Level Compatibility in the Passage from Modular Symbols to Cup Products

by

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For Erin

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Abstract

For a positive integer M and an odd prime p, Sharifi defined a map ϖ_M from the first homology group of the modular curve $X_1(M)$ with $\mathbb{Z}_p$-coefficients to a second Galois cohomology group over $\mathbb{Q}(\mu_M)$ with restricted ramification and $\mathbb{Z}_p(2)$-coefficients that takes Manin symbols to certain cup products of cyclotomic M-units. Fukaya and Kato showed that if $p|M$ and $p \geq 5$, then ϖ_{Mp} and ϖ_M are compatible via the map of homology induced by the quotient $X_1(Mp) \to X_1(M)$ and corestriction from $\mathbb{Q}(\mu_{Mp})$ to $\mathbb{Q}(\mu_M)$. We show that for a prime $\ell \nmid M$, $\ell \neq p \geq 5$, the maps $\varpi_{M\ell}$ and ϖ_M are again compatible under a certain combination of the two standard degeneracy maps from level $M\ell$ to level M and corestriction.

CHAPTER 1

Introduction

It was conjectured by Sharifi in 2011 [**Sha11**] that there exists a correspondence between the geometry of the upper half plane near the cusp at infinity and the arithmetic of cyclotomic fields.

Specifically, for this introduction, let $\mathbb{H}^*$ denote the extended complex upper half plane with $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}^*$ via Möbius transformations. Let N be a positive integer and $\Gamma_1(N)$ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then the modular curve $X_1(N)$ is the quotient $\Gamma_1(N)\backslash\mathbb{H}^*$. By the "geometry of the upper half plane" we now mean the first homology group of $X_1(N)$ with $\mathbb{Z}_p$-coefficients for an odd prime p, $H_1(X_1(N), \mathbb{Z}_p)$. Let $G_{\mathbb{Q}(\mu_N),S_p}$ denote the Galois group of the maximal unramified outside of the set of primes dividing p, S_p, extension of $\mathbb{Q}(\mu_N)$. By the "arithmetic of cyclotomic fields" we mean the second cohomology group of $G_{\mathbb{Q}(\mu_N),S_p}$ with coefficients in $\mathbb{Z}_p(2)$, $H^2(G_{\mathbb{Q}(\mu_N),S_p}, \mathbb{Z}_p(2))$. Taking the fixed part under complex conjugation of both these spaces, denoted with a superscript $^+$, and reducing the modular side modulo the Eisenstein ideal I, the correspondence is between

$$\begin{array}{rcl} \text{Geometry of } X_1(N) \bmod I & \longleftrightarrow & \text{Arithmetic of } \mathbb{Q}(\mu_N) \\ H_1(X_1(N), \mathbb{Z}_p)^+/IH_1(X_1(N), \mathbb{Z}_p)^+ & \longleftrightarrow & H^2(G_{\mathbb{Q}(\mu_N),S_p}, \mathbb{Z}_p(2))^+. \end{array}$$

The work of Fukaya and Kato in 2012 [**FK12**] proves Sharifi's conjecture in many cases. It is this work which provides us with many of the tools we use throughout.

One of the major implications of Sharifi's conjecture is that the so-called Iwasawa Main Conjecture, which was first proven by Mazur and Wiles in [**MW84**] and further

To aid in understanding throughout this introduction we will be a bit imprecise, opting to delay all of the correct specific details for the later chapters.

simplified and/or generalized by Wiles [**Wil90**], Rubin [**Rub91**], and Ohta [**Oht99**], [**Oht00**], [**Oht03**], [**Oht05**], follows as a direct consequence. For further brevity here, and since it plays no role in our subsequent work, we present only a simplified version of the main conjecture.

For $r \geq 1$, let ζ_{p^r} be a primitive p^rth root of unity, and let $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))\{p\}$ denote the p-power part of the ideal class group of the cyclotomic field $\mathbb{Q}(\zeta_{p^r})$. Taking an inverse limit with respect to norm maps, we set

$$X = \varprojlim_{r} \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))\{p\},$$

which has a decomposition $X = X^+ \oplus X^-$ under the action of complex conjugation.

Let $K = \bigcup_r \mathbb{Q}(\zeta_{p^r})$ and define $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(K/\mathbb{Q})]]$. Then we again have a decomposition $\Lambda = \Lambda^+ \times \Lambda^-$ under complex conjugation. Let $\zeta \in \Lambda^-$ denote the equivariant p-adic Riemann zeta function, where we note as in [**FK12**, §0], we have eliminated the pole of the p-adic zeta function and denoted by ζ this modified version of the usual ζ-function for simplicity.

With this, the Iwasawa main conjecture, as proven by Mazur and Wiles, states that the characteristic ideal of X^- equals the characteristic ideal of $\Lambda^-/(\zeta)$. In some sense, the work of Mazur and Wiles considers only a map Υ, whereas Sharifi's conjecture considers both Υ and its conjectural inverse ϖ, giving a much deeper understanding of the spaces involved.

For any integer $N \geq 3$, let $C_1(N) = X_1(N) - Y_1(N)$ denote the cusps of $X_1(N)$ with $C_1^0(N)$ being the subset of "nonzero cusps". Then space $H_1(X_1(N), C_1^0(N), \mathbb{Z}_p)$ has a presentation by Manin symbols of the form $[a : b]_N$ for $a, b \in \mathbb{Z}/N\mathbb{Z} - \{0\}$ with $(a, b) = (1)$, modulo some relations; we note here that these are modifications of usual Manin symbols [**Man72**] by an Atkin-Lehner involution. We consider elements of $H^2(G_{\mathbb{Q}(\mu_N),S_N}, \mathbb{Z}_p(2))$ which are cup products of cyclotomic N-units, $(1-\zeta_N^i, 1-\zeta_N^j)_N$. With this we may describe one of the maps in Sharifi's conjecture [**Sha11**] by

$$\begin{array}{rcl} H_1(X_1(N), C_1^0(N), \mathbb{Z}_p) & \xrightarrow{\varpi_N} & H^2(G_{\mathbb{Q}(\mu_N),S_N}, \mathbb{Z}_p(2)) \\ {[a:b]_N} & \longmapsto & (1-\zeta_N^a, 1-\zeta_N^b)_N. \end{array}$$

The work of Busuioc in [**Bus08**] also considers the analogous map working with usual Manin symbols.

We also note that we may consider the restriction of this map to $H_1(X_1(N), \mathbb{Z}_p)$ (as in [**FK12**, Thm. 5.2.5]), which we also simply denote by ϖ_N:

$$H_1(X_1(N), \mathbb{Z}_p) \xrightarrow{\varpi_N} H^2(G_{\mathbb{Q}(\mu_N),S_p}, \mathbb{Z}_p(2)).$$

In their work on Sharifi's conjecture, Fukaya and Kato consider infinite towers of homology and cohomology groups, ultimately, among other things, giving that the following diagram for $M \geq 1$ divisible by $p \geq 5$ commutes

$$\begin{array}{ccc} H_1(X_1(Mp), C_1^0(Mp), \mathbb{Z}_p) & \xrightarrow{\varpi_{Mp}} & H^2(G_{\mathbb{Q}(\mu_{Mp}),S_M}, \mathbb{Z}_p(2)) \\ \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \mathrm{Cor}} \\ H_1(X_1(M), C_1^0(M), \mathbb{Z}_p) & \xrightarrow{\varpi_M} & H^2(G_{\mathbb{Q}(\mu_M),S_M}, \mathbb{Z}_p(2)), \end{array}$$

where ϵ_1 is the natural map induced by $X_1(Mp) \to X_1(M)$, and Cor is corestriction (see [**FK12**, Thm. 5.2.3 (2)]).

Rather than examining what happens as the level changes with respect to powers of p, we are instead interested in determining a map such that the following diagram commutes

$$\begin{array}{ccc} H_1(X_1(M\ell), C_1^0(M\ell), \mathbb{Z}_p) & \xrightarrow{\varpi_{M\ell}} & H^2(G_{\mathbb{Q}(\mu_{M\ell}),S_{M\ell}}, \mathbb{Z}_p(2)) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Cor}} \\ H_1(X_1(M), C_1^0(M), \mathbb{Z}_p) & \xrightarrow{\varpi_M} & H^2(G_{\mathbb{Q}(\mu_M),S_{M\ell}}, \mathbb{Z}_p(2)) \end{array} \tag{1.0.1}$$

for primes $\ell \nmid M$. A large hurdle here is that we may no longer use the ϵ_1 map for our left vertical arrows. Instead we must rely on a combination of degeneracy maps, as we will now explain.

To describe the degeneracy maps we will be using on homology we first need to describe them on modular curves. For notation, set $N = M\ell$. There are two natural degeneracy maps which change the level of a modular curve from N to M. We begin by viewing the curve $X_1(K)$ for any positive integer K as classifying isomorphism classes of pairs (E, P), where E is an elliptic curve and P is a point of order K on E.

The degeneracy maps from $X_1(N)$ to $X_1(M)$ are given by

$$\begin{aligned}\epsilon_1 : (E, P) &\longmapsto (E/\langle P\rangle, P),\\ \epsilon_\ell : (E, P) &\longmapsto (E, \ell P).\end{aligned}$$

Via the identification $X_1(K) = \Gamma_1(K)\backslash\mathbb{H}^*$, we may view ϵ_1 and ϵ_ℓ as maps from $\Gamma_1(N)\backslash\mathbb{H}^*$ to $\Gamma_1(M)\backslash\mathbb{H}^*$ given by

$$\begin{aligned}\epsilon_1 : z &\longmapsto z,\\ \epsilon_\ell : z &\longmapsto z\ell.\end{aligned}$$

These degeneracy maps induce degeneracy maps on modular symbols $\{\alpha, \beta\}_N$, which we think of as classes of paths in $\mathbb{H}^*$ from α to β:

$$\begin{aligned}\epsilon_1 : \{\alpha, \beta\}_N &\longmapsto \{\alpha, \beta\}_M,\\ \epsilon_\ell : \{\alpha, \beta\}_N &\longmapsto \{\ell\alpha, \ell\beta\}_M.\end{aligned}$$

In turn, these maps on modular symbols induce (nontrivial) degeneracy maps on Manin symbols, which are the objects that provide (modulo some relations) a presentation of $H^1(X_1(N), C_1^0(N), \mathbb{Z}_p)$. Describing the action of ϵ_ℓ on Manin symbols $[u : v]_N$ is quite easy,

$$\epsilon_\ell : [u : v]_N \longmapsto [u : v]_M,$$

but describing the action of ϵ_1 on a general symbol is considerably more difficult. It is, however, much easier to describe this action for ϵ_1 on Manin symbols of the form $[\ell u : v]_N$:

$$\epsilon_1 : [\ell u : v]_N \longmapsto [u : v]_M.$$

We also note here that the diamond operator $\langle d\rangle$ for $d \in (\mathbb{Z}/M\mathbb{Z})^\times$ acts on a Manin symbol via

$$\langle d\rangle[u : v]_M = [d^{-1}u : d^{-1}v]_M.$$

With this, we are now in a position to describe our main result, Theorem 7.2.3:

Theorem. *Let p, ℓ be distinct primes such that $p \geq 5$ with $p|M$ and $\ell \nmid M$. Set $N = M\ell$. The diagram*

$$\begin{array}{ccc} H_1(X_1(N),\mathbb{Z}_p) & \xrightarrow{\varpi_N} & H^2(G_{\mathbb{Q}(\mu_N),S_p},\mathbb{Z}_p(2)) \\ \Big\downarrow{\scriptstyle \epsilon_1-\langle\ell\rangle\epsilon_\ell} & & \Big\downarrow{\scriptstyle \mathrm{Cor}} \\ H_1(X_1(M),\mathbb{Z}_p) & \xrightarrow{\varpi_M} & H^2(G_{\mathbb{Q}(\mu_M),S_p},\mathbb{Z}_p(2)) \end{array}$$

commutes.

In other words, the map $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ provides a compatible degeneracy map between levels in the passage from modular symbols to cup products via ϖ.

1.1. Motivation

The proof of Theorem 7.2.3 will require substantial technical details including numerous auxiliary commutative diagrams as well as passage to the infinite level in two Iwasawa towers. Despite this, one may fairly easily establish evidence as to why the map $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ should be the correct map, as we will do now to conclude our introduction.

One of the biggest obstacles to proving Theorem 7.2.3 lies in describing the corestriction map on the image of an arbitrary symbol $[u : v]_N$ under ϖ_N; however, describing the each of these maps on symbols of the form $[\ell u : v]_N$ is a much simpler task.

Let ζ_N denote a primitive Nth root of unity, and let ζ_M denote the Mth root of unity such that $\zeta_N^\ell = \zeta_M$. Recall that $H_1(X_1(N), C_1^0(N), \mathbb{Z}_p)$ has a presentation by Manin symbols $[u : v]_N$ with $u, v \in \mathbb{Z}/N\mathbb{Z} - \{0\}$ (modulo some relations) and cup products of cyclotomic N-units, $(1 - \zeta_N^i, 1 - \zeta_N^j)_N$, are elements of $H^2(G_{\mathbb{Q}(\mu_N),S_N}, \mathbb{Z}_p(2))$. Then the map from Sharifi's conjecture is:

$$\begin{array}{rcl} H_1(X_1(N), C_1^0(N), \mathbb{Z}_p) & \xrightarrow{\varpi_N} & H^2(G_{\mathbb{Q}(\mu_N),S_N}, \mathbb{Z}_p(2)) \\ {[u : v]_N} & \longmapsto & (1 - \zeta_N^u, 1 - \zeta_N^v)_N. \end{array}$$

Thus, on the Manin symbols we are currently interested in,

$$[\ell u : v]_N \overset{\varpi_N}{\longmapsto} (1 - \zeta_N^{\ell u}, 1 - \zeta_N^v)_N.$$

On elements of the form $(1-\zeta_N^{\ell u}, 1-\zeta_N^v)_N$ the corestriction map is easily described as

$$\begin{array}{ccc} H^2(G_{\mathbb{Q}(\mu_N),S_N}, \mathbb{Z}_p(2)) & \stackrel{\mathrm{Cor}}{\longrightarrow} & H^2(G_{\mathbb{Q}(\mu_M),S_N}, \mathbb{Z}_p(2)) \\ (1-\zeta_N^{\ell u}, 1-\zeta_N^v)_N & \longmapsto & \left(1-\zeta_M^u, N_{\mathbb{Q}(\mu_N)/\mathbb{Q}(\mu_M)}(1-\zeta_N^v)\right)_N, \end{array}$$

where $N_{\mathbb{Q}(\mu_N)/\mathbb{Q}(\mu_M)}$ denotes the norm map. The corestriction map can be defined using a norm in this way since the element $\zeta_N^{\ell u}$ equals ζ_M^u. (See [**CF93**] for details on this norm property as well as the action of corestriction on cup products.)

We also note here that our map $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ can easily be described on Manin symbols of this form:

$$\epsilon_1([\ell u : v]_N) - \langle \ell \rangle \epsilon_\ell([\ell u : v]_N) = [u : v]_M - \langle \ell \rangle [\ell u : v]_M.$$

Before proceeding to our computations we need to explain the details of the norm map on the element $1-\zeta_N^v$. We have

$$\mathbb{Q}(\mu_N) = \mathbb{Q}(\mu_M)\mathbb{Q}(\mu_\ell),$$

and the conjugates of $1-\zeta_N^v$ in $\mathbb{Q}(\mu_N)/\mathbb{Q}(\mu_M)$ will all be of the form $1-\zeta_N^v\zeta_\ell^k$ for some k, where ζ_ℓ is a primitive ℓth root of unity. Alternatively, we can also write the conjugates as $1-\zeta_N^{v\cdot j}$ where $j \equiv 1 \pmod{M}$ and $\ell \nmid j$. Hence we have $j = 1+Mk$ for some k such that $\ell \nmid 1+Mk$. Thus, our conjugates become $1-\zeta_N^{v(1+Mk)}$ for $\ell \nmid 1+Mk$. If $\ell | 1+Mk$, then

$$\begin{aligned} 1-\zeta_N^{v(1+Mk)} &= 1+\zeta_N^{v\ell\left(\frac{1+Mk}{\ell}\right)} \\ &= 1-\zeta_M^{v\left(\frac{1+Mk}{\ell}\right)} \\ &= 1-\zeta_M^{v\ell^{-1}(1+Mk)}. \end{aligned}$$

However, $\zeta_M^{v\ell^{-1}Mk} = \left(\zeta_M^M\right)^{v\ell^{-1}k} = 1$, so the above reduces to just $1-\zeta_M^{v\ell^{-1}}$.

In general, we have that

$$x^\ell - y^\ell = \prod_{k=1}^{\ell}(x-\zeta_\ell^k y),$$

so evaluating this at $x = 1$ and $y = \zeta_N^v$ we see

$$1 - \zeta_N^{v\ell} = \prod_{k=1}^{\ell} (1 - \zeta_N^v \zeta_\ell^k).$$

Since $\zeta_N^\ell = \zeta_M$ it follows that

$$1 - \zeta_M^{v} = \prod_{k=1}^{\ell} (1 - \zeta_N^v \zeta_\ell^k).$$

So we see the product over all $1 \leq k \leq \ell$ is $1 - \zeta_M^v$, but in our norm we are not including the k such that $\ell | 1 + Mk$; therefore, it follows

$$N_{\mathbb{Q}(\mu_N)/\mathbb{Q}(\mu_M)}(1 - \zeta_N^v) = \frac{\prod_{k=1}^{\ell}(1 - \zeta_N^v \zeta_\ell^k)}{1 - \zeta_M^{v\ell^{-1}}} = \frac{1 - \zeta_M^v}{1 - \zeta_M^{v\ell^{-1}}}.$$

Now that we have described our maps we can provide the evidence as to why we believe $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ is the correct map to make the diagram in (1.0.1) commute. Recall the diagram is given by

$$\begin{array}{ccc} H_1(X_1(N), C_1^0(N), \mathbb{Z}_p) & \xrightarrow{\varpi_N} & H^2(G_{\mathbb{Q}(\mu_N), S_N}, \mathbb{Z}_p(2)) \\ \downarrow \epsilon_1 - \langle \ell \rangle \epsilon_\ell & & \downarrow \mathrm{Cor} \\ H_1(X_1(M), C_1^0(M), \mathbb{Z}_p) & \xrightarrow{\varpi_M} & H^2(G_{\mathbb{Q}(\mu_M), S_N}, \mathbb{Z}_p(2)). \end{array}$$

Tracing the images of an element of the form $[\ell u : v]_N$ we see:

$$\begin{array}{ccc} [\ell u : v]_N & \xmapsto{\varpi_N} & (1 - \zeta_N^{\ell u}, 1 - \zeta_N^v)_N \\ & & \downarrow \mathrm{Cor} \\ \downarrow \epsilon_1 - \langle \ell \rangle \epsilon_\ell & & \big(1 - \zeta_M^u, N_{\mathbb{Q}(\mu_N)/\mathbb{Q}(\mu_M)}(1 - \zeta_N^v)\big)_M \\ & & \| \\ [u : v]_M - [u : \ell^{-1} v]_M & \xmapsto{\varpi_M} & \left(1 - \zeta_M^u, \frac{1 - \zeta_M^v}{1 - \zeta_M^{v\ell^{-1}}}\right)_M \end{array}$$

where the image under ϖ_M follows from the properties of cup products. Thus the diagram commutes on elements of the form $[\ell u : v]_N$, and hence Theorem 7.2.3 holds for these elements.

1.2. Plan for Subsequent Chapters

Chapters 2-5 will establish much of the background on the spaces and functions with which we will be primarily concerned. Chapter 2 gives a basic reminder of modular forms and modular curves as well as some of the properties and maps associated with them; we also describe Λ-adic forms in this chapter.

An introduction to modular and Manin symbols appears in Chapter 3, including a description of the action of Hecke operators on these symbols. We close the chapter with the important property which states that $H^1(X_1(N), C_1(N), \mathbb{Z}_p)$ admits a presentation by Manin symbols.

In an effort to reduce the sporadic definition of spaces or elements across multiple sections or chapters we have chosen to include the majority of our notation and conventions in Chapter 4. While this chapter does not encompass all of the notation we may use throughout, the bulk of the integral notation is set here.

Chapter 5 provides the definitions and descriptions of the maps which play a key role in the proof of our main result, Theorem 7.2.3. Descriptions for many of the maps are given at both infinite and finite level, as we will be working in multiple infinite towers.

