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THE EXTRAPOLATING PUPIL, IMAGE SYNTHESIS, AND APPLICATIONS FOR THE FUTURE

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ABSTRACT

A function $P_N(\beta)$ exists whose finite Fourier transform over a specified range of its argument is asymptotic (with N) to an Airy distribution with arbitrary scale compression. Consequently, when the function is applied as a passive coating to a diffraction-limited lens of fixed aperture, the point amplitude response collapses inward as if the lens were physically replaced by a diffraction-limited lens of greater aperture.

Investigating the implications of coating $P_N(\beta)$ to image theory, we find the following: (1) The scalar wave equation has intrinsically a particlelike solution. (2) A modification of $P_N(\beta)$ causes an arbitrarily narrow depth of focus. (3) An arbitrary point amplitude response may be optically produced. (Suppose $g(x)$ to be a required, and arbitrary, point response function with $\hat{G}(\beta)$ its finite Fourier transform. Then pupil $P_N(\beta)\hat{G}(\beta)$ produces $g(x)$, asymptotic with N.) (4) When applied onto any band-limited pupil $G(\beta)$, coating $P_N(\beta)$ effectively extrapolates $G(\beta)$ arbitrarily beyond the bounds of the aperture.

Some amusing analog devices, based on the extrapolating property (No. 4 above), are next developed. These are an optical analog signal extrapolator, a picture extrapolator, and an analog method of band-unlimited image processing. We also suggest the existence of a laser "superposition mode" whose output would be arbitrarily directive, and the possibility of using an acoustical pupil $P_N(\beta)$ to resolve these long wavelengths with near-optical quality. The ultimate limitations on the practical use and fabrication of pupil $P_N(\beta)$ are discussed.

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TABULATION OF SYMBOLS USED
(Approximately in order of appearance)

$U_N(\beta)$	"Extrapolating" pupil, nonnormalized
$P_N(\beta)$	"Extrapolating" pupil, normalized
N	Highest order term in series for $P_N(\beta)$
N_{max}	Largest value of N permitted by scalar, Fraunhofer theory
β	Reduced pupil coordinate (one-dimensional)
β'	Related to β
c	Space-bandwidth product for object-optics
λ_n	Prolate spheroidal eigenvalue
ψ_n	Prolate spheroidal wave function
β_0	"Reduced" pupil halfwidth; proportional to pupil extent
$\delta(x)$	Dirac delta function
$\delta_N(x)$	Point amplitude response to $U_N(\beta)$; asymptotic to Dirac $\delta(x)$
x_0	Twice the field over which $\delta_N(x) \to \delta(x)$
sinc x	$[\sin(x)]/x$
$M\beta_0$	Enlarged aperture, synthesized by $P_N(\beta)$, $\lvert\beta\rvert \le \beta_0$
Δ_N	Extent of central core of $\delta_N(x)$
$\vec{r}$	Position vector in wave field
v	Velocity of light
t	A time or pupil coordinate
f_1, f_2	Arbitrary functions
$a(z)$	Scalar amplitude distribution along optic axis
$W(\beta)$	Any pupil
$W_N(\beta)$	Pupil causing $a(z)$ to approach Dirac $\delta(z)$
V	Pupil related to W
f#	f/number of optical system
$a(x)$	Point amplitude response
$g(x)$	Point amplitude response of arbitrary profile
$g_N(x)$	$g_N(x) \to g(x)$ as $N \to \infty$
$\hat{G}(\beta)$	Finite Fourier transform of $g(x)$
$o(x)$	Limited object scene
x, y	Image space coordinates (one-dimensional)
$\lvert g(z)\rvert^2$	Required intensity spread function along optic axis
$g_r(x)$	Rectangular point amplitude response
$G_r(\beta)$	Infinite pupil required to produce $g_r(x)$
$V(t)$	Voltage waveform
$v(x)$	Fourier transform of $V(t)$
$v_N(x)$	$v_N(x) \to v(x)$ as $N \to \infty$
$t(x)$	An image formed by either coherent or incoherent light
X	Extent of given $t(x)$
X'	Any number $\le X$
Ω	Cutoff frequency in image
Ω_0	Cutoff frequency of lens
$i(x)$	Image transparency
$I(\beta)$	Image spectrum
$\tau(\beta)$	Optical transfer function connecting object and image
$1/\tau(\beta)$	Amplitude transparency placed in Fraunhofer plane
$TEM_{n,q}(x)$	Electromagnetic mode of transverse order n and longitudinal order q
σ_0	Noise variance in pupil before $P_N(\beta)$ is applied
σ	Noise variance in pupil after $P_N(\beta)$ is applied
r_N	Ratio of S/N (across point amplitude response) arising from $P_N(\beta)$ to S/N in absence of $P_N(\beta)$
γ_N	S/N sensitivity factor
α_N	Ratio of central core width in $\delta_N(x)$ to Airy central core; resolution enhancement
Δw_N	Spatial width of narrowest pupil fluctuation
d_0	Aperture extent (units of length)
λ	Wavelength of scalar wave

BACKGROUND

In an earlier report,[1] we described a pupil coating with an unusual property: When applied to a diffraction-limited pupil of modest aperture, it causes the point amplitude diffraction pattern to collapse inward, as if the resultant pupil were still clear and diffraction-limited but now with an *arbitrarily large* aperture. Thus, the image is formed as if the coating had physically extended the given aperture outward in space, and by an arbitrary amount. In this sense, the coating is a pupil enlarger, or extrapolator. It accomplishes this by diffracting outside the field of view all light that would otherwise tend to broaden the pattern into an ordinary Airy disc. Unfortunately, most of the incident light is thereby wasted unless means are found for recycling this light back into the object space. This illumination problem and an equally vexing one of tolerances on fabrication of the coating (both to be treated later in this report) are serious obstacles to current use of the phenomenon.

At present, the importance of the pupil may lie in its implications, if not in its substance. The phenomenon suggests the ultimate capabilities and applications of any system that relays a scalar wave (be it of electromagnetic or acoustical origin) to an image plane. It also implies that manufacturing tolerances, "seeing" conditions, and detection sensitivity (but *not* diffraction) enforce the ultimate limitation on optical quality. These capabilities, applications and, finally, limitations, are the subject of this paper. In particular, we will see that the "extrapolating pupil" solves some of the classic problems of image formation.

