Abstract. We prove that non-trivial representations of the alternating group $A_n$ are reducible over a primitive proper subgroup which is isomorphic to some alternating group $A_m$. A similar result is established for finite simple classical groups embedded in $A_n$ via their standard rank 3 permutation representations.

1. Introduction

If $\Gamma$ is a transitive permutation group with a point stabilizer $X$ then $\Gamma$ is primitive if and only if $X < \Gamma$ is a maximal subgroup. So studying primitive permutation groups is equivalent to studying maximal subgroups. In most problems involving a finite primitive group $\Gamma$, the Aschbacher-O’Nan-Scott theorem [AS] allows one to concentrate on the case where $\Gamma$ is almost quasi-simple, i.e. $L < \Gamma/\Gamma(Z) \leq \text{Aut}(L)$ for a non-abelian simple group $L$. The results of Liebeck-Praeger-Saxl [LPS] and Liebeck-Seitz [LS] then allow one to assume furthermore that $\Gamma$ is a finite classical group.

In the latter case, the possible structure of maximal subgroups $X$ is described by Aschbacher’s theorem [A]: if $X < \Gamma$ is maximal then $X$ belongs to

$$S \cup \bigcup_{i=1}^{8} C_i,$$

where $C_1, \ldots, C_8$ are collections of certain explicit natural subgroups of $\Gamma$, and $S$ is the collection of almost quasi-simple groups that act absolutely irreducibly on the natural module for the classical group $\Gamma$.

It is not true, however, that every subgroup $X$ in (1.1) is actually maximal in $\Gamma$. For $X \in \bigcup_{i=1}^{8} C_i$, the maximality of $X$ has been determined by Kleidman-Liebeck [KIL] (see also [BDR] for a complete classification of maximal subgroups in low-dimensional finite classical groups). So let $X \in S$. If $X$ is not maximal then $X < G < \Gamma$ for a certain maximal subgroup $G$ in $\Gamma$. The most challenging case to handle is when $G \in S$ as well. This motivates the following problem, where $\mathbb{F}$ is an algebraically closed field of characteristic $p \geq 0$:

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Problem 1.1. Classify all triples \((G,V,X)\) where \(G\) is an almost quasi-simple finite group, \(V\) is an \(FG\)-module of dimension greater than one, and \(X\) is a proper subgroup of \(G\) such that the restriction \(\text{Res}_X^G V\) is irreducible.

Many results have been obtained concerning various cases of Problem 1.1 — see for instance [KT2] and references therein. In this paper, we are mostly interested in the case \(G = A_n\) or the symmetric group \(S_n\). In this case, under the assumption \(p > 3\) (or \(p = 0\)), Problem 1.1 has been solved in [BK, KS2] — see also [KT1] for double-covers \(A_n\) and \(S_n\) and [S, KI] for the characteristic zero case. A number of techniques employed in these papers unfortunately break down in the case \((G,X) = (A_n,A_m)\) and \(p = 2,3\) (and especially when \(X\) is a primitive subgroup of \(G\)). On the other hand, this case is of crucial importance in a number of applications. The purpose of this paper is to resolve this important case, and our main result is:

**Theorem 1.2.** Let \(X \cong A_m\) be a primitive subgroup of \(A_n\) with \(n > m \geq 9\). Let \(F\) be an algebraically closed field of arbitrary characteristic and \(V\) be a non-trivial \(FA_n\)-module. Then \(V\) is reducible over \(X\).

The bound \(m \geq 9\) is the best possible — see Remark 8.2 and Lemma 8.1.

We emphasize that our methods also apply to many other primitive subgroups of \(A_n\). To illustrate this, in this paper we handle the simple classical groups \(X\) that embed in \(A_n\) via their standard rank 3 permutation representations:

**Theorem 1.3.** Let \(X = L/Z(L)\) be a finite simple classical group, where \(L\) is one of the following group: \(SL_d(q)\), \(SU_d(q)\), or \(Sp_{d'}(q)\) with \(d \geq 4\), and \(\Omega^+_d(q)\) with \(d \geq 5\). Let \(W\) denote the natural \(d\)-dimensional module for \(L\), and let \(X\) be embedded in \(\text{Sym}(\Omega) = S_n\) via its rank 3 permutation action on the set \(\Omega\) of 2-dimensional subspaces of \(W\) in the case \(L = SL_d(q)\), and of 1-dimensional singular subspaces of \(W\) otherwise. If \(V\) is any \(FA_n\)-module of dimension \(> 1\), then \(\text{Res}_X^{A_n} V\) is reducible.

We plan to extend this result to the remaining simple primitive subgroups of \(A_n\) in a sequel. Together with the results of [BK, KS2] and the current paper, this will completely solve Problem 1.1 for \(G = A_n\) in the cases that are of most interest for the Aschbacher-Scott program.

The paper is organized as follows. Basic notions are recalled in §2. Theorem 3.2 in §3 compares the dimensions of the Hom-spaces of irreducible \(S_n\)-modules in characteristic 2 over certain Young subgroups of \(S_n\) when \(n\) is even. Then Propositions 4.1, 4.3, and 4.4 in §4 show in particular that the \(p\)-modular irreducible representations of \(A_n\) which do not extend to \(S_n\) must have large enough dimension (at least exponential in \(n\)). These results, which we believe are also of independent interest, allow us to discard non-\(S_n\)-extendible \(A_n\)-modules in the proof of Theorem 1.2. In §5 we describe the submodule structure of the permutation modules of \(S_n\) acting on subsets of \(\{1,2,\ldots,n\}\) of cardinality 2 or 3 in characteristic 2, again in the case of even \(n\). This description plays a key role in the proof of Theorem 6.5 in §6, which gives a criterion for a 2-modular irreducible \(S_n\)-representation to be reducible over certain subgroups of \(S_n\). Theorem 6.5 is then used in §7 to show that non-trivial 2-modular irreducible \(A_n\)-representations are reducible over \(A_m\), if \(A_m\) is embedded into \(A_n\) via its actions on subsets or set partitions of \(\{1,2,\ldots,m\}\)
— see Theorem 7.12. Theorem 1.2 is proved in §8, which also contains further results concerning non-primitive embeddings of $A_m$ into $A_n$. The final §9 is devoted to the proof of Theorem 1.3.

2. Preliminaries

Throughout the paper, unless otherwise stated, we assume that the ground field $\Bbb F$ is algebraically closed, and $p := \text{char} (\Bbb F)$. For a group $G$, the trivial $\Bbb F G$-module is denoted $1_G$ or simply $1$ if it is clear what $G$ is. If $V$ is an $\Bbb F G$-module, we denote by $\text{soc}(V)$ the socle of $V$, and for $n = 1, 2, \ldots$, define $\text{soc}^n(V)$ from $\text{soc}^1(V) = \text{soc}(V)$ and $\text{soc}^n(V)/\text{soc}^{n-1}(V) = \text{soc}(V/\text{soc}^{n-1}(V))$ for $n > 1$. We refer to the quotients $\text{soc}^n(V)/\text{soc}^{n-1}(V)$ as the socle layers of $V$ and usually list them from bottom to top, i.e. first $\text{soc}(V)$, then $\text{soc}^2(V)/\text{soc}(V)$, etc.

For $n \in \mathbb{Z}_{> 0}$, let $\Omega := \{1, 2, \ldots, n\}$.

For $r = 1, \ldots, n$, denote by $\Omega_r$ the set of $r$-element subsets of $\Omega$. The symmetric group $S_n$ acts naturally on the sets $\Omega = \Omega_1, \Omega_2, \ldots, \Omega_n$ and the stabilizer of an element of $\Omega_r$ is conjugate to the subgroup $S_{n-r,r} := S_\{1,2,\ldots,n-r\} \times S_{\{n-r+1,\ldots,n\}}$. We write $S_{n-1,1}$ simply as $S_{n-1}$.

We denote by

$$M_r = \mathbb{F} \Omega_r \cong \text{Ind}_{S_{n-r,r}}^{S_n} 1_{S_{n-r,r}}, \quad (1 \leq r \leq n)$$

the permutation module for the action of $S_n$ on $\Omega_r$.

We recall some basic notions of representation theory of symmetric groups referring to [J2] for details. The irreducible $\mathbb{F} S_n$-modules are labeled by $p$-regular partitions of $n$ (if $p = 0$ then $p$-regular partitions are interpreted as all partitions). If $\lambda$ is a $p$-regular partition of $n$, the corresponding irreducible module is denoted $D_\lambda$. The Specht modules over $\mathbb{F} S_n$ are labeled by partitions of $n$. If $\lambda$ is such a partition, the corresponding Specht module is denoted $S_\lambda$.

Let $p = 2$. Consider the partition

$$\alpha_n = \begin{cases} (k + 1, k - 1) & \text{if } n = 2k \text{ is even}, \\ (k + 1, k) & \text{if } n = 2k + 1 \text{ is odd}. \end{cases}$$

The irreducible module $D^{\alpha_n}$ is called the basic spin module for $S_n$. It is known [W, Table III] that

$$\dim D^{\alpha_n} = 2^{[(n-1)/2]}.$$

Let $\text{sgn}_n$ be the sign module over $\mathbb{F} S_n$. For any $p$-regular partition $\lambda$, we have that $D^\lambda \otimes \text{sgn}_n$ is an irreducible $\mathbb{F} S_n$-module, so we can write $D^\lambda \otimes \text{sgn}_n \cong D^{\lambda^M}$, where

$$M : \lambda \mapsto \lambda^M$$

is the Mullineux involution on the set of $p$-regular partitions of $n$. To describe the Mullineux involution, we briefly recall the notion of the Mullineux symbol $G(\lambda)$ of $\lambda$, referring the reader to [FK] for details. Let $h_1$ be the number of nodes in the $p$-rim of $\lambda$, $\ldots$.
and let \( r_1 \) be the number of rows in \( \lambda \). Delete the \( p \)- rim and repeat to obtain sequences \( h_1, h_2, \ldots \) and \( r_1, r_2, \ldots \). Let \( k \) be such that \( h_{k+1} = r_{k+1} = 0 \) but \( h_k \neq 0 \neq r_k \). Then

\[
G(\lambda) := \begin{pmatrix}
  h_1 & h_2 & \cdots & h_k \\
  r_1 & r_2 & \cdots & r_k
\end{pmatrix}.
\]

It was proved in [Mu] that \( \lambda \) is uniquely determined by \( G(\lambda) \). Moreover, we have

\[
G(\lambda^M) = \begin{pmatrix}
  h_1 & h_2 & \cdots & h_k \\
  h_1 - r_1 + \epsilon_1 & h_2 - r_2 + \epsilon_2 & \cdots & h_k - r_k + \epsilon_k
\end{pmatrix},
\]

where \( \epsilon_i := 0 \) if \( p | h_i \) and \( \epsilon_i := 1 \) otherwise. This description of \( M \) is the main result of [FK] (see also [BeO]), which was conjectured by Mullineux.

Given an irreducible representation \( D^\lambda \), either the restriction \( E^\lambda := \text{Res}_{A_n}^{S_n} D^\lambda \) is irreducible or \( \text{Res}_{A_n}^{S_n} D^\lambda \cong E_+^\lambda \oplus E_-^\lambda \), a direct sum of two inequivalent irreducible representations. Moreover, every irreducible \( \mathbb{F}A_n \)-module is isomorphic to one of \( E_+^\lambda \) and the only non-trivial isomorphism of the form \( E_+^\lambda \cong E_{(\pm)}^\mu \) is \( E^\lambda \cong E^{\lambda M} \).

If \( p \neq 2 \), then \( \text{Res}_{A_n}^{S_n} D^\lambda \) is reducible if and only if \( \lambda = \lambda^M \). If \( p = 2 \), then an explicit criterion for reducibility of \( \text{Res}_{A_n}^{S_n} D^\lambda \) is given in [Ben, Theorem 1.1].

**3. Comparing some Hom-spaces**

Throughout this section we assume that \( p = 2 \). In this section we get some results on the dimensions

\[
d_r(V) := \dim \text{Hom}_{\mathbb{F}S_n}(M_r, \text{End}_F(V)) = \dim \text{End}_{\mathbb{F}S_{n-r,r}}(\text{Res}_{S_{n-r,r}}^{S_n} V).
\]

The last equality follows using \( M_r = \text{Ind}_{S_{n-r,r}}^{S_n} 1_{S_{n-r,r}} \) and Frobenius reciprocity.

**Lemma 3.1.** Let \( V \) be an irreducible \( \mathbb{F}S_n \)-module and \( 1 \leq r \leq n \). Then \( d_r(V) = 1 \) if and only if \( \text{Res}_{S_{n-r,r}}^{S_n} V \) is irreducible.

**Proof.** The sufficiency of the condition is clear. Conversely, irreducible \( \mathbb{F}S_n \)-modules are self-dual, so the restriction \( \text{Res}_{S_{n-r,r}}^{S_n} V \) is self-dual. Since irreducible \( \mathbb{F}S_{n-r,r} \)-modules are also self-dual, the head of \( \text{Res}_{S_{n-r,r}}^{S_n} V \) is isomorphic to its socle. So if \( \text{Res}_{S_{n-r,r}}^{S_n} V \) is reducible then \( d_r(V) > 1 \).

The goal of this section is to prove the following result:

**Theorem 3.2.** Let \( V \) be a simple \( \mathbb{F}S_n \)-module and \( 2|n \geq 6 \). Then one of the following statements holds:

(i) \( d_3(V) > d_1(V) \).
(ii) \( V \cong D^{\alpha_n} \) is the basic spin module or \( V \cong 1 \) is the trivial module, in which cases we have \( d_3(V) = d_1(V) \).

Let \( V = D^\lambda \) for a 2-regular partition

\[
\lambda = (\lambda_1 > \ldots > \lambda_s > 0)
\]

of \( n \). If \( s = 1 \), then \( V \) is the trivial module, and Theorem 3.2 holds trivially.
Lemma 3.3. Theorem 3.2 holds if $d_1(V) = 1$.

Proof. Suppose that $d_1(V) = d_3(V) = 1$ and $\dim V > 1$. By Lemma 3.1, $V$ is irreducible over $S_{n-1}$ and $S_{3,n-3}$, and so the lemma follows from [P, Theorem 10].

Lemma 3.4. Theorem 3.2 holds if $s = 2$.

Proof. Since $n$ is even, we have that the restriction $D^\lambda|_{S_{n-1}}$ is irreducible by [K1]. So $d_1(V) = 1$, and we may apply Lemma 3.3.

Lemma 3.5. Let

$$x := \sum_{g \in S_{\{1,2,3\}}, \sigma \in S_{\{1,4\}} \times S_{\{2,5\}} \times S_{\{3,6\}}} (\text{sign} \, \sigma)g \sigma^{-1} \in \mathbb{F}S_n.$$ 

If $s > 2$, then $xD^\lambda \neq 0$.

Proof. By [BaK, Lemma 4.7] with $(m, p) = (3, 2)$, we see that the restriction $V|_{S_6}$ has a composition factor isomorphic to $D^{(3, 2, 1)}$, which for $p = 2$ is isomorphic to the Specht module $S^{(3, 2, 1)}$. Since $x \in \mathbb{F}S_6$, it suffices to prove that $xS^{(3, 2, 1)} \neq 0$. We use the notation of [J2, §4]; in particular, $e_t$ is the polytabloid and $\{t\}$ is the tabloid corresponding to a $(3, 2, 1)$-tableau $t$. Let

$$t := \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 \\ 3 \end{array}, \quad s := \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{array}.$$ 

An explicit calculation shows that $\{s\}$ appears in $xe_t$ with coefficient 1.

We now complete the proof of Theorem 3.2. Recall that $M_3$ is the permutation module on all three element subsets $\{i, j, k\} \subseteq \Omega$, while $M_1$ is the permutation module on the one element subsets $\{i\} \subseteq \Omega$. Consider the $\mathbb{F}S_n$-module homomorphism

$$f : M_3 \to M_1, \quad \{i, j, k\} \mapsto \{i\} + \{j\} + \{k\}.$$ 

It is easy to see that $f$ is surjective. So it induces an injective linear map

$$f^* : \text{Hom}_{\mathbb{F}S_n}(M_1, \text{End}_F(V)) \to \text{Hom}_{\mathbb{F}S_n}(M_3, \text{End}_F(V)), \quad \psi \mapsto \psi \circ f.$$ 

It suffices to prove that $f^*$ is not surjective.

We exhibit an element $\phi \in \text{Hom}_{\mathbb{F}S_n}(M_3, \text{End}_F(V))$ which is not in the image of $f^*$. For $\Theta \subseteq \Omega$ let $S_\Theta \subseteq S_n = S_\Omega$ be the subgroup of all permutations which stabilize the elements of $\Omega \setminus \Theta$. Now, define $\phi$ as follows:

$$\phi(\{i, j, k\})(v) := \sum_{g \in S_{\{i,j,k\}}} gv, \quad (\{i, j, k\} \subseteq \Omega, \ v \in V).$$

(3.2)

If $\phi \in \text{Im} f^*$, then $\phi = \psi \circ f$ for some $\psi \in \text{Hom}_{\mathbb{F}S_n}(M_1, \text{End}_F(V))$. Consider the element

$$E = \sum_{\sigma \in S_{\{1,4\}} \times S_{\{2,5\}} \times S_{\{3,6\}}} (\text{sign} \, \sigma)\sigma \{1, 2, 3\} \in M_3.$$
Note that \( f(E) = 0 \). So \( \phi(E) = \psi(f(E)) = 0 \). On the other hand, we compute \( \phi(E) \) using (3.2):

\[
\phi(E)(v) = \sum_{\sigma \in S_{(1,4)} \times S_{(2,5)} \times S_{(3,6)}} (\text{sign } \sigma) \sum_{g \in S_{\{(1), \sigma(2), \sigma(3)\}}} gv = xv,
\]

where \( x \) is as in Lemma 3.5. Now Lemma 3.5 yields a contradiction.

