline{\rho}$ are

$$\begin{aligned} W_{\mathrm{obv}}(\overline{\rho}) = \{&F(a-2, b-1, c), F(b+p-3, a-1, c), \\ &F(a-2, c-1, b-p+1), F(b-2, c-1, a-p+1), \\ &F(c+p-3, a-1, b), F(c+p-3, b-1, a-p+1)\}. \end{aligned}$$

The shadow weights of $\overline{\rho}$ are

$$\begin{aligned} \mathcal{C}(W_{\mathrm{obv}}(\overline{\rho})) \setminus W_{\mathrm{obv}}(\overline{\rho}) = \{&F(c+p-2, b-1, a-p), \\ &F(a-1, c-1, b-p), F(b+p-2, a-1, c-1)\}. \end{aligned}$$

Conjecture 3.2.7 ([GHS16])**.** *If $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ is semisimple and generic, then $\mathcal{C}(W_{\mathrm{obv}}(\overline{\rho})) = W_{\mathrm{cris}}(\overline{\rho})$.*

The following example shows why we don't have $\mathcal{C}(W_{\mathrm{obv}}(\overline{\rho})) = W_{\mathrm{cris}}(\overline{\rho})$ in general.

Example 3.2.8. Consider the representation $\overline{\rho} = \omega^a \oplus \omega^a \oplus \omega^b$ with $1 \leq a - b \leq p - 1$. Let $\overline{\rho}_1 = \omega^a \oplus \omega^b$. This has a lift with Hodge-Tate weights (a, b), and since $(a-1, b)$ is p-restricted, $\overline{\rho}_1$ has a lift with Hodge type $(a-1, b)$.

By Theorem 2.2.3, we have $F(a-1, b) \in JH(W(a+p-1, b-1))$, so $\overline{\rho}$ conjecturally has a lift with Hodge-Tate weights $(a+p, b-1)$. So $\overline{\rho} = \overline{\rho}_1 \oplus \omega^a$ has a lift with Hodge-Tate weights $(a+p, b-1, a)$. Let $\lambda + \eta = \mathrm{dec}(a+p, b-1, a) = (a+p, a, b-1)$. Then $\lambda = (a+p-2, a-1, b-1)$ is p-restricted, so we expect that $F(\lambda) \in W_{\mathrm{cris}}(\overline{\rho})$, but on the other hand one can check that $F(\lambda) \notin \mathcal{C}(W_{\mathrm{obv}}(\overline{\rho}))$.

3.3 Explicit Weights

Suppose $\lambda \in \mathbb{Z}_+^m$. A set $\{\lambda^{(i)}\}_{i=1,\dots,r}$ with $\lambda^{(i)} \in \mathbb{Z}_+^{m_i}$ is called an *η-partition* of λ if $\{\lambda^{(i)} + \eta_{m_i}\}_{i=1,\dots,r}$ is obtained by partitioning $\lambda + \eta_m$ into r decreasing subsequences of length m_i.

Definition 3.3.1. For a representation $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, define the set of *explicit weights*, $W_{\mathrm{expl}}(\overline{\rho})$, to be the smallest set of Serre weights such that

- $W_{\mathrm{obv}}(\overline{\rho}) \subset W_{\mathrm{expl}}(\overline{\rho})$, and
- if there exists a decomposition $\overline{\rho} = \bigoplus \overline{\rho}^{(i)}$ and an η-partition $\lambda^{(i)}$ of λ such that

$$W_{\mathrm{expl}}(\overline{\rho}^{(i)}) \cap JH(W(\lambda^{(i)})) \neq \varnothing$$

  then $\lambda \in W_{\mathrm{expl}}(\overline{\rho})$.

We call the weights in $W_{\mathrm{expl}}(\overline{\rho}) \setminus \mathcal{C}(W_{\mathrm{obv}}(\overline{\rho}))$ *obscure* weights.

Example 3.3.2 ([GHS16], Example 7.2.5)**.** As in Example 3.2.8, let $\overline{\rho} = \omega^a \oplus \omega^a \oplus \omega^b$ with $1 \leq a - b \leq p - 1$. We show that the weight $F(a+p-2, a-1, b-1) \in W_{\mathrm{expl}}(\overline{\rho})$. Decompose $\overline{\rho} = \overline{\rho}_1 \oplus \overline{\rho}_2$ with $\overline{\rho}_1 = \omega^a \oplus \omega^b$, and $\overline{\rho}_2 = \omega^a$. Choose $\lambda^{(1)} = (a+p-1, b-1)$ and $\lambda^{(2)} = (a)$ as the η-partition of λ. Then $\lambda^{(1)} \in \widehat{C}_1$, so by Theorem 2.2.3, we have $F(a-1, b) \in JH(W(\lambda^{(1)}))$, and thus $F(a-1, b) \in W_{\mathrm{expl}}(\overline{\rho}_1)$. Clearly, $F(a) \in JH(W(a))$ and $F(a) \in W_{\mathrm{expl}}(\overline{\rho}_2)$. This shows that $(a+p-2, a-1, b-1) \in W_{\mathrm{expl}}(\overline{\rho})$.

Chapter 4

The Tensor Product Problem

One method of constructing higher-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ is by taking the tensor product of two lower-dimensional crystalline representations. If we know the Hodge-Tate weights of the lower-dimensional representations, the list of Hodge-Tate weights of the tensor product is found by taking pairwise sums of the original Hodge-Tate weights.

More concretely, suppose that $\overline{\rho}' : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ has a lift ρ' with $\mathrm{HT}(\rho') = \{s_i\}_{1\leq i\leq n}$ and that $\overline{\rho}'' : G_{\mathbb{Q}_p} \to \mathrm{GL}_m(\overline{\mathbb{F}}_p)$ has a lift ρ'' with $\mathrm{HT}(\rho'') = \{t_j\}_{1\leq j\leq m}$. Then $\rho' \otimes \rho''$ is a lift of $\overline{\rho}' \otimes \overline{\rho}''$, and

$$\mathrm{HT}(\rho' \otimes \rho'') = \{s_i + t_j\}_{1\leq i\leq n, 1\leq j\leq m}.$$

Suppose further that the set $\{s_i + t_j\}$ is p-restricted, so that $\rho' \otimes \rho''$ has Hodge type $\{s_i + t_j\} - \eta_{n+m}$. Therefore, we must have $F((s_i + t_j) - \eta_{n+m}) \in W_{\mathrm{cris}}(\overline{\rho})$. However, a priori it is not clear whether this weight is in the set $W_{\mathrm{expl}}(\overline{\rho})$ that we constructed in the previous chapter, or whether we should expand this set.

In this chapter we will show that in the case of four-dimensional representations $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_4(\overline{\mathbb{F}}_p)$ with $\overline{\rho}', \overline{\rho}''$ two-dimensional, all weights obtained in the manner described above are already contained in the set $W_{\mathrm{expl}}(\overline{\rho})$. In particular, we will prove:

Theorem A. *If $\overline{\rho}'$ and $\overline{\rho}''$ are two-dimensional representations with crystalline lifts* *of Hodge-Tate weights $(s,0)$ and $(t,0)$ such that $(s+t,t,s,0)-\eta$ is dominant and p-restricted, then $F((s+t,t,s,0)-\eta)$ is in $W_{\mathrm{expl}}(\overline{\rho}'\otimes\overline{\rho}'')$.*

In this chapter we prove this theorem for $p>3$. In fact, Theorem A is also true for $p=2,3$. For completeness, we will verify this (by hand) in Section 5.1.

4.1 Tensoring 2D Representations

In the remainder of this chapter, we suppose $\overline{\rho}' : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ has a lift ρ' with $\mathrm{HT}(\rho') = \{s,0\}$, and that $\overline{\rho}'' : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ has a lift ρ'' with $\mathrm{HT}(\rho'') = \{t,0\}$. Without loss of generality, we assume that $s \le t$. Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_4(\overline{\mathbb{Z}}_p)$ be the tensor product $\rho = \rho' \otimes \rho''$. By the discussion above, $\mathrm{HT}(\rho) = \{s+t,t,s,0\}$. In addition, we assume that $(s+t,t,s,0)$ is dominant and p-restricted, so that ρ has Hodge type $(s+t-3,t-2,s-1,0)$. In order for this to hold, we must have $1 \le s \le p$ and $s < t \le s+p$.

Our goal is to show that the weight $F((s+t,t,s,0)-\eta)$ is contained in the set of explicit weights $W_{\mathrm{expl}}(\overline{\rho}'\otimes\overline{\rho}'')$. We will benefit from the fact that the representations $\overline{\rho}'$ and $\overline{\rho}''$ restricted to inertia have explicit forms we can use. Indeed, the reduction $\bmod p$ of 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights in the interval $[0,2p]$ is given by Fontaine-Laffaille theory in the $[0,p-1]$ cases, and by [BB] for all cases in the interval $[0,2p]$.

Lemma 4.1.1. *The representation $\overline{\rho}'|_{I_{\mathbb{Q}_p}}$ has one of the following two forms:*

$$\overline{\rho}_1 = \omega^s \oplus 1, \ \ \textit{or } \overline{\rho}_2 = \omega_2^s \oplus \omega_2^{ps}, \tag{4.1}$$

and the representation $\overline{\rho}''|_{I_{\mathbb{Q}_p}}$ *has one of the following four forms:*

$$\overline{\rho}_1 = \omega^t \oplus 1, \quad \overline{\rho}_2 = \omega_2^t \oplus \omega_2^{pt}, \quad \overline{\rho}_{1^+} = \omega^{t-1} \oplus \omega, \quad \overline{\rho}_{2^+} = \omega_2^{t-p+1} \oplus \omega_2^{p(t-p+1)}. \tag{4.2}$$

The forms $\overline{\rho}_{1^+}$ *and* $\overline{\rho}_{2^+}$ *occur only if* $t \geq p+1$*. Also, if* $t = 2p$*, then* $\overline{\rho}_{1^+} = \overline{\rho}_{2^+}$*.*

We write $\overline{\rho}_{i,j}$ for the tensor product $\overline{\rho}_{i,j} = \overline{\rho}_i \otimes \overline{\rho}_j$, where $\overline{\rho}_i$ is one of the forms in (4.1) and $\overline{\rho}_j$ is one of the forms in (4.2). From these possibilities, we see that $\overline{\rho}|_{I_{\mathbb{Q}_p}}$ has one of the following eight forms:

$$\begin{aligned}
\overline{\rho}_{1,1} &= \omega^{s+t} \oplus \omega^s \oplus \omega^t \oplus 1 \\
\overline{\rho}_{1,2} &= \omega_2^{(s+t)+sp} \oplus \omega_2^{p((s+t)+sp)} \oplus \omega_2^t \oplus \omega_2^{pt} \\
\overline{\rho}_{2,1} &= \omega_2^{(s+t)+tp} \oplus \omega_2^{p((s+t)+tp)} \oplus \omega_2^s \oplus \omega_2^{ps} \\
\overline{\rho}_{2,2} &= \omega_2^{s+t} \oplus \omega_2^{p(s+t)} \oplus \omega_2^{s+pt} \oplus \omega_2^{p(s+pt)} \\
\overline{\rho}_{1,1^+} &= \omega^{s+t-1} \oplus \omega^{t-1} \oplus \omega^{s+1} \oplus \omega \\
\overline{\rho}_{2,1^+} &= \omega_2^{(s+t-1)+p(t-1)} \oplus \omega_2^{p(s+t-1)+t-1} \oplus \omega_2^{s+1+p} \oplus \omega_2^{p(s+1)+1} \\
\overline{\rho}_{1,2^+} &= \omega_2^{s+t+1+p(s-1)} \oplus \omega_2^{p(s+t+1)+s-1} \oplus \omega_2^{t+1-p} \oplus \omega_2^{p(t+1)-1} \\
\overline{\rho}_{2,2^+} &= \omega_2^{(s+t+1)-p} \oplus \omega_2^{p(s+t+1)-1} \oplus \omega_2^{s-1+p(t+1)} \oplus \omega_2^{p(s-1)+t+1}
\end{aligned} \tag{4.3}$$

If $\overline{\rho} = \overline{\rho}' \otimes \overline{\rho}''$ has $\overline{\rho}|_{I_{\mathbb{Q}_p}} = \overline{\rho}_{i,j}$, we say that $\overline{\rho}$ has *type* (i,j).