But first, let us review the basic phenomenon and establish notation.

Basic phenomenon

Following the notation used in the earlier report,[1] we denote the pupil coating in question as $P_N(\beta)$, where β is the "reduced"[2] pupil coordinate. Being a passive coating, $P_N(\beta)$ is normalized as

$$P_N(\beta) \;=\; U_N(\beta)\,/\,U_N(\beta_0), \qquad -1 \le P_N(\beta) \le 1, \tag{1a}$$

where

$$U_N(\beta) \;=\; (x_0/2\pi\beta_0)^{\frac{1}{2}} \sum_{n(\text{even})=0}^{N} (-1)^{n/2}\,\lambda_n(c)^{-3/2}\,\psi_n(c,0)\,\psi_n(c,\beta x_0/\beta_0), \tag{1b}$$

$$c \;=\; \beta_0 x_0 .$$

The λ_n, ψ_n are prolate spheroidal eigenvalues and wavefunctions, respectively, as normalized by Slepian and Pollak.[3] The β_0 is the reduced pupil halfwidth.

Parameter x_0 (field) is at the user's disposal and is defined as follows: Let $\delta_N(x)$ represent the point amplitude response to pupil $U_N(\beta)$. For $|x| \le x_0$, amplitude $\delta_N(x)$ is an Nth-order approximation

$$\delta_N(x) = \sum_{n(\text{even})=0}^{N} \lambda_n(c)^{-1}\psi_n(c,0)\psi_n(c,x) \tag{2}$$

to the Dirac delta function $\delta(x)$. The right-hand side is the completeness series for functions $\psi_n(x)$, truncated at term N. This is why $\delta_N(x) \to \delta(x)$ as $N \to \infty$. For values $|x| > x_0$, the amplitude response to $U_N(\beta)$ departs widely from a $\delta(x)$ profile. This defines a useful field of view for delta-function imagery: if, for example, an object scene lies wholly within $|x| \le x_0/2$, the field of view should also be $|x| \le x_0/2$ (at equal conjugates).

Finally, we should note that the Fraunhofer approximation[2] was used in derivation of pupil $P_N(\beta)$, so that $U_N(\beta)$ and $\delta_N(x)$ obey

$$\int_{-\beta_0}^{\beta_0} d\beta U_N(\beta) e^{j\beta x} = \delta_N(x), \qquad |x| \le x_0 \quad (j = (-1)^{\frac{1}{2}}). \tag{3}$$

$P_N(\beta)$ and $\delta_N(x)$ were described further in the earlier report.[1]

<u>Basic extrapolating property</u>

Even more interesting than the asymptotic behavior of $\delta_N(x)$ as $N \to \infty$ is its functional form at any finite N:

$$\delta_N(x) \propto \text{sinc}\,[(M\beta_0)x], \qquad |x| \le x_0 \tag{4a}$$

to a good approximation. This may be recognized as the Airy disc amplitude due to a magnified aperture $M\beta_0$. (Recall that β_0 is the original aperture size.) It was numerically established[1] that M is asymptotically given (with good precision once $N \gtrsim 20$) by

$$M = N\pi/(3\beta_0 x_0). \tag{4b}$$

Since N is merely the highest order term used in series (1b), it is at the user's disposal. Hence, Eq. (4b) states that the effective aperture magnification due to coating $P_N(\beta)$ is arbitrarily large, for fixed values of β_0 (the original aperture) and field x_0.

We now examine properties that follow directly from the phenomenon described by Eq. (3). Because Eq. (3) is based on the scalar, Fraunhofer-Fresnel approximation[2] and because it ignores the quantum nature of radiation, we must be skeptical of these properties near the limit $N \to \infty$. Nevertheless, it is useful to establish what can be accomplished within the framework of classical scalar theory, especially since this is the usual approximation made in studies of apodization[4] and diffraction imagery.[2]

At the close of this work, the requirements for scalar, Fraunhofer theory to be valid will be shown to limit N to a finite, but large, value.

A PARTICLELIKE SOLUTION TO THE WAVE EQUATION

It is instructive to find the superresolution due to pupil $P_N(\beta)$. From Eq. (4a), it can be seen that the extent Δ_N of the central core of δ_N is as follows:

$$\Delta_N = 2\pi/M\beta_0. \tag{5}$$

Combined with Eq. (4b), this yields

$$\Delta_N = 6x_0/N. \tag{6}$$

So, for a fixed field x_0, Δ_N approaches zero as N is increased. Or, the lower limit on Δ_N does not depend on λ, the light wavelength used! Note also from Eq. (3) that, as $N \to \infty$, the amplitude response $\delta_N(x)$ approaches point concentration.

This limit implies that the *wave* equation (from which Eq. (6) is derived) admits of particlelike solutions. This seems paradoxical at first. However, recall that one solution[5] to the wave equation is $f_1(\vec{r} - \vec{v}t) + f_2(\vec{r} - \vec{v}t)$, with $\vec{r}$ the position vector, v the velocity of light, and functions f_1 and f_2 arbitrary. The particular choice $f_1(\vec{r} - \vec{v}t) = \delta(\vec{r} - \vec{v}t)$, $f_2 = 0$ is therefore a valid solution. It represents a particlelike concentration of wave energy at distance vt from the origin.

The foregoing might be considered an "existence proof" for particlelike solutions. That is, the physical circumstances that would cause such a solution are not established by the method of proof. The particlelike result, Eq. (3), is, then, one physical solution of this type.

ON PRODUCING AN INDEFINITELY NARROW DEPTH OF FOCUS

The scalar amplitude distribution a(z) along the optic axis due to any pupil $W(\beta)$ obeys a pure Fresnel law,[2]

$$a(z) = \int_{-\beta_0}^{\beta_0} d\beta W(\beta) e^{j(z/2k)\beta^2}, \qquad k = 2\pi/\lambda. \tag{7}$$

The depth of focus can be measured by the resemblance of a(z) to Dirac $\delta(z)$. If $a(z) \propto \delta(z)$, the point amplitude distribution is longitudinally compressed entirely onto the gauss point, and the depth of focus is zero. We therefore ask whether a pupil $W(\beta)$ exists that causes a(z) to approach $\delta(z)$.