The proof of Theorem 7.2.3 will require decomposing the map ϖ_K into two maps, $z_K^\sharp$ and $\infty_K(0,1)$, then examining several diagrams involving them. Chapter 6 establishes the commutativity of many diagrams related to the zeta maps of Fukaya and Kato [**FK12**, Ch. 3] as well as their finite level counterparts.

Chapter 7 begins our work on the ϖ_K-map. Section 7.1 considers specialization at a cusp, while in Section 7.2 we put the main theorems of Chapter 6 and Section 7.1 together to prove our main result concerning the compatibility of our degeneracy map $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ and the map ϖ_K.

CHAPTER 2

Modular Forms and Modular Curves

Here we review the concepts of both classical and Λ-adic modular forms as well as modular curves. These objects play a central role in the "geometry" portion of the correspondence we are considering.

2.1. Modular Forms

Let $\mathrm{GL}_2^+(\mathbb{R})$ denote the matrix group of 2×2 real-valued matrices with positive determinant. To begin our discussion of classical modular forms we first introduce an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$. For any $\tau \in \mathbb{H}$, the matrix $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ acts via the fractional linear (Möbius) transformation

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d},$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For any $k \in \mathbb{Z}$ we define the weight-k operator $|_k$ on $f : \mathbb{H} \to \mathbb{C}$ to be

$$\left(f|_k\gamma\right)(\tau) = \det(\gamma)^{k-1}(c\tau + d)^{-k} f(\gamma(\tau)),$$

for each $\tau \in \mathbb{H}$. The relationship between f and $f|_k$ for various γ will play a central role in the definition of modular forms.

Let $\mathrm{SL}_2(\mathbb{Z})$ denote the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ consisting of integer-valued matrices of determinant 1. We shall refer to this group as the modular group, and say that a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some N where

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

For a congruence subgroup Γ, the minimal N for which $\Gamma(N) \subseteq \Gamma$ is said to be the level of Γ. We further explicitly define one standard congruence subgroup:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Definition 2.1.1. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let k be an integer. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ if:*

(1) f is holomorphic,

(2) $f|_k\gamma = f$ for each $\gamma \in \Gamma$, and

(3) $f|_k\gamma$ is holomorphic at ∞ for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of level N. Clearly

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma,$$

and hence if f is any weight k modular form with respect to Γ then $f(\tau + N) = f(\tau)$ for all $\tau \in \mathbb{H}$. One may then use this fact (see [**DS07**, §1.1] for details) to show that f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n q_N^n, \quad \text{where } q_N = e^{2\pi i\tau/N}.$$

Using this we can extend Definition 2.1.1:

If, in addition,

(4) $a_0 = 0$ *in the Fourier expansion of* $f|_k\gamma$ *for each* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

then f is a cusp form of weight k with respect to Γ.

The space of modular forms of weight k with respect to Γ is denoted $M_k(\Gamma)$, and the space of cusp forms is $S_k(\Gamma)$. We note that $M_k(\Gamma) = 0$ for $k < 0$. Since we will be concentrating mostly on the congruence subgroup $\Gamma_1(N)$, we will use the convention that $M_k(N) = M_k(\Gamma_1(N))$, and similarly, $S_k(N) = S_k(\Gamma_1(N))$.

We can also define a subspace of modular forms based upon the ring in which the coefficients of the q-expansion lie. Let Γ be a congruence subgroup. For any subring

$A \subset \mathbb{C}$ we define

$$M_k(\Gamma)_A = \{f \in M_k(\Gamma) : a_n(f) \in A \text{ for all } n \geq 0\},$$

and

$$S_k(\Gamma)_A = M_k(\Gamma)_A \cap S_k(\Gamma).$$

2.2. Hecke Operators

There exist a family of operators known as the double coset operators $M_k(N) \to M_k(N)$ which are linear and also take cusp forms to cusp forms. These double coset operators are known as Hecke operators.

Definition 2.2.1. *Let $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. The diamond operator for d is the Hecke operator given by*

$$\begin{aligned} \langle d \rangle : M_k(\Gamma_1(N)) &\longrightarrow M_k(\Gamma_1(N)) \\ \langle d \rangle f &= f|_k \gamma \end{aligned}$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right) \in \Gamma_1(N)$ with $\delta \equiv d \pmod N$.

For any character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$, we have that $M_k(N, \chi)$ is the χ-eigenspace of the diamond operators

$$M_k(N, \chi) = \{f \in M_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f \text{ for each } d \in (\mathbb{Z}/N\mathbb{Z})^\times\}.$$

Definition 2.2.2. *Let p be a prime. The Hecke operator $T(p)$ is given by*

$$T(p) : M_k(\Gamma_1(N)) \longrightarrow M_k(\Gamma_1(N))$$

with

$$T(p)f = \begin{cases} \displaystyle\sum_{j=0}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} & \textit{if } p|N \\ \displaystyle\sum_{j=0}^{p-1} f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} + f|_k \left[\begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right] & \textit{if } p \nmid N \end{cases}$$

where m, n are chosen such that $mp - nN = 1$.

Typically, when $p|N$ the operator $T(p)$ is denoted $U(p)$.

Proposition 2.2.3. *Let $d, e \in (\mathbb{Z}/N\mathbb{Z})^\times$ and let p, q be primes. Then*

(1) $\langle d\rangle T(p) = T(p)\langle d\rangle$,

(2) $\langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle$, and

(3) $T(p)T(q) = T(q)T(p)$.

Proof. See [**DS07**, Prop. 5.2.4]. □

We may further extend the definitions of $\langle d\rangle$ and $T(p)$ removing the requirements that $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and p a prime. For $n \in \mathbb{N}$ with $(n, N) = 1$ we set $\langle n\rangle = \langle n \pmod N\rangle$, and if $(n, N) > 1$, $\langle n\rangle = 0$. It follows that the mapping $n \mapsto \langle n\rangle$ is totally multiplicative, i.e., $\langle nm\rangle = \langle n\rangle\langle m\rangle$ for all $n, m \in \mathbb{N}$.

Next, we set $T(1) = 1$ (the identity operator) and inductively define

$$T(p^r) = T(p)T(p^{r-1}) - p^{k-1}\langle p\rangle T(p^{r-2}) \text{ for } r \geq 2.$$

One should now note that by induction and Proposition 2.2.3 it follows that $T(p^r)T(q^s)$ $= T(q^s)T(p^r)$ for distinct primes p and q, and $r, s \in \mathbb{N}$. With this we may multiplicatively extend to $T(n)$ for any $n \in \mathbb{N}$:

$$T(n) = \prod T(p_i^{e_i}),$$

where $n = \prod p_i^{e_i}$ is the factorization of n into powers of distinct primes. Hence, it follows that $T(nm) = T(n)T(m)$ if $(n, m) = 1$.

Considering the q-expansion of a modular form allows us to provide a simple description of the manner in which the Hecke operators $T(p)$ act.

Proposition 2.2.4. *Let $f \in M_k(N, \chi)$ with*

$$f = \sum_{n=0}^{\infty} a_n(f)q^n.$$

For all integers $m > 0$, the q-expansion of $T(m)f$ is given by

$$T(m)f = \sum_{n=0}^{\infty} a_n(T(m)f)q^n$$

where

$$a_n(T(m)f) = \sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f)$$

for each $n \geq 0$.

Proof. See [**DS07**, Prop. 5.3.1]. □

The space $M_k(N, \chi)$ is naturally a $\mathbb{C}$-vector space and the Hecke operators $T(n)$ act linearly, so we may discuss their corresponding eigenvectors and eigenvalues. We refer to modular forms which are simultaneous eigenvectors for all of the Hecke operators as eigenforms.

Definition 2.2.5. *A modular form* $f \in M_k(N, \chi)$ *is a Hecke eigenform if* $T(n)f = c_n f$ *for some* $c_n \in \mathbb{C}$ *for each* $n > 0$*. If, in addition,* $a_1(f) = 1$ *we say that* f *is a normalized Hecke eigenform.*

We note that if $f = \sum_{n=0}^{\infty} a_n(f) q^n$ is a normalized Hecke eigenform then $T(i)f = a_i(f)f$ for each i. One may see [**RS11**, Def. 3.5.9] for further details.

Next we define an algebra generated by the Hecke operators $T(n)$ and $\langle n \rangle$.

Definition 2.2.6. *Let* $k > 0$ *and* A *a subalgebra of* $\mathbb{C}$*. For any* A*-submodule* V *of* $M_k(N)_A$ *which is stable under the action of* $T(n)$ *and* $\langle n \rangle$ *for each* $n \in \mathbb{N}$*, we define the Hecke algebra* $\mathcal{H}(V)_A$ *to be the* A*-subalgebra of* $\mathrm{End}_A(V)$ *generated by* $\{T(n), \langle n \rangle : n \in \mathbb{N}\}$.

By Proposition 2.2.3 and the definitions of $T(n)$ and $\langle n \rangle$ it follows that $\mathcal{H}(V)_A$ is a commutative algebra; and since $T(1) \in \mathcal{H}(V)_A$ it is actually a commutative algebra with unity.

For notation, we let

$$\mathcal{H}_k(N)_A = \mathcal{H}(M_k(N)_A)_A$$

and

$$h_k(N)_A = \mathcal{H}(S_k(N)_A)_A.$$

If $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a Dirichlet character taking values in A we define the χ-eigenspaces of these Hecke algebras as one would expect. We will use the convention that A is omitted above whenever $A = \mathbb{C}$.

One should also take note that while the Hecke algebra is a priori generated by both the $T(n)$ and $\langle n \rangle$ operators, since $\langle n \rangle$ acts on $M_k(N, \chi)_A$ and $S_k(N, \chi)_A$ as multiplication by a constant, it follows that $\mathcal{H}_k(N, \chi)_A$ and $h_k(N, \chi)_A$ are generated as A-modules by only $\{T(n)\}_{n \in \mathbb{N}}$, and then in turn by $T(p)$ for primes p.

2.3. Ordinary Parts

We begin by presenting a general category-theoretic description of ordinary parts as detailed in [**Eme99**], which in a sense provides a conceptual understanding. Then we return to the realm of modular forms and introduce Hida's original definition as developed in [**Hid81**].

Let U be an indeterminant, and let $\mathscr{C}$ denote the full subcategory of $\mathbb{Z}_p[U]$-modules which are finitely generated as $\mathbb{Z}_p$-modules; this is an abelian category. For an object $M \in \mathscr{C}$, there exists a $\mathbb{Z}_p$-module morphism

$$\mathbb{Z}_p[U] \to \mathrm{End}_{\mathbb{Z}_p}(M).$$

Since M is finitely generated as a $\mathbb{Z}_p$-module, it follows that $\mathrm{End}_{\mathbb{Z}_p}(M)$ is a finite $\mathbb{Z}_p$-algebra, and thus the image of $\mathbb{Z}_p[U]$ in $\mathrm{End}_{\mathbb{Z}_p}(M)$ is also a finite $\mathbb{Z}_p$-algebra; we denote it by A.

Now, any finite $\mathbb{Z}_p$-algebra factors as a product of local rings and hence $A = \prod A_i$ where each A_i is local. One may project U onto these local factors of A; some of the projections will be contained in the maximal ideal while others will not, and will hence be units. Let A^{ord} denote the product of local factors in which the image of U is a unit.

Definition 2.3.1. *The ordinary part of $M \in \mathscr{C}$ is*

$$M^{\mathrm{ord}} = M \otimes_A A^{\mathrm{ord}}.$$

As a direct factor, A^{ord} is a flat A-algebra, and hence it follows that taking ordinary parts is an exact functor. If we take our indeterminant U to be the Hecke operator $T(p)$, we can discuss the ordinary parts of spaces such as M_k and S_k.

Now we will present Hida's original construction from [**Hid81**], which was later refined in [**Hid93**], of the ordinary parts of M_k and S_k.

Lemma 2.3.2. *Let K be a finite extension of $\mathbb{Q}_p$ with $\mathcal{O}_K$ its p-adic ring of integers. For any commutative $\mathcal{O}_K$-algebra A of finite rank over $\mathcal{O}_K$ and for any $x \in A$, the limit $\lim_{n\to\infty} x^{n!}$ exists in A and is an idempotent of A.*

Proof. See [**Hid93**, Lemma 7.2.1]. □

With this in hand, let K be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}$, and let $\chi : (\mathbb{Z}/Np^r\mathbb{Z})^\times \to \mathcal{O}^\times$ be an arbitrary character with $r \geq 0$.

Definition 2.3.3. *The ordinary projector e of the Hecke algebra $\mathcal{H}(Np^r, \chi)_\mathcal{O}$ is given by*

$$e = \lim_{n\to\infty} T(p)^{n!}.$$

By Lemma 2.3.2 it follows that e exists in both $\mathcal{H}_k(Np^r, \chi)_\mathcal{O}$ and $h_k(Np^r, \chi)_\mathcal{O}$ and is an idempotent in each.

If $f \in M_k(Np, \chi)_\mathcal{O}$ is an eigenform of $T(p)$ with eigenvalue c_p then

$$ef = \begin{cases} f & \text{if } |c_p|_p = 1, \\ 0 & \text{if } |c_p|_p < 1, \end{cases}$$

where $|\cdot|_p$ denotes the p-adic absolute value. One should further note that if f is a normalized eigenform then the eigenvalue will be $a_p(f)$, which allows us to rewrite the above as

$$ef = \begin{cases} f & \text{if } |a_p(f)|_p = 1, \\ 0 & \text{if } |a_p(f)|_p < 1. \end{cases}$$

Definition 2.3.4. *We say that $f \in M_k(Np^r, \chi)_\mathcal{O}$ is ordinary if $ef = f$.*

Since e is an idempotent, it is clear that ef is ordinary for every $f \in M_k(Np^r, \chi)_\mathcal{O}$.

Definition 2.3.5. *For every $k \geq 0$, the spaces of weight k, level Np^r ordinary modular forms and ordinary cusp forms are given by*

$$M_k^{\mathrm{ord}}(Np^r, \chi)_\mathcal{O} = eM_k(Np^r, \chi)_\mathcal{O} = \{ef : f \in M_k(Np^r, \chi)_\mathcal{O}\}$$

and

$$S_k^{\mathrm{ord}}(Np^r,\chi)_{\mathcal{O}} = eS_k(Np^r,\chi)_{\mathcal{O}} = \{ef : f \in S_k(Np^r,\chi)_{\mathcal{O}}\},$$

respectively.

Their corresponding ordinary Hecke algebras are

$$\mathcal{H}_k^{\mathrm{ord}}(Np^r,\chi)_{\mathcal{O}} = e\mathcal{H}_k(Np^r,\chi)_{\mathcal{O}} = \{eH : H \in \mathcal{H}_k(Np^r,\chi)_{\mathcal{O}}\},$$

and

$$h_k^{\mathrm{ord}}(Np^r,\chi)_{\mathcal{O}} = eh_k(Np^r,\chi)_{\mathcal{O}} = \{eH : H \in h_k(Np^r,\chi)_{\mathcal{O}}\}.$$

2.4. Modular Curves

Recall from Section 2.1.1 that any element of $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathbb{H}$ via fractional linear transformations. These matrices still act in the same manner if we instead consider $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, the extended upper half-plane. Now, for any congruence subgroup Γ we define the modular curve $Y(\Gamma)$ to be the quotient space of orbits under Γ, i.e.,

$$Y(\Gamma) = \Gamma\backslash\mathbb{H} = \{\Gamma\tau : \tau \in \mathbb{H}\},$$

where

$$\Gamma\tau = \left\{ \frac{a\tau+b}{c\tau+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}.$$

The curve $Y(\Gamma)$ is a (noncompact) Riemann surface. In spirit, we may think of the compact modular curve $X(\Gamma)$ as given by

$$X(\Gamma) = \Gamma\backslash\mathbb{H}^* = \{\Gamma\tau : \tau \in \mathbb{H}^*\};$$

however, care must be taken in specifying the topology used for an actual definition of this space. For details see [**DS07**, Ch. 2]. Of particular interest to us will be the modular curves associated to the congruence subgroup $\Gamma_1(N)$. These curves are denoted $Y_1(N)$ and $X_1(N)$.

2.4.1. Cusps

We can define the cusps of the modular curve $X(\Gamma) = \Gamma\backslash\mathbb{H}^*$ to be the set of orbits stemming from rational points or infinity, i.e., $\{\Gamma\tau : \tau \in \mathbb{Q} \cup \{\infty\}\}$. We will provide a deeper description of the cusps of $X_1(N)$, which will be of importance in the later sections. We proceed as in [**FK12**, §1.3].

Let

$$P_N = \left\{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 : (a,b) = (1) \text{ in } \mathbb{Z}/N\mathbb{Z}\right\},$$

and define an equivalence relation $\sim$ on P_N by $(a,b) \sim (a',b')$ if and only if $a' = ua$ and $b' \equiv ub \pmod{a}$ where $u \in \{\pm 1\}$. Then we make the following identifications:

$$\begin{aligned}\{\text{cusps of } X_1(N)(\mathbb{C})\} = \Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q}) = \Gamma_1(N)\backslash \operatorname{SL}_2(\mathbb{Z})/\{\pm 1\}\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \\ = P_N/\sim,\end{aligned}$$

where the second identification is given by

$$\text{class of } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z}) \longleftrightarrow \text{class of } \frac{a}{c} = g\infty \in \mathbb{P}^1(\mathbb{Q}),$$

and the third is given by

$$\text{class of } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z}) \longleftrightarrow \text{class of } (c,d) \in P_N.$$

Of particular interest to us will be the nonzero cusps of $X_1(N)$, so we define the 0-cusps to be the classes of those a/c with $(c,N) = 1$. Using the identifications above, the 0-cusps of $X_1(N)$ correspond to $(a,b) \in P_N$ with $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, which depend only on a.

2.5. Modular Curves as Moduli Spaces

There are some instances where we will need to interpret a modular curve as a moduli space. For integers $m, M \geq 1$ such that $m + M \geq 5$, we have the closed modular curve

$X(m,M)$ and the open modular curve $Y(m,M)$ defined over $\mathbb{Z}[1/mM]$. These schemes are defined as follows.

The modular curve $Y(m,M)$ is the moduli space which associates for each $\mathbb{Z}[1/mM]$-scheme S a triple (E,e_1,e_2) such that E is an elliptic curve over S and e_1,e_2 are sections of $E\to S$ such that $me_1=Me_2=0$ and such that the map

$$\begin{array}{rcl}\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/M\mathbb{Z} & \longrightarrow & E\\ (a,b) & \longmapsto & ae_1+be_2\end{array}$$

of S-group schemes is injective. The scheme $X(m,M)$ is then the integral closure of the projective j-line $\mathbb{P}^1_{\mathbb{Z}[1/Mm]}$ in $Y(m,M)$. (See [**DR73**], [**KM85**] for more details.)

As a last bit of notation, we have that for $M\geq 3$, $X_1(M)=X(1,M)$ and $Y_1(M)=Y(1,M)$. Also for $M\geq 4$, $X(M)=X(M,M)$, and we have that $\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ acts on $X(M)$ via the map $(E,e_1,e_2)\mapsto(E,e_1',e_2')$ given by

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} e_1\\ e_2\end{pmatrix},$$

for each $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$.

Let L be such that $m|L$ and $M|L$, and define

$$G_L(m,M)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z}): a\equiv 1, b\equiv 0 \bmod m,\ c\equiv 0, d\equiv 1 \bmod M\right\}.$$

Then the modular curve $X(m,M)\otimes\mathbb{Z}[1/L]$ is the quotient of $X(L)$ by the action of $G_L(m,M)$.

Lastly, let us consider the formal Laurent series ring $\mathbb{Z}[1/N,\zeta_N]((q^{1/N}))$. For $(a,b)\in P_N$, we let

$$\infty_N(a,b):\mathrm{Spec}(\mathbb{Z}[1/N,\zeta_N]((q^{1/N})))\longrightarrow Y_1(N)\otimes\mathbb{Z}[1/N]$$

be the map corresponding to the N-torsion point $q^{a/N}\zeta_N^b$ (mod $q^{\mathbb{Z}}$) of the q-Tate elliptic curve E_q over $\mathbb{Z}[1/N,\zeta_N]((q^{1/N}))$. Then the morphism $\infty_N(a,b)$ gives the cusp of $X_1(N)\otimes\mathbb{Q}(\zeta_N)$ corresponding to $(a,b)\in P_N$.

2.5.1. Hecke Operators

As we have given an alternative description of the modular curve above, we should also describe the action of Hecke operators on these spaces.