### 4. Dimension and Extendibility to \( S_n \)

First we prove the following statement, which relies on some results of [Ben] and [GLT]:

**Proposition 4.1.** Let \( p = 2 \), \( n \geq 5 \), and let \( V \) be an irreducible \( \mathbb{F}A_n \)-module. Suppose that \( V \) does not extend to \( S_n \). Then \( \dim V \geq 2^{(n-6)/4} \).

**Proof.** By assumption, \( W := \text{Ind}_{A_n}^{S_n}(V) \) is an irreducible \( \mathbb{F}S_n \)-module, and \( \dim W = 2 \dim V \). Let \( \lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_s > 0) \) be the partition of \( n \) into distinct parts corresponding to \( W \). Since \( W \) is reducible over \( A_n \), by [Ben, Theorem 1.1], we have \( s > 1 \) and \( \lambda_1 - \lambda_2 \in \{1,2\} \). In particular, \( n \geq \lambda_1 + \lambda_2 \geq 2\lambda_1 - 2 \), i.e. \( \lambda_1 \leq (n + 2)/2 \). Now

\[
\dim W \geq 2^{\frac{n-\lambda_1}{2}} \geq 2^{\frac{n-(n+2)/2}{2}} = 2^{\frac{n-2}{4}},
\]

thanks to [GLT, Theorem 5.1]. \( \square \)

We will also need the following branching result which is of interest in its own right:

**Proposition 4.2.** Let \( p = 2 \), and \( \lambda = (\lambda_1 > \ldots > \lambda_s > 0) \neq (n) \) be a non-trivial 2-regular partition of \( n \). If \( 2\lambda_1 - n \geq k \geq 3 \) then the restriction of \( D^\lambda \) to a natural subgroup \( S_k \) of \( S_n \) affords both \( 1 = D^{(k)} \) and \( D^{(k-1,1)} \) as composition factors.

**Proof.** We apply induction on \( m := n - \lambda_1 \geq 1 \). If \( m = 1 \), then \( D^\lambda = D^{(n-1,1)} \) is the heart of the natural permutation module, and the statement follows easily. Let \( m \geq 2 \).

Case 1: \( \lambda_1 - \lambda_2 \) is odd. Then, in the terminology of [K2, Definition 0.3], \( 2 \) is a normal index. Let \( j \geq 2 \) be the largest normal index; in particular \( j \) is a good index in the sense of [K2, Definition 0.3] again. Then by [K2, Theorem 0.5], \( D^\mu \) is a simple submodule of \( D^\lambda|_{S_{n-1}} \), where

\[
\mu = \lambda(j) := (\lambda_1, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_s) \vdash (n-1).
\]

Note that \( 2\lambda_1 - (n-1) \geq k + 1 \) and \( (n-1) - \lambda_1 = m - 1 \). Hence we can apply the induction hypothesis to \( D^\mu \) restricted to \( S_k \).

Case 2: \( \lambda_1 - \lambda_2 \) is even. Now \( 1 \) is a normal index. Then by [K2, Theorem 0.4], \( D^\nu \) is a composition factor of \( D^\lambda|_{S_{n-1}} \), where

\[
\nu = \lambda(1) := (\lambda_1 - 1, \lambda_2, \ldots, \lambda_s) \vdash (n-1).
\]

Since \( (\lambda_1 - 1) - \lambda_2 \) is odd, as in Case 1 we now see that \( 2 \) is a normal index for \( \nu \). Let \( j \geq 2 \) be the highest normal index of \( \nu \); in particular \( j \) is a good index. Then again by [K2, Theorem 0.5], \( D^\mu \) is a simple submodule of \( D^\lambda|_{S_{n-2}} \), where

\[
\mu = \nu(j) := (\lambda_1 - 1, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_s) \vdash (n-2).
\]
Note that $2(\lambda_1 - 1) - (n - 2) \geq k$ and $(n - 2) - (\lambda_1 - 1) = m - 1$. Hence we can apply the induction hypothesis to $D^\mu$ restricted to $S_k$. \hfill ∎

Using the Mullineux involution, we prove an analogue of Proposition 4.1 for $p \neq 2$ (certainly, the most interesting case being $p \leq n$):

**Proposition 4.3.** Let $n \geq 5$ and $p \neq 2$.

(i) Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a $p$-regular partition of $n$. Suppose that $\lambda_1 \geq (n+p+2)/2$. Then $D^\lambda$ is irreducible over $A_n$.

(ii) Let $V$ be an irreducible $\mathbb{F}A_n$-module. Suppose that $V$ does not extend to $S_n$. Then $\dim V \geq 2^{(n-p-5)/4}$.

**Proof.** (i) Recalling the definition of the Mullineux map from Section 2, denote the partitions obtained from $\lambda$ by successively removing $p$-rims as $\lambda^{(j)}$, $1 \leq j \leq k$. We prove the statement by induction on $n - \lambda_1$. Since $\lambda_1 \geq (n-\lambda_1) + p + 2$ by the assumption, the first $p$-segment of the $p$-rim of $\lambda$ has length $p$. Assume for a contradiction that $\lambda = \lambda^M$. Then

(4.1) 
$$h_i + \epsilon_i = 2r_i$$

for $1 \leq i \leq k$.

Suppose first that the $p$-rim of $\lambda' := (\lambda_2, \lambda_3, \ldots)$ has at most $p - 1$ nodes. Write $h_1 = p + x$ and $r_1 = 1 + y$, where $0 \leq x \leq p - 1$ and $y$ is the number of rows of $\lambda'$. Then according to (4.1) we have

\begin{align*}
(4.2) 
 p + x &\leq p + x + \epsilon_1 = h_1 + \epsilon_1 = 2r_1 = 2 + 2y.
\end{align*}

Note that $x$ is the length of the $p$-rim of $\lambda'$. Hence $y \leq x$, and so (4.2) yields $x \geq p - 2 > 0$. In turn, this implies that $p|h_1$, whence $\epsilon_1 = 1$ and (4.2) yields $x = y = p - 1$ (as $x \leq p - 1$).

Recall we are assuming that the $p$-rim of $\lambda'$ has at most $p - 1$ nodes. It follows that the $p$-rim of $\lambda'$ has exactly $p - 1$ nodes and $\lambda'$ also has $p - 1$ rows. This can happen only when $\lambda' = (1^{p-1})$, a column of $p - 1$ nodes. In this case, $\lambda^{(1)} = (\lambda_1 - p)$ has one part, which is of length $\geq 2$. Hence the $p$-rim of $\lambda^{(1)}$ is of length $p$ (if $\lambda_1 \geq 2p$), or $z \geq 2$ (where $2p - 1 \geq \lambda_1 = p + z \geq p + 2$). Correspondingly, $r_2 = 1$ and $(h_2, \epsilon_2) = (p, 0)$ or $(z, 1)$. In either case

$$h_2 + \epsilon_2 \geq z + 1 \geq 3 > 2r_2,$$

contrary to (4.1).

Assume now that the $p$-rim of $\lambda'$ has at least $p$ nodes. Then, aside from the first $p$-segment contained in the first row, the $p$-rim of $\lambda$ contains at least $p$ nodes of $\lambda'$. It follows that the condition $\lambda_1 \geq (n - \lambda_1) + p + 2$ also holds for $\lambda^{(1)}$. By the induction hypothesis, $\lambda^{(1)}$ is not equal to its Mullineux dual, i.e. $h_i - r_i + \epsilon_i \neq r_i$ for some $i \geq 2$, again contradicting (4.1).

(ii) By assumption, $W := \text{Ind}_{A_n}^{S_n}(V)$ is an irreducible $\mathbb{F}S_n$-module and $\dim W = 2 \dim V$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be the $p$-regular partition of $n$ corresponding to $W$. Since $W$ is reducible over $A_n$, $\lambda_1 \leq (n + p + 1)/2$ by (i). It now follows by [GLT, Theorem 5.1] that

$$\dim W \geq 2^{n-\lambda_1}/2 \geq 2^{n-(n-(p+1)/2)/2} = 2^{n-p-1}/4.$$
which implies the result.

Here is another version of Proposition 4.3:

**Proposition 4.4.** Let \( p > 2, \ n \geq 5, \) and \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a \( p \)-regular partition of \( n. \) Suppose that there is some \( s \geq 1 \) such that

\[
\lambda_1 - \lambda_2 \geq \lambda_2 - \lambda_3 \geq \ldots \geq \lambda_s - \lambda_{s+1} \geq p
\]

and

\[
\sum_{i=1}^{s} \left( \frac{\lambda_i - \lambda_{i+1}}{p} \right) > \frac{n}{2p-1}.
\]

Then \( D^\lambda \) is irreducible over \( A_n. \) In particular, if

\[
\lambda_1 \geq \lambda_2 + p \left[ \frac{n+1}{2p-1} \right],
\]

then \( D^\lambda \) is irreducible over \( A_n. \)

**Proof.** Assume that \( D^\lambda \) is reducible over \( A_n. \) Then \( \lambda = \lambda^M \) and so

(4.3)

\[
\sum_{i=1}^{k} (h_i - 2r_i + \epsilon_i) = 0.
\]

We will estimate \( h_1 - 2r_1 + \epsilon_1 \) by going down the \( p \)-segments of the \( p \)-rim of \( \lambda. \) Since \( \lambda_1 - \lambda_2 \geq p, \) the first \( p \)-segment consists of \( p \) nodes of the first row and so contributes \( p-2 \) to \( h_1 - 2r_1 + \epsilon_1. \) More generally, any horizontal \( p \)-segment of length \( p \) contributes \( p-2 \) to \( h_1 - 2r_1 + \epsilon_1. \) On the other hand, since \( \lambda \) is \( p \)-regular, any non-horizontal \( p \)-segment of length \( p \) has height \( \leq (p-1) \) and so it contributes at least \( p - 2(p-1) = 2 - p \) to \( h_1 - 2r_1 + \epsilon_1. \) Suppose the \( p \)-rim also has a \( p \)-segment of length \( j \) less than \( p. \) Then it must be the last \( p \)-segment, and \( \epsilon_1 = 1. \) So the contribution of this \( p \)-segment to \( h_1 - 2r_1 + \epsilon_1 \) is \( \geq j - 2j + 1 = 1 - j. \)

As the \( p \)-rims are removed in succession, let \( a \) be the total number of horizontal \( p \)-segments of length \( p, \) \( b_p \) be the total number of non-horizontal \( p \)-segments of length \( p, \) and \( b_j \) be the total number of \( p \)-segments of length \( 1 \leq j < p, \) so that \( n = pa + \sum_{j=1}^{p} jb_j. \) Applying the above arguments to all successive \( p \)-rims of \( \lambda \) we have that

\[
\sum_{i=1}^{k} (h_i - 2r_i + \epsilon_i) \geq (p-2)a - (p-2)b_p - \sum_{j=1}^{p-1} (j-1)b_j
\]

\[
\geq (2p-2)a + (2b_p + \sum_{j=1}^{p-1} b_j) - (pa + \sum_{j=1}^{p} jb_j)
\]

\[
\geq (2p-2)a + (2b_p + \sum_{j=1}^{p-1} b_j) - n.
\]
Observe that

\[ 2b_p + \sum_{j=1}^{p-1} b_j \geq \frac{\sum_{j=1}^{p} j b_j}{p-1} = \frac{n - pa}{p-1}. \]

Under the hypothesis, we can find an integer \( t \geq (n + 1)/(2p - 1) \) such that

\[ \sum_{i=1}^{s} \left( \frac{\lambda_i - \lambda_{i+1}}{p} \right) \geq t. \]

Now observe that at least \( t \) horizontal \( p \)-segments from the first \( s \) rows of \( \lambda \) belong to these successive \( p \)-rims. Thus \( a \geq t \geq (n + 1)/(2p - 1) \). Hence,

\[ \sum_{i=1}^{k} (h_i - 2r_i + \epsilon_i) \geq (2p - 2)a + \frac{n - pa}{p-1} - n \geq \frac{2p - 2}{2p - 1}((2p - 1)a - n) > 0, \]

contradicting (4.3). \( \square \)

5. Structure of permutation modules

Throughout the section \( n \geq 6 \) is an even integer and \( p = 2 \).

We will study permutation modules \( M_r \), mainly for \( r = 1, 2, 3 \). For \( n \geq 2r \), let \( S_r \subset M_r \) denote the Specht module \( S^{(n-r,r)} \) and (assuming \( n > 2r \)) let \( D_r = D^{(n-r,r)} \) be the unique simple quotient of \( S_r \). Let \( T_r \subset M_r \) be the sum of all \( r \)-element sets. Let \( \eta_{r,s} : M_r \to M_s \)

denote the incidence homomorphism sending an \( r \)-set to the sum of \( s \)-sets incident with (i.e. containing or contained in) it. By [J2, Corollary 17.18],

\[ (5.1) \quad S_r = \bigcap_{t=0}^{r-1} \ker \eta_{r,t}. \]

We denote by \( M'_r \) the augmentation module, i.e. the submodule \( \ker \eta_{r,0} \) of \( M_r \) (spanned by differences of pairs of basis elements).

The space \( M_r \) has a natural bilinear form \( \langle \cdot, \cdot \rangle_r \), with respect to which the standard basis is orthonormal. If we identify \( M_r \) and \( M_s \) with their dual spaces, using the corresponding bilinear forms, then \( \eta_{s,r} \) is the dual map of \( \eta_{r,s} \). In particular, \( \eta_{s,r} \) is injective iff \( \eta_{r,s} \) is surjective and vice versa. Also \( \operatorname{Im} \eta_{s,r} \cong \operatorname{Im} \eta_{r,s} \) as \( F \text{S}_n \)-modules. We have

\[ (5.2) \quad \langle x, \eta_{r,s}(y) \rangle_s = \langle \eta_{s,r}(x), y \rangle_r \quad (x \in M_s, y \in M_r). \]

The ranks of the maps \( \eta_{r,s} \) are given in [Wil]. We state the special cases that we need of this general result.

**Lemma 5.1.** For \( r \leq \min\{s, n-s\} \) we have

\[ \text{rank}_F \eta_{r,s} = \sum_{1 \leq i \leq r, \frac{n}{i} \text{ is odd}} \left( \binom{n}{i} - \binom{n}{i-1} \right). \]

In particular, \( \eta_{1,3} \) is injective, and

\[ \text{rank}_F \eta_{1,2} = n - 1 = \dim S_1, \quad \text{rank}_F \eta_{2,3} = 1 + \frac{n(n-3)}{2} = 1 + \dim S_2. \]
Lemma 5.2. \( M_1 \) is a uniserial \( \mathbb{F}S_n \)-module with socle layers \( \mathbb{F} \cdot T_1 \cong 1, D_1, 1 \).

**Proof.** This is well known, see e.g. [J2, Example 5.1]. \( \square \)

Let the \( \mathbb{F}S_n \)-module \( Q \) be defined by the short exact sequence

\[
0 \to \mathbb{F} \cdot T_1 \to M_1 \to Q \to 0.
\]

(5.3) The composition factors are given by [J1].

In fact, \( Q \cong S_1^* \).

The following lemma can be deduced from [MO, Theorem (1.1)], but we give an independent proof for the reader’s convenience:

Lemma 5.3. As \( \mathbb{F}S_n \)-modules, \( \text{Im} \eta_{1,2} \cong Q \) and \( M_2 \) has the following structure:

(i) If \( n \equiv 0 \pmod{4} \) then the composition factors of \( M_2 \) are \( 1 \) (twice), \( D_1 \) (twice) and \( D_2 \) (once).

(ii) If \( n \equiv 2 \pmod{4} \) then \( M_2 \cong \mathbb{F} \cdot T_2 \oplus M_2' \), and \( M_2' \) is uniserial with socle layers \( D_1, 1, D_2, 1, D_1 \).

**Proof.** The dimension of \( \text{Im} \eta_{1,2} \) is given by Lemma 5.1, from which we see that it is isomorphic to \( Q \). The composition factors are given by [J1].

(i) The second statement in (i) now follows using the facts that the dual Specht module \( S_2^* \) is a quotient of \( M_2 \) and \( S_2^* \) is uniserial with socle layers \( D_2, D_1 \).

(ii) The submodule \( \mathbb{F} \cdot T_2 \) is a direct summand of \( M_2 \) because \( \binom{n}{2} \) is odd. Then \( M_2' = \text{Ker} \eta_{2,0} \) is the complementary summand. The composition factors of \( M_2' \) are \( D_1, 1, D_2, 1, D_1 \). Also we have \( S_2 \subseteq M_2' \). The composition factors of \( S_2 \) are \( D_2, D_1, 1 \) and \( S_2 \) has a simple head isomorphic to \( D_2 \). Since \( M_2^{S_2} = 0 \), it follows that \( S_2 \) must be uniserial with socle layers \( D_1, 1, D_2 \). The uniseriality of \( M_2' \) and its socle layers now follow from the self-duality of \( M_2' \). \( \square \)

Lemma 5.4. The \( \mathbb{F}S_n \)-module \( M_3 \) has the following structure.

(i) If \( n \equiv 0 \pmod{4} \), then \( M_3 \cong M_1 \oplus U \), where \( U \) is uniserial with socle layers \( D_2, D_1, D_3, 1, D_2 \).

(ii) If \( n \equiv 2 \pmod{4} \) then \( \text{Im} \eta_{2,3} \) is uniserial with socle layers \( 1, D_2, 1, D_1 \). The composition factors of \( M_3 \) are \( 1 \) (with multiplicity 4), \( D_1 \) (twice), \( D_2 \) (twice) and \( D_3 \) (once).