Let us first reduce the power of the exponent in Eq. (7) by using a transformation

$$W(\beta) = |\beta| V[(\beta^2 - \beta_0^2/2)/2k], \tag{8}$$

where V is now the unknown function. There results, owing to the evenness of the integrand,

$$a(z) = 2 \int_0^{\beta_0} d\beta\, \beta V[(\beta^2 - \beta_0^2/2)/2k]\, e^{j(z/2k)\beta^2}. \tag{9}$$

Now, change variable β to

$$\beta' = (\beta^2 - \beta_0^2/2)/2k. \tag{10}$$

Then Eq. (9) becomes

$$a(z) = 2ke^{j\beta_0^2 z/4k} \int_{-\beta_0^2/4k}^{\beta_0^2/4k} d\beta' V(\beta') e^{jz\beta'}. \tag{11}$$

The integral is the finite Fourier transform of $V(\beta')$, which we would like to be $\delta(\beta')$. But this is precisely the problem in Eq. (3). The solution is therefore $U_N(\beta)$ once more, provided we substitute $\beta_0^2/4k$ for β_0 and a field-of-view parameter z_0 for x_0. The final result, after re-use of the Eq. (8) transformation, is as follows:

$$W_N(\beta) = |\beta| \sum_{n(\text{even})=0}^{N} (-1)^{n/2} \lambda_n(c)^{-3/2} \psi_n(c,0) \psi_n[c, z_0(1-2\beta^2/\beta_0^2)],$$

$$c = \beta_0^2 z_0/4k. \tag{12a}$$

This causes a longitudinal amplitude

$$a(z) \propto \delta_N(z) \quad \text{for } |z| \le z_0, \tag{12b}$$

as was required.

The longitudinal extent of the central core Δ_N of $a(z)$ may be considered the focal depth. From Eq. (6) we therefore have $\Delta_N = 6z_0/N$, or an indefinitely narrow focal depth.

Parameter c may be related to the classical focal depth $d \equiv 4\lambda f\#^2$, f# being the system f/number. Using the definition of c in Eq. (12a), we have $c = (\pi/2)(z_0/d)$. Thus, c now measures the longitudinal field extent z_0, in units of d, throughout which $a(z) \to \delta_N(z)$.

The graph below shows the normalized pupil $W_{20}(\beta)$ for the case $\beta_0 = z_0 = c = 1$. In its alternate zones of zero and π phase,[1] $W_N(\beta)$ resembles $P_N(\beta)$. However, $W_N(\beta)$ demands twice as many such zones as $P_N(\beta)$.

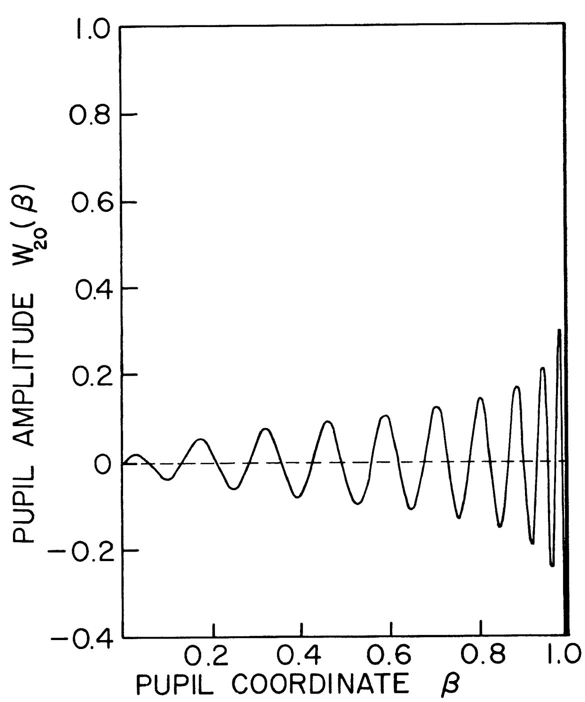

Fig. 1.

A pupil coating $W_{20}(\beta)$ for a required narrow focal depth. Parameters $\beta_0 = c = z_0 = 1$, yielding a classical focal depth $d = (\pi/2)(z_0/c) = \pi/2$ and enhanced focal depth $\Delta_{20} = 6z_0/20 = 0.3$. Such enhancement generally requires a pupil with 2N - 1 fringes of alternating phase. The focal depth Δ_N can be made arbitrarily close to zero as N (and hence the number of pupil fringes) increases indefinitely.

IMAGE SYNTHESIS

We consider the problem of using an optical system to form a point amplitude response $a(x)$ of arbitrary profile. Given the requirement that $a(x) = g(x)$, can a pupil function $\hat{G}(\beta)$ be found? If we require $a(x) = g(x)$ for all x (that is, $-\infty \le x \le \infty$), the well-known answer is "no."[6] However, let us imagine that a limited object scene,

$$o(x), \qquad |x| \le x_0/2, \tag{13a}$$

is to be imaged with point amplitude $g(x)$. At equal conjugates, the image field of concern will then likewise be $|x| \le x_0/2$. Hence, only the central region $|x| \le x_0$ of the point amplitude $g(x)$ contributes to the observed image. The problem of synthesis is now simplified to demanding

$$a(x) = g(x) \qquad \text{for } |x| \le x_0. \tag{13b}$$

This has a solution, as shown next.

Eq. (3) may be written as

$$\int_{-\beta_0}^{\beta_0} d\beta P_N(\beta) e^{j\beta(y-x)} = \delta_N(y-x), \qquad |y-x| \le x_0. \tag{14}$$

Multiply both sides by $g(x)$ and integrate dx from $-x_0/2$ to $x_0/2$. There results

$$\int_{-\beta_0}^{\beta_0} d\beta P_N(\beta) \int_{-x_0/2}^{x_0/2} dx\, e^{j\beta(y-x)}\, g(x) = \int_{-x_0/2}^{x_0/2} dx\, \delta_N(y-x) g(x). \tag{15}$$

The right-hand side may properly be denoted as $g_N(y)$; since $\delta_N(y-x) \to \delta(y-x)$ as $N \to \infty$, likewise $g_N(y) \to g(y)$ as $N \to \infty$. The left-hand side may be cast as

$$\int_{-\beta_0}^{\beta_0} d\beta P_N(\beta) e^{j\beta y} \hat{G}(\beta), \tag{16}$$

where

$$\hat{G}(\beta) = \int_{-x_0/2}^{x_0/2} dx g(x) e^{-j\beta x}. \tag{17}$$

Hence we have established that

$$\int_{-\beta_0}^{\beta_0} d\beta P_N(\beta)\hat{G}(\beta)e^{j\beta y} = g_N(y), \qquad |y| \le x_0/2. \tag{18}$$

But this is the Fraunhofer approximation for forming a point amplitude $g_N(y)$ from a net pupil $P_N(\beta)\hat{G}(\beta)$. We may therefore conclude that any required amplitude profile $g(y)$ may be arbitrarily approximated by use of a pupil $P_N(\beta)\hat{G}(\beta)$, with N sufficiently large.