The action of the diamond operator $\langle a\rangle$ on $Y(m, M)$, with $a \in (\mathbb{Z}/L\mathbb{Z})^\times$ for some L such that $m|L$ and $M|L$, is compatible with the action of $\langle a\rangle \otimes 1$ on $Y_1(M) \otimes \mathbb{Q}(\zeta_m)$, both being induced by the automorphism $\left(\begin{smallmatrix} 1/a & 0 \\ 0 & a \end{smallmatrix}\right)$.

Let ℓ be a prime. We want to describe the Hecke operators $T(\ell)$ and dual Hecke operators $T^*(\ell)$. Let $m, M \geq 1$ with $m+M \geq 5$, and let L be such that $m|L$ and $M\ell|L$. Similar to the above, we may further describe $Y(m, M(\ell))$ as the quotient of $Y(L)$ by the subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_L(m, M) : c \equiv 0 \bmod M\ell \right\}.$$

As mentioned in the introduction, we have the degeneracy map ϵ_ℓ, which is compatible with the multiplication-by-ℓ map: $z \mapsto \ell z$, and the ϵ_1 map, which is compatible with the identity map: $z \mapsto z$, as maps between the modular curves $X_1(N) \to X_1(M)$. Now that we are viewing the modular curve as a moduli space more care must be taken in providing a description of these degeneracy maps.

The moduli space $Y(m, M(\ell))$ over $\mathbb{Z}[1/mM\ell]$ represents the isomorphism classes of quadruples (E, e_1, e_2, C), where (E, e_1, e_2) is the triple as in the description of the modular curve as a moduli space in the start of this section, and C is a subgroup scheme of E which is étale locally isomorphic to $\mathbb{Z}/M\ell\mathbb{Z}$ and étale locally generated by some e_2' with $e_2 = \ell e_2'$.

With this, we define

$$\begin{aligned} \epsilon_1, \epsilon_\ell : Y(m, M\ell) &\longrightarrow Y(m, M) \otimes \mathbb{Z}[1/\ell] \\ (E, e_1, e_2, C) &\overset{\epsilon_1}{\longmapsto} (E, e_1, e_2), \\ (E, e_1, e_2, C) &\overset{\epsilon_\ell}{\longmapsto} (E/(MC), e_1 \bmod MC, e_2' \bmod MC). \end{aligned}$$

With these maps, we may define the Hecke and dual Hecke operators which, among other spaces, act on $H^i(X(m, M)(\mathbb{C}), \mathbb{Z})$ and $H^i(Y(m, M)(\mathbb{C}), \mathbb{Z})$ via

$$T(\ell) = (\epsilon_\ell)_* \epsilon_1^*, \qquad T^*(\ell) = (\epsilon_1)_* \epsilon_\ell^*.$$

The standard properties of Hecke operators follow. If $n = \ell^e$ with $\ell \nmid m$ a prime and $e \geq 0$, we may define both $T(n)$ and $T^*(n)$. If $\ell | M$ then

$$T(\ell^e) = T(\ell)^e, \qquad T^*(\ell^e) = T^*(\ell)^e.$$

And if $\ell \nmid M$, we define the operators inductively: $T(1) = T^*(1) = 1$;

$$\begin{aligned} T(\ell^{e+2}) &= T(\ell)T(\ell^{e+1}) + T(\ell^e)\langle \ell \rangle \ell, \\ T^*(\ell^{e+2}) &= T^*(\ell)T^*(\ell^{e+1}) + T^*(\ell^e)\langle \ell \rangle^{-1} \ell. \end{aligned}$$

Further, if $n = \prod_i \ell_i^{e_i}$ with each ℓ_i a distinct prime not dividing m, then

$$T(n) = \prod_i T(\ell_i^{e_i}), \qquad T^*(n) = \prod_i T^*(\ell_i^{e_i}).$$

Provided $(n, N) = 1$, $(n_1, N) = 1$ and $(n_2, N) = 1$, the actions of T and T^* also commute with themselves and with the diamond operator $\langle a \rangle$ for $(a, M) = 1$ in the following sense:

$$\begin{aligned} T(n_1)T(n_2) = T(n_2)T(n_1), &\qquad T^*(n_1)T^*(n_2) = T^*(n_2)T^*(n_1), \\ T(n)\langle a \rangle = \langle a \rangle T(n), &\qquad T^*(n)\langle a \rangle = \langle a \rangle T^*(n). \end{aligned}$$

We also have the relation

$$T(n) = T^*(n)\langle n \rangle$$

if $(n, mM) = 1$.

As a last point, since we will be focusing more on the dual Hecke operator $T^*(p)$, when we discuss the ordinary component of $H^1_{\text{ét}}(X_1(M) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$ we mean the part of the space for which $T^*(p)$ acts bijectively – not $T(p)$ acting bijectively as usual. Hence, for any $x \in H^1_{\text{ét}}(X_1(M) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$, the ordinary component $x^{\text{ord}} \in H^1_{\text{ét}}(X_1(M) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)^{\text{ord}}$ is given by

$$x^{\text{ord}} = \lim_{n \to \infty} T^*(p)^{n!} x.$$

The definition of $H^1_{\text{ét}}(Y_1(M) \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$ is similar.

Again, since we are mainly using the dual Hecke operators, we now define the dual Hecke algebras. Let $\mathfrak{h}(M)_{\mathbb{Z}}$ be the subring of $\mathrm{End}_{\mathbb{Z}}(H^1(X_1(M))(\mathbb{C}), \mathbb{Z})$ and $\mathfrak{H}(M)_{\mathbb{Z}}$ the subring of $\mathrm{End}_{\mathbb{Z}}(H^1(Y_1(M))(\mathbb{C}), \mathbb{Z})$ generated over $\mathbb{Z}$ by $T^*(n)$ for $n \geq 1$ and $\langle n \rangle$ for

$(n, M) = 1$. We also define the p-adic dual Hecke algebras $\mathfrak{h}(M)_{\mathbb{Z}_p} = \mathfrak{h}(M)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathfrak{H}(M)_{\mathbb{Z}_p} = \mathfrak{H}(M)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which are generated by $T^*(n)$ for $n \geq 1$ over $\mathbb{Z}_p$. We note that the diamond operators are not needed in the $\mathbb{Z}_p$-case since for any prime $q \nmid M$, $\langle q \rangle^{-1} = q^{-1}(T^*(q^2) - T^*(q)^2)$.

2.6. Λ-adic Forms

We now turn our attention to certain families of p-adic modular forms known as Λ-adic forms. We present here only the fundamentals of Λ-adic forms. For a more complete description see [**Hid93**, Ch. 7] or [**BCG**].

Let p be an odd prime with N prime to p. We have a decomposition of the units in $\mathbb{Z}_p$,

$$\mathbb{Z}_p^\times = \mu_{p-1}(\mathbb{Z}_p) \times (1 + p\mathbb{Z}_p)$$

where elements may be written as $x = \omega(x)\langle x \rangle \in \mathbb{Z}_p^\times$ with $\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ the unique Dirichlet character of order p such that $\omega(x) \equiv x \bmod p$ for all $x \in \mathbb{Z}_p^\times$. Let us fix some notation which will be used throughout this section:

- $u = 1 + p$, a topological generator of $1 + p\mathbb{Z}_p$.
- χ, a fixed Dirichlet character of modulus Np.
- χ_ζ, a Dirichlet character of conductor p^r associated to a p-power root of unity as follows: If ζ has exact order p^{r-1} with $r \geq 1$, define χ_ζ by mapping the image of u in $(\mathbb{Z}/p^r\mathbb{Z})^\times$ to ζ.
- ν_p, the p-adic cyclotomic character defined by $\zeta^g = \zeta^{\nu_p(g)}$ for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and all p-power roots of unity ζ.

We will also let $\Lambda = \mathbb{Z}_p[[X]]$; this is the usual Iwasawa algebra. In addition, let K be a finite field extension of the quotient field of Λ, and let I denote the integral closure of Λ in K. We define a character

$$\kappa : (1 + p\mathbb{Z}_p) \longrightarrow \Lambda^\times,$$

which maps u to $1 + X$. The character κ may also be viewed as a Galois character via the natural map $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. In the subject of Λ-adic forms, κ plays the Λ-adic analogue of the cyclotomic character ν_p. Lastly, we define for each integer k and for each p-power root of unity ζ the homomorphism

$$\nu_{k,\zeta} : \Lambda \to \mathbb{Z}_p[\zeta]$$

to be the unique continuous $\mathbb{Z}_p$-algebra map satisfying $\nu_{k,\zeta}(1+X) = \zeta u^{k-2}$.

Definition 2.6.1. *A Λ-adic form F of level N and character $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{C}^\times$ is a formal q-expansion*

$$F = \sum_{n=0}^{\infty} a_n(F) q^n \in I[[q]]$$

such that for all homomorphisms $\nu : I \to \overline{\mathbb{Q}}_p$ extending the homomorphisms $\nu_{k,\zeta}$ the q-expansion

$$f_\nu = \nu(F) = \sum_{n=0}^{\infty} \nu(a_n(F)) q^n \in \overline{\mathbb{Q}}_p[[q]]$$

is the image under a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of the q-expansion in $\overline{\mathbb{Q}}_p[[q]]$ of a classical modular form of weight k, level Np^r and character $\chi_\nu = \chi\omega^{2-k}\chi_\zeta$, i.e., f_ν is an element of $M_k(Np^r, \chi_\nu)$.

It may be beneficial for one to think of the specializations $\nu(F) = f_\nu$ as $F(\zeta u^{k-2} - 1)$ for some $k > 1$, so that $f_\nu = F(\zeta u^{k-2} - 1) \in M_k(Np^r, \chi_\nu)$.

We write $M_\Lambda(N, \chi)$ for the space of Λ-adic forms of level N and character χ, and we let

$$M_\Lambda(N) = \bigoplus_\chi M_\Lambda(N, \chi)$$

denote the space of all Λ-adic forms, where the sum is taken over all characters $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{C}^\times$.

A Λ-adic form F is cuspidal if all the specializations f_ν are cusp forms. We denote the space of Λ-adic cusp forms by $S_\Lambda(N)$. It has the obvious decomposition

$$S_\Lambda(N) = \bigoplus_\chi S_\Lambda(N, \chi).$$

To end our discussion of Λ-adic forms, we introduce a Hecke operator on $M_\Lambda(N)$. First, recall that κ was defined so that

$$\kappa(u^s) = (1+X)^s \in \Lambda^\times.$$

If we define $\hat{n} = \omega(n)^{-1}n = u^{s(n)}$, where $s(n) = \log(\hat{n})/\log(u)$ and log is the p-adic logarithm, we then have, for all integers n prime to p

$$\kappa(\hat{n})(u^k - 1) = \kappa(u^{s(n)})(u^k - 1) = u^{ks(n)} = \omega(n)^{-k} n^k.$$

Then we define for each $F \in M_\Lambda(N)$ the coefficients of a formal q-expansion for $T(n)F$, for $n \geq 1$, by

$$a_m(T(n)F)(X) = \sum_{b|(m,n)} \kappa(\hat{b})(X)\chi(b)b^{-1}a_{mn/b^2}(F)(X).$$

We can now specialize the above equation at $\zeta u^k - 1$ to find

$$\begin{aligned} a_m(T(n)F)(\zeta u^k - 1) &= \sum_{b|(m,n)} \kappa(\hat{b})(\zeta u^k - 1)\chi(b)b^{-1}a_{mn/b^2}(F(\zeta u^k - 1)) \\ &= \sum_{b|(m,n)} \chi\omega^{-k}(b)b^{k-1}a_{mn/b^2}(F(\zeta u^k - 1)) \\ &= a_m(T(n)F(\zeta u^k - 1)), \end{aligned}$$

noting that $F(\zeta u^k - 1)$ is a classical modular form of weight $k+2$. Hence, we have that $(T(n)F)(\zeta u^k - 1) = T(n)(F(\zeta u^k - 1))$; therefore, $T(n)$ is well defined and thus gives us the Hecke operator $T(n)$ acting on $M_\Lambda(N)$.

CHAPTER 3

Modular Symbols

We begin by recalling the definition of the congruence subgroup $\Gamma_1(N)$:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

and note that the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ on $z \in \mathbb{H}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

We also recall that $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and $X_1(N) = \Gamma_1(N)\backslash\mathbb{H}^*$.

Now, we may begin describing modular symbols of weight 2. Let $\mathbb{M}_2$ denote the free abelian group with basis $\{\alpha, \beta\}$ for $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ modulo the relations

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0. \tag{3.0.1}$$

The space $\mathbb{M}_2$ is torsion-free and hence

$$\{\alpha, \alpha\} = 0 \quad \text{and} \quad \{\alpha, \beta\} = -\{\beta, \alpha\};$$

in particular, this combined with equation (3.0.1) gives

$$\{\alpha, \beta\} + \{\beta, \gamma\} = \{\alpha, \gamma\},$$

which provides a useful tool for computations.

There is also an action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{M}_2$ given by

$$g\{\alpha, \beta\} = \{g(\alpha), g(\beta)\},$$

for each $g \in \mathrm{GL}_2(\mathbb{Q})$. This allows us to define the group of modular symbols with respect to our above congruence subgroups:

$$\mathbb{M}_2(\Gamma_1(N)) = \mathbb{M}_2 / \langle x - g(x) : x \in \mathbb{M}_2,\ g \in \Gamma_1(N) \rangle.$$

One of the key elements which will lead us from modular symbols into the more specialized Manin symbols is the following:

Proposition 3.0.2 (Manin)**.** *Let N be a positive integer and $r_0, \ldots, r_m$ be a set of coset representatives for $\Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})$. Every $\{\alpha, \beta\} \in \mathbb{M}_2(\Gamma_1(N))$ is a $\mathbb{Z}$-linear combination of $r_0\{0, \infty\}, \ldots, r_m\{0, \infty\}$.*

Proof. See [**Cre97**, §2.1.6] or [**MTT86**, §2]. □

In other words, the $\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$-span of $\{0, \infty\}$ is $\mathbb{Z}$-finitely generated and equal to $M_2(\Gamma_1(N))$.

Proposition 3.0.3. *For relatively prime $c, d \in \mathbb{Z}/N\mathbb{Z}$, there is a bijection*

$$\begin{array}{ccc} (\mathbb{Z}/N\mathbb{Z})^2 & \longleftrightarrow & \{\textit{right cosets of } \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})\} \\ (c : d) & \longleftrightarrow & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}). \end{array}$$

Proof. See [**Ste07**, Prop. 1.27]. □

Before moving on, we have a final statement on general modular symbols, again due to Manin. The proposition involves the homology of a modular curve relative to the cusps. This the same as the usual homology of the curve, but in addition to the typical closed loops one would consider we also allow paths whose endpoints lie in the set of cusps on the curve.

Proposition 3.0.4 (Manin)**.** *For $N \geq 4$, there is a natural isomorphism*

$$\phi : \mathbb{M}_2(\Gamma_1(N)) \longrightarrow H_1(X_1(N)(\mathbb{C}), C_1(N), \mathbb{Z})$$

that sends a linear combination of symbols to the image as the sum of classes of the geodesic paths on $X_1(N)$ that each symbol represents.

Proof. This follows by [**AS86**, Theorem 4.2] since $\Gamma_1(N)$ has no elements of finite order for $N \geq 4$ (as in [**DS07**, Exercise 2.3.7]). □

The preceding proposition allows us to think of a modular symbol as the homology class, relative to the cusps, of the image of a path from α to β in $\mathbb{H}^*$. Via Poincaré duality, $H_1(X_1(N)(\mathbb{C}), C_1(N), \mathbb{Z}) \cong H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})(1)$, where (1) is the Tate twist $\otimes_{\mathbb{Z}}\mathbb{Z}(1)$ with $\mathbb{Z}(1) = \mathbb{Z} \cdot 2\pi i \subset \mathbb{C}$, we may regard $\{\alpha, \beta\}_N \in H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})(1)$.

3.1. Manin Symbols

For this section we begin by fixing a set of coset representatives, $r_0, \ldots, r_m$ for the space $\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. We now consider formal symbols $[r_k]'$ for $k = 0, \ldots m$; these are Manin symbols. However, we will introduce an alternative notation soon.

Equip $\{[r_k]'\}_{k=0}^m$ with a right $\mathrm{SL}_2(\mathbb{Z})$-action given by

$$[r_i]' \cdot g = [r_j]',$$

where $\Gamma_1(N)r_j = \Gamma_1(N)r_i g$. This can be extended to any $\gamma \in \Gamma_1(N)r_i$ as

$$[\gamma]' = [\Gamma_1(N)\gamma]' = [r_i]',$$

so that $[\gamma]' \cdot g = [\gamma g]'$.

Let

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

and define $\mathbb{M}$ to be the free group on $[r_0]', \ldots, [r_m]'$ modulo the subgroup generated by $[r_i]' + [r_i]'\sigma$ and $[r_i]' + [r_i]'\tau + [r_i]'\tau^2$ for each i.

For notation, let $[r_i] = r_i\{0, \infty\} = \{r_i(0), r_i(\infty)\}$. Then we have the following theorem due to Manin which provides a finite presentation of $\mathbb{M}_2(\Gamma_1(N))$ via Manin symbols.

Theorem 3.1.1 (Manin)**.** *For $N \geq 4$, there is an isomorphism*

$$\Psi : \mathbb{M} \xrightarrow{\sim} \mathbb{M}_2(\Gamma_1(N))$$

given by $[r_i]' \longmapsto [r_i] = r_i\{0, \infty\}$.

Proof. See [**Man72**, §1.5]. □

Proposition 3.0.3 and Theorem 3.1.1 provide us with a more convenient way of writing Manin symbols, i.e., if we let $r_i = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $u \equiv c \pmod{N}$ and $v \equiv d \pmod{N}$, then the Manin symbol $[r_i]'$ can be written $[u : v]$.

3.2. Hecke Operators on Modular Symbols

On modular symbols we have

$$T(p)\left(\{\alpha,\beta\}\right) = \begin{cases} \begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\{\alpha,\beta\} + \displaystyle\sum_{r=0}^{p-1}\begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}\{\alpha,\beta\} & \text{if } p \nmid N, \\ \displaystyle\sum_{r=0}^{p-1}\begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}\{\alpha,\beta\} & \text{if } p|N, \end{cases}$$

where $mp - nN = 1$.

Merel gave a method for computing the action of $T(p)$ on a Manin symbol of weight greater than or equal to 2 in [**Mer94**]. And more specifically, in just the weight 2 case Zagier provided an algorithm for this computation in [**Zag90**], namely $T(p)([x]') = \sum_{g \in C_p} [xg]'$ where C_p is a particular, non-unique, choice of a set of matrices. For example,

$$T(2)([r_i]') = [r_i]'\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + [r_i]'\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + [r_i]'\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + [r_i]'\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

3.3. Presentation of the Homology of a Modular Curve

We now will define a variation of the Manin symbols described above. Let $N \geq 4$ and $u, v \in \mathbb{Z}/N\mathbb{Z}$ with $(u, v, N) = 1$. We define a Manin symbol

$$[u : v]_N = \left\{\frac{-d}{bN}, \frac{-c}{aN}\right\}_N$$

which denotes the class of the geodesic from $-d/bN$ to $-c/aN$ with $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$, and $u \equiv c, v \equiv d \pmod{N}$. These are the usual Manin symbols as described in Section 3.1 with the Atkin-Lehner involution applied, i.e., let $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ act on the usual Manin symbols.

Proposition 3.3.1. *The relative homology group* $H_1(X_1(N), C_1(N), \mathbb{Z})$ *has a presentation as a* $\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times]$*-module for the* $(\mathbb{Z}/N\mathbb{Z})^\times$*-action of diamond operators with generators* $[u:v]_N$ *for* $u, v \in \mathbb{Z}/N\mathbb{Z}$ *with* $(u,v) = (1)$*, subject to the following relations:*

- $[-u:-v]_N = [u:v]_N$,
- $[u:v]_N + [-v:u]_N = 0$,
- $[u:v]_N - [u:u+v]_N - [u+v:v]_N = 0$, *and*
- $\langle j \rangle [u:v]_N = [j^{-1}u : j^{-1}v]_N$.

Proof. See [**Man72**, Thm. 1.9]. □

Complex conjugation provides a decomposition of $H_1(X_1(N), C_1(N), \mathbb{Z}[1/2])$ into (± 1)-eigenspaces, denoted $H_1(X_1(N), C_1(N), \mathbb{Z}[1/2])^\pm$. If we take the $+1$-eigenspace, i.e., the part fixed by complex conjugation, $H_1(X_1(N), C_1(N), \mathbb{Z}[1/2])^+$ then we must introduce one final relation:

- $[u:v]_N - [-u:v]_N = 0$.