For the types (i,j) with $i,j \in \{1,2\}$, it follows immediately from the definitions and the formulas in (4.3) that $\lambda = (s+t,t,s,0) - \eta$ is an obvious weight of $\overline{\rho} = \overline{\rho}' \otimes \overline{\rho}''$ so that $F(\lambda) \in W_{\mathrm{expl}}(\overline{\rho}' \otimes \overline{\rho}'')$. This reduces the proof of Theorem A to the cases with $j \in \{1^+, 2^+\}$. In particular, we will assume that $t \geq p+1$ for the remainder of the chapter.

4.2 Root System and Alcoves for GL_4

Let $G = \mathrm{GL}_4$, and $X(T) \cong \mathbb{Z}^4$ be the character group for GL_4. Let R be the standard root system. Following the convention from Sections 2.2 and 2.3, write α_{ij} for the root $\epsilon_i - \epsilon_j \in R$ so that the reflection $s_{\alpha_{ij}} \in W$ acting on $\lambda \in X(T)$ swaps the ith and jth entries of λ. We follow Florian Herzig's notation in [Her06, §9] for the alcoves we will use. Let $\lambda = (a, b, c, d) - \eta \in X(T) \otimes \mathbb{R}$. Then λ is in C_i if it satisfies the following conditions:

$$\begin{aligned}
C_0 :&\quad 0 < a-b, b-c, c-d;\; a-d < p, \\
C_1 :&\quad 0 < b-c;\; p < a-d;\; a-c, b-d < p, \\
C_2 :&\quad 0 < c-d;\; p < a-c;\; a-b, b-d < p, \\
C_3 :&\quad 0 < a-b;\; p < b-d;\; c-d, a-c < p, \\
C_4 :&\quad p < a-c, b-d;\; b-c < p;\; a-d < 2p, \\
C_5 :&\quad 2p < a-d;\; a-b, b-c, c-d < p, \\
C_{0'} :&\quad 0 < b-c, c-d; p < a-d; a-d < 2p, \\
C_{0''} :&\quad 0 < a-b, b-c; p < c-d; a-d < 2p.
\end{aligned} \tag{4.4}$$

The alcoves C_i with $i = 0, 1, 2, 3, 4, 5$ are the alcoves in the restricted region $X_1(T)$. The alcoves $C_{0'}$ and $C_{0''}$ are the only non-restricted alcoves lying below a restricted alcove in the $\uparrow$ ordering ($C_{0'}, C_{0''} \uparrow C_5$). We will also (as in [Her06]) call C_i *alcove* i.

Recall that for the remainder of this chapter, we have assumed that $1 \leq s \leq p$, $1 \leq t - s \leq p$, and $p < t$.

Lemma 4.2.1. *The weight* $\lambda = (s + t, t, s, 0) - \eta$ *is inside or on the boundary of* *either C_4 or C_5. Moreover, λ is on a boundary of one of these two alcoves under the conditions listed in the table below:*

condition	*boundary*
$s = p$	*upper boundary of* C_5
$t - s = p, s < p/2$	*upper boundary of* C_4
$t - s = p, s > p/2$	*upper boundary of* C_5
$s + t = 2p$	*upper bdry. of* C_4*, and lower bdry. of* C_5

Proof. Setting $(a, b, c, d) = (t + s, t, s, 0)$ in the inequalities defining C_4 and C_5 in (4.4), we have that $\lambda \in C_4, C_5$ if the following inequalities are satisfied:

$$\begin{aligned} C_4: &\qquad p < t;\ t - s < p;\ s + t < 2p \\ C_5: &\qquad 2p < s + t;\ s, t - s < p \end{aligned} \tag{4.5}$$

As we assume the inequalities $p < t, t - s \leq p$, and $s \leq p$, we have that λ is in the closure of C_4 if $s + t \leq 2p$, and in the closure of C_5 if $2p \leq s + t$. In particular, we can read off from (4.5) the boundary conditions given in the table. We note in particular that if $t - s = p$, then $s + t < 2p$ only if $s < p/2$, and $s + t > 2p$ only if $s > p/2$, giving us the first two lines of the table. Also, note that if $s + t = 2p$, then λ is simultaneously on the upper boundary of C_4 and the lower boundary of C_5. In this case, since $s \neq p/2$ we know $t - s \neq p$, and since $p < t$ we must have $s \neq p$. So we never have λ simultaneously on a lower and an upper boundary of C_5. □

To make use of the closure operator $\mathcal{C}$ defined in Section 3.2, we will want to know the Jordan-Hölder factors of $W(\lambda)$ for λ in the closures of C_4 and C_5. For those λ in the interior of one of these alcoves, we know these Jordan-Hölder factors from Prop.

9.3 in [Her06]. If $\lambda_0 \in C_0$ and λ_i is the unique W_p-translate of λ_0 in C_i, then

$$W'(\lambda_4) = L(\lambda_4) + L(\lambda_3) + L(\lambda_2) + L(\lambda_1) + L(\lambda_0), \tag{4.6}$$

$$W'(\lambda_5) = L(\lambda_5) + L(\lambda_{0'}) + L(\lambda_{0''}) + L(\lambda_4) + L(\lambda_3) + L(\lambda_2) + L(\lambda_1). \tag{4.7}$$

Still following the notation of [Her06, (9.2)], write r_{ij} for the affine reflection in W_p that maps a weight in C_i to one in C_j. Explicitly, we have the following formulas:

$$\begin{aligned} r_{01} \bullet \lambda &= s_{\alpha_{14}}(\lambda + \eta) + p(1,0,0,-1) - \eta \\ r_{13} \bullet \lambda = r_{24} \bullet \lambda = r_{0'5} \bullet \lambda &= s_{\alpha_{24}}(\lambda + \eta) + p(0,1,0,-1) - \eta \\ r_{12} \bullet \lambda = r_{34} \bullet \lambda = r_{0''5} \bullet \lambda &= s_{\alpha_{13}}(\lambda + \eta) + p(1,0,-1,0) - \eta \\ r_{45} \bullet \lambda &= s_{\alpha_{14}}(\lambda + \eta) + p(2,0,0,-2) - \eta \end{aligned} \tag{4.8}$$

Lemma 4.2.2. *1. Suppose that $\lambda = (s+t, t, s, 0) - \eta =: \lambda_4$ lies on an upper boundary of C_4. Then*

$$\begin{aligned} W(\lambda_4) &= F(\lambda_4) + F(\lambda_0) & & t - s = p, s < p/2 \\ W(\lambda_4) &= F(\lambda_4) + F(\lambda_3) + F(\lambda_2) & & s + t = 2p \end{aligned}$$

where $\lambda_i = r_{i4} \bullet \lambda$.

2. Suppose that $\lambda = (s+t, t, s, 0) - \eta =: \lambda_5$ lies on an upper boundary of C_5. Then

$$\begin{aligned} W(\lambda_5) &= F(\lambda_5) + F(\lambda_1) & & s = p,\ t - s \neq p \\ W(\lambda_5) &= F(\lambda_5) + F(\lambda_4) & & t - s = p,\ p/2 < s < p \\ W(\lambda_5) &= F(\lambda_5) & & s = p,\ t - s = p \end{aligned}$$

where $\lambda_i = r_{i5} \bullet \lambda$.

Proof. Let $S = \{0,1,2,3,4,5\}$ in the case 1 and $S = \{1,2,3,4,0',0''\}$ in case 2. By applying a translation functor, and using Proposition 2.1.5, it is straightforward to show that

$$W'(\lambda) = \sum_{i \in S \,:\, \lambda_i \in \widehat{C}_i} L(\lambda_i).$$

That is, the decomposition is the same as in (4.6) or (4.7), except that we drop all the terms $L(\lambda_i)$ where λ_i is not on an upper boundary of C_i. We show the details for the case $t - s = p$ and $p/2 < s < p$. The other cases are similar.

Suppose $t - s = p$, and that $p/2 < s < p$. Let $\lambda_0 = (s+p, p, 2s, s) - \eta$, and let μ_0 be in the interior of C_0. Let μ_i be the unique W_p-translate of μ_0 lying in C_i. Suppose that w_i is the element of W_p such that $\mu_i = w_i \bullet \mu_0$, and write $\lambda_i = w_i \bullet \lambda_0$. In fact, let's work out exactly what each of these λ_i's are:

$$\begin{aligned}
\lambda_0 &= (s+p, p, 2s, s) - \eta, \\
\lambda_1 = r_{01} \bullet \lambda_0 &= (s+p, p, 2s, s) - \eta, \\
\lambda_2 = r_{12} \bullet \lambda_1 &= (2s+p, p, s, s) - \eta, \\
\lambda_3 = r_{13} \bullet \lambda_1 &= (s+p, s+p, 2s, 0) - \eta, \\
\lambda_4 = r_{34} \bullet \lambda_3 &= (2s+p, s+p, s, 0) - \eta.
\end{aligned}$$

Notice that λ_0 is on the upper boundary of C_0, λ_4 is on an upper boundary of C_4, and each of the other λ_i are on a lower boundary of its respective C_i. Now, take the decomposition of $W(\mu_4)$ from (4.6), and apply the translation functor $T_{\mu_4}^{\lambda_4}$ to this identity:

$$T_{\mu_4}^{\mu_4} W'(w_4 \cdot \mu_0) = T_{\mu_4}^{\lambda_4} L(w_4 \cdot \mu_0) + T_{\mu_4}^{\lambda_4} L(w_3 \cdot \mu_0) + T_{\mu_4}^{\lambda_4} L(w_2 \cdot \mu_0) + T_{\mu_4}^{\lambda_4} L(w_1 \cdot \mu_0) + T_{\mu_4}^{\lambda_4} L(id \cdot \mu_0)$$

Since $w_3 \cdot \lambda_0 = \lambda_3 \notin \widehat{C}_3, w_2 \cdot \lambda_0 = \lambda_2 \notin \widehat{C}_2$, and $w_1 \cdot \lambda_0 = \lambda_1 \notin \widehat{C}_1$, using Proposition 2.1.5 implies that the middle three terms on the right hand side are 0, and

$$W'(\lambda_4) = L(\lambda_4) + L(\lambda_0).$$

Finally, since λ_4, λ_0 are in the restricted region $X_1(T)$, evaluating on $\mathbb{F}_p$-points gives us

$$W(\lambda_4) = F(\lambda_4) + F(\lambda_0).$$

□

4.3 Shadow Weight Cases

Lemma 4.3.1. *Suppose $\overline{\rho}$ has type $(1, 1^+), (2, 1^+), (1, 2^+)$, or $(2, 2^+)$, and that $\lambda = (s+t, t, s, 0)$ lies in C_4, the boundary of C_4 with $t - s = p$, C_5, or the boundary of C_5 with $s = p$. Then $F(\lambda)$ is a shadow weight of $\overline{\rho}$.*

Proof. If λ lies in C_4, C_5, or on the upper boundary of C_5 with $s = p$, then set $\mu = \lambda_1$, where λ_1 is the weight from (4.6), (4.7), or Lemma 4.2.2 (2), respectively. If λ lies on the upper boundary of C_4 with $t - s = p$, then set $\mu = \lambda_0$ as defined in

Lemma 4.2.2 (1). These values for μ are given by:

$$C_4: \qquad \mu+\eta=(s+p,p,s+t-p,t-p), \tag{4.9}$$

$$C_5: \qquad \mu+\eta=(s+p,s+t-p,p,t-p), \tag{4.10}$$

$$\widehat{C}_4,\ (t-s=p): \qquad \mu+\eta=(s+p,p,2s,s),$$

$$\widehat{C}_5,\ (s=p): \qquad \mu+\eta=(2p,t,p,t-p).$$

Notice that the $\widehat{C}_4,\ t-s=p$ case can be found by substituting $t=s+p$ into (4.9), the $\widehat{C}_5,\ s=p$ case can be found by substituting $s=p$ into (4.10), and that (4.9) and (4.10) are permutations of each each other. So all cases mentioned have $\mu+\eta$ a permutation of

$$\{s+p,p,s+t-p,t-p\}.$$

We will show that $F(\mu)$ is an obvious weight of $\overline{\rho}$, so that by (4.6), (4.7), and Lemma 4.2.2, in each case $F(\lambda)$ is a shadow weight of $\overline{\rho}$.