A similar result holds for point amplitude synthesis along the optic axis. Observing the similarity between Eqs. (11) and (14), we see that any required intensity spread function $|g(z)|^2$ may be produced with arbitrary precision.

THE GENERAL EXTRAPOLATING PROPERTY

We have seen that coating $P_N(\beta)$ has a property of effective pupil extension when it is applied to diffraction-limited optics. Would $P_N(\beta)$ also appear to extrapolate outward an arbitrary pupil when applied to it? Eqs. (17) and (18) provide the answer. Suppose that amplitude $g(x)$ is itself limited in extent,

$$g(x) \quad = \quad 0 \qquad \text{for } |x| > x_0/2. \tag{19}$$

Then the finite limits in Eq. (17) may be replaced by infinite ones, and $\hat{G}(\beta) = G(\beta)$, the Fourier transform of $g(x)$. Then Eq. (18) becomes

$$\int_{-\beta_0}^{\beta_0} d\beta P_N(\beta) G(\beta) e^{j\beta y} \quad = \quad g_N(y), \qquad |y| \le x_0/2. \tag{20a}$$

Compare this with the purely Fourier relation

$$\int_{-\infty}^{\infty} d\beta G(\beta) e^{j\beta y} \quad = \quad g(y). \tag{20b}$$

For N sufficiently large we had $g_N(y) \to g(y)$. The asymptotic equality of Eqs. (20a) and (20b) then shows that the finitely extended pupil $P_N(\beta)G(\beta)$ is equivalent to the infinitely extended pupil $G(\beta)$. Or, when a coating $P_N(\beta)$ is applied to a finite pupil $G(\beta)$, the point amplitude $g(y)$ is formed as if $G(\beta)$ were physically extended beyond its physical aperture. As N is made ever larger, the apparent extension increases without bound.

As an example, suppose we wish to optically produce a rectangular amplitude response,

$$g_r(x) \quad = \quad \begin{cases} 1 & \text{for } |x| \le x_0/4 \\ 0 & \text{for } |x| > x_0/4. \end{cases} \tag{21a}$$

Of course the infinite pupil

$$G_r(\beta) \quad = \quad (.25x_0/\pi)\operatorname{sinc}(.25x_0\beta) \tag{21b}$$

would suffice, but any optical system must have a finite aperture β_0. According to Eqs. (20), we should simply truncate $G_r(\beta)$ at $\beta = \beta_0$ and superimpose upon it coating $P_N(\beta)$. Then, within a field of view $|x| \le x_0/2$, the observed amplitude would approximate $g_r(x)$ as well as is required.

Figure 2 shows the composite pupil $P_{20}(\beta)G_r(\beta)$, $|\beta| \le \beta_0$, and its finite Fourier transform $g_{20}(x)$, $|x| \le x_0/2$. Values $\beta_0 = 1$, $x_0 = 1$ were used. Note that $g_{20}(x)$ does well approximate the required rectangle over the field of view. For comparison, we also show the pupil $G_r(\beta)$ and its finite Fourier transform $\hat{g}_r(x)$. The latter looks nothing like $g_r(x)$, since $G_r(\beta)$ is very nearly flat over the pupil.

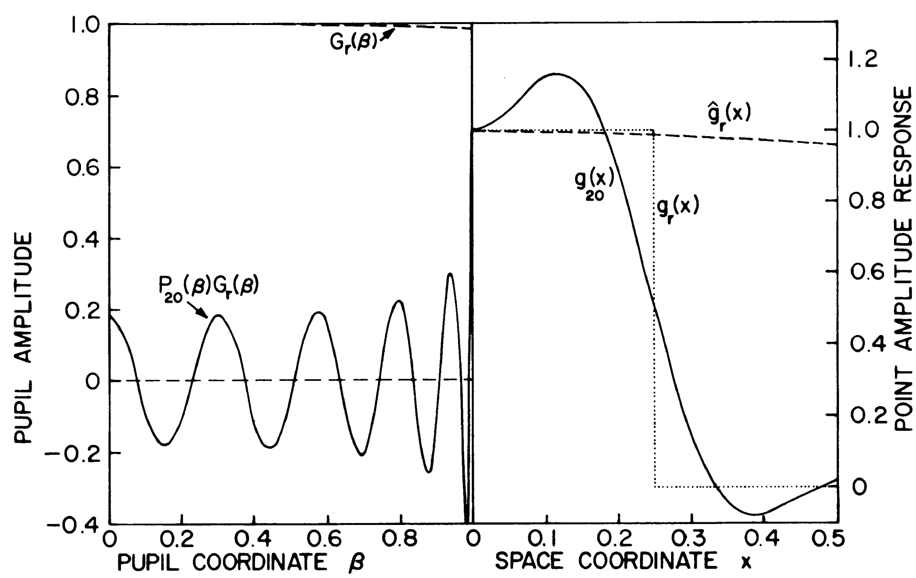

Fig. 2. Example of image synthesis.

Dotted line: The rectangular amplitude response $g_r(x) = 1$ for $|x| \le \frac{1}{4}$ is required. It is well approximated by $g_{20}(x)$, the physical response to pupil $P_{20}(\beta)G_r(\beta)$.

Dashed line: The response $\hat{g}_r(x)$ to the pupil $G_r(\beta)$ alone departs widely from the rectangle.

Regarding field illumination, the Strehl flux ratio was very small in all cases of amplitude synthesis that have been examined: a rectangle, a triangle, and a semicircle, with $\beta_0 = 1$ and $x_0 = 1,10$. It may be that use of larger x_0 will permit larger Strehl, but this has not yet been tried.