**Proof.** The composition factors are given by [J1]. The dimensions of the images of the incidence maps \( \eta_{r,s} \) are given by Lemma 5.1.

(i) A simple computation shows that \( \eta_{3,1} \circ \eta_{1,3} = 1_{M_1} \), so \( \eta_{1,3} \) is a split injection of \( M_1 \) into \( M_3 \). In fact, by computing bilinear forms on the images of basis elements, \( \eta_{1,3} \) is seen to be an isometry. So \( M_3 = \text{Im} \eta_{1,3} \oplus U \), with \( U = (\text{Im} \eta_{1,3})^\perp = \text{Ker} \eta_{3,1} \), where the last equality is by (5.2). The module \( U \) is a self-dual module and its composition factors are \( D_1 \) (twice), \( D_2 \) (twice) and \( D_3 \), and \( U \) contains \( S_3 \). The structure of \( U \) will follow from its self-duality if we prove that \( S_3 \) is uniserial with socle layers \( D_2, D_1, D_3 \). Since we know that the head of \( S_3 \) is isomorphic to \( D_3 \), it suffices to show that \( D_1 \) is not a submodule of \( S_3 \). If it were, then we would have \( \text{Hom}_{\mathbb{F}S_n}(D_1, M_3) \neq 0 \), whence \( \text{Hom}_{\mathbb{F}S_n}(M_3, D_1) \neq 0 \), i.e. the fixed point subspace \( D_1^{S_3} \) is non-trivial, which is easily checked to be false.
(ii) We have $\text{Im} \eta_{1,2} \cong Q$. Also, it is easy to see that $\eta_{2,3} \circ \eta_{1,2} = 0$, whence by dimensions using Lemma 5.1, we conclude that $\text{Im} \eta_{1,2} = \text{Ker} \eta_{2,3}$. Moreover, $\text{Im} \eta_{1,2} \cong Q$ by Lemma 5.3. Since $\text{Im} \eta_{1,2} \not\subseteq M'_2$, the structure of $Q$ implies that $\text{Ker} \eta_{2,3} \cap M'_2$ must be zero or isomorphic to $D_1$. Since we know the rank of $\eta_{2,3}$, we see that the latter holds. Thus from the structure of $M'_2$, we see that $\eta_{2,3}(M'_2) \cong M'_2/\text{soc}(M'_2)$. By dimensions, we see that $\eta_{2,3}(M'_2) = \text{Im} \eta_{2,3}$, so $\text{Im} \eta_{2,3}$ is uniserial with the socle layers as stated.

It remains to show that $\text{soc}(M_3) = F \cdot T_3$. For this, it suffices to prove that $M_3$ has no submodule isomorphic to $D_1$, $D_2$ or $D_3$. For $D_1$ we explicitly check as in (i) that $D_1^{S_3-S_3} = 0$. The unique $D_3$ composition factor of $M_3$ is the head of $S_3$, so since $S_3$ is not simple, it follows that $M_3$ has no submodule isomorphic to $D_3$. Finally, we consider $D_2$.

By the first part of (ii), $\text{Im} \eta_{2,3}$ has one composition factor $D_2$ as its second socle layer, so it suffices to show that $M_3/\text{Im} \eta_{2,3}$, which has a single $D_2$ composition factor, has no submodule isomorphic to $D_2$. However, $M_3/\text{Im} \eta_{2,3}$ has $D_3$ as a composition factor, so maps surjectively onto $S_3^*$. By [J1], $D_2$ is a composition factor of $S_3^*$, so $S_3^*$ has a simple socle isomorphic to $D_3$. So $D_2$ is not a submodule of $M_3$.

Figures 1 and 2 below are given for the reader’s convenience, but they will not be used in proofs. The pictures give partial information on submodule structure of the permutation modules $M_2$ and $M_3$. The edges indicate the existence of uniserial subquotients.

**Figure 1.** Submodule structures for $n \equiv 0 \pmod{4}$

**Figure 2.** Submodule structures for $n \equiv 2 \pmod{4}$

**Lemma 5.5.** We have:
(i) \( \text{Im} \eta_{1,3} \) is the unique submodule of \( M_3 \) that is isomorphic to \( M_1 \) as \( F\mathfrak{S}_n \)-modules.
(ii) \( \text{Ker} \eta_{3,1} \) is the unique submodule \( N \) of \( M_3 \) such that \( M_3/N \cong M_1 \) as \( F\mathfrak{S}_n \)-modules.

**Proof.** Part (ii) follows from (i) by the duality of \( \eta_{1,3} \) and \( \eta_{3,1} \). For (i), note that \( \dim \text{Hom}_{F\mathfrak{S}_n}(M_1, M_3) = 2 \). The map \( \eta_{1,3} \) and the map \( \beta \) sending each 1-set to \( T_3 \) form a basis of this Hom-space. Now \( \eta_{1,3} \) is injective since \( \text{soc}(M_1) \) is spanned by \( T_1 \) and \( \eta_{1,3}(T_1) = 3T_3 = T_3 \neq 0 \). Also we have \( \text{Im} \beta \subset \text{Im} \eta_{1,3} \), so \( \text{Im} \eta_{1,3} \) is the unique submodule of \( M_3 \) isomorphic to \( M_1 \).

In the following two lemmas, \( N \) denotes the submodule of \( M_3 \) specified in Lemma 5.5(ii).

**Lemma 5.6.** Let \( n \equiv 0 \pmod{4} \). Then \( \text{Ker} \eta_{3,2} \cap N = \text{soc}^3(N) \), and

\[
N/\text{soc}^3(N) \cong \eta_{3,2} \cap M_2' = S_2.
\]

**Proof.** By Lemma 5.4, \( N/\text{soc}^3(N) \) is uniserial with socle \( D_1 \) and head \( D_2 \). Since \( M_2 \) has no composition factor isomorphic to \( D_3 \), we have \( \text{soc}^3(N) \subseteq \text{Ker} \eta_{3,2} \).

We claim that the induced map \( N/\text{soc}^3(N) \to M_2 \) is injective. If not, its image is either zero or isomorphic to \( D_2 \). The latter is impossible since \( M_2 \) has no submodule isomorphic to \( D_2 \). The former is also impossible since it forces the rank of \( \eta_{3,2} \) to be at most \( \dim M_1 = n \), contrary to Lemma 5.1, which gives the actual rank as \( 1 + \dim S_2 \).

Thus the map \( \eta_{3,2} \) induces an isomorphism of \( N/\text{soc}^3(N) \) with a submodule \( \eta_{3,2}(N) \subseteq M_2 \). Since \( N \subseteq M_3 \) and \( \eta_{3,0} \circ \eta_{3,2} = \eta_{3,0} \), we have \( \eta_{3,2}(N) \subseteq M_2' \). Comparing the dimensions, we see that \( \eta_{3,2}(N) = \text{Im} \eta_{3,2} \cap M_2' \). This submodule of \( M_2 \) has the same dimension and the same composition factors as \( S_2 \). Since any submodule of \( M_2 \) with \( D_2 \) as a composition factor must contain \( S_2 \), we now conclude that \( \eta_{3,2}(N) = S_2 \).

**Lemma 5.7.** Let \( n \equiv 2 \pmod{4} \), \( W = \text{Im} \eta_{3,2} \subseteq M_3 \), and \( Y := M_2' \subseteq M_2 \). Then:

(i) \( \text{soc}^2(W) \subseteq N \) and \( \text{soc}^2(W) \) is uniserial with socle layers \( 1, D_2 \).
(ii) \( \text{Ker}(\eta_{2,3}|_Y) = \text{soc}(Y) \), and \( Y/\text{soc}(Y) \cong \eta_{2,3}(Y) = W \).
(iii) \( N \cap W = N \cap \text{soc}^3(W) = \text{soc}^2(W) \), \( M_3/(N + \text{soc}^3(W)) \cong Q \).
(iv) We have \( M_1 \cong \text{Im} \eta_{1,3} \subseteq N \), \( \text{Im} \eta_{1,3} \cap W = F \cdot T_3 \), and the submodule

\[
Q' := (\text{Im} \eta_{1,3} + \text{soc}^2(W))/\text{soc}^2(W) \subseteq N'/:= N/\text{soc}^2(W)
\]

is isomorphic to \( Q \). Moreover, \( N' := N'/Q' \) is uniserial with socle layers \( D_3, D_2 \).
(v) We have \( \text{soc}^2(W) \subseteq S_3 \subseteq N \), the submodule \( D' := S_3/\text{soc}^2(W) \subseteq N' \) is isomorphic to \( D_3 \). Moreover, \( \text{soc}(N') = D' \oplus \text{soc}(Q') \cong D_3 \oplus D_1 \) and \( N'/D' \cong N/S_3 \cong \eta_{3,2}(N) \cong S_2 \).

**Proof.** (i) The structure of \( W \) is given in Lemma 5.4, which implies that \( \text{soc}^2(W) \) is uniserial with socle layers \( 1, D_2 \). Any nonzero quotient of \( \text{soc}^2(W) \) has \( D_2 \) as its head. But \( M_3/N \cong M_1 \) and \( D_2 \) is not a composition factor of \( M_1 \). So we see that \( \text{soc}^2(W) \subseteq N \).

(ii) Recall that \( M_2 \cong Y \oplus 1 \) and \( Y \) is uniserial with socle layers \( D_1, 1, D_2, 1, D_1 \) by Lemma 5.3. Next, \( \eta_{2,3}(Y) \) has codimension \( \leq 1 \) in \( W = \text{Im} \eta_{2,3} \). Inspecting the submodule structures of \( W \) and \( Y \) given in Lemma 5.4(ii) and Lemma 5.3, we see that \( \eta_{2,3}(Y) = W \) and \( \text{Ker}(\eta_{2,3}|_Y) = \text{soc}(Y) \).

(iii) Note that \( \eta_{3,1} \circ \eta_{3,2} \neq 0 \), so \( W \not\subseteq \text{Ker} \eta_{3,1} = N \). Moreover, \( \text{soc}^3(W) \not\subseteq N \), since otherwise \( W \cap N = \text{soc}^3(W) \), and \( M_3/N \cong M_1 \) contains \( (W + N)/N \cong W/\text{soc}^3(W) \cong D_1 \).
as a submodule, which is a contradiction. As $W$ is uniserial and $\text{soc}^2(W) \subseteq N$, it now follows that $N \cap W = N \cap \text{soc}^3(W) = \text{soc}^2(W)$. Now, $M_3/(N + \text{soc}^3(W))$ is a quotient of $M_3/N \cong M_1$ by

$$(N + \text{soc}^3(W))/N \cong \text{soc}^3(W)/(N \cap \text{soc}^3(W)) = \text{soc}^3(W)/\text{soc}^2(W) \cong 1,$$

so this quotient must be isomorphic to $Q$.

(iii) We know that $N'$ has composition factors $D_3, D_2, D_1, \text{ and } 1$. It is easy to check that $\eta_{3,1} \circ \eta_{1,3} = 0$, so $\text{Im} \eta_{1,3} \subseteq N = \text{Ker} \eta_{3,1}$. By Lemma 5.5(ii), $\text{Im} \eta_{1,3} \cong M_1$. Using the submodule structure of $W$ and $\text{Im} \eta_{1,3}$ and the fact that $\text{soc}(M_3) = \mathbb{F} \cdot T_3$, we conclude that $\text{Im} \eta_{1,3} \cap W = \mathbb{F} \cdot T_3$. Therefore the image $Q'$ of $\text{Im} \eta_{1,3}$ in $N'$ is isomorphic to $Q$.

Now we know that $N''$ has composition factors $D_3$ and $D_2$. Note that $S_3^*$ is a quotient of $M_3$, so some submodule $S'$ of $S_3^*$ is a quotient of $N$. Also, $D_3$ is the head of $S_3$ and the socle of $S_3^*$. But $D_3$ is not a composition factor of $\text{soc}^2(W) + \text{Im} \eta_{1,3}$. It follows that $S_3^*$ is a quotient of $M_3/(\text{soc}^2(W) + \text{Im} \eta_{1,3})$, whence $S'$ is a quotient of $N/(\text{soc}^2(W) + \text{Im} \eta_{1,3}) = N''$ and also of $N'$. By [31] the composition factors of $S_3$ (and $S_3^*$) are $D_3, D_2, 1$. Among these, only $1$ is a composition factor of $M_1 \cong M_3/N$, so $S'$ has both $D_3$ and $D_2$ as composition factors. We have shown that $S'$ is a quotient of $N''$ which has exactly two composition factors $D_3$ and $D_2$. It follows that in fact $N'' \cong S'$. In this case, $\text{soc}(N'') \cong D_3$ since $\text{soc}(S_3^*) \cong D_3$ is simple.

(v) Recall that $S_3 \subset M_3$ and $\text{head}(S_3) \cong D_3$ is not a composition factor of $M_1 \cong M_3/N$. It follows that $S_3 \subset N$ and, since $D_1$ is not a composition factor of $S_3$, the image $D'$ of $S_3$ in $N'$ intersects $Q'$ trivially. The aforementioned structure of $N''$ implies that $S_3$ has no quotient isomorphic to $N''$. Therefore, under the natural projection $N' \to N''$, $D'$ projects onto a module isomorphic to $D_3$, or $0$. The latter cannot happen since $D_3$ is not a composition factor of $\text{soc}^2(W) + \text{Im} \eta_{1,3}$. So $D'$ projects onto a module isomorphic to $D_3$. This implies that the composition factors of $D' + Q'$ are $D_3, D_1, 1$. Since $D' \cap Q' = 0$ and $Q' \cong Q$, it follows that $D' \cong D_3$. We have shown that that $D_3 + D_1 \cong D' \oplus \text{soc}(Q') \cong \text{soc}(N')$. On the other hand, $N'/Q' = N''$, $\text{soc}(N'') \cong D_3$, and $\text{soc}(Q') \cong D_1$. Together these imply that $\text{soc}(N')$ embeds into $D_3 + D_1$. So $D' \oplus \text{soc}(Q') = \text{soc}(N')$.

Let $\pi$ denote the natural projection $N \to N'$. Then we have shown that $\text{Ker}(\pi|_{S_3})$ has two composition factors $D_3$ and $1$. On the other hand, $\text{Ker} \pi = \text{soc}^2(W)$. It follows that $\text{Ker}(\pi|_{S_3}) = \text{soc}^2(W)$ so that in fact $D' = S_3/\text{soc}^2(W)$ and $N'/D' \cong N/S_3$. By (5.1) we have $S_3 = N \cap \text{Ker} \eta_{3,2}$. So $N/S_3 = N/(\text{Ker} \eta_{3,2} \cap N) \cong \eta_{3,2}(N)$ is a submodule of $M_2$ with composition factors $D_1, 1, D_2$, which are precisely the composition factors of $S_2$. Hence, $N'/D' \cong N/S_3 = \eta_{3,2}(N) = S_2$. \[\square\]

6. Main reduction theorem

The following theorem is the main tool for proving reducibility of various restrictions $\text{Res}^N_X$ in the key case $p = 2|n$. Note by Theorem 3.2 that the assumption $d_3(V) > d_1(V)$ is equivalent to the assumption that $V$ is not trivial and not the basic spin module.

**Theorem 6.1.** Let $p = 2|n \geq 6$, $V$ be an irreducible $\mathbb{F}_n$-module satisfying $d_3(V) > d_1(V)$, and $X$ be a subgroup of $S_n$. Let $N$ be the $\mathbb{F}_n$-submodule of $M_3$ specified in Lemma 5.5(ii). Suppose that for any nonzero $\mathbb{F}_n$-quotient $J$ of $N$ we have $J^X \neq 0$ and if $J^S \neq 0$ then $\dim J^X \geq 2$. Then the restriction $\text{Res}^S_X V$ is reducible.
Proof. Set $E := \text{End}_F(V)$ so that $d_r(V) = \dim \text{Hom}_{\mathbb{F}S_n}(M_r, E)$. By Schur’s Lemma, 
$\dim E^{S_n} = \dim \text{Hom}_{\mathbb{F}S_n}(M_1, E) = 1$, so the $\mathbb{F}S_n$-module $E$ contains a unique submodule $E_1 \cong 1$.

Note that $E^X = \text{End}_F(\text{Res}^X_{\mathbb{F}} V)$, so it suffices to prove that $\dim E^X \geq 2$.

By definition of $N$ in Lemma 5.5(ii), we have an exact sequence 
$$0 \to N \to M_3 \to M_1 \to 0.$$ 
Applying $\text{Hom}_{\mathbb{F}S_n}(-, E)$ to this sequence and using the assumption $d_3(V) > d_1(V)$, we conclude that there is some $f \in \text{Hom}_{\mathbb{F}S_n}(M_3, E)$ such that $J := f(N) \neq 0$.

If $J \cap E_1 = 0$, then $E$ contains a submodule isomorphic to $J \oplus E_1$, whence $\dim E^X \geq 2$ as $J^X \neq 0$ by assumption. On the other hand, if $J \cap E_1 \neq 0$ then $J^{S_n} \neq 0$. In this case $\dim E^X \geq \dim J^X \geq 2$ by assumption again. \hfill $\Box$

Our main goal now will be to obtain permutation group theoretic conditions on the subgroup $X$ which guarantee that the assumptions of Theorem 6.1 hold.

To bound dimensions of various fixed point subspaces, we will frequently use the following well-known estimates:

Lemma 6.2. Let $X$ be a group and $U \supseteq V$ be $FX$-modules. Then:

(i) $\dim(U/V)^X - \dim H^1(X, V) \leq \dim U^X - \dim V^X \leq \dim(U/V)^X$.

(ii) If in addition $X$ acts trivially on $V$ and $\text{Hom}(X, F) = 0$, then
$$\dim U^X = \dim V + \dim(U/V)^X.$$

Proof. (i) follows from the exact sequence $0 \to V^X \to U^X \to (U/V)^X \to H^1(X, V)$.