If we take symbols $[u:v]_N$, where $u, v \in \mathbb{Z}/N\mathbb{Z}$ with each u, v nonzero, subject to the above relations we will obtain a presentation of $H_1(X_1(N), C_1^0(N), \mathbb{Z}[1/2])^+$, i.e., the $+1$-eigenspace of homology relative to the nonzero cusps, as a $\mathbb{Z}[1/2][(\mathbb{Z}/N\mathbb{Z})^\times]$-module. Computing the boundary, which is described below, of Manin symbols of the form $[u:v]_N$ for nonzero u, v in $\mathbb{Z}/N\mathbb{Z}$ with $(u,v) = 1$ one sees that these symbols are elements of the homology relative to the nonzero cusps. On the other hand, the boundary of symbols of the form $[u:0]_N$ for $(u, N) = 1$ with $1 \le u \le N/2$ have nonzero boundary at distinct 0-cusps. From this it follows that the presentation of the relative homology group is given as described.

We end this section with a description of the boundary of a Manin symbol $[u:v]_N$ following [**FK12**, Lemma 3.3.11]. Let R and S be positive divisors of N such that $(u) = (R)$ and $(v) = (S)$ in $\mathbb{Z}/N\mathbb{Z}$. The boundary of $[u:v]_N$ is given by

$$\infty_N\left(\frac{Nv'}{R}, u'\right) - \infty_N\left(\frac{Nu''}{S}, v''\right),$$

where $u', v', u'', v'' \in \mathbb{Z}/N\mathbb{Z}$ with $(u/R)u' \equiv 1 \pmod{N/R}$, $vv' \equiv 1 \pmod{R}$, $uu'' \equiv 1 \pmod{S}$, $(v/S)v'' \equiv 1 \pmod{N/S}$, and here by an abuse of notation $\infty_N(a,b)$ represents the cusp over $\mathbb{C}$.

CHAPTER 4

Notation

This chapter will set the notation which will be used for the remainder of this thesis. We will follow closely the notation used in [**FK12**]. Additionally, to avoid an abuse of notation, K will be used rather than the typical N or M to denote the level. We will also make use of the convention that for a scheme S over $\mathbb{Q}$, $H^n_{\text{ét}}(S \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$ will be denoted simply by $H^n_{\text{ét}}(S)$. Lastly, for the modular curve $X_1(K)$ we note that $H^1(X_1(K)(\mathbb{C}), \mathbb{Z}_p) \cong H^1_{\text{ét}}(K)$.

In some instances we will need to use a variation of the standard tensor product known as a topological tensor product. To define this tensor product let A and B be topological abelian groups such that $A \cong \varprojlim_{A'} A/A'$ and $B \cong \varprojlim_{B'} B/B'$ where A' and B' range over all open subsets of A and B, respectively. The topological tensor product of A and B, denoted $A\hat{\otimes}B$, is then given by

$$A\hat{\otimes}B = \varprojlim_{A',B'} A/A' \otimes_{\mathbb{Z}} B/B',$$

where once again the limit is taken over all open subgroups A', B' of A, B, respectively.

Let p, ℓ be distinct primes with $p \geq 5$, let $r \geq 0$, fix K such that $p|K$, and fix a primitive p^nth root of unity, ζ_{p^n}. We then let

$$\begin{aligned} H &= \varprojlim_{r} H^1_{\text{ét}}(X_1(Kp^r))^{\text{ord}}, \\ \widetilde{H} &= \varprojlim_{r} H^1_{\text{ét}}(Y_1(Kp^r))^{\text{ord}}, \end{aligned}$$

where the inverse limits are taken with respect to the natural maps of modular curves, and we note that $H \subset \widetilde{H}$.

Also, we let

$$\mathfrak{h} = \varprojlim_r \mathfrak{h}(Np^r)^{\text{ord}}_{\mathbb{Z}_p},$$

$$\mathfrak{H} = \varprojlim_r \mathfrak{H}(Np^r)^{\text{ord}}_{\mathbb{Z}_p},$$

where $\mathfrak{h}(Np^r)_{\mathbb{Z}_p}$ and $\mathfrak{H}(Np^r)_{\mathbb{Z}_p}$ are as defined in Section 2.5.1.

Let $H^1_{\text{ét}}(Y_1(Kp^r))_0$ denote the part of $H^1_{\text{ét}}(Y_1(Kp^r))$ generated by Manin symbols of the form $[u : v]_K$ such that $u, v \in \mathbb{Z}/Kp^r\mathbb{Z}$ with $(u, v) = (1)$ and $u, v \neq 0$; this corresponds to the part of $H^1_{\text{ét}}(Y_1(Kp^r))$ whose boundaries at 0-cusps are 0. The space $H^1_{\text{ét}}(X_1(Kp^r))_0$ is defined similarly. With this we may define the projective limits of these spaces

$$H_0 = \varprojlim_r H^1_{\text{ét}}(X_1(Kp^r))^{\text{ord}}_0,$$

$$\widetilde{H}_0 = \varprojlim_r H^1_{\text{ét}}(Y_1(Kp^r))^{\text{ord}}_0.$$

Next, we define a few Iwasawa algebras:

$$\begin{aligned}
\boldsymbol{\Lambda} &= \varprojlim_{n,r} \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times \times (\mathbb{Z}/Kp^r\mathbb{Z})^\times], \\
\Lambda &= \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Kp^r\mathbb{Z})^\times] = \mathbb{Z}_p[[\mathbb{Z}_p^\times \times (\mathbb{Z}/K\mathbb{Z})^\times]], \\
\Lambda(p^\infty) &= \varprojlim_n \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times] = \mathbb{Z}_p[[\mathbb{Z}_p^\times]],
\end{aligned}$$

with $\Lambda \subset \boldsymbol{\Lambda}$.

In $\boldsymbol{\Lambda}$ the following identifications can be made

$$\begin{aligned}
(\mathbb{Z}/p^n\mathbb{Z})^\times &\longleftrightarrow \text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \\
a &\longleftrightarrow \sigma_a,
\end{aligned}$$

where $\sigma_a(\zeta_{p^n}) = \zeta_{p^n}^a$, and

$$\begin{aligned}
(\mathbb{Z}/Kp^r\mathbb{Z})^\times &\longrightarrow (\mathbb{Z}/Kp^r\mathbb{Z})^\times / \langle -1 \rangle \\
a &\longmapsto \langle a \rangle,
\end{aligned}$$

where the right side represents the group of diamond operators.

If we let $\mathbb{Q}(\zeta_{p^\infty}) = \bigcup_n \mathbb{Q}(\zeta_{p^n})$ then, by abuse of notation, we also make the following identification for σ_a:

$$\begin{aligned} \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) &\cong \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times \\ \sigma_a &\leftrightarrow a \end{aligned}$$

with $\sigma_a(\zeta_{p^n}) = \zeta_{p^n}^a$. Using this we may identify $\Lambda(p^\infty)$ with $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]]$.

Let

$$\begin{aligned} \mathfrak{Z} &= \varprojlim_n H^1(\mathbb{Z}[1/K, \zeta_{p^n}], H), \\ \widetilde{\mathfrak{Z}} &= \varprojlim_n H^1(\mathbb{Z}[1/K, \zeta_{p^n}], \widetilde{H}), \end{aligned}$$

where the inverse limits are taken with respect to corestriction, and we note $\mathfrak{Z} \subset \widetilde{\mathfrak{Z}}$ since $\varprojlim_n H^0(\mathbb{Z}[1/K, \zeta_{p^n}], \widetilde{H}/H) = 0$. Via the above identifications, we may regard $\widetilde{\mathfrak{Z}}$ as a $\mathbf{\Lambda}$-module.

Take $c, d \in \mathbb{Z}$ such that $c, d \equiv 1 \pmod p$, $c, d \not\equiv 1 \pmod{p^2}$ and $c, d \equiv 1 \pmod K$. Then we let

$$\begin{aligned} \lambda &= (c - \sigma_c)(d - \sigma_d\langle d\rangle) \in \mathbf{\Lambda}, \\ \mu &= p\left((1+Kp)^2 - \langle 1+Kp\rangle\right) \in \Lambda, \end{aligned}$$

each of which will play the role of the denominators in the codomains of maps in Chapter 5.

As in [**FK12**, 3.2.2], the ideal generated by λ in $\mathbf{\Lambda}$, as well as $\mathbf{\Lambda}\lambda^{-1} \subset Q(\mathbf{\Lambda})$, are independent of the choices of c, d, where for any commutative ring R, we let $Q(R)$ be the total quotient ring of R.

Let $\mathfrak{T}$ be the set of elements of $\widetilde{\mathfrak{Z}}$ killed by $1 - a\sigma_a$ for any a such that $(a, p) = 1$ and $a \equiv 1$ modulo the prime-to-p part of K. We note that $\mathfrak{T}(1)$ is killed by λ.

Now we define two $\Lambda(p^\infty)$-modules:

$$\begin{aligned} \mathfrak{N}_r &= \varprojlim_n H^1(\mathbb{Z}[1/K, \zeta_{p^n}], H^1_{\text{ét}}(X_1(Kp^r))) \\ \widetilde{\mathfrak{N}}_r &= \varprojlim_n H^1(\mathbb{Z}[1/K, \zeta_{p^n}], H^1_{\text{ét}}(Y_1(Kp^r))). \end{aligned}$$

The $\Lambda(p^\infty)$-torsion of $\widetilde{\mathfrak{N}}_r$ is killed by $1 - a\sigma_a$ for any $a \in \mathbb{Z}$ with $a \equiv 1 \pmod{Kp^r}$, as seen in [**FK12**, 3.1.3(4)].

As a final note, when working with multiple levels we will sometimes use a superscript ^{K} to denote the current level for clarity, e.g., $H^{K_1} = \varprojlim_r H^1_{\text{ét}}(X_1(K_1p^r))^{\text{ord}}$, and $\mathbf{\Lambda}^{K_1} = \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times \times (\mathbb{Z}/K_1p^r\mathbb{Z})^\times]$. We omit this superscript when the level is clear from context.

CHAPTER 5

Maps

Here we will present the maps which will play a key role in the subsequent chapters. Many of these maps were defined in [**FK12**]. To ease notation, any time a map is defined on a tensor product, but evaluated only on one component we are implying the other component is 1, e.g., if $f : A \otimes B \to C$ with $b \in B$, then $f(b) = f(1 \otimes b)$.

5.1. Siegel Units and Beilinson Elements

In order to describe the maps z_{1,Kp^∞,p^∞} and $z^\sharp_{Kp^\infty}$ in the next two sections we first need to introduce Siegel units and Beilinson elements. For a more thorough description, see [**Kat04**, §1], [**FK12**, SS2.2-2.4].

To describe these units we begin with a proposition of Kato [**Kat04**, Prop. 1.3].

Proposition 5.1.1. *Let E be an elliptic curve over a scheme S. Let c be an integer prime to* 6*. Then:*

(1) there exists a unique element ${}_c\theta_E$ of $\mathcal{O}(E - {}_cE)^\times$ satisfying the following conditions:

(a) ${}_c\theta_E$ has the Cartier divisor $c^2(0) - {}_cE$ on E, where (0) denotes the zero section of E, and ${}_cE = \ker(c : E \to E)$.

(b) $N_a({}_c\theta_E) = {}_c\theta_E$ for any integer a which is prime to c, where N_a is the norm map $\mathcal{O}(E - {}_{ac}E)^\times \to \mathcal{O}(E - {}_cE)^\times$ associated to the pullback by $a : E \to E$.

(2) if d is also an integer which is prime to 6*, we have an equality in $\mathcal{O}(E - {}_{cd}E)^\times$*

$$({}_d\theta_E)^{c^2}(c^*({}_d\theta_E))^{-1} = ({}_c\theta_E)^{d^2}(d^*({}_c\theta_E))^{-1}$$

where c^* (resp. d^*) denotes the pullback by multiplication by c (resp. d).

(3) *for* $\tau \in \mathbb{H}$ *and* $z \in \mathbb{C} - c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$, *let* ${}_c\theta(\tau, z)$ *be the value at* z *of* ${}_c\theta$ *on the elliptic curve* $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ *over* $\mathbb{C}$. *Then,*

$$ {}_c\theta(\tau, z) = {}_c\theta(t) = q^{\frac{c^2-1}{12}}(-t)^{\frac{c-c^2}{2}}\gamma_q(t)^{c^2}\gamma_q(t^c)^{-1} \in \mathbb{Z}[t, 1/t]((q))^\times, $$

where $q = e^{2\pi i\tau}$, $t = e^{2\pi iz}$, *and*

$$ \gamma_q(t) = \prod_{n\geq 0}(1 - q^n t) \cdot \prod_{n\geq 1}(1 - q^n t^{-1}) \in \mathbb{Z}[t, 1/t][[q]]^\times. $$

For $\alpha, \beta \in \mathbb{Q}/\mathbb{Z} - \{0\}$ and $c \in \mathbb{Z}$ such that c is prime to $6Mm$, the Siegel unit

$$ {}_cg_{\alpha,\beta} \in \mathcal{O}(Y(m, M))^\times $$

where $m, M \geq 1, m + M \geq 5$ such that $m\alpha = 0$ and $M\beta = 0$, is defined as follows. Let E be the universal elliptic curve over $Y(m, M)$. Write $\alpha = a/m \bmod \mathbb{Z}$, and $\beta = b/M \bmod \mathbb{Z}$ for $a, b \in \mathbb{Z}$. Then the Siegel unit ${}_cg_{\alpha,\beta}$ is defined to be

$$ {}_cg_{\alpha,\beta} = \iota^*_{\alpha,\beta}({}_c\theta_E) $$

where $\iota_{\alpha,\beta} = ae_1 + be_2 : Y(m, M) \to E - {}_cE$, and has a q-expansion

$$ {}_cg_{\alpha,\beta} = {}_c\theta(q^{a/m}\zeta_M^b) \in \mathbb{Z}[1/mM, \zeta_M]((q^{1/m}))^\times, $$

where $q^{1/m} = e^{2\pi i\tau/m}$, $\tau \in \mathbb{H}$.

We may further define a Siegel unit $g_{\alpha,\beta} \in \mathcal{O}(Y(m, M))^\times \otimes \mathbb{Q}$ as follows. Take a c such that $c \equiv 1 \pmod m$, $c \equiv 1 \pmod M$ with $c \neq \pm 1$, and set

$$ g_{\alpha,\beta} = {}_cg_{\alpha,\beta} \otimes (c^2 - 1)^{-1}. $$

From Proposition 5.1.1(2) it follows that the definition of $g_{\alpha,\beta}$ is independent of the choice of c.

Let $L \geq 1$, and assume that $(c, L) = 1$. If $m|L$ and $M|L$ then for any $\left(\begin{smallmatrix} s & u \\ t & v \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/L\mathbb{Z})$ we have the following action on Siegel units:

$$ \begin{pmatrix} s & u \\ t & v \end{pmatrix}^* ({}_cg_{\alpha,\beta}) = {}_cg_{\alpha',\beta'}, $$

where $(\alpha', \beta') = (\alpha, \beta) \left(\begin{smallmatrix} s & u \\ t & v \end{smallmatrix}\right)$.

Next, we begin our description of Kato-Beilinson elements, which are constructed via specific Siegel units. For a matrix $R = \left(\begin{smallmatrix} s & u \\ t & v \end{smallmatrix}\right) \in M(2, \mathbb{Z})$ such that the images of (s, u) and (t, v) in $\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ are nonzero, we define Beilinson elements ${}_{c,d}z_{m,M}(R) \in H^2_{\text{ét}}(Y(m, M), \mathbb{Z}_p(2))$ and $z_{m,M}(R) \in H^2_{\text{ét}}(Y(m, M), \mathbb{Z}_p(2)) \otimes \mathbb{Q}$ where $c, d \in \mathbb{Z}$ such that $(cd, 6mM) = 1$ as follows:

$$ {}_{c,d}z_{m,M}(R) = ({}_c g_{s/m,u/M}, {}_d g_{t/m,v/M}) \in H^2_{\text{ét}}(Y(m, M), \mathbb{Z}_p(2)), $$

and

$$ z_{m,M}(R) = (g_{s/m,u/M}, g_{t/m,v/M}) \in H^2_{\text{ét}}(Y(m, M), \mathbb{Q}_p(2)), $$

where $(\cdot, \cdot)$ denotes the cup product. Note that these Kato-Beilinson elements are the usual Beilinson elements in $K_2(Y(m, M))$ $(\otimes\mathbb{Q})$ with the Chern class map applied. These Beilinson elements depend only on $s, t \pmod m$ and $u, v \pmod M$.

Now we can describe the Kato-Beilinson elements which will correspond to modular symbols, and hence Manin symbols (see [**FK12**, 2.4.2] for further details). Let $M \geq 4, m \geq 1$, and $u, v \in \mathbb{Z}/M\mathbb{Z}$ with $(u, v) = (1)$. If $m = 1$ we must also make the assumption that $u, v \neq 0$. These elements

$$ {}_{c,d}z_{1,M,m}(u, v) \in H^2_{\text{ét}}(Y_1(M) \otimes \mathbb{Z}[1/m, \zeta_m], \mathbb{Z}_p(2)), $$

and

$$ z_{1,M,m}(u, v) \in H^2_{\text{ét}}(Y_1(M) \otimes \mathbb{Z}[1/m, \zeta_m], \mathbb{Z}_p(2)) \otimes \mathbb{Q} $$

are defined as follows.

Lift u, v to $\tilde{u}, \tilde{v} \in \mathbb{Z}$, and let s, t be such that $s\tilde{v} - t\tilde{u} = 1$. Then ${}_{c,d}z_{1,M,m}(u, v)$ is the image of ${}_{c,d}z_{m,Mm} \left(\begin{smallmatrix} s & \tilde{u} \\ t & \tilde{v} \end{smallmatrix}\right) \in H^2_{\text{ét}}(Y(m, Mm), \mathbb{Z}_p(2))$ under the corestriction map $H^2_{\text{ét}}(Y(m, Mm), \mathbb{Z}_p(2)) \to H^2_{\text{ét}}(Y_1(M) \otimes \mathbb{Z}[1/m, \zeta_m], \mathbb{Z}_p(2))$. The definition of $z_{1,M,m}(u, v)$ is similar. Once again these elements are independent of the choices of $\tilde{u}, \tilde{v}, s, t$. We also note that, in particular,

$$ {}_{c,d}z_{1,M,1}(u, v) = \left({}_c g_{0,u/M}, {}_d g_{0,v/M}\right). $$

We end this section with a useful norm compatibility relation which will be applied later. For any integer n, let $\mathrm{prime}(n)$ denote the set of prime divisors of n.

Proposition 5.1.2. [**FK12**, Prop. 2.2.2] *Let $m, M \geq 1$ with $m + M \geq 5$. Assume that $\mathrm{prime}(m) \subset \mathrm{prime}(M)$ and let $L \geq 1$. Let $R = \left(\begin{smallmatrix} s & u \\ t & v \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ such that neither of (s,u), $(t,v) \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ is $(0,0)$, and further assume that the image of R in $M_2(\mathbb{Z}/L\mathbb{Z})$ is invertible.*

Let $c, d \in \mathbb{Z}$ such that $(cd, 6mML) = 1$. Then the norm map

$$H^2_{\text{ét}}(Y(mL, ML), \mathbb{Z}_p(2)) \to H^2_{\text{ét}}(Y(m, M) \otimes \mathbb{Z}[1/L], \mathbb{Z}_p(2))$$

sends ${}_{c,d}z_{mL,ML}(R)$ to

$$\left(\prod_{\substack{\ell | L \\ \ell \nmid m \\ \ell \text{ prime}}} P_\ell \right) \cdot {}_{c,d}z_{m,M}(R),$$

where

$$P_\ell = \begin{cases} 1 - T^*(\ell) \begin{pmatrix} 1/\ell & 0 \\ 0 & 1 \end{pmatrix}^* + \begin{pmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{pmatrix}^* \ell & \text{if } \ell \nmid M, \\ 1 - T^*(\ell) \begin{pmatrix} 1/\ell & 0 \\ 0 & 1 \end{pmatrix}^* & \text{if } \ell | M. \end{cases}$$

5.2. z_{1,Kp^∞,p^∞}

A map, z_{1,Kp^∞,p^∞}, will be described which requires traveling up both modular and cyclotomic towers. To begin we define a map z_{1,Kp^r,p^∞} at a finite modular level and infinite cyclotomic level and then use this map to travel up the modular tower and define our desired map.