LOOKING TO THE FUTURE: SOME POSSIBLE APPLICATIONS

If the preceding effects were practicable at high N, they could be used with splendid advantage. However, as $N \to \infty$, the problems of field illumination, fabrication, and "seeing" (where appropriate) grow without bound. With today's technology, low values of N (up to 10) seem the attainable limit. Therefore, mainly with an eye to the future, we suggest the following applications.

An optical analog signal extrapolator

Suppose a signal is observed over a finite interval, and we wish to know the signal outside that interval. To be specific, let the signal be a voltage waveform $V(t)$ extending from time t_a to t_b. To physically produce $V(t)$ at t outside the given interval, we use the setup of Fig. 3. A coating $P_N(t)V(t)$, with t now a pupil coordinate, is placed in the exit pupil of a diffraction-limited lens L, which is illuminated by a collimated, monochromatic wave. We assume $V(t)$ is band-limited, i.e., its Fourier transform $v(x) = 0$ for $|x| > x_0/2$. Then according to Eq. (20a), in the Fraunhofer plane F, distance f away, the amplitude response is $v_N(x) \to v(x)$ for $|x| \le x_0/2$. We simulate $v(x) = 0$ over the remaining space $|x| > x_0/2$ by masking off the spurious amplitude at $|x| > x_0/2$.

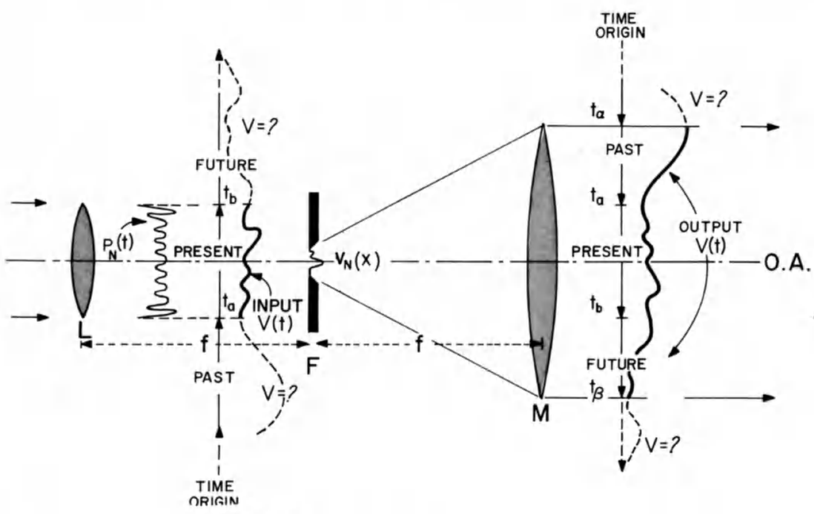

Fig. 3. Signal extrapolation by coherent processing.
The signal $V(t)$ to be extrapolated is coated jointly with $P_N(t)$ upon diffraction-limited lens L. Edges of the lens correspond to the extreme time values at which V has been observed. The spectrum $v(x)$ of $V(t)$ is known to not extend beyond $|x| = x_0/2$.
In Fraunhofer plane F, the amplitude response due to collimated coherent illumination of L is masked for $|x| > x_0/2$. According to the principle illustrated in Fig. 2, $v(x)$ is formed as if $V(t)$ existed upon a lens that arbitrarily exceeds L in extent, thereby extending $V(t)$ outside its domain of observation. The Fourier transform of $v(x)$ should yield this extrapolation.
The light leaving plane F is recollimated by lens M, which performs the Fourier transform by analog. Hence, the amplitude distribution leaving M is the physical extrapolation to $V(t)$ given on L.

A larger, diffraction-limited lens M is placed the same focal distance f from the Fraunhofer plane. What will be the output wave from M? Lens M sees the entire distribution $v_N(x) \to v(x)$ a focal length away. Its output, the Fourier transform of $v_N(x)$, must therefore be asymptotically $V(t)$. However, since the lens is larger than L, with extremes $t_\alpha < t_a$ and $t_\beta > t_b$, the output wave covers a longer time interval than did that of L. Hence, the input wave $V(t)$, $t_a \le t \le t_b$, is now physically extrapolated to $V(t)$, $t_\alpha \le t \le t_\beta$.

As a simple example, consider the case where voltage is observed to obey $V(t) = \text{sinc}(.25t)$ for $|t| \le 1$. Thus, $t_a = -1$, $t_b = +1$. Suppose that the experimenter does not know the analytic formula for $V(t)$, merely its values with great accuracy from time t_a to t_b. Suppose also that he knows that the bandwidth of $V(t)$ is limited to $|x| \le \frac{1}{2}$. Using the apparatus of Fig. 3 with a value $N = 20$, the Fraunhofer amplitude $v_{20}(x)$ will be $g_{20}(x)$ in Fig. 2. Suppose output lens M of Fig. 3 has 10 times the aperture of L. Then the output wave, the Fourier transform of $g_{20}(x)$, will be $V(t)$ defined over time $0 \le |t| \le 10$. We performed this transform numerically, with results in Fig. 4.