(ii) In this case $H^1(X, V) = \text{Hom}(X, F) = 0$, whence the statement follows from (i). \hfill $\Box$

Let $X \leq S_n$ be any subgroup and $1 \leq r \leq n/2$. We set

$$f_r(X) := \dim(M_r)^X, \quad f_0(X) := 0, \quad e_r(X) := f_r(X) - f_{r-1}(X).$$

Note that $f_r(X)$ is the number of $X$-orbits on $\Omega_r$. As in (5.3) let the $\mathbb{F}S_n$-module $Q$ be defined by the short exact sequence 
$$0 \to \mathbb{F} \cdot T_1 \to M_1 \to Q \to 0.$$ 

When $p = 2|n$ we also put

$$h(X) := \dim H^1(X, Q).$$

For any partition $\lambda \vdash n$, let $\chi^\lambda$ denote the irreducible ordinary $S_n$-character labeled by $\lambda$. It is well known (and follows for instance from the Littlewood-Richardson formula [J2, 16.4]) that the $S_n$-character afforded by the permutation module $\mathbb{C}\Omega_r$ is $\sum_{s=0}^r \chi^{(n-s,s)}$. Denote $\alpha := \chi^{(n-1,1)}$, so that $\alpha + 1_{S_n}$ is the permutation character of $S_n$ acting on $\Omega := \{1, 2, \ldots, n\}$. Applying [GT2, Lemma 3.3], we see that
$$\sum_{s=0}^r \chi^{(n-s,s)} = \begin{cases} S^2(\alpha) & \text{if } r = 2, \\
S^3(\alpha) - \wedge^2(\alpha) - \alpha & \text{if } r = 3, \end{cases}$$
where \( S^k \) denotes the \( k \)th symmetric power and \( \wedge^k \) denotes the \( k \)th exterior power. If we know the restriction \( \alpha_X := \text{Res}^S_X \alpha \) explicitly, we can compute \( f_2(X) \) and \( f_3(X) \) by computing the scalar product of \( X \)-characters as follows:

\[(6.3) \quad f_2(X) = [S^2(\alpha_X), 1_X], \quad f_3(X) = [(S^3(\alpha_X) - \wedge^2(\alpha_X) - \alpha_X, 1_X)].\]

Next we record some elementary observations:

**Lemma 6.3.** Let \( X \leq S_n \) be a transitive subgroup. Then:

(i) \( f_2(X) = 1 \) if and only if \( X \) is 2-homogeneous.

(ii) Suppose \( |X| \) is even. Then \( f_2(X) = 1 \) if and only if \( X \) is 2-transitive.

(iii) \( f_2(X) \leq 2 \) if \( X \) is a rank \( \leq 3 \) subgroup of \( S_n \).

**Proof.** (i) is obvious: \( X \) is 2-homogeneous precisely when it acts transitively on \( \Omega_2 \). For (ii), observe that \( X \) contains an involution \( t \), and so we can find \( x, y \in \Omega \) interchanged by \( t \). It follows that \( X \) is 2-homogeneous precisely when it is 2-transitive.

For (iii), note that if \( X \) is a rank 2 subgroup \( f_2(X) = 1 \) by (ii). If \( X \) is a rank 3 subgroup, then the point stabilizer of \( x \in \Omega \) in \( X \) has two orbits on \( \Omega \setminus \{x\} \), whence \( X \) has at most two orbits on \( \Omega_2 \).

Note that since \( p = 2 \) in the following proposition, the condition \( X = O^2(X) \) is equivalent to the condition \( \text{Hom}(X, \mathbb{F}) = 0 \) from Lemma 6.2. In many applications \( X \) will be perfect, in which case this assumption of course holds.

**Proposition 6.4.** Let \( p = 2|n \geq 6, N \) the \( \mathbb{F}S_n \)-submodule of \( M_3 \) specified in Lemma 5.5(ii), and let \( J \) be any nonzero \( \mathbb{F}S_n \)-quotient of \( N \). Suppose that \( X = O^2(X) \leq S_n \) is a subgroup such that

\[ f_1(X) = 1, \quad e_3(X) \geq h(X) + 1, \quad \text{and either} \quad f_2(X) \geq 3 \quad \text{or} \quad S_2^X \neq 0. \]

Then \( J^X \neq 0. \) Moreover, if \( J^{S_n} \neq 0 \), then \( \dim J^X \geq 2. \)

**Proof.** We write \( f_r \) for \( f_r(X) \), \( e_r \) for \( e_r(X) \), and \( h \) for \( h(X) \). Note that \( Q^X = 0 \) and \( D_1^X = 0 \) since \( f_1 = 1 \) and \( X = O^2(X) \). Combining this with the structure of \( M_2 \) given in Lemma 5.3 and applying Lemma 6.2, we see that

\[(6.4) \quad f_2 \geq 1 + \dim D_2^X - \dim H^1(X, Q) = 1 + \dim D_2^X - h.\]

Note that \( D_2^X \neq 0. \) Indeed, if \( f_2 \geq 3 \), this follows by considering composition factors of \( M_2 \) described in Lemma 5.3 and using \( Q^X = D_1^X = 0. \) On the other hand, if \( S_2^X \neq 0 \), this follows by considering composition factors of \( S_2 \) using \( D_1^X = 0. \)

**Case 1:** \( n \equiv 0 \pmod{4} \). Then \( N \) is uniserial by Lemma 5.4, and we are going to check that \( J^X \neq 0 \) for each of its five non-trivial quotients \( J \). This is all we have to do, since \( 1 \) is not a composition factor of \( N \), and so we never have \( J^{S_n} \neq 0. \)

Note that \( \text{soc}(N) \cong D_2 \), so \( N^X \supseteq D_2^X \neq 0. \) By assumption, we have \( f_3 \geq f_2 + h + 1 \), so (6.4) implies \( f_3 \geq \dim D_2^X + 2. \) Since \( M_3 = M_1 \oplus N \) and \( f_1 = 1 \), it follows that \( (N/\text{soc}(N))^X \neq 0. \) Then since \( D_1^X = 0 \), we also get \( (N/\text{soc}^2(N))^X \neq 0. \) Next, \( N/\text{soc}^3(N) \cong S_2 \) by Lemma 5.6. If \( S_2^X \neq 0 \), we are done. Otherwise, the conditions \( f_2 \geq 3 \) and \( D_1^X = 0 \) imply that \( (N/\text{soc}^3(N))^X \neq 0. \) Finally, \( N/\text{rad}(N) \cong D_2 \) and we already have \( D_2^X \neq 0. \).
Case 2: \( n \equiv 0 \pmod{4} \). We are going to use the notation of Lemma 5.7.

Step 1: we prove that \( J^X \neq 0 \) for any nonzero quotient \( J = N/K \) of \( N \).

By Lemma 5.7(i), we have the submodule \( \text{soc}^2(W) \subseteq N \) which is uniserial with socle layers \( 1, D_2 \). So any nonzero quotient of \( \text{soc}^2(W) \) either contains \( 1 \) or is isomorphic to \( D_2 \), hence it contains nonzero \( X \)-fixed points. In particular, \( (N/K)^X > 0 \) if \( \text{soc}^2(W) \nsubseteq K \), and we may now assume that \( \text{soc}^2(W) \subseteq K \). In other words, we are reduced to showing that \( X \) has nonzero fixed points on every nonzero \( \mathbb{F}S_n \)-quotient of \( N'' = N/\text{soc}^2(W) \).

Recall that \( M_2 \cong Y \oplus 1 \), see Lemma 5.7. In particular, \( \dim Y^X = f_2 - 1 \), and so \( \dim(\text{soc}(Y))^X = f_2 - 1 \), since \( X \) has no fixed points on \( U/\text{soc}^4(U) \cong D_1 \). Applying Lemma 6.2(i) to the exact sequence

\[
0 \rightarrow \text{soc}^2(Y) \rightarrow \text{soc}^4(Y) \rightarrow \text{soc}^4(Y)/\text{soc}^2(Y) \rightarrow 0
\]

with \( (\text{soc}^2(Y))^X \cong Q^X = 0 \), we see that

\[
(6.5) \quad \dim(\text{soc}^4(Y)/\text{soc}^2(Y))^X \leq f_2 + h - 1.
\]

By Lemma 5.7(ii), we have \( n_{2,3}(Y) = W \cong Y/\text{soc}(Y) \). So \( \text{soc}^3(W) \cong \text{soc}^4(Y)/\text{soc}(Y) \) is an extension of \( \text{soc}^4(Y)/\text{soc}^2(Y) \) by \( \text{soc}^2(Y)/\text{soc}(Y) \cong 1 \). Together with (6.5) and Lemma 6.2, this implies that

\[
(6.6) \quad \dim(\text{soc}^3(W))^X \leq f_2 + h.
\]

Since \( Q^X = 0 \), Lemma 5.7(iii) yields

\[
(6.7) \quad \dim(N + \text{soc}^3(W))^X = f_3.
\]

Moreover, by the same lemma, we have

\[
N' = N/\text{soc}^2(W) = N/(N \cap \text{soc}^3(W)) \cong (N + \text{soc}^3(W))/\text{soc}^3(W).
\]

Since \( f_3 - f_2 = e_3 \geq h + 1 \) we deduce from (6.6) and (6.7) that \( (N')^X \neq 0 \).

Now we apply Lemma 5.7(iv). Since \( N'' = N'/Q' \) and \( Q'^X \cong Q^X = 0 \), we have that \( (N')^X \neq 0 \) implies \( N''^X \neq 0 \). Recalling that \( D_2^X \neq 0 \) and \( \text{head}(N'') \cong D_2 \), we have now shown that \( X \) has nonzero fixed points on every nonzero quotient of \( N'' \).

It remains to consider quotients of \( N' \) by nonzero submodules \( R' \) which do not contain \( Q' \). Since \( Q' \) has a simple socle and \( Q'/\text{soc}(Q') \cong 1 \), we only need to consider \( R' \) such that \( R' \cap Q' = 0 \). We have shown in Lemma 5.7(v) that \( \text{soc}(N') = D' \oplus \text{soc}(Q') \). Hence the condition \( R' \cap Q' = 0 \) implies that \( R' \supseteq D' \). So we must show that \( X \) has nonzero fixed points on every nonzero quotient of the \( \mathbb{F}S_n \)-module \( N'/D' \cong S_2 \), see Lemma 5.7(v) again. By assumption, \( S_2^X \neq 0 \) unless \( f_2 \geq 3 \), in which case \( D_1^X = 0 \) implies \( S_2^X = 0 \). The only proper quotients of \( S_2 \) have socles \( 1 \) and \( D_2 \) and we have seen already that \( X \) has non-trivial fixed points on these simple modules.

Step 2: we assume that \( J = N/K \) contains a trivial \( S_n \)-submodule \( E \cong 1 \) and prove that \( \dim J^X \geq 2 \). We again use the notation of Lemma 5.7. Since we have shown that \( D_2^X (N'')^X \neq 0 \), by Lemma 6.2(ii) it suffices to show that \( J \) contains an \( \mathbb{F}S_n \)-submodule \( L \) where \( L/E \) is isomorphic to \( D_2 \) or \( N'' \). This is obvious if \( K = 0 \). So we may assume that \( K \supseteq \text{soc}(W) = \text{soc}(N) \cong 1 \).

If \( K \cap \text{soc}^2(W) = \text{soc}(W) \), then \( J \) contains \( \text{soc}^2(W)/\text{soc}(W) \cong D_2 \), and we can take \( L \cong D_2 \oplus E \). Thus we may assume that \( K \supseteq \text{soc}^2(W) \) and so \( J \) can be regarded as
a quotient $N'/K'$ of $N' = N/\text{soc}^2(W)$. Let $P'$ be the preimage of $E$ in $N'$ so that $E \cong P'/K'$. Since $1$ is not a composition factor of $N'/Q'$, we see that $Q' + K' \supseteq P'$, whence $Q'/(Q' \cap K') \cong (Q' + K')/K'$ contains $E = P'/K'$. But $Q'$ has socle layers $D_1, 1$, so $Q' \cap K' = \text{soc}(Q')$ and $P' = Q' + K'$. Recall that $N'/Q' = N''$ is uniserial with socle layers $D_3, D_2$. So if $P' \neq N'$ then $J \cong N'/K'$ is isomorphic to an extension of the nonzero quotient $N'/P'$ of $N'/Q' \cong N''$ by $P'/K' = E$, and we may take $L = J$.

Assume $P' = N'$, so that $N'/K' \cong 1$. Since $D' \cong D_3$ is simple, it follows that $\Delta \subset K'$. Now we see that $N'/K' \cong 1$ is a quotient of $N'/D'$ which is isomorphic to $S_2$ by Lemma 5.7(v), a contradiction since $\text{head}(S_2) \cong D_2$. □

Now we can prove the main result of this section:

**Theorem 6.5.** Let $p = 2|n \geq 6$, $V$ be an irreducible $FS_n$-module satisfying $d_3(V) > d_1(V)$, and $X = O^2(X) \leq S_n$ be a subgroup such that

$$f_1(X) = 1, \quad f_2(X) \geq h(X) + 1, \quad \text{and either} \quad f_2(X) \geq 3 \quad \text{or} \quad S_2^X \neq 0.$$  

Then the restriction $\text{Res}_{X}^{S_n} V$ is reducible.

**Proof.** Apply Theorem 6.1 and Proposition 6.4. □

Next we show that the condition $S_2^X \neq 0$ is a fairly mild condition which always holds for transitive permutation groups of even degree $n$ of rank $\geq 3$:

**Lemma 6.6.** Let $2|n \geq 6$ and let $G \leq S_n$ be a transitive permutation group of rank $\geq 3$. Then $S_2^G \neq 0$.

**Proof.** Since $2|n$, $|G|$ is even and so $G$ contains an involution $j$. Choose $x_0 \in \Omega = \{1, 2, \ldots, n\}$ not fixed by $j$. By assumption, the stabilizer $G_{x_0}$ of $x_0$ has an orbit $\Delta_1 \supset j(x_0)$ on $\Omega \setminus \{x_0\}$, and $\Delta_2 := (\Omega \setminus \{x_0\}) \setminus \Delta_1 \neq \emptyset$. Let $\Phi_1$ denote the $G$-orbit of the $2$-subset $\{x_0, j(x_0)\}$. Since the $G$-orbit of the ordered pair $(x_0, j(x_0))$ has length $n|\Delta_1|$ and $j$ interchanges $x_0$ and $j(x_0)$, we see that $|\Phi_1| = n|\Delta_1|/2$. Let $\Phi_2 := \Omega_2 \setminus \Phi_1$, where $\Omega_2$ denotes the set of all $2$-subsets $\{x, y\}$ of $\Omega$. Then

$$|\Phi_2| = |\Omega_2| - n|\Delta_1|/2 = n|\Delta_2|/2 > 0.$$  

As $|\Delta_1| + |\Delta_2| = n - 1 \equiv 1 (\text{mod} 2)$, there is (a unique) $k \in \{1, 2\}$ such that $|\Delta_k|$ is even.

Since $\Phi_k$ is a non-empty union of $G$-orbits, it suffices to show that the orbit sum $\hat{\Phi}_k := \sum_{(x,y) \in \Phi_k} \{x,y\} \in S_2$. A standard fact about Specht modules following from (5.1) and (5.2) is that

$$S_2 = \langle T_2 \rangle^\perp \cap \eta_{1,2}(M_1)^\perp,$$

where perpendicularity is with respect to the natural inner product $\langle \cdot , \cdot \rangle_2$ on $M_2$. Now,

$$\langle \hat{\Phi}_k, T_2 \rangle_2 = |\Phi_k| = n|\Delta_k|/2 \equiv 0 (\text{mod} 2).$$  

Next, for any $a \in \Omega \setminus \{x_0\}$, $\{x_0, a\} \in \Phi_1$ if and only if there is some $g \in G$ sending $\{x_0, j(x_0)\}$ to $\{x_0, a\}$. Replacing $g$ by $gj$ if necessary, we get $g \in G_{x_0}$ and $a \in \Delta_1$. It follows that $\{x_0, a\} \in \Phi_k$ if and only if $a \in \Delta_k$, whence

$$\langle \hat{\Phi}_k, \eta_{1,2}(x_0) \rangle_2 = |\Delta_k| \equiv 0 (\text{mod} 2).$$

By transitivity of $G$ on $\Delta$, $\langle \hat{\Phi}_k, \eta_{1,2}(x) \rangle_2 = 0$ for all $x \in \Omega$, and the claim follows. □
7. Special embeddings of $A_m$ into $A_n$

Let $X \cong A_m$ with $m \geq 5$. In this section, we consider two special kinds of embeddings of $X$ into symmetric groups $S_n$. The first arises from the action of $X$ on $k$-subsets of

$$\Delta := \{1, 2, \ldots, m\}$$
with $2 \leq k < m/2$, giving rise to an embedding of $X$ into $S_n$, where $n = \binom{m}{k}$. The second embedding comes from the action on set partitions of $\Delta$ into $b$ subsets of size $a$, where $m = ab$. This gives rise to an embedding into $S_n$, where $n = (ab)!/(a!^b \cdot b!)$.