There exists a unique $\Lambda(p^\infty)$-homomorphism

$$z_{1,Kp^r,p^\infty} : \Lambda(p^\infty) \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(Kp^r)(\mathbb{C}), \mathbb{Z}) \longrightarrow \widetilde{\mathfrak{N}}_r \otimes_{\Lambda(p^\infty)} Q(\Lambda(p^\infty))$$

defined as follows. Let $\gamma \in H^1(Y_1(Kp^r)(\mathbb{C}), \mathbb{Z})$ and write $\gamma = \sum_i a_i [u_i : v_i]$ with $a_i \in \mathbb{Z}$ and $u_i, v_i \in \mathbb{Z}/Kp^r\mathbb{Z}$ such that $(u_i, v_i) = (1) \in \mathbb{Z}/Kp^r\mathbb{Z}$. Take any $c, d \in \mathbb{Z} - \{\pm 1\}$ such that $(cd, 6K) = 1$; it will then follow that $c - \sigma_c$ and $d - \sigma_d$ are non-zero-divisors in $\Lambda(p^\infty)$. Let tw_{-1} denote the isomorphism $\mathfrak{N}_r(2) \xrightarrow{\cong} \mathfrak{N}_r(1)$ given by $x \mapsto x \otimes (\zeta_{p^n})_n^{\otimes -1}$, where $\zeta_{p^n} = e^{2\pi i/p^n} \in \overline{\mathbb{Q}} \subset \mathbb{C}$. Then,

$$z_{1,Kp^r,p^\infty}(\gamma) = (c^2 - c\sigma_c)^{-1}(d^2 - d\sigma_d)^{-1} \sum_i a_i \mathrm{tw}_{-1}({}_{c,d}z_{1,Kp^r,p^\infty}(u_i, v_i)).$$

Notice that in our definition of z_{1,Kp^r,p^∞} we used Beilinson elements of the form ${}_{c,d}z_{1,Kp^r,p^\infty}(u_i, v_i)$. To construct these elements as the inverse limit of the finite level elements ${}_{c,d}z_{1,Kp^r,p^n}(u_i, v_i)$ discussed in the previous section, we require the following norm compatibility relation [**FK12**, Prop. 2.4.4].

Proposition 5.2.1. *Let $L \geq 1$. Then the norm map*

$$H^2_{\text{ét}}(Y_1(M) \otimes \mathbb{Z}[1/mL, \zeta_{mL}], \mathbb{Z}_p(2)) \to H^2_{\text{ét}}(Y_1(M) \otimes \mathbb{Z}[1/mL, \zeta_m], \mathbb{Z}_p(2))$$

sends ${}_{c,d}z_{1,M,mL}(u, v)$ to

$$\prod_{\substack{\ell | L \\ \ell \nmid m}} (1 - \sigma_\ell^{-1} \otimes T^*(\ell)) \cdot {}_{c,d}z_{1,M,m}(u, v).$$

Proof. This follows from Proposition 5.1.2. □

The following lemma of Fukaya and Kato [**FK12**, Lemma 5.2.5] establishes a useful isomorphism.

Lemma 5.2.2. *There is a canonical isomorphism*

$$H^2_{\text{ét}}(Y_1(K) \otimes \mathbb{Z}[1/Km, \zeta_m], \mathbb{Z}_p(2))^{\text{ord}} \cong H^1(\mathbb{Z}[1/Km, \zeta_m], H^1_{\text{ét}}(Y_1(K))(2))^{\text{ord}}.$$

Next, we want to move up the modular tower, so once again we have a norm compatibility property [**FK12**, Prop. 3.1.9]:

Proposition 5.2.3. *Assume that $M|M'$, $m|m'$, and* $\mathrm{prime}(M) = \mathrm{prime}(M')$. *Consider the norm map*

$$\begin{aligned} \text{Norm}: \quad & (\varprojlim_n H^1(\mathbb{Z}[1/Mm', \zeta_{m'p^n}], H^1_{\text{ét}}(Y_1(M'))(1))) \otimes_{\Lambda(m'p^\infty)} Q(\Lambda(m'p^\infty)) \\ & \to (\varprojlim_n H^1(\mathbb{Z}[1/Mm', \zeta_{mp^n}], H^1_{\text{ét}}(Y_1(M))(1))) \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty)). \end{aligned}$$

Then for $\gamma \in H^1(Y_1(M')(\mathbb{C}), \mathbb{Z})$,

$$\mathrm{Norm}(z_{1,M',m'p^\infty}(\gamma)) = \left(\prod_{\substack{\ell \mid m' \\ \ell \nmid m \\ \ell \text{ prime}}} P_\ell(\ell^{-1}) \right) \cdot z_{1,M,mp^\infty}(\overline{\gamma}),$$

where $\overline{\gamma}$ *is the image of* γ *in* $H^1_{\text{ét}}(Y_1(M)(\mathbb{C}), \mathbb{Z})$ *and* $P_\ell(u)$ *is defined by*

$$P_\ell(u) = \begin{cases} 1 - \sigma_\ell^{-1} \otimes T^*(\ell)u + \sigma_\ell^{-2} \otimes \langle \ell \rangle^{-1} \cdot \ell u^2 & \text{if } \ell \nmid Mm \\ 1 - \sigma_\ell^{-1} \otimes T^*(\ell)u & \text{if } \ell \mid M, \ell \nmid m \\ 1 & \text{if } \ell \mid m. \end{cases}$$

With this in place we can define the unique $\mathbf{\Lambda}$-homomorphism

$$z_{1,Kp^\infty,p^\infty} : \mathbf{\Lambda} \otimes_{\mathbf{\Lambda}[\{\pm 1\}]} \widetilde{H} \longrightarrow \left(\widetilde{\mathfrak{Z}}/\mathfrak{T}\right)(1) \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}\lambda^{-1}$$

which induces the ordinary component of z_{1,Kp^r,p^∞} for any $r \geq 1$. The -1 in $\{\pm 1\}$ acts on $\mathbf{\Lambda}$ as σ_{-1} and on $\widetilde{H}$ as complex conjugation.

5.3. $z^\sharp_{Kp^r}$ and $z^\sharp_{Kp^\infty}$

Once again, we begin with the map at infinite level, and here we suppose that $p \nmid K$. There exists a unique $\mathfrak{H}$-homomorphism

$$z^\sharp_{Kp^\infty} : \widetilde{H}_0(1) \longrightarrow H^1(\mathbb{Z}[1/Kp], \widetilde{H}(2)) \otimes_\Lambda \Lambda\mu^{-1},$$

which induces the map at finite level

$$z^\sharp_{Kp^r} : H^1_{\text{ét}}(Y_1(Kp^r))_0 \longrightarrow H^1(\mathbb{Z}[1/Kp], H^1_{\text{ét}}(Y_1(Kp^r))(2)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

To fully describe $z^\sharp_{Kp^\infty}$ we need the following lemma [**FK12**, Lemma 3.3.8], where to simplify notation in this section we will denote by $[u : v]_r$ the Manin symbol for $Y_1(Kp^r)$ rather than the usual notation of $[u : v]_{Kp^r}$.

Lemma 5.3.1. *As a* Λ*-module,* $\widetilde{H}_0(1)$ *is generated by the elements* $([p^{r-1}u : v]_r^{\text{ord}})_{r \geq 1}$*, where* u *and* v *range over integers such that* $(u, v, K) = 1$*,* $p \nmid v$*, and* $u \not\equiv 0$ *(mod* Kp*).*

Now, let $\gamma \in \widetilde{H}_0^K(1)$ be written as $\gamma = ([p^{r-1}u : v]_r)_r$ where $(u, v, K) = 1$, $(v, p) = 1$ and $u \not\equiv 0 \pmod{Kp}$. Take $c = d = 1 + Kp$ and define

$$z^{\sharp}_{Kp^\infty}(\gamma) = ({}_{c,d}z_{1,Kp^r,1}(p^{r-1}u, v) \otimes (c^2-1)^{-1}(d^2 - \langle d \rangle)^{-1})_r,$$

with the inverse limit taken with respect to the norm maps, which are compatible by Proposition 5.1.2. It also follows that

$$\mathrm{Cor}\, z_{1,Kp^r,p^\infty}(\gamma) = (1 - T^*(p)) z^{\sharp}_{Kp^\infty}(\gamma)$$

where Cor is the corestriction map.

5.4. $\exp^*$

While [**FK12**] makes use of several dual exponential maps, we will only make use of the following:

$$\exp^* : \widetilde{\mathfrak{N}}_r(1) \longrightarrow \varprojlim_n M_2(Np^r) \otimes_{\mathbb{Q}} \mathbb{Q}_p(\zeta_{p^n}),$$

and its restriction to $\mathfrak{N}_r$

$$\exp^* : \mathfrak{N}_r(1) \longrightarrow \varprojlim_n S_2(Np^r) \otimes_{\mathbb{Q}} \mathbb{Q}_p(\zeta_{p^n}).$$

Note that by [**FK12**, 3.1.4], the kernel of the former map coincides with the $\Lambda(p^\infty)$-torsion of $\widetilde{\mathfrak{N}}_r(1)$ and the latter map is injective. This map is the Bloch-Kato dual exponential map; one should see [**Kat04**, §10] for further details.

5.5. Col and $\mathrm{Col}^\flat$

In this section, a slightly less general version of the Iwasawa-Coates-Wiles-Coleman homomorphism which is presented in [**FK12**, §4.2] is defined.

We are interested in the maps

$$\mathrm{Col} : \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), H_{\mathrm{quo}}(2)) \longrightarrow S_\Lambda[[\mathbb{Z}_p^\times]]$$

and

$$\mathrm{Col}^\flat : H^1(\mathbb{Q}, H_{\mathrm{quo}}(2)) \longrightarrow S_\Lambda.$$

Proceeding as in [**FK12**, 4.2.2], we will first define $\mathrm{Col}^\flat$. Let $L = \mathbb{Q}_p^{\mathrm{ur}} \subset \overline{\mathbb{Q}}_p$ be the maximal unramified extension of $\mathbb{Q}_p$. Let $\mathcal{O}_L$ denote the valuation ring of L, then we have $L^\times = p^{\mathbb{Z}} \otimes \mathcal{O}_L^\times$. Since restriction provides an isomorphism

$$H^1(\mathbb{Q}_p, H_{\mathrm{quo}}(2)) \xrightarrow[\sim]{\mathrm{Res}} H^0(\mathbb{F}_p, H^1(L, H_{\mathrm{quo}}(2))),$$

it follows that

$$H^1(\mathbb{Q}_p, H_{\mathrm{quo}}(2)) \cong (H_{\mathrm{quo}}(1)\hat{\otimes} L^\times)^{\mathrm{Fr}_p=1},$$

where Fr_p is the arithmetic Frobenius which acts by $x \mapsto x^p$.

The homomorphism

$$\begin{aligned} l_\phi : \mathcal{O}_L^\times &\longrightarrow \mathcal{O}_L \\ x &\longmapsto p^{-1}\log(x^p/\mathrm{Fr}_p(x)) \end{aligned}$$

induces $\varprojlim_n \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{p^n} \cong \mathcal{O}_L$ and hence we have an isomorphism

$$H^1(\mathbb{Q}_p, H_{\mathrm{quo}}(2)) = \left(H_{\mathrm{quo}}(1)\hat{\otimes}(\mathbb{Z} \oplus \mathcal{O}_L)\right)^{\mathrm{Fr}_p=1} \cong (H_{\mathrm{quo}}(1))^{\mathrm{Fr}_p=1} \oplus S_\Lambda.$$

The map $\mathrm{Col}^\flat$ is then defined to be the projection onto the second component in this isomorphism. Hence, $\mathrm{Col}^\flat$ is surjective with $\ker \mathrm{Col}^\flat \cong (H_{\mathrm{quo}}(1))^{\mathrm{Fr}_p=1}$. By [**FK12**, Prop. 3.3.3, 4.2.6], we know that $(H_{\mathrm{quo}}(1))^{\mathrm{Fr}_p=1} = 0$, and thus $\mathrm{Col}^\flat$ is an isomorphism.

Now, to describe the map Col, for notation set

$$P = \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), H_{\mathrm{quo}}(2)).$$

There is a canonical isomorphism

$$\mathrm{Col} : P \longrightarrow S_\Lambda[[\mathbb{Z}_p^\times]]$$

given by the composition

$$P \to (H_{\mathrm{quo}}(1)\hat{\otimes}(\varprojlim_n \mathcal{O}_L[\zeta_{p^n}]^\times))^{\mathrm{Fr}_p=1} \to (H_{\mathrm{quo}}(1)\hat{\otimes}\, \mathcal{O}_L[[\mathbb{Z}_p^\times]])^{\mathrm{Fr}_p=1} = S_\Lambda[[\mathbb{Z}_p^\times]],$$

where the second arrow is induced from the homomorphism of Iwasawa-Coates-Wiles-Coleman

$$\varprojlim_n \mathcal{O}_L[\zeta_{p^n}]^\times \longrightarrow \mathcal{O}_L[[\mathbb{Z}_p^\times]]$$

with respect to $(\zeta_{p^n})_{n\geq 1}$.

5.6. $\infty_K(a,b)$

The second key piece in the decomposition of the map ϖ_K is the evaluation at a cusp map, $\infty_K(a,b)$. Let $K \geq 1$ with $p|K$. For any $a, b \in \mathbb{Z}/K\mathbb{Z}$ such that $(a,b) = (1)$, the evaluation map at a cusp will be a homomorphism

$$H^2_{\text{ét}}(Y_1(K) \otimes \mathbb{Z}[1/K], \mathbb{Z}_p(2)) \longrightarrow H^2(\mathbb{Z}[1/K, \zeta_K], \mathbb{Z}_p(2)).$$

To begin, let R be a Noetherian local ring and T an indeterminant. If p is invertible in R then there exists an isomorphism

$$\begin{aligned} H^i(R[[T]], \mathbb{Z}/p^n\mathbb{Z}(r)) \oplus H^{i-1}(R, \mathbb{Z}/p^n\mathbb{Z}(r-1)) \quad &\cong \quad H^i(R((T)), \mathbb{Z}/n^n\mathbb{Z}(r)) \\ (x, y) \quad &\longmapsto \quad x + \{y, T\} \end{aligned}$$

where $\{\cdot, T\}$ is the cup product with the Kummer class of T in $H^1(R((T)), \mathbb{Z}/n^n\mathbb{Z}(1))$, as seen in [**FK12**, 5.1.2].

Consider also the homomorphism given by $T \mapsto 0$,

$$H^i(R[[T]], \mathbb{Z}/p^n\mathbb{Z}(r)) \longrightarrow H^i(R, \mathbb{Z}/p^n\mathbb{Z}(r)).$$

Composing these maps

$$\begin{aligned} H^i(R((T)), \mathbb{Z}/p^n\mathbb{Z}(r)) &\to H^i(R[[T]], \mathbb{Z}/p^n\mathbb{Z}(r)) \oplus H^{i-1}(R, \mathbb{Z}/p^n\mathbb{Z}(r-1)) \\ &\to H^i(R[[T]], \mathbb{Z}/p^n\mathbb{Z}(2)) \to H^i(R, \mathbb{Z}/p^n\mathbb{Z}(r)) \end{aligned}$$

where the middle homomorphism is projection onto the first entry, gives a homomorphism

$$\Psi : H^i(R((T)), \mathbb{Z}/p^n\mathbb{Z}(r)) \longrightarrow H^i(R, \mathbb{Z}/p^n\mathbb{Z}(r)).$$

Now we may describe our evaluation at a cusp map, which we will denote $\infty_K(a,b)$:

$$H^2_{\text{ét}}(Y_1(K),\mathbb{Z}_p(2)) \overset{\infty_K(a,b)^*}{\longrightarrow} H^2(\mathbb{Z}[1/K,\zeta_K]((q^{1/K})),\mathbb{Z}_p(2)) \overset{\Psi}{\longrightarrow} H^2(\mathbb{Z}[1/K,\zeta_K],\mathbb{Z}_p(2))$$

where the first map is the pullback by $\infty_K(a,b)$ as described in Section 2.4.1 and the second is Ψ, where we have taken $i=2$, $R=\mathbb{Z}[1/K,\zeta_K]$ and $T=q^{1/K}$. Thus,

$$\infty_K(a,b) : H^2_{\text{ét}}(Y_1(K),\mathbb{Z}_p(2)) \longrightarrow H^2(\mathbb{Z}[1/K,\zeta_K],\mathbb{Z}_p(2)).$$

For notation, let $\infty_K = \infty_K(0,1)$, and we omit the subscript K whenever the level is clear.

5.7. ϖ and ϖ_K

Here the maps of main interest to us, the infinite level ϖ and its finite level counterpart ϖ_K, will be defined. Again, we will begin with the definition of ϖ since it induces the map at finite level. This map will be the composition of our previously defined map, $z^\sharp_{Kp^\infty}$, as well as the map ∞_{Kp^∞} defined as follows.

By [**FK12**, Prop. 5.1.8] the maps ∞_{Kp^r} as defined in Section 5.6 are compatible with respect to varying r on ordinary parts. Thus, we define

$$\infty_{Kp^\infty} = (\infty_{Kp^r})_r$$

as a map on $\varprojlim_r H^2_{\text{ét}}(Y_1(Kp^r),\mathbb{Z}_p(2))^{\text{ord}}$.

Now, we define ϖ to be the composition

$$\begin{aligned}\widetilde{H}_0(1) \overset{z^\sharp_{Kp^\infty}}{\longrightarrow} H^1(\mathbb{Z}[1/K],\widetilde{H}(2))\otimes_\Lambda \Lambda\mu^{-1} &\cong \varprojlim_r H^2_{\text{ét}}(Y_1(Kp^r),\mathbb{Z}_p(2))^{\text{ord}}\otimes_\Lambda \Lambda\mu^{-1} \\ &\overset{\infty_{Kp^\infty}}{\longrightarrow} \varprojlim_r H^2(\mathbb{Z}[1/K,\zeta_{Kp^r}],\mathbb{Z}_p(2))\otimes_\Lambda \Lambda\mu^{-1},\end{aligned}$$

so that

$$\varpi : \widetilde{H}_0(1) \longrightarrow \varprojlim_r H^2(\mathbb{Z}[1/K,\zeta_{Kp^r}],\mathbb{Z}_p(2))\otimes_\Lambda \Lambda\mu^{-1}.$$

By [**FK12**, 5.2.9] the image of ϖ actually lands in $\varprojlim_r H^2(\mathbb{Z}[1/K, \zeta_{Kp^r}], \mathbb{Z}_p(2))$ without the $\otimes_\Lambda \Lambda\mu^{-1}$. Hence,

$$\varpi : \widetilde{H}_0(1) \longrightarrow \varprojlim_r H^2(\mathbb{Z}[1/K, \zeta_{Kp^r}], \mathbb{Z}_p(2)).$$

As in [**FK12**, 5.2.11], ϖ induces the homomorphism ϖ_{Kp^r} at finite level:

$$\begin{aligned} \varpi_{Kp^r} : H^1_{\text{ét}}(Y_1(Kp^r))_0(1) &\longrightarrow H^2(\mathbb{Z}[1/K, \zeta_{Kp^r}], \mathbb{Z}_p(2)) \\ \varpi_{Kp^r}([u:v]_r) &= \left(1 - \zeta^u_{Kp^r}, 1 - \zeta^v_{Kp^r}\right)_{Kp^r}, \end{aligned}$$

with $\varpi_{Kp^r} = \infty_{Kp^r} \circ z^\sharp_{Kp^r}$.