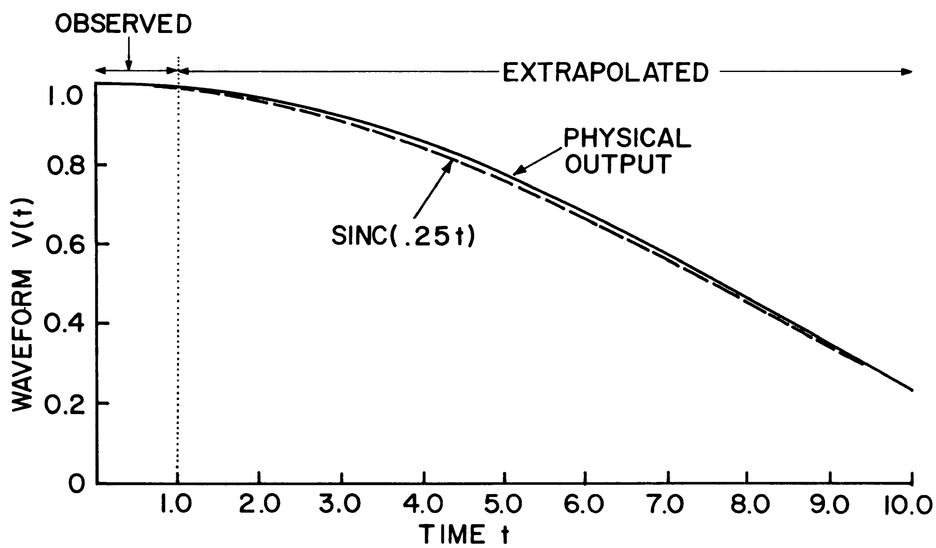

Fig. 4. Example of an extrapolated signal. Since, from Fig. 2, $g_r(x) = 0$ for $|x| \ge \frac{1}{4}$, its spectrum $G_r(t)$ is a fitting signal for extrapolation. $G_r(t) = \text{sinc}(.25t)$ is assumed known for $0 \le |t| \le 1$, and is coated with $P_{20}(t)$ upon lens L. The response in plane F is $g_{20}(x)$ of Fig. 2. The output lens M is assumed to extend over $0 \le |t| \le 10$, or 10 times the extent of L. Its output is the finite (due to the mask) Fourier transform of $g_{20}(x)$. This is not exactly sinc(.25t) because $g_{20}(x)$ is not precisely a rectangle. However, the degree of approximation is quite good throughout the large region of extrapolation.

The physical output wave from lens M is seen to hardly depart from the analytic function sinc(.25t) throughout the entire region of extrapolation $1 \le |t| \le 10$. This interval is 9 times the time interval over which V(t) was observed! Evidently V(t) would have to be known with extraordinary precision (eight figures for this numerical example) over $0 \le |t| \le 1$ for such an extensive extrapolation to be possible.

Since this analog technique requires the signal to be band-limited, it is important to inquire when this is so. Although waveforms do not usually cut off absolutely, there is always a frequency beyond which spectral contributions are insignificant for certain applications.[7] If our requirements on accuracy and extent of extrapolation are not overly severe, a usefully small cutoff frequency probably can be employed.

On the other hand, some optical signals (for example, the coherent and incoherent images) are absolutely band-limited.[2] We turn next to their use in the extrapolating scheme.

Picture extrapolation

Consider an image t(x), $|x| \le X$, which has been formed by either coherent or incoherent light. Assume that there is no light contributed to t(x) from object points lying outside the field of view (hypothesis of a limited object). Now suppose that, by some mishap, a portion $X' \le |x| \le X$ of the image is irreversibly damaged or cut away. Can we physically re-create the lost piece from the remaining one?

Since t(x) is band-limited,[2] it may be used as input to the signal extrapolator of Fig. 3. If the cutoff frequency is Ω, the opening in mask F is $(\lambda/\pi)f\Omega$. And if output lens M is of extent $|x| \le X$, the output wave will be asymptotic with N to the entire image t(x), $|x| \le X$. The lost image piece has been recovered.

We can see why image extrapolation is possible. The optical image is a superposition of sinc(x) functions,[2] each of which extends to infinity and is modulated by the image value at one point. Therefore, at any one image point of a transparency, there is contribution from, and information about, every other image point, even after the latter have been physically cut from the image.

An analog method of band-unlimited image processing

In an earlier report,[8] we described a numerical method for restoring an object scene from its image with arbitrary accuracy. By this method, the object spectrum at frequencies outside the optical passband is recovered by extrapolation from within the passband. The final restoration is then obtained by Fourier inversion.

This method is extremely sensitive to error in measurement of the image. If the method could be accomplished completely by analog operation upon the physical image, the latter would not have to be measured, and a serious obstacle to use of the method would be avoided. A slight modification of the Fraunhofer-plane processor due to Maréchal and Croce[9] will accomplish just this. (See Fig. 3.)

Suppose the image transparency $i(x)$ alone is placed upon lens L, now without superimposing coating $P_N(\beta)$. Also, let lens M have the same aperture size as L. In Fraunhofer plane F the image spectrum $I(\beta)$ is produced. In plane F, place an amplitude transparency $1/\tau(\beta)$, where $\tau(\beta)$ is the optical transfer function connecting object and image. The net amplitude leaving F is now $I(\beta)/\tau(\beta)$, which is just the object spectrum.[2] However, because $\tau(\beta) = 0$ for all $|\beta| > \Omega_0$, the optical cutoff, the transparency $1/\tau(\beta)$ must be masked for values $|\beta| > \Omega_0$. The output wave from lens M would now be a band-limited approximation to the object scene--Maréchal and Croce's result.

To physically remove the band limit, we further superimpose in plane F the coating $P_N(\beta)$. Recall the extrapolating property of pupil $P_N(\beta)$ when it multiplies a band-limited function. If the object scene $o(x)$ is of limited extent, $I(\beta)/\tau(\beta)$ is indeed band-limited. Hence, the output wave from lens M is now formed as if $I(\beta)/\tau(\beta)$ existed in plane F for values $|\beta| > \Omega_0$. As $N \to \infty$, effectively the upper limit to $|\beta|$ approaches infinity. Therefore the output from lens M is asymptotic to the infinite Fourier transform of $I(\beta)/\tau(\beta)$, which is just the object scene to be restored.

Although the serious problem of image measurement is thereby avoided, the problems of fabricating $P_N(\beta)$ and of final field illumination replace it. Perhaps some combination of analog and digital processing would effectively accomplish the extrapolation.

A laser superposition mode with an indefinitely directive output

In view of the severely low illumination[1] within field $|x| \le x_0$, caused by both absorption and scattering at pupil $P_N(\beta)$, it is natural to consider optical systems that both amplify and recycle light to and fro between object and image. This is indeed what happens within the confocal laser resonator, if by "object" and "image" we now mean the amplitude distributions on the end plates. Toward this end, we further note that amplitude $\delta_N(x)$ of Eq. (2) is precisely a superposition of the $TEM_{n,q}(x)$ modes for rectangular endplates.[10] The nth weight factor is $\lambda_n(c)^{-1}\psi_n(c,0)$ for n even, and 0 for n odd.

Hence, a laser "superposition mode" exists whose amplitude is asymptotic to a Dirac delta function over $|x| \le x_0$. This means that, by choosing N large enough and masking off the output at $|x| \le x_0$, an arbitrarily narrow and directive beam can be made to emerge from the laser!