**Lemma 7.1.** Suppose that $m \geq 8$. Then for any of the two special embeddings, we have that $n \geq m(m-1)/2$.

**Proof.** For the first embedding, observe that the sequence $\binom{m}{k}$ is increasing for $2 \leq k < m/2$, whence $\binom{m}{k} \geq \binom{m}{2}$. For the second embedding, denote $N_{a,b} := (ab)!/(a!^b \cdot b!)$ and observe that

$$\frac{N_{a,b+1}}{N_{a,b}} = \frac{(ab+1)(ab+2)\ldots(ab+a-1)}{(a-1)!} \geq ab + 1 \geq 5$$

as long as $a, b \geq 2$. An induction on $a$ shows that $N_{a,2} > \binom{2a}{2}$ for $a \geq 4$. Now another induction on $b \geq 2$ using (7.1) shows that $N_{a,b} > \binom{ab}{2}$ whenever $a \geq 4$ and $b \geq 2$. Next, an induction on $a$ shows that $N_{a,3} \geq \binom{3a}{2}$ for $a \geq 2$, with equality only when $a = 2$. Now another induction on $b \geq 3$ using (7.1) shows that $N_{a,b} > \binom{ab}{2}$ if $a \geq 2, b \geq 3$, and $ab \geq 8$. \hfill \Box

Recall that $h(X) := \dim H^1(X, Q)$, where the $S_n$-module $Q$ is defined in (5.3).

**Lemma 7.2.** Let $p = 2$. We have:

(i) $\dim H^2(X, \mathbb{F}) = 1$.

(ii) If $n$ is even, then one of the following statements holds:

(a) $\dim H^1(X, M_1) = 1$ and $h(X) \leq 2$.

(b) $4|m, A_m$ embeds into $A_n$ via its action on partitions $(m/2, m/2)$ of $\Delta$, and $\dim H^1(X, M_1) = 2, h(X) \leq 3$.

**Proof.** (i) is a well-known fact about the Schur multiplier of $A_m$ (see e.g. [KIL, Theorem 5.1.4(ii)]).

(ii) By Frobenius reciprocity we have that

$$H^1(X, M_1) \cong H^1(X_1, \mathbb{F}) \cong \text{Hom}(X_1, (\mathbb{F}, +)),$$

where $X_1$ is the stabilizer in $X$ of a point on the set $\Omega$, and $(\mathbb{F}, +)$ is the additive group of the field $\mathbb{F}$. First we consider the case where $X$ is acting on $k$-sets of $\Delta$. Then

$$X_1 = (S_k \times S_{m-k}) \cap A_m \cong (A_k \times A_{m-k}) \cdot 2.$$

Since $p = 2$, we have that $\text{Hom}(A_k, (\mathbb{F}, +)) = 0$ for all $s \geq 1$. Denoting by $C_2$ the group of order $2$, it follows that $\text{Hom}(X_1, (\mathbb{F}, +)) \cong \text{Hom}(C_2, (\mathbb{F}, +))$ is one-dimensional.

Next we consider the case $X$ is acting on set partitions of $\Delta$ into $b \geq 2$ subsets of size $a = m/b \geq 2$. Then

$$X_1 = (S_a \wr S_b) \cap A_m.$$
We may assume that the transposition \((1, 2)\) fixes the set partition fixed by \(X_1\). Then \((1, 2)\) belongs to the base subgroup \(B = S^b_n\), whence \([B : B \cap X_1] = 2\) and \(X_1 \cong (B \cap X_1) \cdot S^b_n\). As mentioned above, \(\text{Hom}(A_\alpha, (\mathbb{F}, +)) = 0\). Hence
\[
\text{Hom}(X_1, (\mathbb{F}, +)) \cong \text{Hom}(X_1/A^b_\alpha, (\mathbb{F}, +)) = \text{Hom}(Y, (\mathbb{F}, +)),
\]
where \(Y = X_1/A^b_\alpha \cong 2^{b-1} \cdot S^b_n\). If \(b \geq 3\), then one can check that \([Y, Y]\) contains the normal subgroup \(2^{b-1}\), and so
\[
\text{Hom}(Y, (\mathbb{F}, +)) \cong \text{Hom}(S_n, (\mathbb{F}, +)) \cong \text{Hom}(C_2, (\mathbb{F}, +)) \cong \mathbb{F}
\]
as \(\text{Hom}(A_\alpha, (\mathbb{F}, +)) = 0\). Assume that \(b = 2\), i.e. \(X_1\) fixes the partition \(\Delta = \{1, 2, \ldots, a\} \cup \{a + 1, \ldots, m\}\). If \(a \geq 3\) is odd, then the permutation
\[
g : (1, a + 1, 2, a + 2)(3, a + 3) \ldots (a, 2a)
\]
belongs to \(X_1\) and \(g^2 = (1, 2)(a + 1, a + 2)\) gives rise to an involution in \(Y\). Thus \(Y\) is cyclic of order 4, and so again \(\text{Hom}(Y, (\mathbb{F}, +)) \cong \mathbb{F}\). If \(a\) is even, then \(Y\) (of order 4) is generated by two involutions \((1, 2)(a + 1, a + 2)\) and \((1, a + 1)(2, a + 2) \ldots (a, 2a)\), whence \(\text{Hom}(Y, (\mathbb{F}, +)) \cong \mathbb{F}^2\). This proves the claims on \(\dim H^1(X, M_1)\) in (ii).

Now the bounds on \(h(X)\) in (ii) follow immediately from the portion
\[
0 = H^1(X, \mathbb{F}) \rightarrow H^1(X, M_1) \rightarrow H^1(X, Q) \rightarrow H^2(X, \mathbb{F})
\]
of the long exact sequence arising from (5.3).

\[\square\]

**Lemma 7.3.** Let \(X\) embed into \(S_n\) via its action on \(k\)-subsets of \(\Delta\) for \(2 \leq k < m/2\). Then \(f_2(X) = k\).

**Proof.** We claim that the orbits of \(X\) or of \(S_m\) on pairs \(\{A, B\}\) of distinct \(k\)-subsets of \(\Delta\) are labeled by \(j := |A \cap B|\) for \(0 \leq j \leq k - 1\), hence \(f_2(X) = k\). Indeed, the claim is obvious for \(S_m\). Since \(A \neq B\), we can find \(i \in A \setminus B\) and \(j \in B \setminus A\). Now the transposition \((i, j)\) fixes the pair \(\{A, B\}\), and so \(S_m\) and \(A_m\) have the same orbits on pairs \(\{A, B\}\). \[\square\]

Next we handle the embedding of \(A_m\) into \(A_n\) via its action on \(2\)-subsets:

**Corollary 7.4.** Let \(p = 2\) and \(m \geq 6\) be such that \(n := \binom{m}{2}\) is even. Let \(X = A_m\) embed into \(A_n\) via its action on \(2\)-subsets of \(\{1, 2, \ldots, m\}\). Suppose that an irreducible \(\mathbb{F}S_n\)-module \(V\) satisfies the condition \(d_3(V) > d_1(V)\). Then \(\text{Res}^S_X V\) is reducible.

**Proof.** By Lemma 7.2(ii), we have that \(h(X) \leq 2\). On the other hand, \(f_1(X) = 1\), and \(f_2(X) = 2\) by Lemma 7.3. Also, \(f_3(X) \geq 5\). Indeed, we can regard the \(\mathbb{F}S_n\)-permutation module \(M_1\) as having a basis consisting of all \(2\)-subsets \(\{i, j\}\) of \(\Delta = \{1, 2, \ldots, m\}\). Then the module \(M_3\) has a basis consisting of unordered triples of distinct pairs, and \(S_m\) has 5 orbits on this set represented by the triples
\[
\begin{align*}
\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}, & \quad \{\{1, 2\}, \{3, 4\}, \{4, 5\}\}, \quad \{\{1, 2\}, \{2, 4\}, \{3, 4\}\} \\
\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}, & \quad \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.
\end{align*}
\]
In particular, \(e_3(X) \geq 3 \geq h(X) + 1\); furthermore, \(S^X_2 \neq 0\) by Lemma 6.6. Hence we are done by Theorem 6.5. \[\square\]
Lemma 7.5. Suppose that $m \geq 11$. Then $e_2(X) \geq 2$, unless $A_m$ embeds in $A_n$ via its action on 2-subsets of $\Delta$, in which case $e_2(X) = 1$.

Proof. Recall that $f_1(X) = 1$ for the special embeddings of $X$ into $S_n$ in question. Now for the action of $X$ on $k$-subsets of $\Delta$ the result follows from Lemma 7.3.

Now let $X$ act on partitions $P = \{P_1, \ldots, P_b\}$ of $\Delta$ into $b$ $a$-subsets $P_1, \ldots, P_b$. We will exhibit at least 3 orbits of $X$ on pairs of partitions $\{P, Q\}$. Note that $\Delta$ admits two partitions with no common subset between them. It follows that, for each $j = 0, 1, \ldots, b - 2$, $\Delta$ admits a pair of partitions $\{P, Q\}$, where $P$ and $Q$ contain exactly $j$ common subsets. Certainly, such pairs with different parameters $j$ belong to different $S_m$-orbits. In particular, we are done if $b \geq 4$.

Suppose $b = 3$ and $a \geq 4$. Then we get at least one orbit with the above parameter $j = 0$. For $j = 1$, we get at least two orbits with representatives $\{P, Q\}$, where $P = \{P_1, P_2, R\}$, $Q = \{Q_1, Q_2, R\}$, and $|P_1 \cap Q_1| = 1$, respectively $|P_1 \cap Q_1| = 2$.

Suppose $b = 2$ and $a \geq 6$. Then for each $s = 1, 2, 3$ we get at least one orbit with representatives $\{P, Q\}$, where $P = \{P_1, P_2\}$, $Q = \{Q_1, Q_2\}$, and $|P_1 \cap Q_1| = s$. □

Remark 7.6. It is easy to check that for the embedding via the action of $X$ on $b$ $a$-subsets, we have $e_2(X) = 0$ if $(a, b) = (3, 2)$, and $e_2 = 1$ if $(a, b) = (2, 3), (4, 2), (5, 2)$. On the other hand, $e_2(X) = 3$ if $(a, b) = (3, 3)$, as one can compute using (6.3) above and [GAP].

Lemma 7.7. Suppose $m \geq 6$ and $X = A_m$ embeds into $S_n$ via its action on $k$-subsets of $\Delta$ for $2 \leq k < m/2$. Then either $e_3(X) \geq 4$, or $k = 2$ and $e_3(X) = 3$.

Proof. By Lemma 7.3, we have $f_2(X) = k$. So we will try to exhibit at least $(k + 4)$ $S_m$-orbits on triples $\{A, B, C\}$ of $k$-subsets. We may assume that

$$|A \cap B| \geq |A \cap C| \geq |B \cap C|$$

and call $(|A \cap B|, |A \cap C|, |B \cap C|)$ the mark of the triple $\{A, B, C\}$. Certainly, triples with different marks belong to different $S_m$-orbits.

Recall that $m \geq 2k + 1$. First let $|A \cap B| = k - 1$, so we may assume

$$A = \{1, 2, \ldots, k - 1, k\}, \quad B = \{1, 2, \ldots, k - 1, k + 1\}.$$ 

For $0 \leq j \leq k - 1$, by choosing

$$C = \{1, 2, \ldots, j, k + 2, k + 3, \ldots, 2k + 1 - j\}$$

we get a triple with the mark $(k - 1, j, j)$. Similarly, for $1 \leq j \leq k - 1$, by choosing

$$C = \{1, 2, \ldots, j - 1, k, k + 2, k + 3, \ldots, 2k + 1 - j\}$$

we get a triple with the mark $(k - 1, j, j - 1)$.

Next we consider the case $|A \cap B| = k - 2$, say

$$A = \{1, 2, \ldots, k - 2, k - 1, k\}, \quad B = \{1, 2, \ldots, k - 2, k + 1, k + 2\}.$$ 

For $1 \leq j \leq k - 2$, by choosing

$$C = \{1, 2, \ldots, j, k + 3, k + 4, \ldots, 2k + 2 - j\}$$

and call $(|A \cap B|, |A \cap C|, |B \cap C|)$ the mark of the triple $\{A, B, C\}$. Certainly, triples with different marks belong to different $S_m$-orbits.

Recall that $m \geq 2k + 1$. First let $|A \cap B| = k - 1$, so we may assume

$$A = \{1, 2, \ldots, k - 1, k\}, \quad B = \{1, 2, \ldots, k - 1, k + 1\}.$$ 

For $0 \leq j \leq k - 1$, by choosing

$$C = \{1, 2, \ldots, j, k + 2, k + 3, \ldots, 2k + 1 - j\}$$

we get a triple with the mark $(k - 1, j, j)$. Similarly, for $1 \leq j \leq k - 1$, by choosing

$$C = \{1, 2, \ldots, j - 1, k, k + 2, k + 3, \ldots, 2k + 1 - j\}$$

we get a triple with the mark $(k - 1, j, j - 1)$.

Next we consider the case $|A \cap B| = k - 2$, say

$$A = \{1, 2, \ldots, k - 2, k - 1, k\}, \quad B = \{1, 2, \ldots, k - 2, k + 1, k + 2\}.$$ 

For $1 \leq j \leq k - 2$, by choosing

$$C = \{1, 2, \ldots, j, k + 3, k + 4, \ldots, 2k + 2 - j\}$$

and call $(|A \cap B|, |A \cap C|, |B \cap C|)$ the mark of the triple $\{A, B, C\}$. Certainly, triples with different marks belong to different $S_m$-orbits.
we get a triple with the mark \((k - 2, j, j)\). Similarly, for \(1 \leq j \leq k - 2\), by choosing
\[
C = \{1, 2, \ldots, j - 1, k, k + 2, k + 3, \ldots, 2k + 2 - j\}
\]
we get a triple with the mark \((k - 2, j, j - 1)\).

We have produced at least \(4k - 5\) different marks. So we have \(e_3(X) \geq 3k - 5 \geq 4\) if \(k \geq 3\).

Finally, consider the case \(k = 2\). Then the triples
\[
\{\{1, 2\}, \{1, 3\}, \{2, 4\}\}, \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}, \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}
\]
have marks \((1, 1, 0), (1, 0, 0), \text{and} \ (0, 0, 0)\). Furthermore, the triples \(\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}\)
and \(\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}\) have the same mark \((1, 1, 1)\), but different cardinality of \(A \cup B \cup C\), so they produce two more \(S_m\)-orbits. \(\square\)

To estimate \(e_3(X)\) for the second special embedding of \(X\) into \(S_n\), we need the following observation:

**Lemma 7.8.** Let \(Y\) be any group and \(K\) any field. Suppose that \(A_1, A_2, B_1, B_2\) are \(K\)-modules such that there is an injective \(f \in \text{Hom}_{K}(A, B)\) with \(A = A_1 \oplus A_2, B = B_1 \oplus B_2\), and \(f(A_2) \subseteq B_2\). Then
\[
\dim B^Y - \dim A^Y \geq \dim B_1^Y - \dim A_1^Y.
\]

**Proof.** Clearly, \(A^Y = A_1^Y \oplus A_2^Y\) and \(B^Y = B_1^Y \oplus B_2^Y\). Now \(f\) embeds \(A_2^Y\) in \(B_2^Y\), whence the claim. \(\square\)

The following statement is also well known and follows for example from the formula for rank\(_K\eta_{2,3}\) given in [Wil]:

**Lemma 7.9.** Let \(p \neq 2, 3\) and \(n \geq 4\). Then \(\eta_{2,3} : M_2 \to M_3\) is injective. \(\square\)

Now we can prove a reduction lemma to help estimate \(e_3(X)\) for the second special embedding.

**Lemma 7.10.** Let \(X = A_m\) and \(Y := S_m\) embed into \(S_n\) via their actions on partitions of \(\Delta = \{1, 2, \ldots, m\}\) into \(a\) \(a\)-subsets, with \(a, b \geq 2\). Suppose that \(b > s \geq 2\). Set \(n' := (sa)!/((al)^s \cdot s!)\) and let \(Z := S_{sa}\) embed in \(S_{n'}\) via its action on partitions of \(\Delta' = \{1, 2, \ldots, sa\}\) into \(s\) \(a\)-subsets. Also denote by \(N_r\) the permutation \(S_{n'}\)-module corresponding to its action on \(r\)-subsets of \(\{1, 2, \ldots, n'\}\). Then
\[
\dim M_3^X - \dim M_2^X \geq \dim N_3^Z - \dim N_2^Z.
\]

**Proof.** 1) Since \(M_r\) is a permutation module, \(\dim M_r^X\) remains unchanged when we replace \(F\) by any other field. So throughout this proof we may assume \(F = \mathbb{C}\).

With respect to \(X\) and \(Y\), any pair \(\{P, Q\}\) in \(\Omega_2\) is a pair of two partitions
\[
P = \{P_1, \ldots, P_a\}, \ Q = \{Q_1, \ldots, Q_a\}.
\]
Call \(\{P, Q\}\) a **good pair** if at least \(b - s\) subsets \(P_i\) occur among the \(Q_j\). Next, we call a triple \(\{P, Q, R\} \in \Omega_3\) with \(R = \{R_1, \ldots, R_a\}\) a **good triple** if all three pairs \(\{P, Q\}, \{P, R\}, \{Q, R\}\) are good. We also call \(\{P, Q, R\}\) a very **good triple** if there are at least \(b - s\) subsets \(P_i\) which occur both among the \(Q_j\) and among the \(R_j\). Then \(Y > X\) acts
on the following sets: $\Omega_{21}$ of all good pairs, $\Omega_{22} := \Omega_2 \setminus \Omega_{21}$, $\Omega_{31}$ of all good triples, and $\Omega_{32} := \Omega_3 \setminus \Omega_{31}$. Thus

$$A_1 := \mathbb{C}\Omega_{21}, \ A_2 := \mathbb{C}\Omega_{22}, \ B_1 := \mathbb{C}\Omega_{31}, \ B_2 := \mathbb{C}\Omega_{32}$$

are CY-submodules of $M_2 = A_1 \oplus A_2$ and $M_3 = B_1 \oplus B_2$. Note that $Y > X$ also acts on the set $\Omega_{311}$ of very good triples, and so $B_{11} := \mathbb{C}\Omega_{311}$ is an $X$-submodule of $B_1$.