CHAPTER 6

Compatibility of $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ with Respect to $z_K^\sharp$

As seen in Chapter 5, the map ϖ_K is given by the composition of two functions, $z_K^\sharp$ and ∞_K. To prove that $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ provides a compatible degeneracy map with ϖ_K we first need to show it is compatible with respect to $z_K^\sharp$. In this chapter we will establish this compatibility, i.e., we will prove that the following diagram commutes:

$$\begin{array}{ccc}
(H_r^N)_0^{\mathrm{ord}}(1) & \xrightarrow{z^\sharp_{Np^r}} & H^1(\mathbb{Z}[1/N], (H_r^N)^{\mathrm{ord}}(2)) \otimes \mathbb{Q}_p \\
\Big\downarrow{\scriptstyle \epsilon_1 - \langle \ell \rangle \epsilon_\ell} & & \Big\downarrow{\scriptstyle \mathrm{Cor}} \\
 & & H^1(\mathbb{Z}[1/N], (H_r^M)^{\mathrm{ord}}(2)) \otimes \mathbb{Q}_p \\
 & & \Big\uparrow{\scriptstyle \mathrm{Inf}} \\
(H_r^M)_0^{\mathrm{ord}}(1) & \xrightarrow{z^\sharp_{Mp^r}} & H^1(\mathbb{Z}[1/M], (H_r^M)^{\mathrm{ord}}(2)) \otimes \mathbb{Q}_p.
\end{array}$$

6.1. Relation between Coleman Maps at Infinite Level

Let M, K, r be positive integers and p, ℓ primes such that $\ell \neq p$, $\ell \nmid M$, $p|M$ and $p|K$. Set $N = M\ell$, and let

$$\Lambda(p^\infty) = \varprojlim_{n \geq 1} \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times] = \mathbb{Z}_p[[\mathbb{Z}_p^\times]],$$

$$H_r^K = H^1_{\text{ét}}(X_1(Kp^r))^{\mathrm{ord}},$$

$$H^K = \varprojlim_r H_r^K.$$

Also, to shorten notation, the following notation will be used for the spaces of modular and cusp forms of weight 2 and level Kp^r: $M_r^K = M_2(Kp^r)_{\mathbb{Z}_p}$ and $S_r^K = S_2(Kp^r)_{\mathbb{Z}_p}$.

In this section we will be considering the map

$$1 \otimes \epsilon_1 - \sigma_\ell^{-1} \otimes \langle \ell \rangle \epsilon_\ell : \Lambda(p^\infty) \otimes H_r^N \longrightarrow \Lambda(p^\infty) \otimes H_r^M,$$

and we will show that this map provides a compatible degeneracy map with the composition of z_{1,Kp^r,p^∞} and Col maps. To do so, the use of numerous auxiliary commutative diagrams will be required.

6.1.1. Auxiliary Diagrams with Nontrivial Commutativity

To begin, let $\psi : \mathbb{Z}_p^\times \to \overline{\mathbb{Z}}_p^\times$ be a continuous homomorphism with finite image, and let n be the smallest integer such that ψ factors through $(\mathbb{Z}/p^n\mathbb{Z})^\times$. Note that ϕ induces a map $S_2^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] \to S_2 \otimes \overline{\mathbb{Q}}_p$. Define for any $\omega \in S_r^N \otimes \mathbb{Q}(\zeta_{p^n})$,

$$f_\psi(\omega) = G(\psi, \zeta_{p^n})^{-1} p^n (\phi^n \otimes 1) \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(a)\sigma_a(\omega),$$

where $G(\psi, \zeta_{p^n})$ is the Gauss sum $\sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(b)\zeta_{p^n}^b$. Additionally, we define a twist map, $\mathrm{Tw} : H_r(2) \to H_r(1)$, by $x \mapsto x \otimes (\zeta_{p^n})_n^{\otimes -1}$.

Proposition 6.1.1. *The following diagram is commutative*

$$\begin{array}{ccc}
\varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r)_{\mathrm{quo}}(2)) & \xrightarrow{\mathrm{Col}} & S_r^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] \\
\downarrow \mathrm{Tw} & & \\
\varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r)_{\mathrm{quo}}(1)) & & \\
\downarrow \mathrm{Cor} & & \downarrow \psi \\
H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r)_{\mathrm{quo}}(1)) & & \\
\downarrow \exp^* & & \\
S_r \otimes \mathbb{Q}_p(\zeta_{p^n}) & \xrightarrow{f_\psi} & S_r \otimes \overline{\mathbb{Q}}_p.
\end{array}$$

Proof. This follows from [**FK12**, 4.2.10] after moving to finite level. □

For a modular form f in M_r^K we define a map

$$\begin{array}{rcl}
\Phi_f : \mathbb{M}_2(K) & \longrightarrow & \mathbb{C} \\
\{\alpha, \beta\}_K & \longmapsto & \dfrac{1}{2\pi i} \displaystyle\int_\alpha^\beta f(z)dz,
\end{array}$$

(see [**Ste07**, §10.1]) which induces a period map

$$\mathrm{per} : M_2(K) \to H^1_{\text{ét}}(Y_1(K)(\mathbb{C}), \mathbb{C}). \tag{6.1.1}$$

As in [**FK12**, 2.4.7], there is a another period map, also denoted per, given by

$$\begin{aligned} \mathrm{per} : S_r^K \otimes \mathbb{Q}(\zeta_{mp^n}) &\longrightarrow \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1_{\text{ét}}(X_1(K)(\mathbb{C}), \mathbb{C}) \\ x \otimes y &\longmapsto \sum_{\sigma \in G} \sigma \otimes \sigma^{-1}(y)\mathrm{per}(x), \end{aligned}$$

where per is the period map of (6.1.1). By [**FK12**, 1.5.6] this period map is injective.

Lemma 6.1.2. *The diagram*

$$\begin{array}{ccc} S_r^N \otimes \mathbb{Q}_p(\zeta_{p^n}) & \xhookrightarrow{\mathrm{per}} & H^1_{\text{ét}}(X_1(Np^r)(\mathbb{C}), \mathbb{C}) \otimes \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \\ \big\downarrow \mathrm{Tr} & & \big\downarrow \mathrm{Tr} \\ S_r^M \otimes \mathbb{Q}_p(\zeta_{p^n}) & \xhookrightarrow{\mathrm{per}} & H^1_{\text{ét}}(X_1(Mp^r)(\mathbb{C}), \mathbb{C}) \otimes \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \end{array}$$

is commutative.

Proof. This follows by the definition of each map. Both are given by sums of the original elements with coset representatives in $\Gamma_1(Np^r)/\Gamma_1(Mp^r)$ applied. □

Next, consider the following composition of functions, denoted h_K:

$$\begin{aligned} h_K :\ \Lambda(p^\infty) \otimes H_r &\longrightarrow \varprojlim_n H^1(\mathbb{Z}[1/K, \zeta_{p^n}], H_r(2)) \otimes_{\Lambda(p^\infty)} \Lambda(p^\infty)\mu^{-1} \\ &\longrightarrow \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r)_{\mathrm{quo}}(1)) \otimes \mathbb{Q}_p \\ &\xrightarrow{\mathrm{Cor}} H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r)_{\mathrm{quo}}(1)) \otimes \mathbb{Q}_p \xrightarrow{\exp^*} S_r^K \otimes \mathbb{Q}_p(\zeta_{p^n}) \\ &\xhookrightarrow{\mathrm{per}} H^1_{\text{ét}}(X_1(Kp^r)(\mathbb{C}), \mathbb{C}) \otimes \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})], \end{aligned}$$

where the first arrow is the map z_{1,Kp^r,p^∞} and the second is the composition of restriction, inflation, and a twist.

By [**FK12**, Theorem 3.1.5], h_K is given by multiplication by $Z_{1,Kp^r,p^\infty}(1)$. Here $Z_{1,M,m}(s)$ acting on $H^1_{\text{ét}}(Y_1(M)(\mathbb{C}), \mathbb{C}) \otimes_{\mathbb{Z}[\{\pm 1\}]} \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]$ is the operator valued

L-series defined for $\Re(s) > 2$ by

$$Z_{1,M,m}(s) = \sum_{(n,m)=1} \sigma_n^{-1} \otimes T^*(n) n^{-s} = \prod_{q \text{ prime}} P_{q,M,m}(q^{-s})^{-1},$$

with

$$P_{q,M,m}(u) = \begin{cases} 1 - \sigma_q^{-1} \otimes T^*(q)u + \sigma_q^{-2} \otimes \langle q \rangle^{-1} q u^2 & \text{if } q \nmid Mm, \\ 1 - \sigma_q^{-1} \otimes T^*(q)u & \text{if } q|M \text{ but } q \nmid m, \\ 1 & \text{if } q|m, \end{cases}$$

and for $n \in (\mathbb{Z}/m\mathbb{Z})^\times$, σ_n is the element of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ characterized by $\sigma_n(\zeta_m) = \zeta_m^n$, which is the same as in Proposition 5.2.3. We also note that $Z_{1,Kp^r,p^n}(s)$ has a unique meromorphic continuation to the entire complex plane.

Next, we let h'_K denote the composition of h_K with the automorphism of $\Lambda(p^\infty) \otimes H^1_{\text{ét}}(X_1(Kp^r)(\mathbb{C}), \mathbb{C})$ induced by $\sigma_n \mapsto n^{-1}\sigma_n$; call it τ.

Proposition 6.1.3. *The automorphism τ maps $Z_{1,Kp^r,p^n}(s)$ to $Z_{1,Kp^r,p^n}(s-1)$ in $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \otimes \mathfrak{H}(Kp^r)$.*

Proof. We note that

$$\begin{aligned} \tau\left(P_{q,Kp^r,p^n}(q^{-s})\right) &= \begin{cases} 1 - q\sigma_q^{-1} \otimes T^*(q) q^{-s} + q^2 \sigma_q^{-2} \otimes \langle q \rangle^{-1} q (q^{-s})^2 & \text{if } q \nmid Kp, \\ 1 - q\sigma_q^{-1} \otimes T^*(q) q^{-s} & \text{if } q|K, \\ 1 & \text{if } q = p, \end{cases} \\ &= \begin{cases} 1 - \sigma_q^{-1} \otimes T^*(q) q^{-(s-1)} + \sigma_q^{-2} \otimes \langle q \rangle^{-1} q (q^{-(s-1)})^2 & \text{if } q \nmid Kp, \\ 1 - \sigma_q^{-1} \otimes T^*(q) q^{-(s-1)} & \text{if } q|K, \\ 1 & \text{if } q = p, \end{cases} \\ &= P_{q,Kp^r,p^n}(q^{-(s-1)}). \end{aligned}$$

Hence, it follows by the uniqueness of the meromorphic continuation that $\tau\left(Z_{1,Kp^r,p^n}(s)\right)$ equals $Z_{1,Kp^r,p^n}(s-1)$ as desired. $\square$

We write $Z_{1,Kp^r,p^n}(0)$ to mean the specialization at $s = 0$ of the meromorphic continuation of $Z_{1,Kp^r,p^n}(s)$. With this, we define

$$Z_K(0) = (Z_{1,Kp^r,p^n}(0))_n \in \varprojlim_n \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \otimes \mathcal{H}(Kp^r).$$

As an immediate corollary we then have,

Corollary 6.1.4. *The map h'_K is given by multiplication by $Z_K(0)$.*

By $Z_K(0)$ in the above corollary we mean the specialization at $s = 0$ of the meromorphic continuation of $Z_K(s)$.

Remark 6.1.5. We see that

$$P_{q,Kp^r,p^n}(1) = \begin{cases} 1 - \sigma_q^{-1} \otimes T^*(q) + \sigma_q^{-2} \otimes \langle q \rangle^{-1} q & \text{if } q \nmid Kp, \\ 1 - \sigma_q^{-1} \otimes T^*(q) & \text{if } q | K, \\ 1 & \text{if } q = p, \end{cases}$$

and since $N = M\ell$ it is clear that the only factor which differs between $Z_N(0)$ and $Z_M(0)$ corresponds to $q = \ell$, i.e.,

$$\frac{Z_M(0)}{Z_N(0)} \text{``}=\text{''} \frac{\left(1 - \sigma_\ell^{-1} \otimes T_M^*(\ell) + \sigma_\ell^{-2} \otimes \langle \ell \rangle^{-1} \ell\right)^{-1}}{\left(1 - \sigma_\ell^{-1} \otimes U_N^*(\ell)\right)^{-1}} = \frac{1 - \sigma_\ell^{-1} \otimes U_N^*(\ell)}{1 - \sigma_\ell^{-1} \otimes T_M^*(\ell) + \sigma_\ell^{-2} \otimes \langle \ell \rangle^{-1} \ell},$$

where the subscript on the Hecke operator denotes the level at which the operator acts; noting $T_N^*(\ell) = U_N^*(\ell)$ since $\ell | N$. Here we should be a bit more specific. By the quotient $Z_M(0)/Z_N(0)$ we mean the specialization at $s = 0$ of the explicit ratio formed as the quotient of $Z_M(s)/Z_N(s)$. Additionally, $Z_M(0)$ and $Z_N(0)$ are not elements of the same space as the Hecke operators differ between H_r^M and H_r^N. So, if we truly wanted to compare them, as we will indirectly below, we must apply some combination of the degeneracy maps ϵ_1 and ϵ_ℓ to numerators.

Proposition 6.1.6. *Let $\mathscr{K} = \ker \epsilon_1 \cap \ker \epsilon_\ell$. On $\Lambda(p^\infty) \otimes H_r^N/\mathscr{K}$ the map $1 - \sigma_\ell^{-1} \otimes U_N^*(\ell) \in \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \otimes \mathfrak{H}(Np^r)_{\mathbb{Z}_p}$ for any $n \geq 0$ is injective.*

Proof. Let $(S_r^N)^{\ell\text{-old}}$ denote the ℓ-old part of S_r^N. By the arguments given in the proofs of [**CE98**, Thm. 4.1, Thm. 4.2], the operator $U_N(\ell)$ has eigenvalues of complex absolute value $\ell^{1/2}$ on $(S_r^N)^{\ell\text{-old}}$, and hence $1 - U_N(\ell)$ is injective on the ℓ-old part.

By [**Lan87**, Thm. 2.2], $(S_r^N)^{\ell\text{-old}}$ is dual, under the Petersson inner product, to $S_r^N/\mathscr{K}$. As $U_N^*(\ell)$ is the adjoint to $U_N(\ell)$, again under this inner product, $1 - U_N^*(\ell)$ is injective on $S_r^N/\mathscr{K}$.

We have an identification

$$H^1(X_1(N),\mathbb{C}) \cong S_r^N \oplus \overline{S}_r^N,$$

where the bar denotes complex conjugation, see [**DS07**, Pf. of Thm. 6.5.4]. Hence, it follows that $1 - U_N^*(\ell)$ is injective on $H^1(X_1(N),\mathbb{C})/\mathscr{K}$. By fixing an embedding of $\mathbb{Z}_p$ in $\mathbb{C}$ it then follows that $1-U_N^*(\ell)$ is injective on $H^1(X_1(N),\mathbb{Z}_p)/\mathscr{K} \cong H^1_{\text{ét}}(X_1(N))/\mathscr{K}$ and thus $1 - \sigma_\ell^{-1} \otimes U_N^*(\ell)$ is injective on $\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \otimes H_r^N/\mathscr{K}$.

□

Proposition 6.1.7. *The following diagram commutes*

$$\begin{array}{ccc}
\left(1-\sigma_\ell^{-1}\otimes U_N^*(\ell)\right)\cdot\Lambda(p^\infty)\otimes H_r^N & \xrightarrow{\cdot Z_N(0)} & H^1_{\text{ét}}(X_1(Np^r)(\mathbb{C}),\mathbb{C})\otimes\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \\
\Big\downarrow {\scriptstyle 1\otimes\epsilon_1-\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell} & & \Big\downarrow {\scriptstyle \epsilon_1} \\
\Lambda(p^\infty)\otimes H_r^M & \xrightarrow{\cdot Z_M(0)} & H^1_{\text{ét}}(X_1(Mp^r)(\mathbb{C}),\mathbb{C})\otimes\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})].
\end{array}$$

Proof. To begin, we will show

$$1-\sigma_\ell^{-1}\otimes U_N^*(\ell) \xmapsto{1\otimes\epsilon_1-\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell} 1-\sigma_\ell^{-1}\otimes T_M^*(\ell)+\sigma_\ell^{-2}\otimes\langle\ell\rangle^{-1}\ell$$

To this end we note,

$$\begin{aligned}
&(1\otimes\epsilon_1)\left[\left(1-\sigma_\ell^{-1}\otimes U_N^*(\ell)\right)\ \{\alpha,\beta\}_N\right] \\
&\qquad = \{\alpha,\beta\}_M - \sigma_\ell^{-1}\otimes\epsilon_1\left(\sum_{i=0}^{\ell-1}\left\{\frac{\alpha+i}{\ell},\frac{\beta+i}{\ell}\right\}_N\right) \\
&\qquad = \{\alpha,\beta\}_M - \sigma_\ell^{-1}\otimes\left(\sum_{i=0}^{\ell-1}\left\{\frac{\alpha+i}{\ell},\frac{\beta+i}{\ell}\right\}_M\right) \\
&\qquad = \{\alpha,\beta\}_M - \sigma_\ell^{-1}\otimes\left(T_M^*(\ell)\{\alpha,\beta\}_M - \langle\ell\rangle\{\ell\alpha,\ell\beta\}_M\right) \\
&\qquad = \{\alpha,\beta\}_M - \sigma_\ell^{-1}\otimes T_M^*(\ell)\{\alpha,\beta\}_M + \sigma_\ell^{-1}\otimes\langle\ell\rangle\{\ell\alpha,\ell\beta\}_M,
\end{aligned}$$

and

$$(\sigma_\ell^{-1} \otimes \langle \ell \rangle \epsilon_\ell) \left[(1- \ \sigma_\ell^{-1} \otimes U_N^*(\ell) \right) \{\alpha, \beta\}_N]$$

$$= \sigma_\ell^{-1} \otimes \langle \ell \rangle \{\ell\alpha, \ell\beta\}_M - \sigma_\ell^{-2} \otimes \langle \ell \rangle \epsilon_\ell \left(\sum_{i=0}^{\ell-1} \left\{ \frac{\alpha+i}{\ell}, \frac{\beta+i}{\ell} \right\}_N \right)$$

$$= \sigma_\ell^{-1} \otimes \langle \ell \rangle \{\ell\alpha, \ell\beta\}_M - \sigma_\ell^{-2} \otimes \langle \ell \rangle \left(\sum_{i=0}^{\ell-1} \{\alpha+i, \beta+i\}_M \right)$$

$$= \sigma_\ell^{-1} \otimes \langle \ell \rangle \{\ell\alpha, \ell\beta\}_M - \sigma_\ell^{-2} \otimes \langle \ell \rangle \left(\sum_{i=0}^{\ell-1} \{\alpha, \beta\}_M \right)$$

$$= \sigma_\ell^{-1} \otimes \langle \ell \rangle \{\ell\alpha, \ell\beta\}_M - \sigma_\ell^{-2} \otimes \langle \ell \rangle \ell \{\alpha, \beta\}_M.$$

From this we see

$$\left(1 \otimes \epsilon_1 - \sigma_\ell^{-1} \otimes \langle \ell \rangle \epsilon_\ell\right) \left[\left(1 - \sigma_\ell^{-1} \otimes U_N^*(\ell)\right) \{\alpha, \beta\}_N \right]$$

$$= \{\alpha, \beta\}_M - \sigma_\ell^{-1} \otimes T_M^*(\ell) \{\alpha, \beta\}_M + \sigma_\ell^{-2} \otimes \langle \ell \rangle \ell \{\alpha, \beta\}_M$$

$$= (1 - \sigma_\ell^{-1} \otimes T_M^*(\ell) + \sigma_\ell^{-2} \otimes \langle \ell \rangle^{-1} \ell) \{\alpha, \beta\}_M.$$

Applying h'_M, which amounts to multiplying by $Z_M(0)$, then gives:

$$\left(\prod_{q|M} \left(1 - \sigma_q^{-1} \otimes U_M^*(q)\right)^{-1} \right) \left(\prod_{q \nmid M} \left(1 - \sigma_q^{-1} \otimes T_M^*(q) + \sigma_q^{-2} \otimes \langle q \rangle^{-1} q\right)^{-1} \right) \{\alpha, \beta\}_M,$$

where the products are taken over all primes $q \neq \ell$.

On the other hand, the image of $(1 - \sigma_\ell^{-1} \otimes U_N^*(\ell))\{\alpha, \beta\}_N$ under h'_N, which is multiplication by $Z_N(0)$, is given by

$$\left(\prod_{q|N} \left(1 - \sigma_q^{-1} \otimes U_N^*(q)\right)^{-1} \right) \left(\prod_{q \nmid N} \left(1 - \sigma_q^{-1} \otimes T_N^*(q) + \sigma_q^{-2} \otimes \langle q \rangle^{-1} q\right)^{-1} \right) \{\alpha, \beta\}_N,$$

$$= \left(\prod_{q|M} \left(1 - \sigma_q^{-1} \otimes U_N^*(q)\right)^{-1} \right) \left(\prod_{q \nmid M} \left(1 - \sigma_q^{-1} \otimes T_N^*(q) + \sigma_q^{-2} \otimes \langle q \rangle^{-1} q\right)^{-1} \right) \{\alpha, \beta\}_N,$$

where again, the products are taken over all primes $q \neq \ell$, and we note the equality follows since $N = M\ell$. Applying the degeneracy map ϵ_1 to this, the result will then follow as $\epsilon_1 T_N^*(q) = T_M^*(q)\epsilon_1$ and $\epsilon_1 U_N^*(q) = U_M^*(q)\epsilon_1$ since $q \neq \ell$. □

The above then leads to the following theorem.