Methods are already being developed for forcing a laser to simultaneously resonate, with required energy, in each of a series of modes.[11] A difficulty with the particular multimode requirement of Eq. (2) is that, since $\lambda_n(c) \to 0$ rapidly with n once n is greater than a critical value, the relative amount of energy in each mode must increase rapidly with n. Furthermore, all odd modes must be absent. Nevertheless, there seems to be no physical reason that rules out such a mode superposition.

<u>Attainment of near-optical resolution with nonoptical waves</u>

We have noted that the extrapolating pupil works to produce an amplitude response $\delta_N(x)$ for any phenomenon that obeys Fraunhofer imagery. This includes transverse waves such as microwaves and radar, and even longitudinal waves such as sonar. In particular, we note Eq. (6), which states that the central core width is arbitrarily narrow (with N), independent of wavelength. This implies that the above waves, all of whose wavelengths greatly exceed the optical case, can theoretically be used to produce images with optical resolution.

In practice, these waves can be produced with energy densities greatly exceeding those of optical waves. Therefore, considering the upper limit on N to be caused by the field illumination losses previously spoken of, these longer waves can be used with higher N, and hence greater resolution enhancement, than can the optical waves. It may even be possible to attain near-optical resolution with such waves, depending on the available energy in the scene to be imaged.

Thus, sonar waves, which are much more penetrative of sea water than is light, might be used to image large areas of the ocean with near-optical quality. The acoustical analog to pupil $P_N(\beta)$ is simply a one-dimensional array of speakers, all vibrating at a common tone, where volume varies with position β as $|P_N(\beta)|$, and phase is either 0 or π.

ULTIMATE LIMITATIONS ON USE OF $P_N(\beta)$

We have noted the intrinsically low level of field illumination for N sufficiently large.[1] Other limitations arise from (a) fabrication and "seeing" tolerance, (b) the approximate nature of scalar, Fraunhofer-Fresnel theory, and ultimately (c) the quantum nature of light. We will limit our present investigation to points (a) and (b) although (c) may yet prove the most severe limitation of the three.

Fabrication and "seeing" tolerance

The analysis in this paper has assumed that a deterministic coating $P_N(\beta)$ is applied with perfect accuracy to a deterministic pupil $G(\beta)$, which equals 1 in most cases. Of course, both the phase and amplitude of both $P_N(\beta)$ and $G(\beta)$ will vary randomly because of fabrication error and possibly atmospheric turbulence or vibration during actual use. We shall limit this investigation to the case of $G(\beta) = 1$, that is, initially an uncoated, diffraction-limited pupil.

We make the usual assumptions of white, uncorrelated phase noise at each point in the pupil. Assume that in the absence of coating $P_N(\beta)$ the noise variance is σ_0, and that after $P_N(\beta)$ is applied the net phase noise is σ. What will be the average S/N across the point amplitude response both before and after $P_N(\beta)$ is applied? The ratio of the two S/N values will be a measure of the expected sensitivity to pupil noise of the enhanced image.

Assuming small phase variations and using the convenient orthogonality[3] of functions $\psi_n(x)$, we find that the ratio r_N of the S/N arising from $P_N(\beta)$ to the S/N in the absence of $P_N(\beta)$ is simply

$$r_N = (\sigma_0/\sigma)\gamma_N \,, \tag{22a}$$

where

$$\gamma_N^2 = \frac{\sum_{n(\text{even})=0}^{N} \lambda_n^{-1}\,\psi_n(0)^2}{\sum_{n(\text{even})=0}^{N} \lambda_n^{-2}\,\psi_n(0)^2} \tag{22b}$$

We would normally expect proportionality between r_N and σ_0/σ. Hence, factor γ_N describes the extra noise sensitivity characteristic of the coating process $P_N(\beta)$. Because of the fact[3] that $\lambda_n(c) \rightarrow 0$ rapidly for $n > 2c/\pi$, Eqs. (22) show that there will be severe S/N degradation for use of $P_N(\beta)$ with $N > 2c/\pi$. However, this degradation can be overcome, theoretically, by making $\sigma \leq \gamma_N \sigma_0$, that is, by fabricating $P_N(\beta)$ with better tolerance than the uncoated pupil. Of course, when the pupil phase noise arises from "seeing" or other causes subsequent to manufacture, the ratio σ_0/σ is unadjustable.

A measure of the utility of $P_N(\beta)$ is α_N, the ratio of the central core width in $\delta_N(x)$ to the Airy central core. From Eq. (6), α_N varies as $1/N$, so any required small α_N is possible. It is useful now to establish the price paid in S/N loss for attaining a required gain in resolution. Fig. 5 shows the variation of S/N sensitivity factor γ_N with α_N, at various c. Formula (22b) was used. It is immediately apparent that a tradeoff exists between S/N and resolution enhancement, or alternatively, that a useful increase in resolution demands extreme manufacturing tolerance and extremely good seeing.