Certainly, $B_{11}^X \subset B_1^Y$.

Since $\text{char}(\mathbb{C}) = 0$, $\eta_{2,3}$ is injective by Lemma 7.9. Next, if $\{P, Q\} \in \Omega_{22}$, then

$$\eta_{2,3}(\{P, Q\}) = \sum_{R \neq P, Q} \{P, Q, R\},$$

where all occurring triples $\{P, Q, R\}$ are not good (since $\{P, Q\}$ is not good). Thus $\eta_{2,3}(\{P, Q\}) \in B_2$. We have shown that $\eta_{2,3}(A_2) \subset B_2$. Applying Lemma 7.8 to the homomorphism $f = \eta_{2,3}$, we see that

$$\dim M_3^X - \dim M_2^X \geq \dim B_1^X - \dim A_1^X \geq \dim B_{11}^X - \dim A_1^X.$$

Recall that $b > s$. Hence, for any good pair $\{P, Q\}$, $P$ and $Q$ have at least one common $a$-subset $P_1$, and since $a \geq 2$, some transposition $(i, j) \in Y \setminus X$ fixes both $P$ and $Q$. It follows that $X$ and $Y$ have the same orbits on good pairs. Similarly, $X$ and $Y$ have the same orbits on very good triples. Thus

$$\dim B_{11}^X - \dim A_1^X = \dim B_{11}^Y - \dim A_1^Y.$$

2) It remains to prove that

$$(7.2) \quad \dim B_{11}^Y - \dim A_1^Y = \dim N_3^Z - \dim N_2^Z.$$  

To do so, we will count the $X$-orbits on all good pairs and very good triples.

Let $r \in \{2, 3\}$. Suppose the good pairs (if $r = 2$), respectively the very good triples (if $r = 3$), $\{P^1, \ldots, P^r\}$ and $\{Q^1, \ldots, Q^r\}$ belong to the same $X$-orbit. Then, without loss we may assume that

$$(7.3) \quad P^i = \{R_1, \ldots, R_b, P_1^i, \ldots, P_s^i\}, \ Q^i = \{R_1, \ldots, R_b, Q_1^i, \ldots, Q_s^i\}$$

for $1 \leq i \leq r$. Denote

$$\tilde{P}^i := \{P_1^i, \ldots, P_s^i\}, \ \tilde{Q}^i := \{Q_1^i, \ldots, Q_s^i\}$$

for $1 \leq i \leq r$.

By assumption, there is some $\sigma \in S_m$ sending $\{P^1, \ldots, P^r\}$ to $\{Q^1, \ldots, Q^r\}$. Let $t$ be the number of common $a$-subsets that occur in all $P^1, \ldots, P^r$. Then $t$ is also the number of common $a$-subsets that occur in all $Q^1, \ldots, Q^r$, and $t \geq b - s$.

Consider the case $t = b - s$. Then the $t$ common $a$-subsets among all $P^i$ are precisely $R_1, \ldots, R_{b-s}$, and similarly, the $t$ common $a$-subsets among all $Q^i$ are precisely $R_1, \ldots, R_{b-s}$. It follows that $\sigma$ acts on the set $\{R_1, \ldots, R_{b-s}\}$, and preserves the set

$$\bigcup_{j=1}^{s-1} P_j^i = \bigcup_{j=1}^{b-s} Q_j^i$$

which can be identified with $\Delta'$. Now we can write $\sigma = \mu \tau$, where

$$\mu \in S_{m-sa} = \text{Sym}(\bigcup_{j=1}^{b-s} R_j)$$
acts on the set \( \{R_1, \ldots, R_{b-s}\} \), and \( \tau \in Z = \text{Sym}(\Delta') \) sends \( \{P^1, \ldots, P^r\} \) to \( \{Q^1, \ldots, Q^r\} \).

Thus, the two pairs, respectively triples, \( \{P^1, \ldots, P^r\} \) and \( \{Q^1, \ldots, Q^r\} \) of partitions of \( \Delta' \) belong to the same \( Z \)-orbit.

Next assume that \( t > b - s \) and set \( v = t + s - b \). Then the \( t \) common \( a \)-subsets among all \( P^i \) are precisely \( R_1, \ldots, R_{b-s}, S_1, \ldots, S_v \), and similarly, the \( t \) common \( a \)-subsets among all \( Q^i \) are precisely \( R_1, \ldots, R_{b-s}, T_1, \ldots, T_v \), for some \( a \)-subsets \( S_j \) and \( T_j \) of \( \Delta \). It follows that \( \sigma \) sends \( \{R_1, \ldots, R_{b-s}, S_1, \ldots, S_v\} \) to \( \{R_1, \ldots, R_{b-s}, T_1, \ldots, T_v\} \), and \( \Sigma \) to \( \Theta \), where

\[
\Sigma = \Delta \setminus \left( \bigcup_{j=1}^{b-s} R_j \cup \bigcup_{j=1}^{v} S_j \right), \quad \Theta = \Delta \setminus \left( \bigcup_{j=1}^{b-s} R_j \cup \bigcup_{j=1}^{v} T_j \right).
\]

Set \( R_j' := R_j \) for \( 1 \leq j \leq b - s \). Also set \( R_{b-s+j} := S_j \) and \( R'_{b-s+j} := T_j \) for \( 1 \leq j \leq v \).

Then there is a permutation \( \pi \in S_t \) such that \( \sigma(R_j) = R'_{\pi(j)} \). Now we can find \( \gamma \in S_{m} = \text{Sym}(\Delta) \) such that

\[
\gamma_\Sigma = I_{\Sigma}, \quad \gamma(R_j) = R_{\pi^{-1}(j)}.
\]

Clearly, \( \sigma_\gamma \) sends \( R_j \) to \( R_j \) for \( 1 \leq j \leq b - s \) (and \( S_j \) to \( T_j \)), and sends \( \{P^1, \ldots, P^r\} \) to \( \{Q^1, \ldots, Q^r\} \). Now we can repeat the argument of the preceding case \( t = b - s \) to show that the two pairs, respectively triples, \( \{P^1, \ldots, P^r\} \) and \( \{Q^1, \ldots, Q^r\} \) of partitions of \( \Delta' \) belong to the same \( Z \)-orbit.

Conversely, it is obvious that if \( \{P^1, \ldots, P^r\} \) and \( \{Q^1, \ldots, Q^r\} \) belong to the same \( Z \)-orbit, then \( \{P^1, \ldots, P^r\} \) and \( \{Q^1, \ldots, Q^r\} \), defined as in (7.3), belong to the same \( Y \)-orbit. We have therefore proved that \( \dim B_{11}^Y = \dim N_3^Z \) and \( \dim A_1^Y = \dim N_2^Z \), whence (7.2) holds.

\[\Box\]

**Theorem 7.11.** Let \( p = 2 \) and \( n \) be even. Suppose that \( n > m \geq 11 \) and \( X \cong A_m \) embeds into \( A_n \) via its actions on subsets or partitions of \( \{1, 2, \ldots, m\} \). Then \( f_2(X) \geq 3 \) and \( e_3(X) = h(X) + 2 \), unless \( A_m \) embeds into \( A_n \) via its action on 2-subsets of \( \{1, 2, \ldots, m\} \).

**Proof.** 1) The inequality \( e_2(X) \geq 2 \), and hence \( f_2(X) \geq 3 \), is proved in Lemma 7.5.

If \( A_m \) embeds into \( A_n \) via its action of \( k \)-subsets of \( \Delta \) with \( 2 < k < m/2 \), then \( h(X) \leq 2 \) by Lemma 7.2, and \( e_3(X) \geq 4 \) by Lemma 7.7. Thus \( e_3(X) = h(X) + 2 \) as stated.

2) From now on, we assume that \( A_m \) embeds in \( A_n \) via its action on partitions of \( \Delta \) into \( b \)-\( a \)-subsets, \( a, b \geq 2 \). By Lemma 7.2, either \( h(X) \leq 2 \), or \( h(X) = 3 \) and \( b = 2(m/2) \).

Here we consider the case \( a \geq 5 \) and show that \( e_3(X) \geq 5 \geq h(X) + 2 \) in this case. Applying Lemma 7.10 with \( s = 2 \), we are reduced to prove that \( \dim M_3^Y - \dim M_2^Y \geq 5 \) for \( b = 2 \) and \( Y = S_m \). In this base case \( b = 2 \), each partition is a pair

\[
[A] := \{A, \Delta \setminus A\}
\]

with \( |A| = a = |\Delta|/2 \). Hence the \( Y \)-orbits on pairs of partitions \( \{[A], [B]\} \) are labeled by \( 1 \leq j \leq t := \lfloor a/2 \rfloor \), where \( |A \cap B| = a - j \). In particular, \( \dim M_3^Y = t \).

So we need to produce at least \( (t + 5) \) \( Y \)-orbits on triples of partitions \( \{[A], [B], [C]\} \in \Omega_3 \). As in the proof of Lemma 7.7, we will label \( A, B, C \) so that

\[A \cap B \geq |A \cap C| \geq |B \cap C| \geq a/2\]

and call \( (|A \cap B|, |A \cap C|, |B \cap C|) \) the mark of the triple \( \{A, B, C\} \). Certainly, triples with different marks belong to different \( S_m \)-orbits.
First let \(|A \cap B| = a - 1\), so we may assume
\[ A = \{1, 2, \ldots, a - 1, a\}, \quad B = \{1, 2, \ldots, a - 1, a + 1\}. \]
For \(1 \leq j \leq t\), by choosing
\[ C = \{1, 2, \ldots, a - j, a + 2, a + 3, \ldots, a + j + 1\} \]
we get a triple with the mark \((a - 1, a - j, a - j)\). In fact, for \(j = 1\), we have two choices for \(C\):
\[ \{1, 2, \ldots, a - 1, a + 2\}, \text{ and } \{1, 2, \ldots, a - 2, a, a + 1\} \]
which lead to two different \(S_m\)-orbits with mark \((a - 1, a - 1, a - 1)\) for any \(a \geq 4\) (since \(|A \cap B \cap C| = a - 1\) for the first choice and \(|A \cap B \cap C| = a - 2\) for the second choice).
Similarly, for \(1 \leq j \leq t - 1\), by choosing
\[ C = \{1, 2, \ldots, a - j - 1, a, a + 2, a + 3, \ldots, a + j + 1\} \]
we get a triple with the mark \((a - 1, a - j, a - j - 1)\).
Suppose in addition that \(a \geq 6\). Then we choose
\[ A = \{1, 2, \ldots, a - 2, a - 1, a\}, \quad B = \{1, 2, \ldots, a - 2, a + 1, a + 2\}. \]
Taking
\[ C = \{1, 2, \ldots, a - 2, a + 3, a + 4\} \text{ or } \{1, 2, \ldots, a - 3, a + 3, a + 4, a + 5\} \]
we get triples with the mark \((a - 2, a - 2, a - 2)\) and \((a - 2, a - 3, a - 3)\). We have produced at least \((t + 1) + (t - 1) + 2 \geq t + 5\) orbits on triples, as desired.

Next assume that \(a = 5\), and so \(t = 2\). We have already produced 4 orbits with marks \((4, 4, 4)\) (two orbits), \((4, 4, 3)\), and \((4, 3, 3)\). We can also exhibits 3 more orbits with
\[ \{A, B, C\} = \{12345, 12346, 12578\}, \quad \{12345, 12367, 12389\}, \quad \{12345, 12367, 12589\} \]
and mark \((4, 3, 2)\), \((3, 3, 3)\), and \((3, 3, 2)\), respectively. (Note that the last mark deviates from the convention \((7.4)\), but it does not cause any problem since \(a = 5\), as one can check.)

For the next part of the proof, we also consider the case \(a = 4\), and so \(s = 2\). We have already produced 4 orbits with marks \((3, 3, 3)\) (two orbits), \((3, 3, 2)\), and \((3, 2, 2)\). Next, the triples with
\[ \{P, Q, R\} = \{1234, 1256, 3456\} \text{ and } \{1234, 1256, 1357\} \]
have the same mark \((2, 2, 2)\), but belong to different orbits. (Indeed, the former satisfies the identity \(R = P + Q := (P \setminus Q) \cup (Q \setminus P)\), but for the latter no member of \([R]\) can be the sum of a member of \([P]\) with a member of \([Q]\).) Thus we get at least 6 orbits on triples of partitions. (One can show by [GAP] that the number of orbits on triples is indeed 6 for both \(A_m\) and \(S_m\).)

3) It remains to consider the case \(2 \leq a \leq 4\). Since \(m \geq 11\), we must have that \(b \geq 3\), and so \(h \leq 2\) by Lemma 7.2. For \(a = 2, 3, 4, or 5\), we set \(s = 5, 3, 2, or 2\), respectively. Applying Lemma 7.10, we are reduced to prove that \(e' := \dim M_3^Y - \dim M_2^Y \geq 4\) for \(b = s\) and \(Y = S_{sa}\). This has been done in 2) for \((a, s) = (4, 2)\). Using (6.3) and [GAP], one can check that \(e' = 35\) for \((a, s) = (3, 3)\). Finally, assume that \((a, s) = (2, 5)\). Using
(6.3) and [GAP], we see that the number of $A_{10}$-orbits on 2-subsets, respectively on 3-subsets of $\Omega = \{1, 2, \ldots, 945\}$ is 6, respectively 139. Since $A_{10}$ has index 2 in $Y = S_{10}$, it follows that $\dim M^Y_2 \leq 6$ and $\dim M^Y_3 \geq 139/2$, whence $e' \geq 64$. \hfill \Box

Now we can prove the main result concerning the special embeddings of $A_m$ into $A_n$:

**Theorem 7.12.** Let $X = A_m$ be embedded in $A_n$ via its actions on partitions or on $k$-subsets of $\{1, 2, \ldots, m\}$ with $2 \leq k < m/2$, and $11 \leq m < n$. Let $p = 2$ and let $V$ be any $\mathbb{F}A_n$-module of dimension greater than 1. Then $\text{Res}^A_X V$ is reducible.

**Proof.** Assume for a contradiction that $V$ is irreducible over $X$. By Lemma 7.1, $n \geq m(m-1)/2$. Since $m \geq 11$, we have

$$|X| = \frac{m!}{2} < 2^{\frac{m(m-1)-12}{4}} \leq \left(2^{\frac{n-6}{4}}\right)^2.$$  

The irreducibility of $V$ on $X$ forces that $\dim V < \sqrt{|X|} < 2^{(n-6)/4}$. This bound implies by Proposition 4.1 that $V$ extends to $S_n$. Thus, without loss we may assume that $V$ is an irreducible $\mathbb{F}S_n$-module. Also, since $2^{(n-6)/4} < 2^{[(n-2)/2]}$, $V$ cannot be a basic spin module.

Now if $n$ is odd, then the $X$-module $V$ is reducible by [KS1, Theorem 3.10]. Hence $2 | n$, and $V$ satisfies the conclusion (i) of Theorem 3.2. By Corollary 7.4, $X$ does not embed into $A_n$ via its action on 2-subsets of $\{1, 2, \ldots, m\}$. Hence, by Theorem 7.11, $f_2(X) \geq 3$ and $e_3(X) \geq h(X) + 2$. Clearly, $X$ is a perfect subgroup and $f_1(X) = 1$. Therefore, the $X$-module $V$ is reducible by Theorem 6.5, a contradiction. \hfill \Box

8. **General embeddings of $A_m$ into $A_n$**

Throughout this section, $X \cong A_m$ is a subgroup of $A_n$, with $n > m \geq 5$. Recall the notation $\Omega = \{1, 2, \ldots, n\}$ and $\Delta = \{1, 2, \ldots, m\}$.

First, we deal with the small cases $5 \leq m \leq 10$:

**Lemma 8.1.** Let $X \cong A_m$ be a transitive subgroup of $A_n$ with $n > m$ and $5 \leq m \leq 10$, $V$ be an $\mathbb{F}A_n$-module of dimension $> 1$ such that $\text{Res}^A_X V$ is irreducible. Then $(m, n) = (5, 6), (6, 10), (7, 15)$, or $(8, 15)$.

**Proof.** 1) First we consider the case where $V$ is not the heart of the natural permutation module of $Y := A_n$. Suppose for instance that $m = 10$. The assumptions that $X$ is a transitive subgroup of $Y$ and $n > m$ imply by [Atl] that $n \geq 45$. This in turn implies by [GT1, Lemma 6.1] that $\dim V \geq (n^2 - 5n + 2)/2 = 901 > 567 \geq b_\varphi(X)$, where $b_\varphi(X)$ denotes the largest degree of irreducible $\mathbb{F}X$-representations. It follows that $\text{Res}^A_X V$ is reducible. The same arguments apply to the case $m = 9$, where we have $n \geq 36$ and $b_\varphi(X) \leq 216$.

