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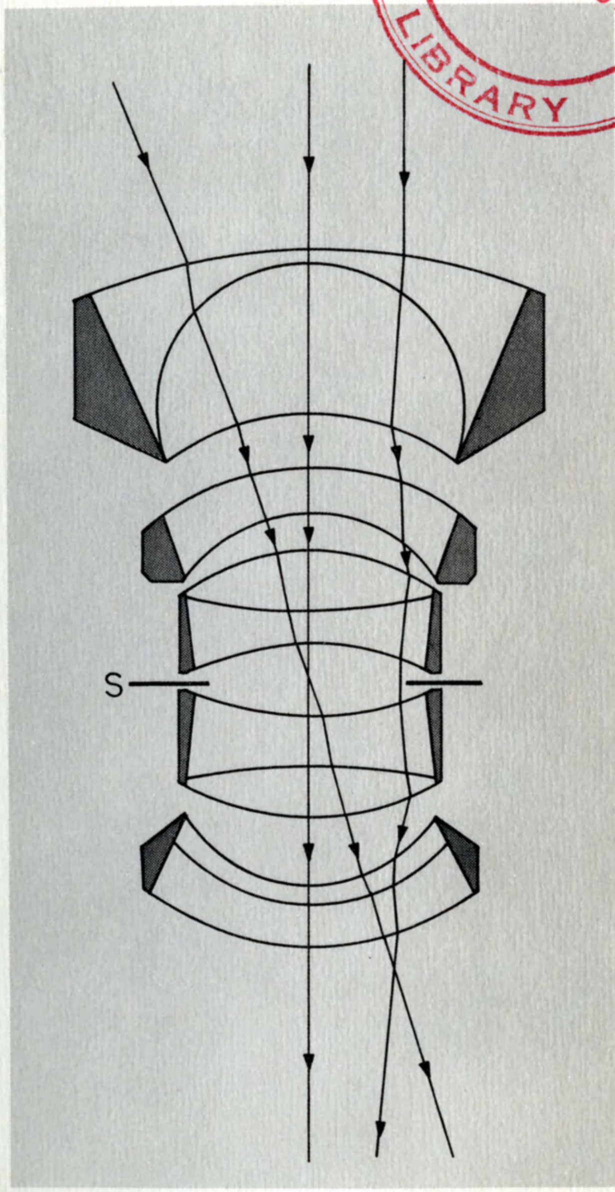

Properties of Optical Design Modules

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PROPERTIES OF OPTICAL DESIGN MODULES

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FOREWORD

This technical report is adapted from a thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Optical Sciences at the University of Arizona. The thesis was completed and approved in February 1971.

Part of the chapter on the Imaginary-Case Modules is part of a dissertation being prepared in partial fulfillment of the requirements for the PhD degree in Optical Sciences. Part of this chapter was published in the Journal of the Optical Society of America.

ABSTRACT

In the first part of this report the class of two-surface optical systems designated as modules, which have zero third-order spherical aberration relative to a pair of conjugate planes one of which is at infinity, has been further analyzed using the parameters of the Delano $y,\bar{y}$ diagram. For a given set of three indices of refraction n_1, n_2, and n_3, functional relationships among the $y,\bar{y}$ diagram parameters that eliminate simultaneously other Seidel aberrations are derived. Expressions for zero coma, astigmatism, and Petzval curvature are also given. Criteria for selecting the non-optical parameter k, which defines the desired properties of modules, are described. A one-to-one correspondence between the canonical optical parameters defined in previous studies of modules and certain quantities derivable from the $y,\bar{y}$ diagram representation is shown. Critical values of the free parameters of modules for both the real and the imaginary cases are derived and defined relative to the $y,\bar{y}$ diagram parameters.

In the second part of this report an analysis is made of a class of modules referred to as the imaginary-case family depending on the new parameter ϕ. The critical values ϕ_0, ϕ_∞, and ϕ^*, which correspond to those obtained for real-case modules, are defined, and the conditions for their existence in the domain of ϕ are derived. These critical values, whose counterparts in the real case exist for both refracting and reflecting systems, do not exist for refracting imaginary-case modules when the indices of refraction are restricted to commonly available optical glasses. The critical values of ϕ exist and have fixed values for all reflecting module systems. A method is proposed for classifying imaginary-case modules, which would permit comparison for coupling purposes.