Theorem 6.1.8. *The diagram*

$$\begin{array}{ccc}
\Lambda(p^\infty) \otimes H_r^N & \xrightarrow{\cdot Z_N(0)\ (=h'_N)} & H^1_{\text{ét}}(X_1(N)(\mathbb{C}), \mathbb{C}) \otimes \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \\
{\scriptstyle 1\otimes\epsilon_1 - \sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\
\Lambda(p^\infty) \otimes H_r^M & \xrightarrow{\cdot Z_M(0)\ (=h'_M)} & H^1_{\text{ét}}(X_1(M)(\mathbb{C}), \mathbb{C}) \otimes \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})]
\end{array}$$

is commutative.

Proof. Let $\mathcal{K} = \ker \epsilon_1 \cap \ker \epsilon_\ell$. Clearly, if $\{\alpha, \beta\}_N \in \mathcal{K}$, then the diagram commutes as $\epsilon_1 = \mathrm{Tr}$. Hence, it suffices to prove commutativity on $\Lambda(p^\infty) \otimes H_r^N/\mathcal{K}$.

We know from Proposition 6.1.6 that $1 - U_N^*(\ell)$ acts injectively on $H_r^N/\mathcal{K}$. Since $H_r^N \otimes \mathbb{Q}_p$ is a finite-dimensional $\mathbb{Q}_p$-vector space this implies that the operator $1 - U_N^*(\ell)$ is invertible on $H_r^N/\mathcal{K}$. Now, $\Lambda(p^\infty) \otimes H_r^N/\mathcal{K}$ is an inverse limit of these finite-dimensional spaces $\mathbb{Q}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})] \otimes H_r^N/\mathcal{K}$ where $1 - \sigma_\ell^{-1} \otimes U_N^*(\ell)$ acts invertibly. It follows that $1 - \sigma_\ell^{-1} \otimes U_N^*(\ell)$ is invertible on $\Lambda(p^\infty) \otimes H_r^N/\mathcal{K}$ as well. Hence, the result follows from Proposition 6.1.7.

$\square$

The commutativity of the following two diagrams is trivial due to the nature of the maps contained within. Hence, we forgo writing proofs for them.

Let ψ, f_ψ be defined as in the previous section, and let Tr denote the trace map. We begin with a diagram which moves between levels.

Lemma 6.1.9. *The diagram*

$$\begin{array}{ccc}
S_r^N \otimes \mathbb{Q}_p(\zeta_{p^n}) & \xrightarrow{f_\psi} & S_r^N \otimes \overline{\mathbb{Q}}_p \\
\downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\
S_r^M \otimes \mathbb{Q}_p(\zeta_{p^n}) & \xrightarrow{f_\psi} & S_r^M \otimes \overline{\mathbb{Q}}_p,
\end{array}$$

is commutative.

Lemma 6.1.10. *The diagram*

$$\begin{array}{ccc} (S_r^N)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] & \xrightarrow{\psi} & S_r^N \otimes \overline{\mathbb{Q}}_p \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ (S_r^M)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] & \xrightarrow{\psi} & S_r^M \otimes \overline{\mathbb{Q}}_p \end{array}$$

is commutative.

We end this section with a proposition which will be used in conjunction with Lemma 6.1.10 in the proof of Theorem 6.1.12.

Proposition 6.1.11. *Let $x, y \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. If $\psi(x) = \psi(y) \in \overline{\mathbb{Q}}_p$ for all finite order characters $\psi : \mathbb{Z}_p^\times \to \overline{\mathbb{Z}}_p^\times$, then $x = y$.*

Proof. The homomorphism ψ factors through $(\mathbb{Z}/p^n\mathbb{Z})^\times$ for some $n \geq 1$. Let

$$\psi : \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times] \to \overline{\mathbb{Q}}_p$$

be the induced map on the finite group ring, and let e_ψ denote the idempotent attached to ψ. For $a \in \mathbb{Z}[(\mathbb{Z}/p^n\mathbb{Z})^\times]$ we write

$$a = \sum_\psi \psi(a) e_\psi,$$

where the sum is taken over all characters of $(\mathbb{Z}/p^n\mathbb{Z})^\times$. Let $\overline{x}, \overline{y}$ denote the images in $\mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times]$ of $x, y \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, respectively, and suppose that for all such ψ we have $\psi(\overline{x}) = \psi(\overline{y})$, then it follows

$$\overline{x} = \sum_\psi \psi(\overline{x}) e_\psi = \sum_\psi \psi(\overline{y}) e_\psi = \overline{y}.$$

Hence, the images of x and y in $\mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times]$ are equal for all n, and therefore $x = y$. □

6.1.2. Compatible Degeneracy Map

We piece all of the commutative diagrams from the previous sections together in Figure 1 which allows us to arrive at the following theorem.

Theorem 6.1.12. *The map $1 \otimes \epsilon_1 - \sigma_\ell^{-1} \otimes \langle \ell \rangle \epsilon_\ell$ provides a compatible degeneracy map with respect to the compositions of z_{1,Kp^r,p^∞} and* Col*, i.e., the following diagram is*

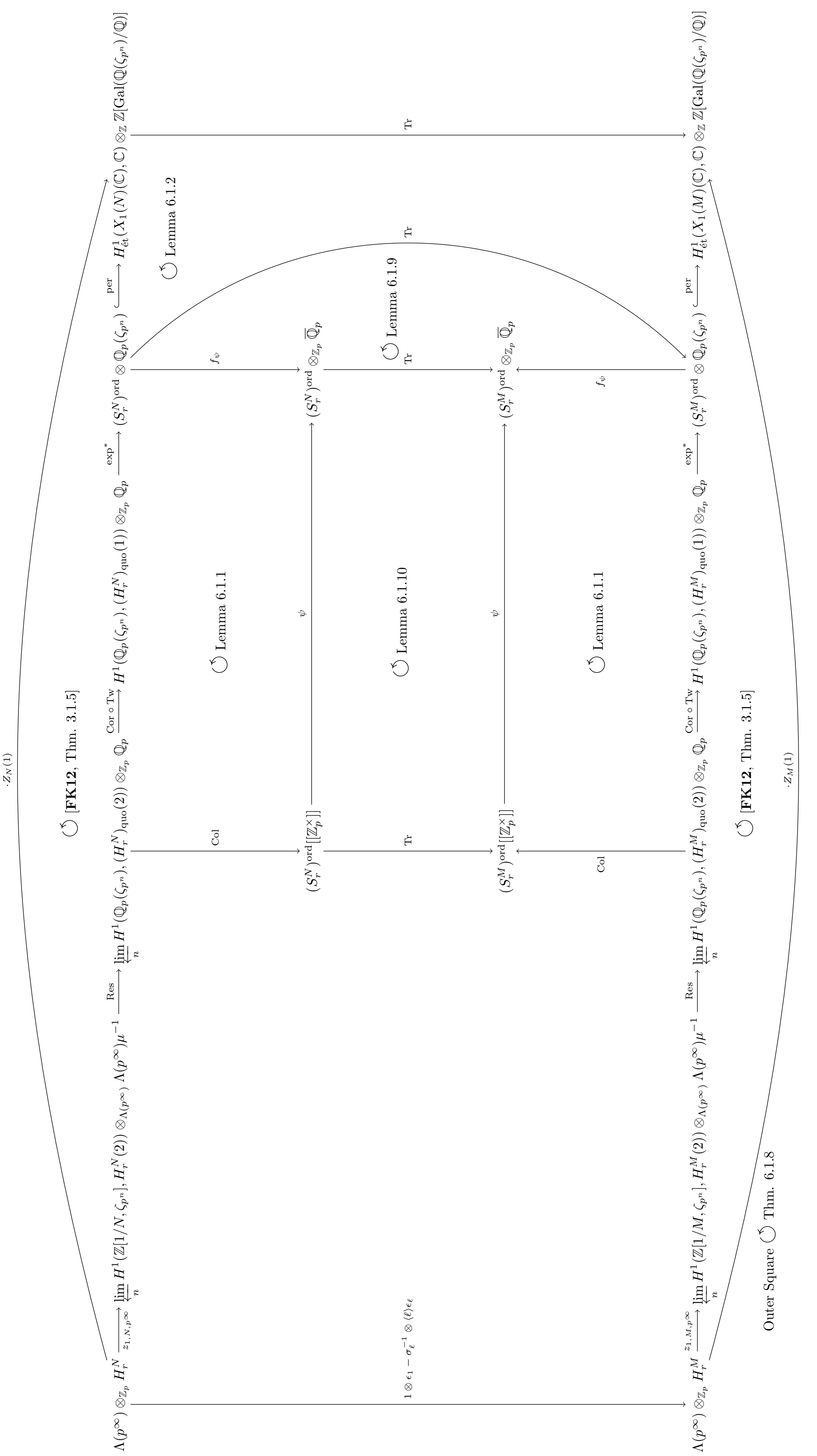

Figure 1. Compatibility with Respect to Col

commutative:

$$\begin{array}{ccc}
\Lambda(p^\infty) \otimes H_r^N & \xrightarrow{z_{1,Np^r,p^\infty}} & \varprojlim_n H^1(\mathbb{Z}[1/Np, \zeta_{p^n}], H_r^N(1)) \otimes_{\Lambda(p^\infty)} \Lambda(p^\infty)\mu^{-1} \\
\Big\downarrow & & \Big\downarrow \text{Res} \\
 & & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^N)_{\mathrm{quo}}(1)) \otimes \mathbb{Q}_p \\
 & & \Big\downarrow \text{Col} \\
 & & (S_r^N)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] \\
(1 \otimes \epsilon_1) - (\sigma_\ell^{-1} \otimes \langle \ell \rangle \epsilon_\ell) & & \Big\downarrow \text{Tr} \\
 & & (S_r^M)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] \\
 & & \text{Col} \Big\uparrow \\
 & & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^M)_{\mathrm{quo}}(1)) \\
\Big\downarrow & & \text{Res} \Big\uparrow \\
\Lambda(p^\infty) \otimes H_r^M & \xrightarrow{z_{1,Mp^r,p^\infty}} & \varprojlim_n H^1(\mathbb{Z}[1/Mp, \zeta_{p^n}], H_r^M(1)) \otimes_{\Lambda(p^\infty)} \Lambda(p^\infty)\mu^{-1}.
\end{array} \tag{6.1.2}$$

Proof. The diagram in (6.1.2) fits as an intermediate diagram in Figure 1. We will use the properties already established in this figure to prove the commutativity in (6.1.2).

To begin, by Theorem 6.1.8 the maps forming the exterior of Figure 1 establish a commutative diagram. As previously mentioned, the map per of (6.1.1) is injective. This combined with Lemma 6.1.2 allows us to shorten the diagram in Figure 1 to

$$\begin{array}{ccccc}
\Lambda(p^\infty) \otimes_{\mathbb{Z}_p} H_r^N & \longrightarrow & \cdots & \longrightarrow & S_r^N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n}) \\
\Big\downarrow (1 \otimes \epsilon_1) - (\sigma_\ell^{-1} \otimes \langle \ell \rangle \epsilon_\ell) & & & & \Big\downarrow \text{Tr} \\
\Lambda(p^\infty) \otimes_{\mathbb{Z}_p} H_r^M & \longrightarrow & \cdots & \longrightarrow & S_r^M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n})
\end{array} \tag{6.1.3}$$

while still maintaining commutativity.

Next, applying the commutativity established by Lemma 6.1.9 to the diagram in (6.1.3) gives us yet another commutative diagram, which is again a reduction of Figure

1:

$$\begin{array}{ccccccc}
\Lambda(p^\infty)\otimes_{\mathbb{Z}_p} H_r^N & \longrightarrow \cdots \longrightarrow & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^N)^{\mathrm{ord}}_{\mathrm{quo}}(2)) & \longrightarrow \cdots \longrightarrow & S_r^N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n}) \\
\Big\downarrow {\scriptstyle (1\otimes \epsilon_1) - (\sigma_\ell^{-1}\otimes \langle \ell\rangle \epsilon_\ell)} & & & & \Big\downarrow {\scriptstyle f_\psi} \\
 & & & & S_r^N \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \\
 & & & & \Big\downarrow {\scriptstyle \mathrm{Tr}} \\
 & & & & S_r^M \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \\
 & & & & {\scriptstyle f_\psi} \Big\uparrow \\
\Lambda(p^\infty)\otimes_{\mathbb{Z}_p} H_r^M & \longrightarrow \cdots \longrightarrow & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^M)^{\mathrm{ord}}_{\mathrm{quo}}(2)) & \longrightarrow \cdots \longrightarrow & S_r^M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n}).
\end{array} \tag{6.1.4}$$

Via the commutativity in Lemma 6.1.1 we can instead consider the diagram in (6.1.4) to be the commutative diagram

$$\begin{array}{ccccccc}
\Lambda(p^\infty)\otimes_{\mathbb{Z}_p} H_r^N \to \cdots \to & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^N)^{\mathrm{ord}}_{\mathrm{quo}}(2)) & \overset{\mathrm{Col}}{\to} & (S_r^N)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] & \overset{\psi}{\to} & S_r^N \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \\
\Big\downarrow {\scriptstyle (1\otimes \epsilon_1) - (\sigma_\ell^{-1}\otimes \langle \ell\rangle \epsilon_\ell)} & & & \Big\downarrow {\scriptstyle \mathrm{Tr}} & & \Big\downarrow {\scriptstyle \mathrm{Tr}} \\
\Lambda(p^\infty)\otimes_{\mathbb{Z}_p} H_r^M \to \cdots \to & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^M)^{\mathrm{ord}}_{\mathrm{quo}}(2)) & \overset{\mathrm{Col}}{\to} & (S_r^M)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]] & \overset{\psi}{\to} & S_r^M \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p.
\end{array} \tag{6.1.5}$$

Thus, we have that the diagram in (6.1.5) commutes for every finite order character ψ. Therefore, by Lemma 6.1.10 and Proposition 6.1.11 we can remove the right square in (6.1.5) which establishes the desired commutativity in (6.1.2). □

6.2. Relation between z_{1,Kp^r,p^∞} and $z^\sharp_{Kp^r}$

The goal of this section is to prove Theorem 6.3.2 which states that the following diagram is commutative:

$$\begin{array}{ccc}
H_r^N(1) & \xrightarrow{z^\sharp_{Np^r}} & H^1(\mathbb{Z}[1/N], H_r^N(2))\otimes \mathbb{Q}_p \\
 & & \Big\downarrow {\scriptstyle \mathrm{Cor}} \\
\Big\downarrow {\scriptstyle \epsilon_1 - \langle \ell\rangle \epsilon_\ell} & & H^1(\mathbb{Z}[1/N], H_r^M(2))\otimes \mathbb{Q}_p \\
 & & {\scriptstyle \mathrm{Inf}} \Big\uparrow \\
H_r^M(1) & \xrightarrow{z^\sharp_{Mp^r}} & H^1(\mathbb{Z}[1/M], H_r^M(2))\otimes \mathbb{Q}_p.
\end{array}$$

We proceed here in a similar fashion as to Section 6.1 by using several auxiliary commutative diagrams.

Lemma 6.2.1. *The following diagram commutes*

$$\begin{array}{ccc} \Lambda(p^\infty)\otimes H_r^N & \xrightarrow{(1\otimes\epsilon_1)-(\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell)} & \Lambda(p^\infty)\otimes H_r^M \\ \Big\downarrow{\scriptstyle \mathrm{proj}} & & \Big\downarrow{\scriptstyle \mathrm{proj}} \\ H_r^N & \xrightarrow{\epsilon_1-\langle\ell\rangle\epsilon_\ell} & H_r^M. \end{array}$$

Lemma 6.2.2. *The following diagram commutes*

$$\begin{array}{ccc} \varprojlim_n H^1(\mathbb{Z}[1/N,\zeta_{p^n}],H_r^N(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} & \xrightarrow{\mathrm{proj}} & H^1(\mathbb{Z}[1/N],H_r^N(2))\otimes\mathbb{Q}_p \\ & & \Big\downarrow{\scriptstyle \mathrm{Tr}} \\ \Big\downarrow{\scriptstyle \mathrm{Tr}} & & H^1(\mathbb{Z}[1/N],H_r^M(2))\otimes\mathbb{Q}_p \\ & & \Big\uparrow{\scriptstyle \mathrm{Inf}} \\ \varprojlim_n H^1(\mathbb{Z}[1/M,\zeta_{p^n}],H_r^M(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} & \xrightarrow{\mathrm{proj}} & H^1(\mathbb{Z}[1/M],H_r^M(2))\otimes\mathbb{Q}_p. \end{array}$$

The next Theorem is due to Fukaya and Kato [**FK12**, Thm. 3.3.9 (ii)]:

Theorem 6.2.3. *The following diagram is commutative*

$$\begin{array}{ccc} \Lambda(p^\infty)\otimes H_r^K(1) & \xrightarrow{z_{1,Kp^r,p^\infty}} & \varprojlim_n H^1(\mathbb{Z}[1/K,\zeta_{p^n}],H_r^K(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} \\ \Big\downarrow{\scriptstyle \mathrm{proj}} & & \Big\downarrow{\scriptstyle \mathrm{proj}} \\ H_r^K(1) & \xrightarrow{(1-U_K^*(p))z^\sharp_{Kp^r}} & H^1(\mathbb{Z}[1/K],H_r^K(2))^{\mathrm{ord}}\otimes\mathbb{Q}_p. \end{array}$$

To write our last auxiliary diagram we need the following result.

Proposition 6.2.4. *The injection given by inflation*

$$\varprojlim_n H^1(\mathbb{Z}[1/M,\zeta_{p^n}],H_r^M(2)) \overset{\sim}{\hookrightarrow} \varprojlim_n H^1(\mathbb{Z}[1/N,\zeta_{p^n}],H_r^M(2))$$

is an isomorphism.

Proof. See [**PR00**, §1.3]. ☐

In light of Proposition 6.2.4, we have the trace map

$$\varprojlim_n H^1(\mathbb{Z}[1/N,\zeta_{p^n}], H_r^N(2)) \xrightarrow{\mathrm{Tr}} \varprojlim_n H^1(\mathbb{Z}[1/N,\zeta_{p^n}], H_r^M(2))$$
$$\cong \varprojlim_n H^1(\mathbb{Z}[1/M,\zeta_{p^n}], H_r^M(2)),$$

and the following lemma

Lemma 6.2.5. *The following diagram commutes*

$$\begin{array}{ccc}
\varprojlim_n H^1(\mathbb{Z}[1/N,\zeta_{p^n}], H_r^N(2)) \otimes \Lambda(p^\infty)\mu^{-1} & \to & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^N)_{\mathrm{quo}}(2)) \otimes \Lambda(p^\infty)\mu^{-1} \\
\downarrow \mathrm{Tr} & & \downarrow \mathrm{Tr} \\
\varprojlim_n H^1(\mathbb{Z}[1/M,\zeta_{p^n}], H_r^M(2)) \otimes \Lambda(p^\infty)\mu^{-1} & \to & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), (H_r^M)_{\mathrm{quo}}(2)) \otimes \Lambda(p^\infty)\mu^{-1}
\end{array}$$

where the tensor products are taken over $\Lambda(p^\infty)$ and the horizontal maps are given by restriction.

6.3. Compatibility of $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ with $z^\sharp_{Kp^r}$

For reference, we note in Figure 2 the manner in which the previous diagrams from the previous section are connected.

The previous results of this chapter now bring us to a key theorem which will be used to prove Theorem 6.3.2.