If $\overline{\rho}$ has type $(1,1^+)$, then

$$\overline{\rho}|_{I_{\mathbb{Q}_p}}=\omega^{s+t-1}\oplus\omega^{t-1}\oplus\omega^{s+1}\oplus\omega.$$

The permutation $(x',y',z',w')=(s+t-p,t-p,s+p,p)$ of $\mu+\eta$ satisfies

$$\begin{aligned}
x'&=s+t-p\equiv s+t-1 \pmod{p-1},\\
y'&=t-p \qquad \equiv t-1 \pmod{p-1},\\
z'&=s+p \qquad \equiv s+1 \pmod{p-1},\\
w'&=p \qquad\quad\ \equiv 1 \pmod{p-1}.
\end{aligned}$$

If $\overline{\rho}$ has type $(2,1^+)$, then

$$\overline{\rho}|_{I_{\mathbb{Q}_p}} = \omega_2^{(s+t-1)+p(t-1)} \oplus \omega_2^{p(s+t-1)+(t-1)} \oplus \omega_2^{s+1+p} \oplus \omega_2^{p(s+1)+1}.$$

The permutation $(x', y', z', w') = (s+t-p, t-p, s+p, p)$ of $\mu + \eta$ satisfies

$$x' + py' \equiv s + t - 1 + p(t-1) \pmod{p^2 - 1},$$

$$z' + pw' \equiv s + p + 1 \pmod{p^2 - 1}.$$

If $\overline{\rho}$ has type $(1,2^+)$, then

$$\overline{\rho}|_{I_{\mathbb{Q}_p}} = \omega_2^{s+t+1+p(s-1)} \oplus \omega_2^{p(s+t+1)+s-1} \oplus \omega_2^{t+1-p} \oplus \omega_2^{p(t+1)-1}.$$

The permutation $(x', y', z', w') = (s+t-p, s+p, t-p, p)$ of $\mu + \eta$ satisfies

$$x' + py' \equiv (s+t+1) + p(s-1) \pmod{p^2 - 1},$$

$$z' + pw' \equiv (t+1) - p \pmod{p^2 - 1}.$$

Finally, if $\overline{\rho}$ has type $(2,2^+)$, then

$$\overline{\rho}|_{I_{\mathbb{Q}_p}} = \omega_2^{s+t+1-p} \oplus \omega_2^{p(s+t+1)-1} \oplus \omega_2^{s-1+p(t+1)} \oplus \omega_2^{p(s-1)+t+1}.$$

The permutation $(x', y', z', w') = (s+t-p, p, s+p, t-p)$ of $\mu + \eta$ satisfies

$$x' + py' \equiv s + t + 1 - p \pmod{p^2 - 1},$$

$$z' + pw' \equiv s - 1 + p(t+1) \pmod{p^2 - 1}.$$

In each case, $F(\lambda_1)$ is an obvious weight of $\overline{\rho}$, which means $F(\lambda)$ is a shadow weight of $\overline{\rho}$. $\square$

4.4 Explicit Weight Cases

Recall that $W_{\text{expl}}(\overline{\rho})$ is the smallest set satisfying $W_{\text{obv}}(\overline{\rho}) \subset W_{\text{expl}}(\overline{\rho})$ and $F(\lambda) \in W_{\text{expl}}(\overline{\rho})$ if there exists a decomposition $\overline{\rho} = \bigoplus \overline{\rho}^{(i)}$ and an η-partition $\lambda^{(i)}$ of λ such that

$$W_{\text{expl}}(\overline{\rho}^{(i)}) \cap JH(W(\lambda^{(i)})) \neq \varnothing.$$

The proofs in this chapter will break $\overline{\rho}$ into two components $\overline{\rho} = \overline{\rho}^{(1)} \oplus \overline{\rho}^{(2)}$ and λ into a corresponding η-partition with two entries $\lambda^{(1)}$ and $\lambda^{(2)}$.

The interesting cases (the partitions that give us something different than obvious weights or shadow weights) are when the $\lambda^{(i)}$ chosen are not p-restricted. Suppose $\lambda = (a,b,c,d) - \eta$, and that $\overline{\rho}^{(1)}, \overline{\rho}^{(2)}$ are both two-dimensional mod p representations. We will want to choose $\lambda^{(1)}$ and $\lambda^{(2)}$ in one of the following two ways:

$$\lambda + \eta_4 = (\overset{\lambda^{(1)}+\eta_2}{\overset{\downarrow}{a}}, \underset{\lambda^{(2)}+\eta_2}{\underset{\uparrow}{b}}, \overset{\lambda^{(1)}+\eta_2}{\overset{\downarrow}{c}}, \underset{\lambda^{(2)}+\eta_2}{\underset{\uparrow}{d}}), \qquad \text{or} \qquad \lambda + \eta_4 = (\overset{\lambda^{(1)}+\eta_2}{\overset{\downarrow}{a}}, \underset{\lambda^{(2)}+\eta_2}{\underset{\uparrow}{b}}, \underset{\lambda^{(2)}+\eta_2}{\underset{\uparrow}{c}}, \overset{\lambda^{(1)}+\eta_2}{\overset{\downarrow}{d}}).$$

If instead $\overline{\rho}^{(1)}$ and $\overline{\rho}^{(2)}$ are three-dimensional and one-dimensional (respectively), then we will want to choose $\lambda^{(1)}$ and $\lambda^{(2)}$ in one of the following two ways:

$$\lambda + \eta_4 = (\overset{\lambda^{(1)}+\eta_3}{\overset{\downarrow}{a}}, \overset{\lambda^{(1)}+\eta_3}{\overset{\downarrow}{b}}, \underset{\lambda^{(2)}}{\underset{\uparrow}{c}}, \overset{\lambda^{(1)}+\eta_3}{\overset{\downarrow}{d}}), \qquad \text{or} \qquad \lambda + \eta_4 = (\overset{\lambda^{(1)}+\eta_3}{\overset{\downarrow}{a}}, \underset{\lambda^{(2)}}{\underset{\uparrow}{b}}, \overset{\lambda^{(1)}+\eta_3}{\overset{\downarrow}{c}}, \overset{\lambda^{(1)}+\eta_3}{\overset{\downarrow}{d}}).$$

As these $\lambda^{(i)}$ may not be p-restricted, we will use the results of Chapter 2 in order to find the Jordan-Hölder factors of $W(\lambda^{(i)})$. Recall that in Chapter 2 we assumed

$p > 3$. However, we wish for Theorem A to hold for all primes p, so we handle the cases $p = 2, 3$ separately. Then for the remainder of the chapter we assume $p > 3$ and we use the results of Chapter 2 to prove that $\lambda = (t + s, t, s, 0) - \eta$ is an explicit weight in the case $t - s = p$ and $s > p/2$, and then in the case $s + t = 2p$.

4.4.1 Boundary Case $t - s = p$.

Suppose that $t - s = p$. Recall from Lemma 4.2.1 that if $s < p/2$, then $\lambda = (s + t, t, s, 0) - \eta$ is on an upper boundary of C_4, and if $s > p/2$, then $\lambda = (s+t, t, s, 0) - \eta$ is on an upper boundary of C_5. We showed in Lemma 4.3.1 that in the $s < p/2$ case, $F(\lambda)$ is a shadow weight of $\overline{\rho}$ for types $(1, 1^+), (1, 2^+), (2, 1^+)$ and $(2, 2^+)$. In this section, we treat the $s > p/2$ case.

We will write λ_5 for λ to emphasize the fact that λ is on an upper boundary of alcove 5. If $p/2 < s < p$, let λ_4 be the unique W_p-translate of λ lying on the upper boundary of C_4. From Lemma 4.2.2, we have

$$W(\lambda_5) = F(\lambda_5) + F(\lambda_4) \qquad t - s = p,\ p/2 < s < p$$

$$W(\lambda_5) = F(\lambda_5) \qquad s = p,\ t = 2p$$

Lemma 4.4.1. *If $\overline{\rho}$ has type $(1, 1^+), (1, 2^+), (2, 1^+)$, or $(2, 2^+)$ and if $t - s = p$ and $p/2 < s < p$, then $\lambda = (s + t, t, s, 0) - \eta$ is an explicit weight of $\overline{\rho}$.*

Proof. Assume that $t - s = p$, and $p/2 < s < p$. Then by Lemma 4.2.2, $W(\lambda_5) =$

$F(\lambda_5) + F(\lambda_4)$, where

$$\lambda_5 = (2s+p, s+p, s, 0) - \eta, \quad \text{and} \quad \lambda_4 = (2p, s+p, s, 2s-p) - \eta.$$

Using $t - s = p$, we have $\overline{\rho}_{1,1^+} = \omega^{2s} \oplus \omega^s \oplus \omega^{s+1} \oplus \omega$. Choose $\overline{\rho} = \overline{\rho}^{(1)} \oplus \overline{\rho}^{(2)}$ with

$$\overline{\rho}^{(1)} = \omega^{2s} \oplus \omega^{s+1}, \qquad \overline{\rho}^{(2)} = \omega^s \oplus \omega$$

as the decomposition of $\overline{\rho}$. And choose the following η-decomposition of $\lambda = \lambda_5$:

$$\lambda_5^{(1)} = (2s+p-1, s), \quad \lambda_5^{(2)} = (s+p-1, 0).$$

Theorem 2.2.3 implies that $F(2s-1, s+1) \in JH(W(\lambda^{(1)}))$. We also have $F(2s-1, s+1)$ is an obvious weight of $\overline{\rho}^{(1)}$. Similarly, $F(s-1,1) \in JH(W(\lambda_5^{(2)}))$ and $F(s-1,1)$ is an obvious weight of $\overline{\rho}^{(2)}$.

In the $(2, 1^+)$ case, we have $\overline{\rho}_{2,1^+} = \omega_2^{2s+ps} \oplus \omega_2^{2ps+s} \oplus \omega_2^{s+1+p} \oplus \omega_2^{p(s+1)+1}$. Choose $\overline{\rho} = \overline{\rho}^{(1)} \oplus \overline{\rho}^{(2)}$ with

$$\overline{\rho}^{(1)} = \omega_2^{s+1+p} \oplus \omega_2^{p(s+1)+1}, \qquad \overline{\rho}^{(2)} = \omega_2^{2s+ps} \oplus \omega_2^{2ps+s}$$

as the decomposition of $\overline{\rho}$. And choose the following η-decomposition of λ_4:

$$\lambda_4^{(1)} = (2p-1, s), \quad \lambda_4^{(2)} = (s+p-1, 2s-p).$$

Using Theorem 2.2.3 as before, we have

$$F(p, s) \in JH(W(\lambda_4^{(1)})) \cap W_{\mathrm{obv}}(\overline{\rho}^{(1)}),$$

$$F(2s-1, s) \in JH(W(\lambda_4^{(2)})) \cap W_{\mathrm{obv}}(\overline{\rho}^{(2)}).$$

So $F(\lambda_4)$ is an explicit weight of $\overline{\rho}$, which means $F(\lambda_5)$ is an explicit weight of $\overline{\rho}$.

In the $(1,2^+)$ case, we have $\overline{\rho}_{1,2^+} = \omega_2^{2s+1+ps} \oplus \omega_2^{p(2s+1)+s} \oplus \omega_2^{s+1} \oplus \omega_2^{p(s+1)}$. Choose $\overline{\rho} = \overline{\rho}^{(1)} \oplus \overline{\rho}^{(2)}$ with

$$\overline{\rho}^{(1)} = \omega_2^{2s+1+ps} \oplus \omega_2^{p(2s+1)+s}, \qquad \overline{\rho}^{(2)} = \omega_2^{s+1} \oplus \omega_2^{p(s+1)}$$

as the decomposition of $\overline{\rho}$. And choose the following η-decomposition of λ_5:

$$\lambda_5^{(1)} = (2s+p-1, s), \qquad \lambda_5^{(2)} = (s+p-1, 0).$$

Then

$$F(2s,s) \in JH(W(\lambda_5^{(1)})) \cap W_{\text{obv}}(\overline{\rho}^{(1)}),$$
$$F(s,0) \in JH(W(\lambda_5^{(2)})) \cap W_{\text{obv}}(\overline{\rho}^{(2)}).$$

So $F(\lambda) = F(\lambda_5)$ is an explicit weight of $\overline{\rho}$.