CONTENTS

INTRODUCTION

Optical design modules are defined as a class of two-surface optical systems with fixed focal lengths and having the property that, relative to a pair of conjugate planes, one finite and the other infinite, the third-order spherical aberration is zero. Such systems have been described and analyzed by Stavroudis (1967, 1969a, 1969b). It is thought that these systems might find an application in the early stages of the process of optical design. For example, if two or more could be arranged so that the rear and front foci of adjacent systems coincide, then the resulting optical system also would have zero third-order spherical aberration. For this reason, these two-surface optical systems are called *modules.* These modules constitute two two-parameter families of lenses, one family called the *real case* and the other the *imaginary case.*

Using conventional optical parameters of curvatures and axial separations, Stavroudis analyzed modules having either refracting or reflecting spherical surfaces. To obtain a one-parameter family of lenses meeting the required conditions for modules, he defined a nonoptical parameter and expressed his four conventional optical parameters as functions of this new parameter. He defined aperture planes relative to which third-order astigmatism is zero, and he derived an expression for coma. He also indicated a means of defining the domains of his free parameters for constructible modules.

Powell (1970) analyzed the two-surface systems described by Stavroudis (1969b), using the first-order parameters of the $y,\bar{y}$ diagram that was introduced by Delano (1963) and used by Pegis et al. (1967). The module analyzed by Powell in terms of the $y,\bar{y}$ diagram parameters was a normalized two-surface refracting system that had its object plane at infinity and had fixed focal length. Powell expressed the third-order coefficients of spherical aberration, astigmatism, and coma of modules in terms of the first-order $y,\bar{y}$ diagram parameters. He also defined a parameter to obtain a functional relationship of the free parameters of two interconnected modules that varies the axial separation between the two systems.

This report amplifies the $y,\bar{y}$ diagram analysis initiated by Powell and extends the work already published on the general properties of modules.

In this report, the third-order aberrations of modules have been further analyzed to include Petzval contribution and distortion. Functional relationships among the $y,\bar{y}$ diagram parameters that provide conditions for modules to eliminate other

Seidel aberrations are derived and analyzed. The canonical optical parameters defined by Stavroudis (1969b) and certain quantities derivable from the $y,\bar{y}$ diagram are compared, and the relationship between his parameter f and the Lagrange invariant (Ж) is established. The critical values of the free parameters for both the real and the pure imaginary cases are derived. Numerical examples of modules are incorporated to illustrate their properties. In this report we also explore the properties of imaginary case modules with particular attention to the critical values of the free parameter. We anticipate that the results of this study when combined with similar results obtained from real case modules will provide valuable analytical tools for the first-order layout of optical systems using modules.

Symbols, Definitions, and Conventions

The terms and symbols used in this report are defined as they appear in the text. In general, the nomenclature, definitions, and conventions follow those given in the Military Standardization Handbook of Optical Design (MIL-HDBK-141, 1962), with the following exceptions and modifications: The Smith-Helmholtz-Lagrange invariant is called simply the Lagrange invariant and is denoted by the Russian letter Ж (zhe), t_j is the axial thickness of the space between the $j-1$ and jth surface, and n_j is the refractive index of the space between the $j-1$ and the jth surface.

With the above exceptions and changes, light is considered to travel from left to right through the system. The optical system is regarded as a series of surfaces starting with an object surface and ending with an image surface. The surfaces are numbered consecutively, in the order in which light is incident on them, starting with zero for the object surface and ending with $k+1$ for the image surface. A general surface is called the jth surface. All quantities between surfaces are given the number of the immediately succeeding surface. The radius of the jth surface is r_j and its curvature is c_j, the reciprocal of r_j. The quantities c_j and r_j are considered positive when the center of curvature lies to the right of the surface. The thickness t_j is positive if the jth surface physically lies to the right of the $(j-1)$th surface and is negative if it lies to the left. The refractive index n_j is positive if the physical ray travels from left to right. Otherwise it is negative. The right-handed cartesian coordinate system is used with the optical axis coincident with the z axis. Light travels initially toward larger values of z. Lower case letters, y_j and $\bar{y}_j$, are used to represent the paraxial heights at the jth surface of the marginal and principal rays, respectively. The slope angles of the marginal and principal rays in the space between the $j-1$ and the jth surfaces are denoted by u_j and $\bar{u}_j$, respectively, where u_j is equal to $(y_j - y_{j-1})/t_j$ and $\bar{u}_j$ is equal to $(\bar{y}_j - \bar{y}_{j-1})/t_j$.

Some changes in notation were made for the $y,\bar{y}$ diagram parameters as used by Delano (1963). The nomenclature used for these parameters is defined as it appears in the text. The figures in the text show the symbols and quantities used in this report.