Theorem 6.3.1. *The following diagram is commutative*

$$\begin{array}{ccc}
H_r^N(1) & \xrightarrow{(1-U_N^*(p))z^\sharp_{Np^r}} & H^1(\mathbb{Z}[1/N], H_r^N(2)) \otimes \mathbb{Q}_p \\
 & & \downarrow \mathrm{Tr} \\
\downarrow \epsilon_1 - \langle \ell \rangle \epsilon_\ell & & H^1(\mathbb{Z}[1/N], H_r^M(2)) \otimes \mathbb{Q}_p \\
 & & \uparrow \mathrm{Inf} \\
H_r^M(1) & \xrightarrow{(1-U_M^*(p))z^\sharp_{Mp^r}} & H^1(\mathbb{Z}[1/M], H_r^M(2)) \otimes \mathbb{Q}_p.
\end{array} \tag{6.3.1}$$

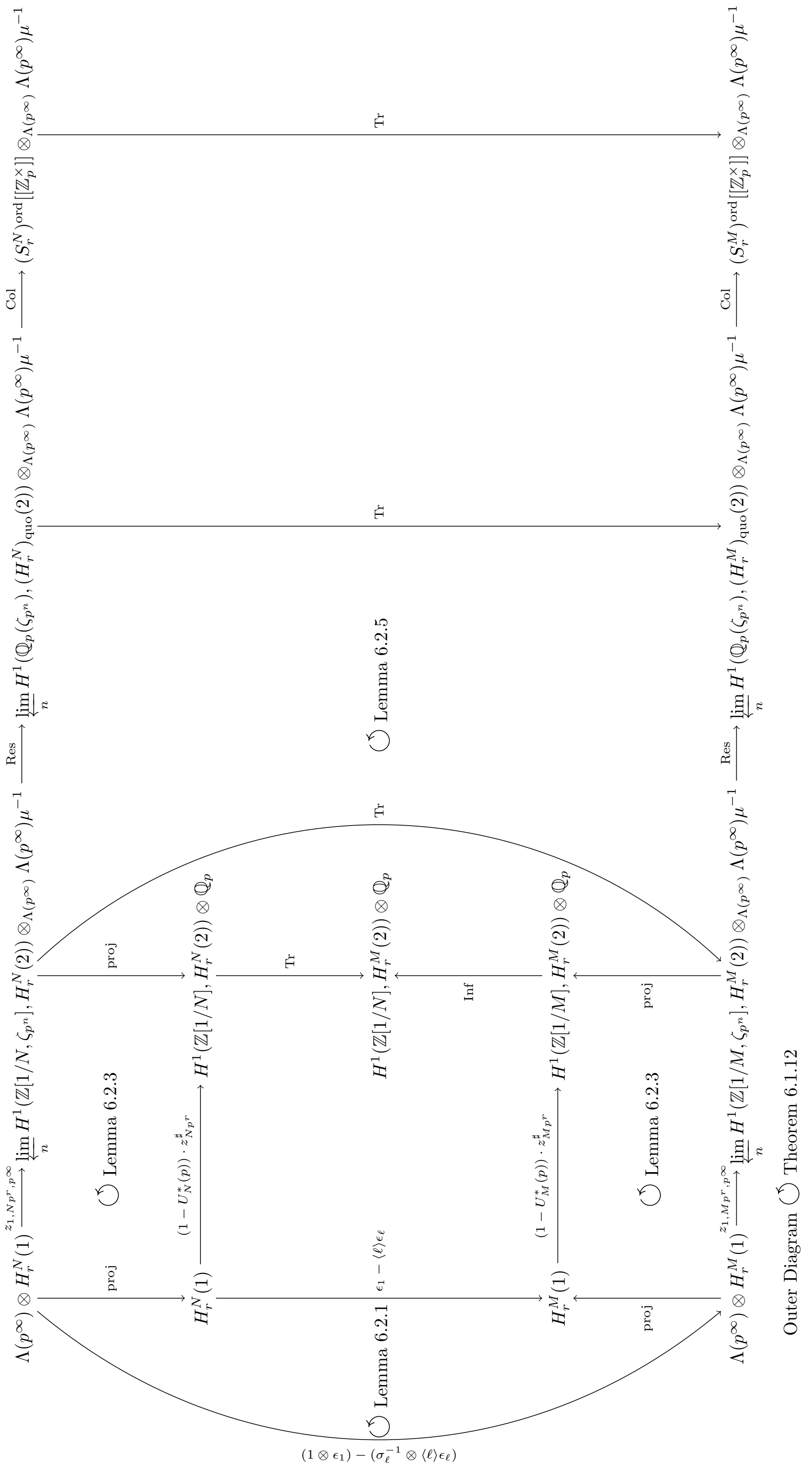

Figure 2. Compatibility with Respect to $(1 - U_K^*(p))z^\sharp_{Kp^r}$

Proof. From Theorem 6.1.12 we have that the exterior maps in Figure 2 form a commutative diagram, i.e.,

$$
\begin{array}{ccc}
\Lambda(p^\infty)\otimes H_r^N(1) \longrightarrow \cdots \longrightarrow & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}),(H_r^N)_{\mathrm{quo}}(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} \\
\Big\downarrow {\scriptstyle (1\otimes\epsilon_1)-(\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell)} & \Big\downarrow {\scriptstyle \mathrm{Col}} \\
 & (S_r^N)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]]\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} \\
 & \Big\downarrow {\scriptstyle \mathrm{Tr}} \\
 & (S_r^M)^{\mathrm{ord}}[[\mathbb{Z}_p^\times]]\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} \\
 & \Big\uparrow {\scriptstyle \mathrm{Col}} \\
\Lambda(p^\infty)\otimes H_r^M(1) \longrightarrow \cdots \longrightarrow & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}),(H_r^M)_{\mathrm{quo}}(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1}
\end{array}
\tag{6.3.2}
$$

commutes.

Now, the map Col is injective by [**FK12**, Prop. 4.2.7 (2)] and the rightmost diagram clearly commutes, so we can shorten diagram (6.3.2) to the following commutative diagram

$$
\begin{array}{ccc}
\Lambda(p^\infty)\otimes H_r^N(1) \longrightarrow \cdots \xrightarrow{\mathrm{Res}} & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}),(H_r^N)_{\mathrm{quo}}(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} \\
\Big\downarrow {\scriptstyle (1\otimes\epsilon_1)-(\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell)} & \Big\downarrow {\scriptstyle \mathrm{Tr}} \\
\Lambda(p^\infty)\otimes H_r^M(1) \longrightarrow \cdots \xrightarrow{\mathrm{Res}} & \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}),(H_r^M)_{\mathrm{quo}}(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1}.
\end{array}
\tag{6.3.3}
$$

A priori, the map Res is not necessarily injective; however, the dual exponential map $\exp^*$ is injective, and $\exp^*$ is given as a composition of functions, the first of which is Res. Hence, Res is injective and thus by Lemma 6.2.5 we can shorten (6.3.3) to obtain the following commutative diagram:

$$
\begin{array}{ccc}
\Lambda(p^\infty)\otimes H_r^N(1) & \xrightarrow{z_{1,Np^r,p^\infty}} & \varprojlim_n H^1(\mathbb{Z}[1/N,\zeta_{p^n}],H_r^N(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} \\
\Big\downarrow {\scriptstyle 1\otimes\epsilon_1-\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell} & & \Big\downarrow {\scriptstyle \mathrm{Tr}} \\
\Lambda(p^\infty)\otimes H_r^M(1) & \xrightarrow{z_{1,Mp^r,p^\infty}} & \varprojlim_n H^1(\mathbb{Z}[1/M,\zeta_{p^n}],H_r^M(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1}.
\end{array}
\tag{6.3.4}
$$

Applying the commutativity provided by Lemma 6.2.2 to (6.3.4) then allows us to once again shorten the diagram to

$$\begin{array}{ccc}
\Lambda(p^\infty)\otimes H_r^N(1) & \longrightarrow & \varprojlim_n H^1(\mathbb{Z}[1/N,\zeta_{p^n}],H_r^N(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1} \\
\Big\downarrow & & \Big\downarrow \text{proj} \\
 & & H^1(\mathbb{Z}[1/N],H_r^N(2))\otimes\mathbb{Q}_p \\
 & & \Big\downarrow \text{Tr} \\
1\otimes\epsilon_1-\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell & & H^1(\mathbb{Z}[1/N],H_r^M(2))\otimes\mathbb{Q}_p \\
 & & \text{Inf}\Big\uparrow \\
 & & H^1(\mathbb{Z}[1/M],H_r^M(2))\otimes\mathbb{Q}_p \\
\Big\downarrow & & \text{proj}\Big\uparrow \\
\Lambda(p^\infty)\otimes H_r^M(1) & \longrightarrow & \varprojlim_n H^1(\mathbb{Z}[1/M,\zeta_{p^n}],H_r^M(2))\otimes_{\Lambda(p^\infty)}\Lambda(p^\infty)\mu^{-1}
\end{array} \qquad (6.3.5)$$

which still commutes.

Next, Lemma 6.2.3 allows us to further reduce the diagram of (6.3.4) into the commutative diagram

$$\begin{array}{ccc}
\Lambda(p^\infty)\otimes H_r^N(1) & \xrightarrow{\text{proj}} & H_r^N(1) \\
\Big\downarrow & & \Big\downarrow (1-U_N^*(p))\cdot z^\sharp_{Np^r} \\
 & & H^1(\mathbb{Z}[1/N],H_r^N(2))\otimes\mathbb{Q}_p \\
 & & \Big\downarrow \text{Tr} \\
(1\otimes\epsilon_1)-(\sigma_\ell^{-1}\otimes\langle\ell\rangle\epsilon_\ell) & & H^1(\mathbb{Z}[1/N],H_r^M(2))\otimes\mathbb{Q}_p \\
 & & \text{Inf}\Big\uparrow \\
 & & H^1(\mathbb{Z}[1/M],H_r^M(2))\otimes\mathbb{Q}_p \\
\Big\downarrow & & \Big\uparrow (1-U_M^*(p))\cdot z^\sharp_{Mp^r} \\
\Lambda(p^\infty)\otimes H_r^M(1) & \xrightarrow{\text{proj}} & H_r^M(1).
\end{array} \qquad (6.3.6)$$

Finally, Lemma 6.2.1 essentially removes the maps inducing $\Lambda(p^\infty)$ in (6.3.6) and replaces them with our degeneracy map $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ to establish the commutativity of

$$\begin{array}{ccc} H_r^N(1) & \xrightarrow{(1-U_N^*(p))z^\sharp_{Np^r}} & H^1(\mathbb{Z}[1/N], H_r^N(2)) \otimes \mathbb{Q}_p \\ & & \downarrow \mathrm{Tr} \\ \downarrow \epsilon_1 - \langle \ell \rangle \epsilon_\ell & & H^1(\mathbb{Z}[1/N], H_r^M(2)) \otimes \mathbb{Q}_p \\ & & \uparrow \mathrm{Inf} \\ H_r^M(1) & \xrightarrow{(1-U_M^*(p))z^\sharp_{Mp^r}} & H^1(\mathbb{Z}[1/M], H_r^M(2)) \otimes \mathbb{Q}_p, \end{array}$$

which proves the theorem. □

We are now ready to proceed with the main theorem of this chapter.

Theorem 6.3.2. *The map $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ is compatible with $z^\sharp_{Kp^r}$; i.e., the following diagram commutes:*

$$\begin{array}{ccc} H_r^N(1) & \xrightarrow{z^\sharp_{Np^r}} & H^1(\mathbb{Z}[1/N], H_r^N(2)) \otimes \mathbb{Q}_p \\ & & \downarrow \mathrm{Tr} \\ \downarrow \epsilon_1 - \langle \ell \rangle \epsilon_\ell & & H^1(\mathbb{Z}[1/N], H_r^M(2)) \otimes \mathbb{Q}_p \\ & & \uparrow \mathrm{Inf} \\ H_r^M(1) & \xrightarrow{z^\sharp_{Mp^r}} & H^1(\mathbb{Z}[1/M], H_r^M(2)) \otimes \mathbb{Q}_p. \end{array}$$

Proof. By Theorem 6.3.1 we have that the following diagram commutes:

$$\begin{array}{ccc} H_r^N(1) & \xrightarrow{(1-U_N^*(p))z^\sharp_{Np^r}} & H^1(\mathbb{Z}[1/N], H_r^N(2)) \otimes \mathbb{Q}_p \\ & & \downarrow \mathrm{Cor} \\ \downarrow \epsilon_1 - \langle \ell \rangle \epsilon_\ell & & H^1(\mathbb{Z}[1/N], H_r^M(2)) \otimes \mathbb{Q}_p \\ & & \uparrow \mathrm{Inf} \\ H_r^M(1) & \xrightarrow{(1-U_M^*(p))z^\sharp_{Mp^r}} & H^1(\mathbb{Z}[1/M], H_r^M(2)) \otimes \mathbb{Q}_p. \end{array}$$

By [**FK12**, 3.3.4, 3.3.6], for $K = M, N$ the kernel of $1 - U_K^*(p)$ on $H^1(\mathbb{Z}[1/K], H^K(2))$ is killed by μ. Hence, $1 - U_K^*(p)$ acts injectively on $H^1(\mathbb{Z}[1/K], H^K(2)) \otimes_\Lambda \Lambda\mu^{-1}$. Descending to finite level, $H^1(\mathbb{Z}[1/K], H_r^K(2)) \otimes \mathbb{Q}_p$ is finite dimensional over $\mathbb{Q}_p$ so it follows that $1 - U_K^*(p)$ acts invertibly, and therefore the diagram commutes with the $(1 - U_K^*(p))$-operator removed. □

We end this chapter with an immediate corollary to Theorem 6.3.2 extending the results to infinite level.

Corollary 6.3.3. *The following diagram is commutative*

$$\begin{array}{ccc}
H^N(1) & \xrightarrow{z^{\sharp}_{Np^{\infty}}} & H^1(\mathbb{Z}[1/M], H^N(2)) \otimes_{\Lambda} \Lambda\mu^{-1} \\
\Big\downarrow{\scriptstyle \epsilon_1 - \langle \ell \rangle \epsilon_\ell} & & \Big\downarrow{\scriptstyle \mathrm{Tr}} \\
 & & H^1(\mathbb{Z}[1/N], H^M(2)) \otimes_{\Lambda} \Lambda\mu^{-1} \\
 & & \Big\uparrow{\scriptstyle \mathrm{Inf}} \\
H^M(1) & \xrightarrow{z^{\sharp}_{Mp^{\infty}}} & H^1(\mathbb{Z}[1/M], H^M(2)) \otimes_{\Lambda} \Lambda\mu^{-1}.
\end{array}$$

CHAPTER 7

Compatibility with respect to ϖ_K

7.1. Commutativity with ∞-maps

In this section we show that several diagrams involving the evaluation at infinity map, ∞_K, are commutative. Namely, we want to prove the following theorem.

Theorem 7.1.1. *The following diagram commutes:*

$$\begin{array}{ccc}
H^2_{\text{ét}}(X_1(N) \otimes \mathbb{Z}[1/N], \mathbb{Z}_p(2)) & \xrightarrow{\infty_N} & H^2(\mathbb{Z}[1/N, \zeta_N], \mathbb{Z}_p(2)) \\
\downarrow{\scriptstyle \mathrm{Cor}} & & \downarrow{\scriptstyle \mathrm{Cor}} \\
H^2_{\text{ét}}(X_1(M) \otimes \mathbb{Z}[1/N], \mathbb{Z}_p(2)) & \xrightarrow{\infty_M} & H^2(\mathbb{Z}[1/N, \zeta_M], \mathbb{Z}_p(2)) \\
{\scriptstyle \mathrm{Inf}}\uparrow & & {\scriptstyle \mathrm{Inf}}\uparrow \\
H^2_{\text{ét}}(X_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2)) & \xrightarrow{\infty_M} & H^2(\mathbb{Z}[1/M, \zeta_M], \mathbb{Z}_p(2)).
\end{array}$$

Proof. To establish the desired commutativities it suffices to show that each of the following squares commute since over the modular curve $X_1(K)$ the map ∞_K is given by pullback on cohomology.

$$\begin{array}{ccc}
\operatorname{Spec} \mathbb{Z}[1/M, \zeta_M] & \longrightarrow & X_1(M) \\
\uparrow & & \uparrow \\
\operatorname{Spec} \mathbb{Z}[1/N, \zeta_M] & \longrightarrow & X_1(M) \otimes \mathbb{Z}[1/N] \\
\uparrow & & {\scriptstyle \epsilon_1}\uparrow \\
\operatorname{Spec} \mathbb{Z}[1/N, \zeta_N] & \longrightarrow & X_1(N).
\end{array} \tag{7.1.1}$$

The top diagram in (7.1.1) simply represents a base change, and hence trivially commutes. The lower diagram in (7.1.1) commutes as the cusp at infinity in $X_1(N)$ lies over the cusp at infinity in $X_1(M)$.

$\square$

Once again, we have the following immediate corollary to Theorem 7.1.1 which gives commutativity at the infinite level.

Corollary 7.1.2. *The following diagram commutes:*

$$\begin{array}{ccc}
\varprojlim_r H^2_{\text{ét}}(X_1(Np^r), \mathbb{Z}_p(2)) \otimes \Lambda\mu^{-1} & \xrightarrow{\infty_{Np^\infty}} & \varprojlim_r H^2(\mathbb{Z}[1/N, \zeta_{Np^r}], \mathbb{Z}_p(2)) \otimes \Lambda\mu^{-1} \\
\downarrow \text{Cor} & & \downarrow \text{Cor} \\
\varprojlim_r H^2_{\text{ét}}(X_1(Mp^r) \otimes \mathbb{Z}[1/N], \mathbb{Z}_p(2)) \otimes \Lambda\mu^{-1} & \xrightarrow{\infty_{Mp^\infty}} & \varprojlim_r H^2(\mathbb{Z}[1/N, \zeta_{Mp^r}], \mathbb{Z}_p(2)) \otimes \Lambda\mu^{-1} \\
\uparrow \text{Inf} & & \uparrow \text{Inf} \\
\varprojlim_r H^2_{\text{ét}}(X_1(Mp^r), \mathbb{Z}_p(2)) \otimes \Lambda\mu^{-1} & \xrightarrow{\infty_{Mp^\infty}} & \varprojlim_r H^2(\mathbb{Z}[1/M, \zeta_{Mp^r}], \mathbb{Z}_p(2)) \otimes \Lambda\mu^{-1},
\end{array}$$

where all of the tensor products are taken over Λ.

7.2. Commutativity with ϖ_K

We begin by working at the infinite level. First, recall that the map ϖ was defined to be the composition of the maps $z^\sharp_{Kp^\infty}$ and ∞_{Kp^∞}. Combining the commutativities provided by the corollaries to our two main theorems, Corollary 6.3.3 and Corollary 7.1.2, we establish the following theorem.

Theorem 7.2.1. *The following diagram commutes*

$$\begin{array}{ccc}
H^N(1) & \xrightarrow{\varpi} & \varprojlim_r H^2(\mathbb{Z}[1/N, \zeta_{Np^r}], \mathbb{Z}_p(2)) \otimes_\Lambda \Lambda\mu^{-1} \\
\downarrow \epsilon_1 - \langle \ell \rangle \epsilon_\ell & & \downarrow \text{Cor} \\
H^M(1) & \xrightarrow{\varpi} & \varprojlim_r H^2(\mathbb{Z}[1/M, \zeta_{Mp^r}], \mathbb{Z}_p(2)) \otimes_\Lambda \Lambda\mu^{-1}.
\end{array}$$

Once again, the work of Fukaya and Kato [**FK12**, 5.2.9] shows that the image of ϖ is integral, and hence the tensor product may be removed. Thus we have the following corollary.

Corollary 7.2.2. *The following diagram commutes*

$$\begin{array}{ccc} H^N(1) & \xrightarrow{\varpi} & \varprojlim_r H^2(\mathbb{Z}[1/N, \zeta_{Np^r}], \mathbb{Z}_p(2)) \\ \Big\downarrow{\scriptstyle \epsilon_1 - \langle \ell \rangle \epsilon_\ell} & & \Big\downarrow{\scriptstyle \mathrm{Cor}} \\ H^M(1) & \xrightarrow{\varpi} & \varprojlim_r H^2(\mathbb{Z}[1/M, \zeta_{Mp^r}], \mathbb{Z}_p(2)). \end{array}$$

Now, descending to finite level in Corollary 7.2.2 proves our main theorem:

Theorem 7.2.3. *The map $\epsilon_1 - \langle \ell \rangle \epsilon_\ell$ provides a compatible degeneracy map between levels in the passage from modular symbols to cup products via ϖ. In other words, the diagram*

$$\begin{array}{ccc} H^1_{\text{ét}}(X_1(N), \mathbb{Z}_p) & \xrightarrow{\varpi_N} & H^2(\mathbb{Z}[1/N, \zeta_N], \mathbb{Z}_p(2)) \\ \Big\downarrow{\scriptstyle \epsilon_1 - \langle \ell \rangle \epsilon_\ell} & & \Big\downarrow{\scriptstyle \mathrm{Cor}} \\ H^1_{\text{ét}}(X_1(M), \mathbb{Z}_p) & \xrightarrow{\varpi_M} & H^2(\mathbb{Z}[1/M, \zeta_M], \mathbb{Z}_p(2)) \end{array}$$

commutes.

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