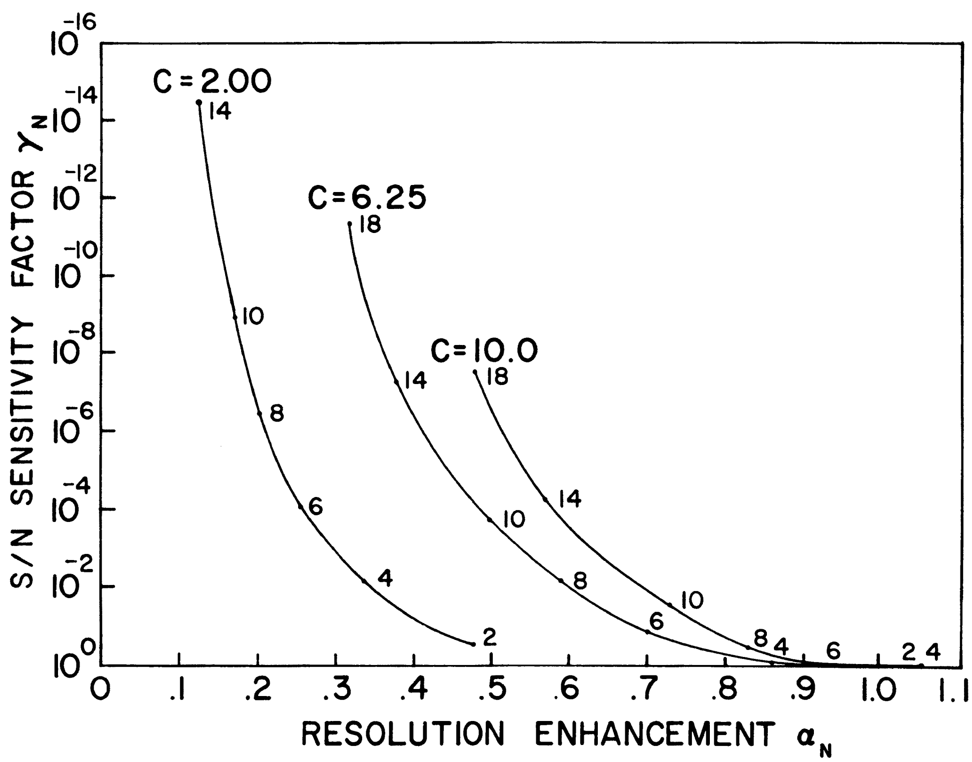

Fig. 5. Tradeoff between S/N degradation, measured by sensitivity factor γ_N, and resolution enhancement ratio α_N; c is the space-bandwidth product for the optics. A useful enhancement of resolution (α_N small) results in a very low S/N for the imagery unless both the manufacturing tolerance on $P_N(\beta)$ and the seeing are extremely good.

If the pupil suffers random amplitude variations, without phase variation, then Eqs. (22) still hold, and σ_0 and σ now refer to rms variations in the log amplitude transmittance. Thus, the tolerances on phase, or on log amplitude, are the same.

The case of simultaneous phase and amplitude noise will not be examined.

Limitations due to scalar, Fraunhofer-Fresnel theory

All of the preceding results depend on the validity of scalar theory and of the Fraunhofer-Fresnel approximation to scalar theory. These effects will be seen to limit N, and hence the attainable extrapolation, to a large but finite value.

Scalar theory has been verified experimentally[12] when the aperture greatly exceeds a wavelength and the image plane is not too close to the aperture. As an example, a system with a very large numerical aperture would violate the latter requirement. These requirements do not seriously restrict our results. In fact, we would normally use a small numerical aperture since the whole point of using coating $P_N(\beta)$ is to make a small aperture act like a large one.

On the other hand, coating $P_N(\beta)$ has $N-1$ fluctuations within a fixed aperture. As N is made to increase to the point that the narrowest fluctuation approaches a wavelength, $P_N(\beta)$ looks to the incident light like a grating. Gratings must be analyzed with a vector theory.[12] Empirical indications are that for $N \gtrsim 10$ the spatial width Δw_N of the narrowest pupil fluctuation (which occurs at the margin[1]) is given roughly by

$$\Delta w_N \simeq 3d_0/N^2, \tag{23}$$

where d_0 is the aperture extent. Setting $\Delta w_N = \lambda$, we find a limiting value of

$$N_{max} = (3d_0/\lambda)^{\frac{1}{2}}. \tag{24}$$

This is normally a very large number at optical wavelengths.

Once scalar theory is permitted, we must see whether the Kirchoff boundary condition[12] is possibly violated by $P_N(\beta)$. Because P_N has an extremely sharp rise (a slope of roughly $1/\Delta w_N$) directly at the margin,[1] once $\Delta w_N \simeq \lambda$ it is conceivable that the light amplitude just beyond the margin is sufficiently nonzero to violate the Kirchoff requirement. However, for Δw_N greater than a few wavelengths, it is hard to imagine such a violation.

Once the Kirchoff boundary condition is allowed, the Fraunhofer-Fresnel formalism follows[12] from the prior assumption of a small numerical aperture and the assumption that the image coordinates x and z are much smaller than the focal length.

In summary, then, for values $N < N_{max}$ given by Eq. (24), the results of this paper do not violate the classical theory from which they derive.

In closing, we note that even for $N < N_{max}$, the resolution Δ_N might be good enough to define the position of a source point better than that allowed by the uncertainty principle of quantum mechanics. This problem is currently under study.

SUMMARY AND DISCUSSION

We have investigated the implications to image formation of an "extrapolating" pupil $P_N(\beta)$. When coated upon any band-limited pupil, $P_N(\beta)$ synthetically extends the pupil into the region beyond its physical aperture. This apparent extension is arbitrary, and may be measured by the pupil magnification factor M of Eqs. (4).

The price paid for this ideal property is twofold: (1) severe loss of object radiation over the field of view, and (2) severe loss of S/N in the image unless the pupil $P_N(\beta)$ can be made with great precision and can be used under very good "seeing" conditions (pages 16-18). These effects may rule out optical use of $P_N(\beta)$ for the time being. However, the theory works for any scalar wave, so applications to microwaves, radar, and even sonar are possible. Use of these longer, and usually more energetic, wavelengths may be possible with today's technology (see page 15). Because the resolution due to $P_N(\beta)$ is independent of wavelength (see Eq. (6)), it is theoretically possible to use these long wavelengths with optical resolution.

The implications of pupil $P_N(\beta)$ to image theory are quite dramatic. Thus, the existence of $P_N(\beta)$ permits an *arbitrary* point amplitude response to be optically produced (see pp. 7-8), albeit with very small Strehl flux ratio in many cases.

Another problem which has plagued image theorists for some time is that of producing an indefinitely narrow depth of focus. A variant of pupil $P_N(\beta)$ accomplishes this (pp. 5-6).

The existence of point amplitude response $\delta_N(x)$ of Eq. (3) shows that the scalar wave equation has particlelike solutions. The wave equation is usually thought to describe a diffuse wave in its purest form.

The extrapolation principle may be combined with the principle of coherent processing to accomplish the *physical extrapolation* of signals. This leads to such exciting prospects as an optical analog signal extrapolator (pp. 11-13), a picture extrapolator (p. 13), and an optical analog method for band-unlimited image processing (pp. 13-14). These inventions are only possibilities for the future, for the current state of optical technology probably will not allow their use with extreme extrapolation (high N or M).

An application that may be possible today is the production of a laser superposition mode with an indefinitely narrow output. This possibility is suggested by the definition (Eq. (2)) of $\delta_N(x)$, which is precisely a superposition of TEM_n laser modes for the confocal cavity. A laser beam whose irradiance profile spreads in space according to $\delta_N(x)$ would be ideal for space communications.

Finally, we show that all of the above effects, which are derived from the scalar, Fraunhofer approximation, are consistent with this approximation provided the extrapolation is limited to N_{max} of Eq. (24).

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