Suppose now that $m = 8$. According to [Atl] and [JLPW], $n = 15$ or $n \geq 28$; furthermore, $b_\varphi(X) \leq 70$. As above, $\dim V > b_\varphi(X)$ if $n \geq 28$. It follows that $n = 15$. The same arguments apply to the case $5 \leq m \leq 7$. (In fact, one can show that we must have either $(m, n) = (5, 6)$ or $(m, n, p, \dim V) = (8, 15, 2, 64).$)

2) Now we may assume that $V$ is the heart of the natural permutation module $\mathbb{F}\Omega$ of $Y$. By the assumption, $X$ acts transitively on $\Omega$. If this action is not doubly transitive,
then the restriction of the permutation character of $\mathbb{C}\Omega$ is $1_X + \alpha + \beta$ where $\alpha$, $\beta$ are (not necessarily irreducible) $X$-characters of degree $>1$. Since the Brauer character of $\mathbb{F}\Omega$ is $\varphi + e \cdot 1_Y$ with $e \in \{1,2\}$ and $\varphi \in \operatorname{IBr}_p(Y)$, it follows that $V$ is reducible over $X$. Thus $X$ acts doubly transitively on $\Omega$. Now we can read off the possible value of $n$ using [Atl].

\begin{remark}
\textbf{Remark 8.2.} The special cases $(m, n)$ listed in the statement of Lemma 8.1 are indeed exceptional. For these pairs, we can embed $A_m$ into $A_n$ so that $A_m$ acts doubly transitively on $\Omega$. Take $V$ to be the heart of the permutation module $\mathbb{C}\Omega$. It is well known (and follows for instance from the Mackey’s formula) that any doubly transitive group is irreducible on $V$. This is also almost always true for $p > 0$ — see [Mo].
\end{remark}

We will need the following group-theoretic result:

\begin{lemma}
\textbf{Lemma 8.3.} Let $X \cong A_m$ be a transitive subgroup of $A_n$ with $n > m \geq 11$. Then $n \geq m(m-1)/2$ and $X$ cannot act doubly transitively on $\Omega$. If $X$ is imprimitive, then $n \geq m(m-1)/2$. If $X$ is primitive, then one of the following statements holds:

(i) $X$ acts on $\Omega$ via its action on $k$-subsets of $\Delta$ with $2 \leq k < m/2$ or on partitions of $\Delta$, and $n \geq m(m-1)/2$.

(ii) $n > |X|^{3/10} > m(m-1)$.
\end{lemma}

\begin{proof}
From the classification of doubly transitive permutation representations of $A_m$ [Ma] and the assumption $m \geq 11$, it follows that $A_m$ has no doubly transitive permutation representation of degree $n > m$. Next, consider the transitive action of $X$ on $\Omega$ and let $X_1$ be a point stabilizer of this action. Then $[X : X_1] = n > m$. We can find a maximal subgroup $M$ of $X$ containing $X_1$.

Now we consider the action of $M$ on $\Delta$. If this action is intransitive, then by maximality of $M$, $M$ is the stabilizer in $X$ of a $k$-subset of $\Delta$ for some $1 \leq k < m/2$. On the other hand, if this action is transitive but imprimitive, then $M$ is the stabilizer in $X$ of a partition of $\Delta$ into $b$-subsets, with $a, b \geq 2$ and $ab = m$. By Lemma 7.1, in either case we have $n \geq [X : M] \geq m(m-1)/2$ (whence $[X : X_1] \geq m(m-1)$ if $X$ is imprimitive), unless $M$ is intransitive and $k = 1$. In this exceptional case, $M = A_{m-1}$ and $[X : X_1] = n > m = [X : M]$. Thus $X_1$ is a proper subgroup of $M$, and so, as is well known (see e.g. [Kil, p. 175]), $[M : X_1] \geq m-1$ and $n \geq m(m-1)$.

Suppose now that $M$ acts primitively on $\Delta$. By Bochert’s Theorem 14.2 of [Wie], $|S_m : M| \geq [(m + 1)/2]!$. Since $m \geq 11$, we have $\frac{(m+1)/2}{2}! > \left(\frac{m}{2}\right)^{3/10}$. Therefore $n \geq [X : M] > |X|^{3/10} > m(m-1)$ as $m \geq 11$.

We have proved the bound $n \geq m(m-1)/2$ and also that $n \geq m(m-1)$ if $X$ is imprimitive. Assume now that $X$ is primitive. Then $X_1 = M$, and the above analysis yields the final claim of the lemma.
\end{proof}

\begin{theorem}
\textbf{Theorem 8.4.} Let $X \cong A_m$ be a primitive subgroup of $A_n$ with $n > m \geq 9$. Let $p = 2$ and let $V$ be an $\mathbb{F}A_n$-module of dimension $> 1$. Then the restriction $\operatorname{Res}^X V$ is reducible.
\end{theorem}

\begin{proof}
By Lemma 8.1, we may assume that $m \geq 11$. Assume for a contradiction that $V$ is irreducible over $X$. We apply Lemma 8.3. In the case (i) of Lemma 8.3 we are done by Theorem 7.12.
Now suppose that the case (ii) of Lemma 8.3 holds. Then $|X| < n^{10/3}$ and $n \geq 155$ as $m \geq 11$. Since $V$ is irreducible over $X$, we must have that

$$\dim V < n^{5/3} < \frac{1}{2} (n - 1)(n - 2).$$

In particular, $\dim V < 2^{(n-6)/4}$ (as $n \geq 155$), whence $V$ extends to $S_n$ by Proposition 4.1. We denote the (unique up to isomorphism) extension of $V$ to $S_n$ by the same letter $V$.

By Lemma 8.3, the action of $X = A_m$ on $\Omega$ is not doubly transitive. Hence, if $n$ is odd then, we are done by appealing to [KS1, Theorem 3.10]. Assume that $2|n \geq 155$. Then the condition $1 < \dim V < (n - 1)(n - 2)/2$ for the irreducible $\mathbb{F}S_n$-module $V$ implies by [J3, Theorem 7] that $V \cong D(n-k,k)$ with $k = 1$ or 2. As before, let $1 + \alpha$ denote the (complex) permutation character of $S_n$ on $\Omega$.

Suppose $k = 1$. Then $V = D^{(n-1,1)}$ is irreducible over $X$. Denoting by $\alpha^0$ the restriction of $\alpha$ to $2'$-elements in $S_n$, we see that the Brauer character of $V$ is $\alpha^0 - 1$. Furthermore, $X$ is perfect and $\text{Res}_{X}^{S_n} \alpha, 1_X|X = 0$ by transitivity of $X$ on $\Omega$. Hence the irreducibility of $V$ over $X$ forces that $\alpha$ is also irreducible over $X$. In other words, $X$ acts doubly transitive on $\Omega$, a contradiction.

Assume now that $k = 2$. Then $\wedge^2(D^{(n-1,1)})$ has a composition series with composition factors $V$ (once), and $1$ (once if $4|n$ and twice if $n \equiv 2 \pmod{4}$). The same is true for $\wedge^2(D^{(n-1,1)})$ considered as an $X$-module. Suppose in addition that $D^{(n-1,1)}$ is reducible over $X$. Then we can write the Brauer character of the $\mathbb{F}X$-module $\text{Res}_{X}^{S_n} D^{(n-1,1)}$ as $\beta + \gamma$, where $\beta$ and $\gamma$ are Brauer characters of $X$, $\beta(1) \geq 1$ and $\gamma(1) \geq 3$. Note that

$$\wedge^2(\beta + \gamma) = \wedge^2(\beta) + \wedge^2(\gamma) + \beta\gamma.$$

It follows that the Brauer character of the $X$-module $\wedge^2(D^{(n-1,1)})$ is the sum of three Brauer characters, with at least two of degree $\geq 3$. This contradicts the aforementioned composition structure of the $X$-module $\wedge^2(D^{(n-1,1)})$. Thus $D^{(n-1,1)}$ is irreducible over $X$. But then, arguing as in the previous paragraph, we again arrive at the contradiction that $X$ acts doubly transitively on $\Omega$. \hfill $\Box$

The case $p \neq 2$ can be done using results of [KS1], [KS2] and Proposition 4.3:

**Proposition 8.5.** Let $X \cong A_m$ be a transitive subgroup of $A_n$ with $n > m \geq 9$. Let $p \neq 2$ and $V$ be an $\mathbb{F}A_n$-module of dimension $> 1$. Then $\text{Res}_{X}^{A_n} V$ is reducible.

**Proof.** By Lemma 8.1, we may assume that $m \geq 11$. Assume for a contradiction that $\text{Res}_{X}^{A_n} V$ is irreducible. By Lemma 8.3, $X$ is not doubly transitive, and $n \geq m(m - 1)/2$.

If $V$ is extendible to $S_n$, we can apply [KS1, Main Theorem] to get a contradiction. Suppose $V$ is not extendible to $S_n$. If $p \neq 3$, we again arrive at a contradiction by using [KS2, Main Theorem]. So we may assume that $p = 3$. By Proposition 4.3(ii) we have that

$$\dim V \geq 2^{n-8} \geq 2^{m^2-m-16} =: c_m.$$

Now, if $m \geq 12$ then $c_m > \sqrt{m!/2} = \sqrt{|X|}$. If $m = 11$, then $c_m > 3444$ whereas the largest degree of complex irreducible representations of $X = A_{11}$ is 2310. In either case, $X$ cannot be irreducible on $V$. \hfill $\Box$
Now Theorem 1.2 follows immediately from Theorem 8.4 and Proposition 8.5.

Note that for $p \neq 2$, Proposition 8.5 shows that any proper transitive subgroup $X \cong A_m$ of $A_n$ acts reducibly on all non-trivial modules over $\mathbb{F}A_n$ (provided $m \geq 9$). For $p = 2$, if $X$ is primitive, the same result holds by Theorem 8.4. Now we handle imprimitive embeddings of $A_m$ into $A_n$ for $p = 2$. Here our result is a little weaker:

**Proposition 8.6.** Let $X \cong A_m$ be a (transitive) imprimitive subgroup of $A_n$ with $n > m \geq 9$. Let $p = 2$ and let $V$ be an irreducible $\mathbb{F}A_n$-module of dimension $> 1$. Then the $X$-module $\text{Res}^X_A V$ cannot be primitive irreducible.

**Proof.** By Lemma 8.1, we may assume that $m \geq 11$. Assume for a contradiction $V$ is irreducible and primitive over $X$. Consider the action of $X$ as a subgroup of $A_n$ on $\Omega$ and let $X_1$ be a point stabilizer. Since $X$ is imprimitive, there is a maximal subgroup $M > X_1$ of $X$. Now $b := [X : M] \geq m \geq 11$, and we may assume that $X < Y := (S_n \wr S_b) \cap A_n$ for $a := n/b > 1$. By Lemma 8.3, $n \geq m(m - 1)$. If $a = 2$ or 4, then $O_2(Y) > 1$, and so $V$ cannot be irreducible over $Y > X$, a contradiction. So we have $a \geq 3$ and $a \neq 4$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a $2$-regular partition of $n$ such that $V$ is a simple submodule of $\text{Res}^S_{A_n} D^\lambda$. Observe that $m! < (m/2)^m$ for $m \geq 6$. Since $\text{Res}^S_{A_n} V$ is irreducible, by [GLT, Theorem 5.1] we have

$$2^{\frac{n-\lambda_1}{2} - 1} \leq \dim V \leq \sqrt{|A_m|} < \sqrt{\frac{m^m}{2m+1}},$$

whence $n - \lambda_1 < m \log_2 m - m + 1$. This in turn implies that $2\lambda_1 - n \geq n/m$. Indeed, otherwise we would have $\lambda_1 < n/2 + n/2m$ and so

$$n - \lambda_1 > \frac{n}{2} - \frac{n}{2m} = n \cdot \frac{m - 1}{2m} \geq \frac{(m - 1)^2}{2} > m \log_2 m - m + 1,$$

a contradiction.

We have shown that $2\lambda_1 - n \geq n/m \geq n/b = a \geq 3$. Also, $n - \lambda_1 \geq 1$ since $\dim V > 1$. Applying Proposition 4.2, we see that the restriction of $D^\lambda$ to a natural subgroup $S_n$ of $S_n$ affords both 1 and $D^{(a-1,1)}$ as composition factors. If $a \geq 5$, then $\dim D^{(a-1,1)} \geq 4$ and so any irreducible summand of the $A_n$-module $D^{(a-1,1)}$ has dimension $\geq 2$ (in fact $D^{(a-1,1)}$ is irreducible over $A_n$). On the other hand, if $a = 3$, then $D^{(a-1,1)}$ splits into a direct sum of two non-trivial irreducible $A_n$-submodules. Thus $\text{Res}^S_{A_n} V$ affords both the trivial module and also another non-trivial irreducible module as composition factors. It follows that $\text{Res}^S_{A_n} V$ cannot be homogeneous for $Z := A_a \times \cdots \times A_a = (A_a)^b \triangleleft Y$. But this is a contradiction, as $Z \triangleleft Y$ and the $Y$-module $V$ is primitive. 

The following lemma deals with tensor indecomposable irreducible representations of $A_m$. These were studied extensively in [Z, BeK1, BeK2, GoK, GrJ, BeK3].

**Lemma 8.7.** Let $X \cong A_m$ be a subgroup of $A_n$ with $n \geq m \geq 5$. Let $V$ be an irreducible $\mathbb{F}A_n$-module of dimension $> 1$. Assume that the $X$-module $\text{Res}^X_A V$ is irreducible and tensor indecomposable. Then one of the following holds:

(i) $X$ is a transitive subgroup of $A_n$. 

(ii) There is some \( t \in \{1, 2\} \) such that \( X \) fixes \( t \) points and acts transitively on \( n - t \) remaining points of \( \Omega \). Furthermore, \( V \) is irreducible over a natural subgroup \( A_{n-t} \) of \( A_n \).

Proof. Suppose that \( X \) acts intransitively on \( \Omega \). Each non-trivial orbit of \( X \) has at least \( m \geq 5 \) points. If \( X \) has at least two non-trivial orbits on \( \Omega \), then we may assume that there is some \( 5 \leq k \leq n/2 \) such that \( X \) acts non-trivially on both \( \Omega^{(1)} := \{1, 2, \ldots, k\} \) and \( \Omega^{(2)} := \{k + 1, \ldots, n\} \). These actions induce embeddings \( \pi_i : X \to Alt(\Omega^{(i)}) \), namely, \( g \in X \) acts on \( \Omega^{(i)} \) as \( \pi_i(g) \). Now, \( X < Y := A_k \times A_{n-k} \) and \( \text{Res}^A_X V \) is irreducible. Hence \( V|_Y \cong V_1 \boxtimes V_2 \) is an outer tensor product of irreducible modules \( V_1 \) over \( A_k \) and \( V_2 \) over \( A_{n-k} \). Note that \( \dim V_1 > 1 \), as otherwise the 3-cycles in \( A_k \) would act trivially on \( V \). Similarly \( \dim V_2 > 1 \). It follows that \( \text{Res}^A_X V = U_1 \otimes U_2 \), where \( U_i \cong \text{Res}^{A_{k_i}}_{\pi_i(X)} V_i \) for \( i = 1, 2 \) and \( (k_1, k_2) := (k, n - k) \). In particular, \( \text{Res}^A_X V \) is tensor decomposable, contrary to the assumption.

Thus \( X \) has only one non-trivial orbit on \( \Omega \), say of length \( n - t \) for some \( t > 0 \), and \( X \) fixes the remaining \( t \) points. Furthermore, \( t \leq 2 \), as otherwise \( X \) is centralized by a 3-cycle and so cannot act irreducibly on \( V \). □

Finally, in connection with Lemma 8.7, we bound the number of extensions of irreducible representations of \( A_{n-1} \) to \( A_n \). For this type of question, it is convenient to use the following simple observation:

Lemma 8.8. Let \( Y \) be a subgroup of \( X \) and \( U \) be a finite-dimensional vector space over \( \mathbb{F} \). Let \( \Psi : Y \to GL(U) \) be an irreducible representation. Suppose that \( \Phi_i : X \to GL(U), i = 1, 2 \), are two isomorphic representations of \( X \) with \( \text{Res}^X_Y(\Phi_i) = \Psi \). Then in fact \( \Phi_1 = \Phi_2 \).

Proof. Fix a basis of the vector space \( U \) and write \( \Phi_1 \) and \( \Phi_2 \) as matrix representations with respect to this basis. By the assumption, there is some invertible matrix \( A \) such that \( \Phi_2(x) = A \Phi_1(x) A^{-1} \) for all \( x \in X \). Since both \( \Phi_i \) extend \( \Psi \), we have that \( \Psi(y) = A \Psi(y) A^{-1} \) for all \( y \in Y \). By Schur’s lemma, \( A \) is scalar, and so \( \Phi_2 = \Phi_1 \). □

Now we deal with the aforementioned question for symmetric groups:

Lemma 8.9. Assume \( n \geq 3 \) and fix a natural embedding of \( S_{n-1} \) into \( S_n \). Then every irreducible \( \mathbb{F}S_{n-1} \)-representation \( S_{n-1} \to GL(U) \) has at most one extension to \( S_n \).

Proof. Consider \( U \) as an \( \mathbb{F}S_{n-1} \)-module, and suppose that there are two distinct \( p \)-regular partitions \( \lambda, \mu \vdash n \) such that

\[
\text{Res}^{S_n}_{S_{n-1}}(D^\lambda) \cong \text{Res}^{S_n}_{S_{n-1}}(D^\mu) \cong U.
\]

Since \( \text{soc}(\text{Res}^{S_n}_{S_{n-1}}(D^\lambda)) \cong \text{soc}(\text{Res}^{S_n}_{S_{n-1}}(D^\mu)) \), by [K3, Corollary 5.3], precisely one of the modules \( \text{Res}^{S_n}_{S_{n-1}}(D^\lambda) \), \( \text{Res}^{S_n}_{S_{n-1}}(D^\mu) \) is reducible, a contradiction. Now apply Lemma 8.8. □

Proposition 8.10. Assume \( n \geq 5 \) and fix a natural embedding of \( A_{n-1} \) into \( A_n \). Then every irreducible \( \mathbb{F}A_{n-1} \)-representation \( A_{n-1} \to GL(U) \) has at most three distinct extensions to \( A_n \).
Proof. We assume that $U$ extends to an $\mathbb{F}A_n$-module $V_1$ and find all possible other extensions $V_2$ of $U$ to $A_n$. Consider $A_{n-1}$ as the derived subgroup of a natural subgroup $S_{n-1}$ of $S_n$. By Lemma 8.8, it suffices to bound the number of extensions up to isomorphism. For each $i = 1, 2$, we can find an irreducible $\mathbb{F}S_n$-module $W_i$ such that $V_i \hookrightarrow \text{soc}(\text{Res}_{A_n}^S(W_i))$, and an irreducible $\mathbb{F}S_{n-1}$-module $T$ such that $U \hookrightarrow \text{soc}(\text{Res}_{A_{n-1}}^S(T))$. We also fix a transposition $g \in S_{n-1} \setminus A_{n-1}$, and distinguish the following cases.