Finally, in the $(2,2^+)$ case, we have $\overline{\rho}_{2,2^+} = \omega_2^{2s+1} \oplus \omega_2^{p(2s+1)} \oplus \omega_2^{s+p(s+1)} \oplus \omega_2^{ps+s+1}$. Choose $\overline{\rho} = \overline{\rho}^{(1)} \oplus \overline{\rho}^{(2)}$ with

$$\overline{\rho}^{(1)} = \omega_2^{2s+1} \oplus \omega_2^{p(2s+1)}, \qquad \overline{\rho}^{(2)} = \omega_2^{s+p(s+1)} \oplus \omega_2^{ps+s+1},$$

and the following η-decomposition of λ_5:

$$\lambda_5^{(1)} = (s+p-1, s), \qquad \lambda_5^{(2)} = (2s+p-1, 0).$$

Then

$$\begin{gathered}F(s+p-1,s)\in JH(W(\lambda_5^{(1)}))\cap W_{\mathrm{obv}}(\overline{\rho}^{(1)}),\\ \begin{cases}F(2s-p,1)\in JH(W(\lambda_5^{(2)}))\cap W_{\mathrm{obv}}(\overline{\rho}^{(2)}) & \text{if } s\neq (p+1)/2,\\ F(p,1)\in JH(W(\lambda_5^{(2)}))\cap W_{\mathrm{obv}}(\overline{\rho}^{(2)}) & \text{if } s=(p+1)/2.\end{cases}\end{gathered}$$

So $F(\lambda)=F(\lambda_5)$ is an explicit weight of $\overline{\rho}$.

□

Lemma 4.4.2. *If $\overline{\rho}$ has type $(1,1^+),(1,2^+),(2,1^+)$, or $(2,2^+)$ and if $t-s=p,\ s=p$, then $F(\lambda)$ with $\lambda=(s+t,t,s,0)-\eta$ is an explicit weight of $\overline{\rho}$.*

Proof. Assume that $t-s=p$ and $s=p$ so that $t=2p$ and $\lambda=(3p,2p,p,0)-\eta$. We will address types $(1,1^+),(1,2^+)$, and then $(2,1^+),(2,2^+)$ separately, but in all cases we will use the same η-partition of λ. Let

$$\lambda^{(1)}=(3p-1,0),\qquad \lambda^{(2)}=(2p-1,p).$$

For types $(1,1^+)$ and $(1,2^+)$, setting $s=p$ and $t=2p$ gives us

$$\overline{\rho}_{1,1^+}=\overline{\rho}_{1,2^+}=\omega^2\oplus\omega\oplus\omega^2\oplus\omega.$$

Choose $\overline{\rho}^{(1)}=\overline{\rho}^{(2)}=\omega^2\oplus\omega$ as the decomposition of $\overline{\rho}$. The Serre weights $F(1,1)$ and $F(p,1)$ are both obvious weights of $\omega^2\oplus\omega$. So

$$F(1,1)\in JH(W(\lambda^{(1)}))\cap W_{\mathrm{obv}}(\overline{\rho}^{(1)})$$

$$F(p,1)\in JH(W(\lambda^{(2)}))\cap W_{\mathrm{obv}}(\overline{\rho}^{(2)}).$$

This shows that $F(\lambda)$ is an explicit weight of $\overline{\rho}$ for types $(1,1^+)$ and $(1,2^+)$.

For $\overline{\rho}$ of types $(2,1^+),(2,2^+)$, we have

$$\overline{\rho}_{2,1^+} = \overline{\rho}_{2,2^+} = \omega_2^{2p+1} \oplus \omega_2^{p+2} \oplus \omega_2^{2p+1} \oplus \omega_2^{p+2}.$$

Choose $\overline{\rho}^{(1)} = \overline{\rho}^{(2)} = \omega_2^{2p+1} \oplus \omega_2^{p+2}$ as the decomposition of $\overline{\rho}$. The Serre weights $F(p,1)$ and $F(2p-1,p)$ are both obvious weights of $\omega_2^{2p+1} \oplus \omega_2^{p+2}$. So

$$F(p,1) \in JH(W(\lambda^{(1)})) \cap W_{\text{obv}}(\overline{\rho}^{(1)}),$$
$$F(2p-1,p) \in JH(W(\lambda^{(2)})) \cap W_{\text{obv}}(\overline{\rho}^{(2)}).$$

This shows that $F(\lambda)$ is an explicit weight of $\overline{\rho}$ for types $(2,1^+)$ and $(2,2^+)$, and this completes the proof. □

4.4.2 Boundary Case $s+t=2p$.

Suppose that $s+t=2p$, so that $\lambda = (2p, 2p-s, s, 0) - \eta$. In order for this to be p-restricted, we must have $s > p/2$. We assume this condition also for the remainder of this section. Recall from Lemma 4.2.2 that if $s+t=2p$, then

$$W(\lambda_4) = F(\lambda_4) + F(\lambda_3) + F(\lambda_2), \tag{4.11}$$

where

$$\lambda = \lambda_4 = (2p, 2p-s, s, 0) - \eta,$$
$$\lambda_3 = (s+p, 2p-s, p, 0) - \eta,$$
$$\lambda_2 = (2p, p, s, p-s) - \eta.$$

Lemma 4.4.3. *If $\overline{\rho}$ has type $(1,1^+),(2,1^+)$, or $(1,2^+)$ and if $s+t=2p$, then $F(\lambda)$ with $\lambda=(s+t,t,s,0)-\eta$ is an explicit weight of $\overline{\rho}$.*

Proof. Assume that $s+t=2p$, so that $\lambda=(2p,2p-s,s,0)-\eta$. We choose the same η-partition of λ for each of the types $(1,1^+),(2,1^+)$, and $(1,2^+)$. Let

$$\lambda^{(1)}=(2p-1,s), \qquad \lambda^{(2)}=(2p-s-1,0).$$

Using Theorem 2.2.3, we see that

$$JH(W(2p-1,s))=\{F(p,s),F(s+p-1,p),F(p-1,s+1)_{(s\neq p-1)}\},$$
$$JH(W(2p-s-1,0))=\{F(p-s,0),F(p-1,p-s),F(p-s-1,1)_{(s\neq p-1)}\}.$$

The final term in each of these lists appears only of $s\neq p-1$. We will take care to choose only from the first two terms as candidates for being in the intersection from the explicit weight condition, so that this proof works for all $p/2<s<p$.

Suppose $\overline{\rho}$ has type $(1,1^+)$. Then $\overline{\rho}_{1,1^+}=\omega\oplus\omega^{1-s}\oplus\omega^{s+1}\oplus\omega$. Choose $\overline{\rho}=\overline{\rho}^{(1)}\oplus\overline{\rho}^{(2)}$ with $\overline{\rho}^{(1)}=\omega^{s+1}\oplus\omega$ and $\overline{\rho}^{(2)}=\omega^1\oplus\omega^{1-s}$ as the decomposition of $\overline{\rho}$. Then

$$F(s+p-1,p)\in JH(W(\lambda^{(1)}))\cap W_{\text{obv}}(\overline{\rho}^{(1)}),$$
$$F(p-1,p-s)\in JH(W(\lambda^{(2)}))\cap W_{\text{obv}}(\overline{\rho}^{(2)}).$$

This shows that for $\overline{\rho}$ of type $(1,1^+)$, $F(\lambda)$ is an explicit weight of $\overline{\rho}$.

Suppose $\overline{\rho}$ has type $(2,1^+)$ or $(1,2^+)$. Then

$$\overline{\rho}_{2,1^+}=\overline{\rho}_{1,2^+}=\omega_2^{1+p(1-s)}\oplus\omega_2^{p+(1-s)}\oplus\omega_2^{s+1+p}\oplus\omega_2^{p(s+1)+1}$$

Choose

$$\overline{\rho}^{(1)} = \omega_2^{s+1+p} \oplus \omega_2^{p(s+1)+1}, \quad \overline{\rho}^{(2)} = \omega_2^{1+p(1-s)} \oplus \omega_2^{p+(1-s)}$$

as the decomposition of $\overline{\rho}$. Then

$$F(p,s) \in JH(W(\lambda^{(1)})) \cap W_{\text{obv}}(\overline{\rho}^{(1)}),$$

$$F(p-s,0) \in JH(W(\lambda^{(2)})) \cap W_{\text{obv}}(\overline{\rho}^{(2)}).$$

This shows that for $\overline{\rho}$ of type $(2,1^+)$ or $(1,2^+)$, then $F(\lambda)$ is an explicit weight of $\overline{\rho}$. This completes the proof. □

The case where $s+t=2p$, and $\overline{\rho}$ has type $(2,2^+)$ is the only one remaining. This is when we will finally make use of the GL_3 results from Section 2.3.

Lemma 4.4.4. *If $\overline{\rho}$ has type $(2,2^+)$ and if $s+t=2p$, then $F(\lambda)$ is an explicit weight of $\overline{\rho}$.*

Proof. Setting $t = 2p-s$, we have

$$\overline{\rho}_{22^+} = \omega_1 \oplus \omega_1 \oplus \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}.$$

To show that $F(\lambda)$ is an explicit weight of $\overline{\rho}$, we will first show that $F(\lambda_2) \in W_{\text{expl}}(\overline{\rho})$, where $\lambda_2 = (2p,p,s,p-s)-\eta$ is that from (4.11). Choose

$$\overline{\rho}^{(1)} = \omega_1, \quad \overline{\rho}^{(2)} = \omega_1 \oplus \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}.$$

as the decomposition of $\overline{\rho}$, and

$$\lambda^{(1)} = (p), \quad \lambda^{(2)} = (2p,s,p-s) - \eta_3$$

as the η-partition of λ_2. The first of these is immediately an obvious weight of $\overline{\rho}^{(1)}$, since $\omega_1^p = \omega_1$. Using $p/2 < s < p$, we know that $\lambda^{(2)}$ is in alcove 2 from (2.24). Theorem 2.3.3 says that

$$\begin{aligned} JH(W(2p-2, s-1, p-s)) = \{&F(s+p-2, p-1, p-s), \\ &F(p-1, s-1, p-s), F(p-2, s, p-s), F(p-2, s-1, p-s+1)\}, \end{aligned}$$

where $F(p-2, s, p-s)$ occurs only if $s \neq p-1$, and $F(p-2, s-1, p-s+1)$ occurs only if $s \neq \frac{p+1}{2}$.

Then, $F(s+p-2, p-1, p-s)$ is an obvious weight of $\overline{\rho}^{(2)}$. To see this, choose the obvious lift $\Psi = \Psi_{\{p\}} \oplus \Psi_{\{s+p, p-s\}}$, so that

$$\begin{aligned} \overline{\Psi} &= \omega^p \oplus \omega_2^{(s+p)+p(p-s)} \oplus \omega_2^{p(s+p)+(p-s)} \\ &= \omega \oplus \omega_2^{(p+1)+s(1-p)} \oplus \omega_2^{(p+1)+s(p-1)} \\ &= \overline{\rho}^{(2)}. \end{aligned}$$

Thus, we have

$$F(s+p-2, p-1, p-s) \in JH(W(\lambda^{(2)})) \cap W_{\text{obv}}(\overline{\rho}^{(2)}),$$

and $F(\lambda_2)$ is an explicit weight of $\overline{\rho}$. Since $W_{\text{expl}}(\overline{\rho})$ is closed under the closure operation, and $F(\lambda_2) \in JH(W(\lambda))$, we have $F(\lambda) \in W_{\text{expl}}(\overline{\rho})$. □

4.5 Proof of Theorem A

Theorem 4.5.1. *Suppose $p > 3$. If $\overline{\rho}'$ and $\overline{\rho}''$ are two-dimensional representations with crystalline lifts of Hodge-Tate weights $(s, 0)$ and $(t, 0)$ such that $(s+t, t, s, 0) - \eta$ is dominant and p-restricted, then $F((s+t, t, s, 0) - \eta)$ is in $W_{\text{expl}}(\overline{\rho}' \otimes \overline{\rho}'')$.*

Proof. Case 1: $\lambda = (s + t - 3, t - 2, s - 1, 0)$ lies inside an alcove.

By Lemma 4.2.1, λ lies in either alcove 4 or alcove 5. If $\overline{\rho}$ has Type $(1, 1)$,$(1, 2)$, $(2, 1)$, or $(2, 2)$, then $F(\lambda)$ is immediately an obvious weight of $\overline{\rho}$. By Lemma 4.3.1, if $\overline{\rho}$ has type $(1, 1^+)$,$(2, 1^+)$,$(1, 2^+)$, or $(2, 2^+)$, then $F(\lambda)$ is a shadow weight of $\overline{\rho}$.

Case 2: $\lambda = (s + t - 3, t - 2, s - 1, 0)$ lies on a boundary of an alcove.

By Lemma 4.2.1, the possibilities for λ lying on a boundary are: $s = p$, $t - s = p$, and $s + t = 2p$. Lemma 4.3.1 shows that if λ is on the upper boundary of C_4 with $t - s = p$ and $s < p/2$, or if λ is on the upper boundary of C_5 with $s = p$, then $F(\lambda)$ is a shadow weight of $\overline{\rho}$. Lemmas 4.4.1 and 4.4.2 show that if λ is on the upper boundary of C_5 with $t - s = p$ and $s > p/2$, then $F(\lambda)$ is an obscure weight. Finally, Lemma 4.4.3 shows that if $s + t = 2p$, then $F(\lambda)$ is an obscure weight of $\overline{\rho}$. In all cases, we have $F(\lambda) \in W_{\text{expl}}(\overline{\rho})$. □

Chapter 5

More Details

5.1 Theorem A for Small Primes

In Chapter 4, we proved Theorem A for $p > 3$. In this section, we show that this theorem also holds for $p = 2, 3$.

Theorem 5.1.1. *If $p = 2$ or $p = 3$, and if $\overline{\rho}'$ and $\overline{\rho}''$ are two-dimensional representations with crystalline lifts of Hodge-Tate weights $(s, 0)$ and $(t, 0)$ such that $(s + t, t, s, 0) - \eta$ is dominant and p-restricted, then $F((s+t, t, s, 0)-\eta)$ is in $W_{\mathrm{expl}}(\overline{\rho}' \otimes \overline{\rho}'')$.*

We remark that in Chapter 4, the only arguments that depend on the assumption $p > 3$ were the explicit weight arguments in Section 4.4. So in order to prove Theorem 5.1.1, we only need to consider the two cases from that section. Namely, the cases

(i) where $\lambda = (s + t, t, s, 0) - \eta$ lies on the upper boundary of C_4 with $s + t = 2p$, and

(ii) where $\lambda = (s + t, t, s, 0) - \eta$ lies on the upper boundary of C_5 with $t - s = p$ and $s > p/2$.

For these small values of p, the restrictions in (i) and (ii) mean that there are few values s can take on, and we are able to just check all these cases directly.

5.1.1 Theorem A for $p = 2$

First, when $p = 2$, all Serre weights are obvious weights of $\overline{\rho}$ if $\overline{\rho}$ has type $(1, 1^+)$. This is because the character ω is trivial when $p = 2$. So on one hand, $\overline{\rho}_{1,1^+} = 1 \oplus 1 \oplus 1 \oplus 1$ regardless of the choice of s, t. And on the other hand, $\Psi_{\{a\}} \oplus \Psi_{\{b\}} \oplus \Psi_{\{c\}} \oplus \Psi_{\{d\}}$ reduces to $1 \oplus 1 \oplus 1 \oplus 1$ for any choice of a, b, c and d. In particular, choosing $\Psi = \Psi_{\{s+t\}} \oplus \Psi_{\{t\}} \oplus \Psi_{\{s\}} \oplus \Psi_{\{0\}}$ gives us the following lemma.

Lemma 5.1.2. *If $p = 2$ and $\overline{\rho}$ has type $(1, 1^+)$, then $F(\lambda)$ with $\lambda = (s+t, t, s, 0) - \eta$ is an obvious weight of $\overline{\rho}$.*

Suppose that $s + t = 2p$. Then $\lambda = (2p, 2p - s, s, 0) - \eta$. For λ to be dominant, the difference between the first and second entries, and between the second and third entries must be nonnegative. We get:

$$(4) - (4 - s) - 1 \geq 0 \qquad \Rightarrow \qquad s \geq 1,$$

$$(4 - s) - (s) - 1 \geq 0 \qquad \Rightarrow \qquad s \leq 3/2.$$

So $s = 1$ is the only possibility, which means $t = 3$, and $\lambda = (4, 3, 1, 0) - \eta$.

Next, suppose that $t - s = p$, and that $p = 2$. Then $\lambda = (2s + 2, s + 2, s, 0) - \eta$. For this λ to be dominant and p-restricted, the only restriction is the original one: $1 \leq s \leq p$. So there are two cases when $t - s = p$: either $s = 1$ or $s = 2$. If $s = 1$, then $t = 3$ and $\lambda = (4, 3, 1, 0) - \eta$. Notice that these values for s, t, and therefore λ are the same as the ones in the $s + t = 2p$ case. If $s = 2$, then $t = 4$ and $\lambda = (6, 4, 2, 0) - \eta$.

Lemma 5.1.3. *Suppose $p = 2$, and that $\overline{\rho}$ has type $(1, 2^+), (2, 1^+)$, or $(2, 2^+)$. If* *(i)* $s + t = 2p$, *or*

(ii) $t - s = p$ *and* $s = 1$, *or*

(iii) $t - s = p$ *and* $s = 2$,

then $F(\lambda)$ *with* $\lambda = (s + t, t, s, 0) - \eta$ *is an obvious weight of* $\overline{\rho}$.

Proof. In both the cases (i) and (ii), we have $s = 1$ and $t = 3$. So $\lambda = (4, 3, 1, 0) - \eta$. We will want to know the specific forms for $\overline{\rho}_{2,1^+}, \overline{\rho}_{1,2^+}$, and $\overline{\rho}_{2,2^+}$ in this case. We substitute $p = 2, s = 1$, and $t = 3$ into the forms in (4.3), and then use the fact $\omega_2^3 = 1$ to simplify. We obtain:

$$\begin{aligned} \overline{\rho}_{2,1^+} &= \omega_2 \oplus \omega_2^2 \oplus \omega_2 \oplus \omega_2^2, \\ \overline{\rho}_{1,2^+} &= \omega_2^2 \oplus \omega_2 \oplus \omega_2^2 \oplus \omega_2, \\ \overline{\rho}_{2,2^+} &= 1 \oplus 1 \oplus \omega_2^2 \oplus \omega_2. \end{aligned} \tag{5.1}$$

To handle types $(1, 2^+)$ and $(2, 1^+)$, set $\Psi = \Psi_{\{4,3\}} \oplus \Psi_{\{1,0\}}$, where $\Psi_{\{a_i\}}$ is defined as in Proposition 3.1.1. Then

$$\begin{aligned} \overline{\Psi} &= \omega_2^{4+2\cdot 3} \oplus \omega_2^{2\cdot 4+3} \oplus \omega_2^{1+2\cdot 0} \oplus \omega_2^{2\cdot 1+0} \\ &= \omega_2 \oplus \omega_2^2 \oplus \omega_2 \oplus \omega_2^2 \\ &= \overline{\rho}_{2,1^+} = \overline{\rho}_{1,2^+}. \end{aligned}$$

For type $(2,2^+)$, let $\Psi = \Psi_{\{4\}} \oplus \Psi_{\{3\}} \oplus \Psi_{\{1,0\}}$. Then

$$\begin{aligned}\overline{\Psi} &= \omega_1^4 \oplus \omega_1^3 \oplus \omega_2^{1+2\cdot 0} \oplus \omega_2^{2\cdot 1+0} \\ &= 1 \oplus 1 \oplus \omega_2 \oplus \omega_2^2 \\ &= \overline{\rho}_{2,2^+}.\end{aligned}$$

This shows that in cases (i) and (ii), $F(\lambda)$ is an obvious weight of $\overline{\rho}$.

In case (iii), we have $s = 2$ and $t = 4$. So $\lambda = (6,4,2,0) - \eta$. To find the specific forms for $\overline{\rho}_{2,1^+}, \overline{\rho}_{1,2^+}$, and $\overline{\rho}_{2,2^+}$ in this case, substitute $p = 2, s = 2$, and $t = 4$ into the forms in (4.3) and simplify as before. We obtain:

$$\begin{aligned}\overline{\rho}_{2,1^+} &= \omega_2^2 \oplus \omega_2 \oplus \omega_2^2 \oplus \omega_2, \\ \overline{\rho}_{1,2^+} &= 1 \oplus 1 \oplus 1 \oplus 1, \\ \overline{\rho}_{2,2^+} &= \omega_2^2 \oplus \omega_2 \oplus \omega_2^2 \oplus \omega_2.\end{aligned} \tag{5.2}$$

For types $(2,1^+)$ and $(2,2^+)$, set $\Psi = \Psi_{\{6,4\}} \oplus \Psi_{\{2,0\}}$. Then

$$\begin{aligned}\overline{\Psi} &= \omega_2^{6+2\cdot 4} \oplus \omega_2^{2\cdot 6+4} \oplus \omega_2^{2+2\cdot 0} \oplus \omega_2^{2\cdot 2+0} \\ &= \omega_2^2 \oplus \omega_2 \oplus \omega_2^2 \oplus \omega_2^1 \\ &= \overline{\rho}_{2,1^+} = \overline{\rho}_{2,2^+}.\end{aligned}$$

For type $(1,2^+)$, set $\Psi = \Psi_{\{6\}} \oplus \Psi_{\{4\}} \oplus \Psi_{\{2\}} \oplus \Psi_{\{0\}}$. Then

$$\begin{aligned}\overline{\Psi} &= \omega^6 \oplus \omega^4 \oplus \omega^2 \oplus \omega^0 \\ &= 1 \oplus 1 \oplus 1 \oplus 1 \\ &= \overline{\rho}_{1,2^+}.\end{aligned}$$

This shows that in case (iii), $F(\lambda)$ is an obvious weight of $\overline{\rho}$, and completes the proof.

□

5.1.2 Theorem A for $p = 3$

Let $p = 3$. If $s + t = 2p$, then $\lambda = (6, 6 - s, s, 0) - \eta$. For λ to be dominant and p-restricted, the difference between the second and third entries must satisfy:

$$0 \leq (6) - (6 - s) - 1 \leq 2 \quad \Rightarrow \quad \frac{3}{2} \leq s \leq \frac{5}{2} \quad \Rightarrow \quad s = 2.$$

So $s = 2$ and $t = 4$.

If $t - s = p$, then $\lambda = (2s + 3, s + 3, s, 0) - \eta$. Requiring this λ to be dominant and p-restricted does not give us any additional criteria – we have only the original restriction $1 \leq s \leq 3$. However, if $s = 3$, then this is the case $s = p$ on the upper boundary of C_5. And if $s = 1$, then this is the case $t - s = p$, $s < p/2$ on the upper boundary of C_4. We showed in Lemma 4.3.1 that in both of these cases, $F(\lambda) \in \mathcal{C}(W_{\text{obv}}(\overline{\rho})) \subset W_{\text{expl}}(\overline{\rho})$. So $s = 2$ is the only case remaining.

Lemma 5.1.4. *Suppose* $p = 3$ *and that* $\overline{\rho}$ *has type* $(1, 1^+), (1, 2^+), (2, 1^+)$, *or* $(2, 2^+)$. *If*

(i) $s + t = 2p$, *or*

(ii) $t - s = p$ *and* $s = 2$,

then $F(\lambda)$ *with* $\lambda = (s + t, t, s, 0) - \eta$ *is either an obvious or a shadow weight of* $\overline{\rho}$.

Proof. In case (i), we have $s = 2, t = 4$, and $\lambda = (6, 4, 2, 0) - \eta$. Substituting these values into (4.3) and using $\omega_2^8 = 1,\ \omega_2^4 = \omega_1$ to simplify, we have

$$\overline{\rho}_{1,1^+} = \omega \oplus \omega \oplus \omega \oplus \omega,$$

$$\overline{\rho}_{2,1^+} = \omega_2^6 \oplus \omega_2^2 \oplus \omega_2^6 \oplus \omega_2^2,$$

$$\overline{\rho}_{1,2^+} = \omega_2^2 \oplus \omega_2^6 \oplus \omega_2^2 \oplus \omega_2^6,$$

$$\overline{\rho}_{2,2^+} = \omega_1 \oplus \omega_1 \oplus 1 \oplus 1.$$

For type $(1, 1^+)$, set $\Psi = \Psi_{\{6,2\}} \oplus \Psi_{\{4,0\}}$. Then using $\omega_2^4 = \omega_1$, we have

$$\begin{aligned}\overline{\Psi} &= \omega_2^{6+3\cdot 2} \oplus \omega_2^{3\cdot 6+2} \oplus \omega_2^4 \oplus \omega_2^{3\cdot 4} \\ &= \omega_1 \oplus \omega_1 \oplus \omega_1 \oplus \omega_1 \\ &= \overline{\rho}_{1,1^+}.\end{aligned}$$

For types $(2, 1^+)$ and $(1, 2^+)$, set $\Psi = \Psi_{\{6,4\}} \oplus \Psi_{\{2,0\}}$ to obtain $\overline{\Psi} = \overline{\rho}_{2,1^+} = \overline{\rho}_{1,2^+}$. For type $(2, 2^+)$, set $\Psi = \Psi_{\{6,2\}} \oplus \Psi_{\{4\}} \oplus \Psi_{\{0\}}$. Then use $\omega_1^2 = 1$ to see that $\overline{\Psi} = \overline{\rho}_{2,2^+}$.