**Case I:** $U$ is not $S_{n-1}$-invariant. Then for $i = 1, 2$ we have that

$$\text{Res}_{A_{n-1}}^S(V_i^g) = U^g \cong U = \text{Res}_{A_{n-1}}^S(V_i),$$

and so $V_i$ is not $S_n$-invariant. Hence $\text{Res}_{A_n}^S(W_i) \cong V_i \oplus V_i^g$, and similarly $\text{Res}_{A_{n-1}}^S(T) \cong U \oplus U^g$. Now $\text{Res}_{A_{n-1}}^S(W_i) \cong U \oplus U \cong \text{Res}_{A_{n-1}}^S(T)$. It follows that $\text{Res}_{A_{n-1}}^S(W_i) \cong T \cong \text{Res}_{A_{n-1}}^S(W_2)$. By Lemma 8.9, $W_1 \cong W_2$, whence $V_2$ is isomorphic to $V_1$ or $V_1^g$. Since $\text{Res}_{A_{n-1}}^S(V_i^g) = U^g \not\cong U$, we must have that $V_2 \cong V_1$.

**Case IIa:** $U$ is $S_{n-1}$-invariant and both $V_1, V_2$ are $S_n$-invariant. In particular, we have $\text{Res}_{A_n}^S(W_i) = V_i$ for $i = 1, 2$ and similarly $\text{Res}_{A_{n-1}}^S(T) = U$. Now if $p = 2$, then the Brauer character of any extension of $U$ to $S_n$ is uniquely determined by its restriction to $A_{n-1}$ which is the Brauer character of $U$. Thus $U$ has a unique extension to $S_n$, and so $\text{Res}_{A_{n-1}}^S(W_1) \cong \text{Res}_{A_{n-1}}^S(W_2)$. It follows by Lemma 8.9 that $W_1 \cong W_2$ and so $V_1 \cong V_2$.

Next suppose that $p \neq 2$. Then $U$ has two distinct extensions $T$ and $T^g$ to $S_n$ and $V_i$ has two distinct extensions $W_i$ and $W_i^g$ to $S_n$. It follows that $\text{Res}_{A_{n-1}}^S(W_i) \in \{T, T^g\}$ and so $\text{Res}_{A_{n-1}}^S(W_2) \in \{\text{Res}_{A_{n-1}}^S(W_1), \text{Res}_{A_{n-1}}^S(W_1^g)\}$. By Lemma 8.9, $W_2 \cong W_1$ or $W_1^g$, and so $V_2 \cong V_1$.

Thus Case IIa shows that among the extensions of $U$ to $A_n$, at most one of them is $S_n$-invariant.

**Case IIb:** $U$ is $S_{n-1}$-invariant, but neither $V_1$ nor $V_2$ is $S_n$-invariant. In this case for $i = 1, 2$ we have that $W_i = \text{Ind}_{A_n}^{S_n}(V_i)$, and so

$$\text{Res}_{S_{n-1}}^{S_n}(W_i) \cong \text{Ind}_{A_{n-1}}^{A_n}(\text{Res}_{A_{n-1}}^S(V_i)) \cong \text{Ind}_{A_{n-1}}^{A_n}(U)$$

is reducible. Also, $\text{soc}(\text{Res}_{S_{n-1}}^{S_n}(W_1)) \cong \text{soc}(\text{Res}_{S_{n-1}}^{S_n}(W_2))$. If moreover $W_1 \not\cong W_2$, then the latter implies by [K3, Corollary 5.3] that (precisely) one of the modules $\text{Res}_{S_{n-1}}^{S_n}(W_i)$, $i = 1, 2$, is irreducible, a contradiction. It follows that $W_1 \cong W_2$ and so $V_2 \in \{V_1, V_1^g\}$ since $\text{Res}_{A_n}^S(W_1) = V_1 \oplus V_1^g$.

Thus Case IIb shows that among the extensions of $U$ to $A_n$, at most two of them can be non-$S_n$-invariant. Hence in the case where $U$ is $S_{n-1}$-invariant, there are at most three extensions to $A_n$. \hfill \square

9. Rank 3 permutation groups

To illustrate applicability of Theorem 6.5 to other primitive subgroups of $S_n$, in this section we consider finite simple classical groups $X$ acting as rank 3 permutation groups on $\Omega$, where $(\Omega, X)$ is one of the following:
(i) $\Omega$ is the set of 2-dimensional subspaces of $W = \mathbb{F}_q^d$ and $X = PSL(W) = PSL_d(q)$ with $d \geq 4$;
(ii) $\Omega$ is the set of singular 1-dimensional subspaces of $W = \mathbb{F}_q^d$ and $X = PSp(W)' = PSp_d(q)'$ with $2|d \geq 4$ (note that $Sp_d(q)' \neq Sp_d(q)$ only when $(d, q) = (4, 2)$);
(iii) $\Omega$ is the set of singular 1-dimensional subspaces of $W = \mathbb{F}_q^d$ and $X = P\Omega(W) = P\Omega_0^{\pm}(q)$ with $d \geq 5$;
(iv) $\Omega$ is the set of singular 1-dimensional subspaces of $W = \mathbb{F}_q^d$ and $X = PSU(W) = PSU_d(q)$ with $d \geq 4$.

According to the main result of [KaL], these families account for all the standard rank 3 permutation representations of finite simple classical groups.

**Lemma 9.1.** Let $X < \text{Sym}(\Omega) = S_n$, where $(\Omega, X)$ is as listed above, and $2|n$.

(i) Assume furthermore that $d \geq 6$ when $X = PSp_d(q)$ or $X = PSU_d(q)$, and $d \geq 7$ when $X = P\Omega_0^{\pm}(q)$. Then $f_3(X) \geq 6$.

(ii) Also, $f_3(X) \geq 5$ if $X = PSp_4(q)$, $PSU_4(q)$, $PSU_5(q)$, $\Omega_5(q)$, or $P\Omega_6^{\pm}(q)$.

**Proof.** (a) First we consider the case $X = PSL(W)$ with $W = \langle e_1, \ldots, e_d \rangle_{\mathbb{F}_q}$ of dimension $d \geq 4$. Then $X$ has at least six orbits on $\Omega_3$, with the following representatives:

- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_2, e_3 \rangle_{\mathbb{F}_q}$ (note that $\dim_{\mathbb{F}_q}(A + B + C) = 3$ and $A \cap B \cap C = 0$ here);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_1, e_2 + e_3 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 3$ and $A \cap B \cap C \neq 0$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_3, e_4 \rangle_{\mathbb{F}_q}$, $C = \langle e_1 + e_3, e_2 + e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\dim_{\mathbb{F}_q}(A \cap B), \dim_{\mathbb{F}_q}(A \cap C), \dim_{\mathbb{F}_q}(B \cap C) = \{0, 0, 0\}$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_2 + e_3, e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\dim_{\mathbb{F}_q}(A \cap B), \dim_{\mathbb{F}_q}(A \cap C), \dim_{\mathbb{F}_q}(B \cap C) = \{1, 0, 0\}$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_2, e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\dim_{\mathbb{F}_q}(A \cap B), \dim_{\mathbb{F}_q}(A \cap C), \dim_{\mathbb{F}_q}(B \cap C) = \{1, 1, 0\}$);
- $A = \langle e_1, e_2 \rangle_{\mathbb{F}_q}$, $B = \langle e_1, e_3 \rangle_{\mathbb{F}_q}$, $C = \langle e_1, e_4 \rangle_{\mathbb{F}_q}$ (here, $\dim_{\mathbb{F}_q}(A + B + C) = 4$ and $\dim_{\mathbb{F}_q}(A \cap B \cap C) = 1$).

(b) In the remaining cases of (i), the assumption on $d$ implies that $W$ contains a non-degenerate 6-dimensional subspace with hyperbolic (Witt) basis $(e_1, e_2, e_3, f_1, f_2, f_3)$. Also, since $n = |\Omega|$ is even, $q$ is odd. We will write any element in $\Omega$ as $(a)$, the 1-space generated by a vector $a \in W$. In the $\Omega$-case, for any unordered triple $\pi = \{A, B, C\}$, with $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle \in \Omega$ and $\dim(A + B + C) = 3$, we can associate to it the Gram matrix $\Gamma$ of the bilinear form $(\cdot, \cdot)$ written in the basis $(a, b, c)$. Since changing to another basis of $A + B + C$ changes $\det(\Gamma)$ by a factor which belongs to the subgroup $F_0 := \mathbb{F}_q^2$ of $F := \mathbb{F}_q^\times$, we can associate to such $\pi$ a canonical element $\delta := \det(\Gamma)F_0 \in F/F_0$. Then the following unordered triples $\{A, B, C\} \in \Omega_3$ belong to disjoint $X$-orbits:

- $A = \langle e_1 \rangle$, $B = \langle e_2 \rangle$, $C = \langle e_1 + e_2 \rangle$ (note that $A + B + C$ is a 2-dimensional totally singular subspace here);
- In the $Sp/SU$-case: $A = \langle e_1 \rangle$, $B = \langle f_1 \rangle$, $C = \langle e_1 + \lambda f_1 \rangle$, where $\lambda = 1$ in the $Sp$-case and $\lambda^{q-1} = -1$ in the $SU$-case. Note that $A + B + C$ is a 2-dimensional non-degenerate subspace here;
• \( A = \langle e_1 \rangle, \quad B = \langle e_2 \rangle, \quad C = \langle e_3 \rangle \) (here, \( A + B + C \) is a 3-dimensional totally singular subspace);

• \( A = \langle e_1 \rangle, \quad B = \langle f_1 \rangle, \quad C = \langle e_2 \rangle \). Note that \( U := A + B + C \) is a 3-dimensional subspace with \( C = \text{rad}(U) \);

• \( A = \langle e_1 \rangle, \quad B = \langle f_1 \rangle, \quad C = \langle e_1 + e_2 \rangle \). Note that \( U := A + B + C \) is a 3-dimensional subspace with \( \dim \text{rad}(U) = 1 \) but \( A, B, C \neq \text{rad}(U) \). In fact, in the \( Sp \)-case, we get one more triple with the same \( A, B \), but with \( C' = \langle e_1 + f_1 + e_2 \rangle \) – note that \( A \perp C \), but no two of \( A, B, C' \) are orthogonal to each other;

• In the \( \Omega/SU \)-case: \( A = \langle e_1 \rangle, \quad B = \langle f_1 \rangle, \quad C = \langle e_2 + \lambda f_2 + e_1 - \lambda f_1 \rangle \), where \( \lambda \in F \) in the \( \Omega \)-case and \( \lambda = 1 \) in the \( SU \)-case. Here, \( W := A + B + C \) is a 3-dimensional non-degenerate subspace. Furthermore, in the \( \Omega \)-case, we have \( \delta = -2\lambda f_0 \), whence we can choose \( \lambda \) so that \( \delta = F_0 \), respectively \( \delta \neq F_0 \).

(c) Ignoring the vectors \( e_3, f_3 \), the arguments in (b) also show that \( f_2(X) \geq 5 \) if \( X = PSp_4(q), \quad PSU_4(q), \quad PSU_5(q), \quad \Omega_5(q), \) or \( \Omega_6^2(q) \).

\[ \square \]

**Lemma 9.2.** Let \( X < \text{Sym}(\Omega) = S_n \), where \( (\Omega, X) \) is as listed above, that is, \( d \geq 4 \) when \( X = PSL_d(q), \quad PSU_d(q), \) or \( PSp_d(q), \) and \( d \geq 5 \) when \( X = P\Omega_4^\pm(q) \). Suppose that \( p = 2|n. \) Then \( h(X) \leq 3 \) if \( X = P\Omega_4^+, \quad 4|d \geq 8, \) and \( h(X) \leq 2 \) otherwise.

**Proof.** As mentioned in the proof of Lemma 9.1, \( 2|n \) implies that \( q \) is odd. We follow the proof of Lemma 7.2 and its notation. First, \( \dim H^2(X, F) \), which is the 2-rank of the Schur multiplier of \( X \), is \( \leq 2 \) if \( X = P\Omega_4^+(q) \) with \( 4|d \geq 8, \) and \( \leq 1 \) otherwise by [KIL, Theorem 5.1.4]. Hence it suffices to show that

\[ \dim \text{Hom}(X_1, (F, +)) \leq 1, \]

where \( X_1 \) is the point stabilizer in \( X \) of a point in \( \Omega \). Without loss we may replace \( X \) by its central cover \( SL_d(q), \quad SU_d(q), \quad Sp_d(q), \) or \( \Omega_d^2(q) \). We also fix a basis \( (e_1, \ldots, e_d) \) of the natural module \( W \) of \( X \).

Consider the case \( X = SL_d(q) \). Then \( X_1 = \text{Stab}_X(\langle e_1, e_2 \rangle) = Q \rtimes Y, \) where \( |Q| \) is a \( q \)-power and \( Y = (SL_d(q) \times SL_{d-2}(q)) \rtimes C_2^{-2}. \) Since \( q \) is odd, we have that \( O^2(Q) = Q \) and \( O^2(SL_e(q)) = SL_e(q) \) for any \( e \geq 2 \). Hence (9.1) follows.

From now on we may assume \( X \neq SL_d(q) \). We can then choose \( e_1 \) to be singular and let \( X_1 = \text{Stab}_X(\langle e_1 \rangle) \).

Let \( X = SU_d(q) \). Then \( X_1 = Q \rtimes Y, \) where \( |Q| \) is a \( q \)-power and \( Y = SU_{d-2}(q) \rtimes C_2^{-2}. \) As \( d \geq 4 \) and \( q \) is odd, we see that \( O^2(SU_{d-2}(q)) = SU_{d-2}(q) \), and so (9.1) follows.

Suppose \( X = Sp_d(q) \). Then \( X_1 = Q \rtimes Y, \) where \( |Q| \) is a \( q \)-power and \( Y = Sp_{d-2}(q) \rtimes C_2^{-1}. \) As \( d \geq 4 \) and \( q \) is odd, we see that \( O^2(Sp_{d-2}(q)) = Sp_{d-2}(q), \) yielding (9.1).

Suppose \( X = \Omega_d^\pm(q) \). Then \( X_1 = Q \rtimes Y, \) where \( |Q| \) is a \( q \)-power and \( Y = \Omega_d^{-}(q) \rtimes C_2^{-1}. \) As \( d \geq 5 \) and \( q \) is odd, we see that \( O^2(\Omega_d^{-2}(q)) = \Omega_d^{-2}(q) \), and so we are done. \( \square \)

**Proof of Theorem 1.3.** Assume the contrary: \( \text{Res}_X^A V \) is irreducible; in particular, \( V \) is irreducible. Note that \( X \) is not 2-transitive on \( \Omega \). Hence, by the main results of [KS1, KS2] we must have that \( p = 2 \) or 3.

(i) In the case \( X = PSp_4(2)^5 \cong A_6, \) we have \( n = 15, \) and so \( \dim V \geq 13 \) (see [Dec]), whereas the largest dimension of \( b(X) \) of irreducible \( \mathbb{F}X \)-modules is at most 10, cf. [JLPW].
Thus $V$ is reducible over $X$. Next, in the cases $X = SU_4(2) \cong PSp_4(3)$, respectively $SL_4(2), Sp_6(2)$, we have $n \geq 27, 35, 63$, and $b(X) \leq 81, 70, 512$, respectively, according to [JLPW]. Certainly, $Res^A_X V$ is reducible if $dim V > b(X)$. So we must have that $dim V \leq b(X)$. Therefore, $Res^A_X V$ is reducible if $dim V > b(X)$. Since $Res^A_X V$ is reducible if $dim V > b(X)$, we have $n \geq 27, 35, 63$, and $b(X) \leq 81, 70, 512$, respectively, according to [JLPW]. Certainly, $Res^A_X V$ is reducible if $dim V > b(X)$. So we must have that $dim V \leq b(X)$. Since $b(X) < (n^2 - 5n + 2)/2$, by [GT1, Lemma 6.1] we see that $V \cong D^{(n-1,1)}$ is isomorphic to the heart of the natural permutation module $\Omega$. As in the proof of Lemma 8.1, we conclude that $X$ is 2-transitive on $\Omega$, a contradiction.

(ii) We may now assume that $X$ is not isomorphic to any of the groups considered in (i). Direct computation shows that $2^{(n-8)/4} > |X|^{1/2}$. Since $Res^A_X V$ is irreducible, we must have that

$$ (9.2) \quad dim(V) < 2^{(n-8)/4}, $$

which implies by Propositions 4.1 and 4.3 that $V$ extends to $S_n$. Applying the Main Theorem and Theorem 3.10 of [KS1], we again arrive at the contradiction that $X$ is 2-transitive in the case $p = 3$, as well as in the case $p = 2 \nmid n$.

Thus we have shown that $p = 2|n$. The upper bound (9.2) implies by Theorem 3.2 that $d_3(V) > d_1(V)$. Also, $f_1(X) = 1$, $f_2(X) = 2$ by Lemma 6.3, and $e_3(X) \geq h(X) + 1$ by Lemmas 9.1 and 9.2. Now we can apply Theorem 6.5.

Non-standard rank 3 permutation representations of finite classical groups, as well as other primitive subgroups of $A_n$, will be considered elsewhere.

References


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