This shows that if $\overline{\rho}$ has type $(1, 1^+), (2, 1^+), (1, 2^+)$, or $(2, 2^+)$, then $F(\lambda)$ is an obvious weight of $\overline{\rho}$.

In case (ii), we have $s = 2, t = 5$, and $\lambda = (7, 5, 2, 0) - \eta$. Substituting these values into (4.3) and using $\omega_2^8 = 1,\ \omega_2^4 = \omega_1$, and $\omega_1^2 = 1$ to simplify again, we have

$$\overline{\rho}_{1,1^+} = 1 \oplus 1 \oplus \omega_1 \oplus \omega_1,$$

$$\overline{\rho}_{2,1^+} = \omega_2^2 \oplus \omega_2^6 \oplus \omega_2^6 \oplus \omega_2^2,$$

$$\overline{\rho}_{1,2^+} = \omega_2^3 \oplus \omega_2 \oplus \omega_2^3 \oplus \omega_2,$$

$$\overline{\rho}_{2,2^+} = \omega_2^5 \oplus \omega_2^7 \oplus \omega_2^3 \oplus \omega_2.$$

For type $(1, 1^+)$, set $\Psi = \Psi_{\{2\}} \oplus \Psi_{\{0\}} \oplus \Psi_{\{7\}} \oplus \Psi_{\{5\}}$ to see that $\overline{\Psi} = \overline{\rho}_{1,1^+}$. For type $(2, 1^+)$, set $\Psi = \Psi_{\{7,5\}} \oplus \Psi_{\{2,0\}}$, then $\overline{\Psi}_{2,1^+} = \overline{\rho}_{2,1^+}$. And for type $(2, 2^+)$, set $\Psi = \Psi_{\{7,0\}} \oplus \Psi_{\{5,2\}}$, then $\overline{\Psi} = \overline{\rho}_{1,2^+}$. In each of these cases, this shows that $F(\lambda)$ is an obvious weight of $\overline{\rho}$.

For type $(1, 2^+)$, no permutation of $\{7, 5, 2, 0\}$ works to give us an obvious weight of $\overline{\rho}$. Instead, recall the closure operator from Section 3.2. Since we are in the case $t - s = p$ and $p/2 < s < p$, Part 2 of Lemma 4.2.2 says that $JH(W(\lambda)) = \{F(\lambda), F(\lambda_4)\}$, where

$$\lambda_4 = (6, 5, 2, 1) - \eta.$$

Finally, set $\Psi = \Psi_{\{6,1\}} \oplus \Psi_{\{5,2\}}$. Then $\overline{\Psi} = \overline{\rho}_{1,2^+}$ and $F(\lambda_4)$ is an obvious weight of $\overline{\rho}$. Since $F(\lambda_4) \in JH(W(\lambda))$, we have $F(\lambda) \in \mathcal{C}(W_{\text{obv}}(\overline{\rho}))$, and $F(\lambda)$ is a shadow weight of $\overline{\rho}$. □

5.2 Obscure Weights Are Really Necessary

In this section, we show that obscure weights really do occur in the tensor product setting discussed in Chapter 4. That is, in this tensor product setting, we cannot make do with the set $\mathcal{C}(W_{\text{obv}}(\overline{\rho}))$, and it is necessary to define the expanded set $W_{\text{expl}}(\overline{\rho})$.

In particular, we will show that when $p > 3$ and λ lies on the upper boundary of C_4 with $s+t = 2p$, then the weight $F(\lambda)$ is an *obscure* weight for $\overline{\rho}$ of type $(2, 2^+)$. That is, it is neither obvious nor shadow. We showed in Lemma 4.4.4 that $F(\lambda) \in W_{\text{expl}}(\overline{\rho})$. We need to show (for $p > 3$) that $F(\lambda)$ is not in $W_{\text{obv}}(\overline{\rho})$, and not in $\mathcal{C}(W_{\text{obv}}(\overline{\rho}))$.

In this section, let $p > 3$ and let $s + t = 2p$. Let $\overline{\rho}$ be a mod p representation of type $(2, 2^+)$ so that

$$\overline{\rho}|_{I_{\mathbb{Q}_p}} = \omega_1 \oplus \omega_1 \oplus \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)},$$

and let $\lambda = (2p, 2p - s, s, 0) - \eta$.

Some proofs in this section will also make use of the following two lemmas:

Lemma 5.2.1. *If* $\frac{p+1}{2} \leq s \leq p - 1$, *then*

$$\frac{(p+1)^2}{2} \leq (p+1) + (p-1)s \leq p^2 - p - 2,$$

$$3p - 1 \leq (p+1) + (p-1)(p+1-s) \leq \frac{(p+1)^2}{2}.$$

In particular, if $p > 3$, *then*

$$2p < (p+1) + (p-1)s, (p+1) + (p-1)(p+1-s) < p^2 - 1.$$

The proof of this lemma is trivial.

Lemma 5.2.2. *If* p *is odd, the representation* $\omega_2^{(p+1)+(p-1)s}$ *simplifies to a niveau one* *representation if and only if* $s = \frac{p+1}{2}$*. In this case,*

$$\omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)} = \omega_1^{\frac{p+1}{2}} \oplus \omega_1^{\frac{p+1}{2}}. \tag{5.3}$$

Proof. First, $\omega_2^{(p+1)+(p-1)s}$ simplifies to a niveau one representation if and only if the exponent $(p+1)+(p-1)s$ is divisible by $1+p$. This happens if and only if $s(1-p)$ is divisible by $1+p$. Since p is odd, and $\frac{1+p}{2}$ and $\frac{1-p}{2}$ are relatively prime, we must have s divisible by $\frac{1+p}{2}$. We assume that $p/2 \leq s < p$, so the only option is $s = \frac{1+p}{2}$.

Substituting $s = \frac{p+1}{2}$ and using $\omega_2^{p+1} = \omega_1$ gives us

$$\omega_2^{1+s+p(1-s)} = \omega_1^{\frac{p+1}{2}}, \qquad \omega_2^{p(1+s)+1-s} = \omega_1^{\frac{p+1}{2}}.$$

This shows Equation (5.3). □

5.2.1 Type $(2, 2^+)$, $F((2p, 2p-s, s, 0) - \eta)$ Is Not Obvious.

Recall that an obvious lift of $\overline{\rho}$ is a representation of the form $\Psi = \Psi_{\Lambda_1} \oplus \cdots \oplus \Psi_{\Lambda_d}$, where Λ_i is a collection of n_i integers with $n_1 + \cdots + n_d = 4$, and Ψ_{Λ_i} is a representation from Proposition 3.1.1 with Hodge-Tate weights Λ_i. Each partition of 4 gives us a different form that an obvious lift of $\overline{\rho}$ may take. These can be summarized as:

$$\begin{aligned}
(4): &\quad \Psi_{\{a,b,c,d\}} \\
(3+1): &\quad \Psi_{\{a,b,c\}} \oplus \Psi_{\{d\}} \\
(2+2): &\quad \Psi_{\{a,b\}} \oplus \Psi_{\{c,d\}} \\
(2+1+1): &\quad \Psi_{\{a,b\}} \oplus \Psi_{\{c\}} \oplus \Psi_{\{d\}} \\
(1+1+1+1): &\quad \Psi_{\{a\}} \oplus \Psi_{\{b\}} \oplus \Psi_{\{c\}} \oplus \Psi_{\{d\}}
\end{aligned} \tag{5.4}$$

Lemma 5.2.3. *If* $\overline{\rho}$ *has type* $(2, 2^+)$*, then* $\overline{\rho}$ *does not have an obvious lift of the form* (4) *or* $(3+1)$*.*

Proof. Let's first recall that if $\overline{\rho}$ has type $(2, 2^+)$, then

$$\overline{\rho} = \omega_1 \oplus \omega_1 \oplus \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}.$$

Suppose that $\overline{\rho}$ lifts to a representation $\Psi_{\{a,b,c,d\}}$. Then the first term lifts to $\omega_4^{p^3+p^2+p+1}$. In the reduction of $\Psi_{\{a,b,c,d\}}$, one of the terms simplifies to ω_1, which means all four terms must simplify to ω_1. However, by Lemma 5.2.2, the only way the final two terms simplify to niveau 1 is if $s = \frac{p+1}{2}$, and they simplify to $\omega_1^{\frac{p+1}{2}}$, a contradiction.

If $\overline{\rho}$ lifts to a niveau-3 representation Ψ plus a niveau-1 representation Ψ', then at least one of the terms ins in the reduction of Ψ is ω_1, so by the same argument as in the previous paragraph, all the terms in Ψ must simplify to ω_1. On the other hand, at least one of the terms in the reduction of Ψ must be one of $\omega_2^{(p+1)+(p-1)s}$ or $\omega_2^{(p+1)+(p-1)(p+1-s)}$. But by the same reasoning as in the previous paragraph, this gives us a contradiction. □

Lemma 5.2.4. *If $\overline{\rho}$ has type $(2, 2^+)$, then $\overline{\rho}$ does not have an obvious lift of the form $(2+1+1)$ or $(1+1+1+1)$ that witnesses the weight $F(\lambda)$ with $\lambda = (2p, 2p-s, s, 0) - \eta$.*

Proof. By Lemma 5.2.2, $\overline{\rho}_{2,2^+} = \omega_1 \oplus \omega_1 \oplus \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}$ has an obvious lift of the form $(1 + 1 + 1 + 1)$ only if $s = \frac{p+1}{2}$. In this case, $\overline{\rho}_{2,2^+} = \omega \oplus \omega \oplus \omega^{\frac{p+1}{2}} \oplus \omega^{\frac{p+1}{2}}$. In order for this representation to have the obvious weight $\lambda = (2p, 2p-s, s, 0) - \eta$, there must be a permutation (x', y', z', w') of $(2p, 2p-s, s, 0)$

such that

$$x', y' \equiv 1 \pmod{p-1},$$
$$z', w' \equiv \frac{p+1}{2} \pmod{p-1}.$$

Since 0 must be one of x', y', z', w', and $0 \not\equiv 1 \pmod{p-1}$ or $\frac{p+1}{2} \pmod{p-1}$, this is not possible.

If $\Psi = \Psi_{\{a,b\}} \oplus \Psi_{\{c\}} \oplus \Psi_{\{d\}}$ is an obvious lift of $\overline{\rho}_{2,2^+}$ of the form $(2+1+1)$, then we must have either:

(i) $\overline{\Psi}_{\{a,b\}} = \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}$, $\overline{\Psi}_{\{c\}} = \overline{\Psi}_{\{d\}} = \omega$, or

(ii) $\overline{\Psi}_{\{a,b\}} = \omega_2^{p+1} \oplus \omega_2^{p+1}$, $\overline{\Psi}_{\{c\}} = \overline{\Psi}_{\{d\}} = \omega^{(p+1)/2}$.

Option (i) is not possible, since there is no permutation (x', y', z', w') of $(2p, 2p-s, s, 0)$ with $z', w' \equiv 1 \pmod{p-1}$. The case $2p-s, s \equiv 1 \pmod{p-1}$ is eliminated, since we assume $s > p/2$.

To see that option (ii) is not possible, notice that for $p > 3$, the equivalence $2p \equiv \frac{p+1}{2} \pmod{p-1}$ is false. So the only permutations (x', y', z', w') of $(2p, 2p-s, s, 0)$ with $z', w' \equiv \frac{p+1}{2} \pmod{p-1}$ have $\{z', w'\} = \{2p-s, s\}$. But then $\{x', y'\} = \{0, 2p\}$, and $x' + py' \equiv p+1 \pmod{p^2-1}$ is not possible. □

Lemma 5.2.5. *Let $p > 3$. If $\overline{\rho}$ has type $(2, 2^+)$, then $F(\lambda)$ is not an obvious weight of $\overline{\rho}$.*

Proof. By Lemmas 5.2.4 and 5.2.3, an obvious lift of $\overline{\rho}$ witnessing the weight $F(\lambda)$ must have the form $(2+2)$. The weight $F(\lambda)$ with $\lambda = (2p, 2p-s, s, 0) - \eta$ is an obvious weight of $\overline{\rho}$ if there is a permutation (x', y', z', w') of $(2p, 2p-s, s, 0)$ such that

$$x' + py' \equiv p+1 \pmod{p^2-1}, \tag{5.5}$$

$$z' + pw' \equiv s+1+p(1-s) \pmod{p^2-1}. \tag{5.6}$$

Note that one of x', y', z', w' must be zero. If $x' = 0$, then $y' \equiv p+1 \pmod{p^2-1}$. The only possibility is $y' = 2p - s = p+1$, so $s = p-1$. Similarly, if $y' = 0$, then $x' = 2p - s = p + 1$. Now $\{x', y'\}$ is a permutation of $\{0, 2p-s\}$, so $\{z', w'\}$ is a permutation of $\{2p, s\} = \{2p, p-1\}$.

But then $z' + pw' \equiv p+1 \pmod{p^2-1}$, and $\omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}$ simplifies to a niveau-1 representation. By Lemma 5.2.2, this implies that $p+1 = s = \frac{p+1}{2}$. As we assume $p > 3$, we get a contradiction.

If $z' = 0$, then $w' \equiv (p+1) + (p+1-s)(p-1) \pmod{p^2-1}$. By Lemma 5.2.1,

$$0 < (p+1) + (p+1-s)(p-1) < p^2 - 1,$$

so $w' = (p+1)+(p+1-s)(p-1)$, and $w' > 2p$, so w' cannot be among $\{2p, 2p-s, s\}$. Contradiction.

If $w' = 0$, then $z' \equiv (p+1) + (p-1)s \pmod{p^2-1}$. Again by Lemma 5.2.1, $z' = p+1+s(p-1)$ and $z' > 2p$. Contradiction. $\square$

5.2.2 Type $(2, 2^+)$, $F((2p, 2p-s, s, 0)-\eta)$ Is Not Shadow.

To show that $F(\lambda)$ is not a shadow weight of $\overline{\rho}$, recall that from the closure condition given in (3.3), we need to show that none of the Jordan-Hölder factors of $W(\lambda)$ are predicted weights of $\overline{\rho}$, i.e. that they are neither obvious nor shadow. From part (1) of Lemma 4.2.2,

$$JH(W(\lambda)) = \{F(\lambda), F(\lambda_3), F(\lambda_2)\},$$

where

$$\begin{aligned} \lambda &= (2p, 2p-s, s, 0) - \eta, \\ \lambda_3 &= (s+p, 2p-s, p, 0) - \eta, \\ \lambda_2 &= (2p, p, s, p-s) - \eta. \end{aligned} \tag{5.7}$$

We need to check that $F(\lambda_2), F(\lambda_3) \notin \mathcal{C}(W_{\mathrm{obv}}(\overline{\rho}))$. Since $W(\lambda_2), W(\lambda_1)$ are irreducible , $F(\lambda_2), F(\lambda_3)$ are not shadow weights, so we only need to check that they are not obvious. And, by the following lemma, it is enough to check only that $F(\lambda_2) \notin W_{\mathrm{obv}}(\overline{\rho})$.

Lemma 5.2.6. *If $\overline{\rho}$ has type $(2, 2^+)$, then $F(\lambda_3)$ is an obvious weight of $\overline{\rho}$ if and only if $F(\lambda_2)$ is an obvious weight of $\overline{\rho}$.*

Proof. The representation $\overline{\rho}_{2,2^+}$ has $\lambda_2 = (2p, p, s, p-s) - \eta$ as an obvious weight if and only if $\overline{\rho}^{\vee}_{2,2^+}$ has $\lambda_2' = (s-p, -s, -p, -2p) - \eta$ as an obvious weight. This weight is equivalent to

$$\lambda_2'' = \lambda_2 + 2(p-1)(1,1,1,1) = (p+s-2, 2p-s-2, p-2, -2).$$

And λ_2'' is an obvious weight of $\overline{\rho}^\vee_{2,2^+}$ if and only if $\lambda_3 = (p+s, 2p-s, p, 0) - \eta$ is an obvious weight of $\overline{\rho}^\vee_{2,2^+} \otimes \omega^2$. Finally, note that

$$\begin{aligned}
\overline{\rho}^\vee_{2,2^+} \otimes \omega^2 &= (\omega_1^{-1} \oplus \omega_1^{-1} \oplus \omega_2^{-1-s-p(1-s)} \oplus \omega_2^{-p(1+s)-1+s}) \otimes \omega^2 \\
&= \omega_1^1 \oplus \omega_1^1 \oplus \omega_2^{(2+2p)+-1-s-p(1-s)} \oplus \omega_2^{(2+2p)+-p(1+s)-1+s} \\
&= \omega_1 \oplus \omega_1 \oplus \omega_2^{1-s+p(1+s)} \oplus \omega_2^{p(1-s)+1+s} \\
&= \overline{\rho}_{2,2^+}.
\end{aligned}$$

This shows that $F(\lambda_2)$ is an obvious weight of $\overline{\rho}_{2,2^+}$ if and only if $F(\lambda_3)$ is an obvious weight of $\overline{\rho}_{2,2^+}$. □

Lemma 5.2.7. *If $\overline{\rho}$ has type $(2, 2^+)$, then $\overline{\rho}$ does not have an obvious lift of the form $(1+1+1+1)$ or $(2+1+1)$ that witnesses the weight $F(\lambda_2)$.*

Proof. By Lemma 5.2.2, $\overline{\rho}$ can only have an obvious lift of the form $(1+1+1+1)$ if $s = \frac{p+1}{2}$. In this case, $\{2p, p, s, p-s\}$ must be a permutation of $\{1, 1, \frac{p+1}{2}, \frac{p+1}{2}\}$ $\bmod p-1$. But for $p > 3$,

$$2p, p, p-s \not\equiv \frac{p+1}{2} \pmod{p-1}.$$

Suppose $\overline{\rho}$ has an obvious lift $\Psi = \Psi_{\{a,b\}} \oplus \Psi_{\{c\}} \oplus \Psi_{\{d\}}$ of the form $(2+1+1)$. Then we have either

(i) $\overline{\Psi}_{\{a,b\}} = \omega_2^{1+s+p(1-s)} \oplus \omega_2^{p(1+s)+1-s}$, $\overline{\Psi}_{\{c\}} = \overline{\Psi}_{\{d\}} = \omega$, or

(ii) $\overline{\Psi}_{\{a,b\}} = \omega_2^{p+1} \oplus \omega_2^{p+1}$, $\overline{\Psi}_{\{c\}} = \overline{\Psi}_{\{d\}} = \omega^{(p+1)/2}$ if $s = \frac{p+1}{2}$.

For option (i) to occur, two of $\{2p, p, s, p-s\}$ must be congruent to $1 \pmod{p-1}$. This is possible only if $s = p-1$, in which case $p, p-s \equiv 1 \pmod{p-1}$. So $\{2p, s\} = \{2p, p-1\}$ must be the parameters involved in the niveau-2 part. But $\omega_2^{(p-1)+p\cdot 2p} = \omega_2^{p+1} = \omega_1$. By Lemma 5.2.2, this must mean $s = \frac{p+1}{2}$. Contradiction with $p \neq 3$.

Option (ii) occurs when $s = \frac{p+1}{2}$, and if some permutation (x', y', z', w') of $(2p, p, s, p-s)$ has $x', y' \equiv \frac{p+1}{2} \pmod{p-1}$. This is not possible by the same argument as in case (1+1+1+1). So option (ii) cannot occur either, and $\overline{\rho}$ does not have an obvious lift of the form (1+1+1+1) or (2+1+1). □

Lemma 5.2.8. *If $\overline{\rho}$ has type $(2, 2^+)$, then $\overline{\rho}$ does not have an obvious lift of the form $(2+2)$ that witnesses the obvious weight $F(\lambda_2)$.*

Proof. Any obvious lift of the form $(2+2)$ that witnesses the obvious weight $F(\lambda_2)$ has the form $\Psi = \Psi_{\{x',y'\}} \oplus \Psi_{\{z',w'\}}$ with $\overline{\Psi} = \overline{\rho}_{2,2^+}$. So that

$$\omega_2^{x'+py'} \oplus \omega_2^{p(x'+py')} \oplus \omega_2^{z'+pw'} \oplus \omega_2^{p(z'+pw')} = \omega_2^{p+1} \oplus \omega_2^{p+1} \oplus \omega_2^{1+s+p(1-s)} \oplus \omega_2^{p(1+s)+1-s}.$$

Such a lift will exist if and only if there exists a permutation (x', y', z', w') of $\lambda_2 + \eta = (2p, p, s, p-s)$ such that

$$x' + py' \equiv p + 1 \pmod{p^2 - 1}, \tag{5.8}$$

$$z' + pw' \equiv s + 1 + p(1-s) \pmod{p^2 - 1}. \tag{5.9}$$

If $x' = p$, then $y' \equiv p \pmod{p^2-1}$. So $y' = p$, contradiction. Similarly for $y' = p$. If $w' = p$, then $z' \equiv p + (p-1)s \pmod{p^2-1}$. By Lemma 5.2.1, $p + (p-1)s < p^2 - 1$, so $z' = p + (p-1)s$. But $p+(p-1)s > p$ and $p+(p-1)s \neq 2p$. So $z' \notin \{2p, s, p-s\}$. Contradiction.

If $z' = p$, then $w' \equiv p + (p-1)(p+1-s) \pmod{p^2-1}$. Again by Lemma 5.2.1, $w' = p+(p+1)(p+1-s)$. But $p+(p+1)(p+1-s) > p$ and $p+(p+1)(p+1-s) \neq 2p$. So $w' \notin \{2p, s, p-s\}$. Contradiction.

□

Lemma 5.2.9. *If $\overline{\rho}$ has type $(2, 2^+)$, then $F(\lambda_2)$ is not an obvious weight of $\overline{\rho}$.*

Proof. Recall that for a four-dimensional mod p representation, an obvious lift has one of the forms (4), (3+1), (2+2), (2+1+1), or (1+1+1+1). Lemma 5.2.3 says that forms (4) and (3+1) are not possible. Lemma **??** says that (2+2) is not possible for $F(\lambda_2)$. Finally, Lemma 5.2.7 shows that (2+1+1) and (1+1+1+1) are not possible for $F(\lambda_2)$. So $\overline{\rho}$ cannot have an obvious lift that witnesses the weight $F(\lambda_2)$. □

Lemma 5.2.10. *If $\overline{\rho}$ has type $(2, 2^+)$, then $F(\lambda) \notin \mathcal{C}(W_{\text{obv}}(\overline{\rho}))$.*

Proof. By Lemma 5.2.5, $F(\lambda) \notin W_{\text{obv}}(\overline{\rho})$. Then, Lemmas 5.2.9 and 5.2.6 show that $F(\lambda_2), F(\lambda_3) \notin W_{\text{obv}}(\overline{\rho})$. Since none of the Jordan-Hölder factors of $W(\lambda)$ are in $W_{\text{obv}}(\overline{\rho})$, we have also that $F(\lambda) \notin \mathcal{C}(W_{\text{obv}}(\overline{\rho}))$. □

5.3 On the Proof of the $(2, 2^+)$, $s + t = 2p$ Case.

In Section 4.4, most of the explicit weight calculations decomposed $\overline{\rho}$ into the sum of two two-dimensional representations, and the η-partition of λ was into two pieces each with two components. Two things were different about the proof in the $(2, 2^+)$, $s + t = 2p$ case.

First, instead of showing directly that an η-partition $\{\lambda^{(1)}, \lambda^{(2)}\}$ of λ satisfies the explicit weight condition,

$$W_{\text{expl}}(\overline{\rho}^{(i)}) \cap JH(W(\lambda^{(i)})) \neq \varnothing,$$

we showed that an η-partition of λ_2 (that λ_2 from $W(\lambda) = F(\lambda) + F(\lambda_3) + F(\lambda_2)$) satisfies the explicit weight condition.

Second, we chose to break $\overline{\rho}$ into a one-dimensional and a three-dimensional representation (instead of two two-dimensional representations). This meant λ_2 was broken into two pieces: $\lambda^{(1)}$ with one component, and $\lambda^{(2)}$ with three components. (The results in Section 2.3 were required just so that we could find the Jordan-Hölder factors of $W(\lambda^{(2)})$ in this one case!)

In this section, we will show that both of these differences in proof were necessary. We assume throughout the section that $s \neq \frac{p+1}{2}$, and show these results for the case $\frac{p+1}{2} < s < p$.

When making use of the obscure weight definition, we first decompose $\overline{\rho}$ as a sum of smaller dimensional representations $\overline{\rho} = \bigoplus \overline{\rho}^{(i)}$. If $s \neq \frac{p+1}{2}$, then a decomposition of

$$\overline{\rho}_{2,2^+} = \omega_1 \oplus \omega_1 \oplus \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}$$

is one of the following:

$$\textbf{Case (2-2):} \quad \overline{\rho}^{(1)} = \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}, \qquad (5.10)$$

$$\overline{\rho}^{(2)} = \omega_1 \oplus \omega_1$$

$$\textbf{Case (2-1-1):} \quad \overline{\rho}^{(1)} = \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}, \qquad (5.11)$$

$$\overline{\rho}^{(2)} = \omega_1, \quad \overline{\rho}^{(3)} = \omega_1$$

$$\textbf{Case (3-1):} \quad \overline{\rho}^{(1)} = \omega_1 \oplus \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}, \qquad (5.12)$$

$$\overline{\rho}^{(2)} = \omega_1$$

We say that $F(\lambda)$ is directly obscure for $\overline{\rho}$ *in case (-)* if there exists an η-decomposition $\{\lambda^{(i)}\}$ of λ itself satisfying the condition

$$JH(W(\lambda^{(i)})) \cap W_{\text{expl}}(\overline{\rho}^{(i)}) \neq \varnothing \qquad (5.13)$$

with the decomposition $\overline{\rho} = \bigoplus \overline{\rho}^{(i)}$ given in Case (-) above.

5.3.1 $F(\lambda)$ Is Not Directly Obscure

Lemma 5.3.1. $F(\lambda)$ *is not directly obscure for* $\overline{\rho}$ *in case (2-2).*

Proof. The decomposition in this case looks like $\overline{\rho} = \overline{\rho}^{(1)} \oplus \overline{\rho}^{(2)}$ with

$$\overline{\rho}^{(1)} = \omega_1 \oplus \omega_1, \qquad \overline{\rho}^{(2)} = \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}.$$

The possible η-partitions of $\lambda = (2p, 2p-s, s, 0) - \eta$ (up to a reordering) are:

(A) $\lambda^{(1)} = (2p, s) - \eta_2$, and $\lambda^{(2)} = (2p-s, 0) - \eta_2$, or

(B) $\lambda^{(1)} = (2p, 0) - \eta_2$, and $\lambda^{(2)} = (2p - s, s) - \eta_2$.

Below, we check that in each of these cases, $JH(W(\lambda^{(i)})) \cap W_{\text{expl}}(\overline{\rho}^{(j)}) = \varnothing$.

Consider the η-partition given in (A). As was noted in the proof of Lemma 4.4.3,

$$\begin{aligned} JH(W(2p-1, s)) &= \{F(p, s), F(s+p-1, p), F(p-1, s+1)_{(s \neq p-1)}\}, \\ JH(W(2p-s-1, 0)) &= \{F(p-s, 0), F(p-1, p-s), F(p-s-1, 1)_{(s \neq p-1)}\}. \end{aligned} \tag{5.14}$$

A Serre weight $F(a, b)$ is an obvious weight of $\overline{\rho} = \omega_2^m \oplus \omega_2^{pm}$ if there exists a permutation (x', y') of $(a, b) + \eta_2$ such that $x' + py' \equiv m \pmod{p^2 - 1}$. Since $p(x' + py') = px' + y'$, we only need to check one permutation. The following table summarizes these calculations for each of the Jordan-Hölder factors in (5.14).

$F(x'-1, y')$	$x' + py'$
$F(p, s)$	$1 + p(1 + s) \pmod{p^2 - 1}$
$F(s + p - 1, p)$	$s + 1 + p \pmod{p^2 - 1}$
$F(p - 1, s + 1)$	$p(2 + s) \pmod{p^2 - 1}$
$F(p - s, 0)$	$1 - s + p \pmod{p^2 - 1}$
$F(p - 1, p - s)$	$p(1 - s) \pmod{p^2 - 1}$
$F(p - s - 1, 1)$	$2p - s \pmod{p^2 - 1}$

Notice that none of these Serre weights has $x' + py' \equiv p + 1 \pmod{p^2 - 1}$. (In the third and sixth options, we do have $p(2+s), 2p-s \equiv p+1 \pmod{p^2-1}$, but only when $s = p - 1$, and the corresponding Serre weights $F(p-1, s+1), F(p-s-1, 1)$ do not occur as Jordan-Hölder factors when $s \neq p - 1$.) So $JH(W(\lambda^{(i)})) \cap W_{\text{expl}}(\overline{\rho}^{(1)}) = \varnothing$ for $i = 1, 2$.

Now, consider the η-partition given in (B). By Theorem 2.2.3,

$$JH(W(2p-1,0)) = \{F(1,0), F(p-1,1)\},$$
$$JH(W(2p-s-1,s)) = \{F(2p-s-1,s)\}.$$

For each of these Serre weights $F(x'-1,y')$, we compute $x'+py'$ modulo p^2-1.

$F(x'-1,y')$	$x'+py'$
$F(1,0)$	$2 \pmod{p^2-1}$
$F(p-1,1)$	$2p-1 \pmod{p^2-1}$
$F(2p-s-1,s)$	$p(2+s)-s \pmod{p^2-1}$

Again, none of these Serre weights has $x'+py' \equiv p+1 \pmod{p^2-1}$. So $JH(W(\lambda^{(i)})) \cap W_{\text{expl}}(\overline{\rho}^{(1)}) = \varnothing$ for $i=1,2$ for this η-partition as well. This finishes the proof that $F(\lambda)$ is not directly obscure in case (2-2).

□

Lemma 5.3.2. *The weight $F(\lambda)$ is not directly obscure for $\overline{\rho}$ in case (2-1-1), or in case (3-1).*

Proof. In both cases (2-1-1) and (3-1), the decomposition of $\overline{\rho}$ (given in (5.11) and (5.11), respectively), have $\overline{\rho}^{(2)} = \omega_1$. By Example 7.2.4 in [GHS16], the only explicit weight of $\overline{\rho}^{(2)} = \omega_1$ is $F(1)$. Since $s > p/2$, each of $2p, 2p-s, s, 0 \not\equiv 1 \pmod{p-1}$. So each η-partition of λ has

$$JH(F(\lambda^{(i)})) \cap W_{\text{expl}}(\overline{\rho}^{(2)}) = \varnothing. \tag{5.15}$$

□

Lemmas 5.3.1 and 5.3.2 together show that $F(\lambda)$ cannot satisfy the condition (5.13) directly. If $F(\lambda)$ is to be an explicit weight, we will need to use the fact that $W_{\text{expl}}(\overline{\rho})$ is closed under the $\mathcal{C}$ operation from Section 3.2.

5.3.2 $F(\lambda_2)$ Is Not Directly Obscure in Case (2-2) or (2-1-1)

To show that the decomposition in case (3-1) is the only one that works in the proof of Lemma 4.4.4 (thus requiring the results of Section 2.3), we show that $F(\lambda_2)$ is not directly obscure for $\overline{\rho}$ in case (2-2) or in case (2-1-1).

Lemma 5.3.3. *$F(\lambda_2)$ is not directly obscure in case (2-2).*

Proof. For reference, here is the case (2-2) decomposition of $\overline{\rho}$ again:

$$\overline{\rho}^{(1)} = \omega_1 \oplus \omega_1, \qquad \overline{\rho}^{(2)} = \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)}.$$

The possible η-partitions of $\lambda_2 = (2p, p, s, p-s) - \eta$ are:

(A) $\lambda^{(1)} = (2p, s) - \eta_2$ and $\lambda^{(2)} = (p, p-s) - \eta_2$, or

(B) $\lambda^{(1)} = (2p, p-s) - \eta_2$ and $\lambda^{(2)} = (p, s) - \eta_2$.

Consider the η-partition given in (A). From the proof of Lemma 5.3.1, we already know that $JH(W(2p-1, s)) \cap W_{\text{expl}}(\overline{\rho}^{(1)}) = \varnothing$. To see that $F(p-1, p-s) \notin W_{\text{expl}}(\overline{\rho}^{(1)}) = \varnothing$, note that

$$p + p(p-s) \equiv p + 1 - ps \pmod{p^2-1} \not\equiv p + 1 \pmod{p^2-1}.$$

So $JH(W(\lambda^{(1)})) \cap W_{\text{expl}}(\overline{\rho}^{(1)}) = \varnothing$ for $i = 1, 2$.

Next, consider the η-partition given in (B). By Theorem 2.2.3,

$$JH(W(2p-1,p-s)) = \{F(p,p-s), F(p-1,p-s+1), F(p-s,1)\},$$

$$JH(W(p-1,s)) = \{F(p-1,s)\}.$$

For each of the Serre weights $F(x'-1,y')$ on the right hand side, we calculate $x'+py'$ modulo p^2-1. The results are summarized in the following table.

$F(x'-1,y')$	$x'+py'$
$F(p,s)$	$p+1+ps \pmod{p^2-1}$
$F(p-1,p-s+1)$	$2p-ps+1 \pmod{p^2-1}$
$F(p-s,1)$	$2p-s+1 \pmod{p^2-1}$
$F(p-1,s)$	$p(1+s) \pmod{p^2-1}$

None of these have $x'+py' \equiv p+1 (\operatorname{mod} p^2-1)$, so

$$JH(W(\lambda^{(i)})) \cap W_{\text{expl}}(\overline{\rho}^{(1)}) = \varnothing$$

for $i = 1, 2$ for this η-partition as well. So $F(\lambda_2)$ is not directly obscure for $\overline{\rho}$ in case (2-2). □

Lemma 5.3.4. *$F(\lambda_2)$ is not obscure in case (2-1-1).*

Proof. For reference, here is the case (2-1-1) decomposition of $\overline{\rho}$ again:

$$\overline{\rho}^{(1)} = \omega_2^{(p+1)+(p-1)s} \oplus \omega_2^{(p+1)+(p-1)(p+1-s)},$$

$$\overline{\rho}^{(2)} = \omega_1, \quad \overline{\rho}^{(3)} = \omega_1.$$

Any η-partition which has $JH(W(\lambda^{(2)})) \cap W_{\text{expl}}(\overline{\rho}^{(2)}) \neq \varnothing$ and $JH(W(\lambda^{(3)})) \cap W_{\text{expl}}(\overline{\rho}^{(3)}) \neq$

$\varnothing$ must have $\lambda^{(2)} = (z')$, $x' \equiv 1 \pmod{p-1}$ and $\lambda^{(3)} = (w')$, $w' \equiv 1 \pmod{p-1}$ for some permutation (x', y', z', w') of $(2p, p, s, p-s)$. Since $s > p/2$, $s \not\equiv 1 \pmod{p-1}$ and clearly $2p \not\equiv 1 \pmod{p-1}$. If $s \neq p-1$, then $p - s \not\equiv 1 \pmod{p-1}$ and $F(\lambda_2)$ is not an obscure weight in case (2-1-1).

If $s = p-1$, then $\lambda_2 = (2p, p, p-1, 1)$. Without loss of generality, we may choose $\lambda^{(2)} = (p), \lambda^{(3)} = (1)$, so that $\lambda^{(1)} = (2p-1, p-1)$. Substituting $s = p-1$ into the formula for $\overline{\rho}^{(1)}$, we have $\overline{\rho}^{(1)} = \omega_2^{3-p} \oplus \omega_2^{3p-1}$. Theorem 2.2.3 gives us $JH(W(2p-1, p-1)) = \{F(1,0), F(p-1,1)\}$. Neither $F(x'-1, y')$ of these satsfies $x' + py' \equiv 3 - p \pmod{p^2-1}$. So $JH(W(\lambda^{(1)})) \cap W_{\mathrm{expl}}(\overline{\rho}^{(1)}) = \varnothing$. This finishes the proof that $F(\lambda_2)$ is not directly obscure in case (2-1-1). $\square$

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