Basic Relationships

Delano (1963) has shown that, given the value of the Lagrange invariant, $\mathcal{H}$, the $y_j,\bar{y}_j$ parameters, and the axial coordinate z_j at every surface of an axially symmetric optical system, the following related quantities may be derived:

$$t_j = z_j - z_{j-1} \qquad \text{axial thickness} \tag{1}$$

$$n_j = t_j/\tau_j \qquad \text{refractive index} \tag{2}$$

$$\omega_j = n_j u_j = (y_j - y_{j-1})/\tau_j \qquad \text{reduced marginal ray angle} \tag{3}$$

$$\bar{\omega}_j = n_j \bar{u}_j = (\bar{y}_j - \bar{y}_{j-1})/\tau_j \qquad \text{reduced principal ray angle} \tag{4}$$

$$r_j = (n_{j+1} - n_j)/\phi_j \qquad \text{radius of curvature} \tag{5}$$

$$\tau_j = (y_{j-1}\bar{y}_j - y_j\bar{y}_{j-1})/\mathcal{H} \qquad \text{reduced axial thickness} \tag{6}$$

$$\phi_j = (\omega_j\bar{\omega}_{j+1} - \omega_{j+1}\bar{\omega}_j)/\mathcal{H} \qquad \text{surface power.} \tag{7}$$

If Eqs. (3), (4), and (7) are divided by the Lagrange invariant, the resulting equations are as follows:

$$\Omega_j = \omega_j/\mathcal{H} = (y_j - y_{j-1})/\mathcal{H}\,\tau_j \tag{8}$$

$$\bar{\Omega}_j = \bar{\omega}_j/\mathcal{H} = (\bar{y}_j - \bar{y}_{j-1})/\mathcal{H}\,\tau_j \tag{9}$$

$$\Phi_j = \phi_j/\mathcal{H} = \Omega_j\bar{\Omega}_{j+1} - \Omega_{j+1}\bar{\Omega}_j\,. \tag{10}$$

Therefore, given the Lagrange invariant and the $\Omega,\bar{\Omega}$ parameters at each surface, the optical system is defined by the following set of equations written for the jth surface:

$$\phi_j = (\Omega_j\bar{\Omega}_{j+1} - \Omega_{j+1}\bar{\Omega}_j)\mathcal{H} \qquad \text{power of surface} \tag{11}$$

$$y_j = (\Omega_j - \Omega_{j+1})/\Phi_j \qquad \text{marginal ray height} \tag{12}$$

$$\bar{y}_j = (\bar{\Omega}_j - \bar{\Omega}_{j+1})/\Phi_j \qquad \text{principal ray height} \tag{13}$$

$$\tau_j = (y_{j-1}\bar{y}_j - y_j\bar{y}_{j-1})/\mathcal{H} \qquad \text{reduced thickness.} \qquad (6)$$

Equations (6), (8), (9), (10), (11), (12), and (13) form the set of tools to be used in the analysis of the general properties of modules.

SEIDEL ABERRATIONS AND THE MODULE

In this chapter the third-order aberrations of the two-surface systems, schematically shown in Fig. 1, are analyzed in terms of the $y,\bar{y}$ diagram parameters. These two-surface systems have fixed focal lengths and one of the conjugate planes located at infinity. Familiarity of the reader with the first-order properties of such optical systems is assumed. A full discussion of these is given by Powell (1970, pp. 5-11).

In the foregoing discussion and derivation of equations, rotational symmetry is assumed, and the indices of refraction n_1, n_2, and n_3 are fixed. It is also assumed that the two-surface system consists of spherical surfaces and the image plane is at infinity. All $y,\bar{y}$ coordinates and $\Omega,\bar{\Omega}$ parameters used in this chapter are referred to the normalized $y,\bar{y}$ diagram for a two-surface system shown in Fig. 2. This normalization scheme follows that proposed by López-López (1970).

Auxiliary Quantities

To compute the coefficients of the third-order aberrations in an optical system, several auxiliary quantities should first be determined.

The paraxial angles of incidence at the jth surface are given by

$$i_j = u_j + c_j y_j \qquad \text{for the marginal ray} \tag{14}$$

and

$$\bar{i}_j = \bar{u}_j + c_j \bar{y}_j \qquad \text{for the principal ray.} \tag{15}$$

These may be written in terms of the Lagrange invariant, the refractive indices, and the $\Omega,\bar{\Omega}$ parameters as

$$i_j = \frac{\mathcal{H}}{n_j(n_{j+1} - n_j)} (\Omega_j n_{j+1} - \Omega_{j+1} n_j) \tag{16}$$

$$\bar{i}_j = \frac{\mathcal{H}}{n_j(n_{j+1} - n_j)} (\bar{\Omega}_j n_{j+1} - \bar{\Omega}_{j+1} n_j). \tag{17}$$

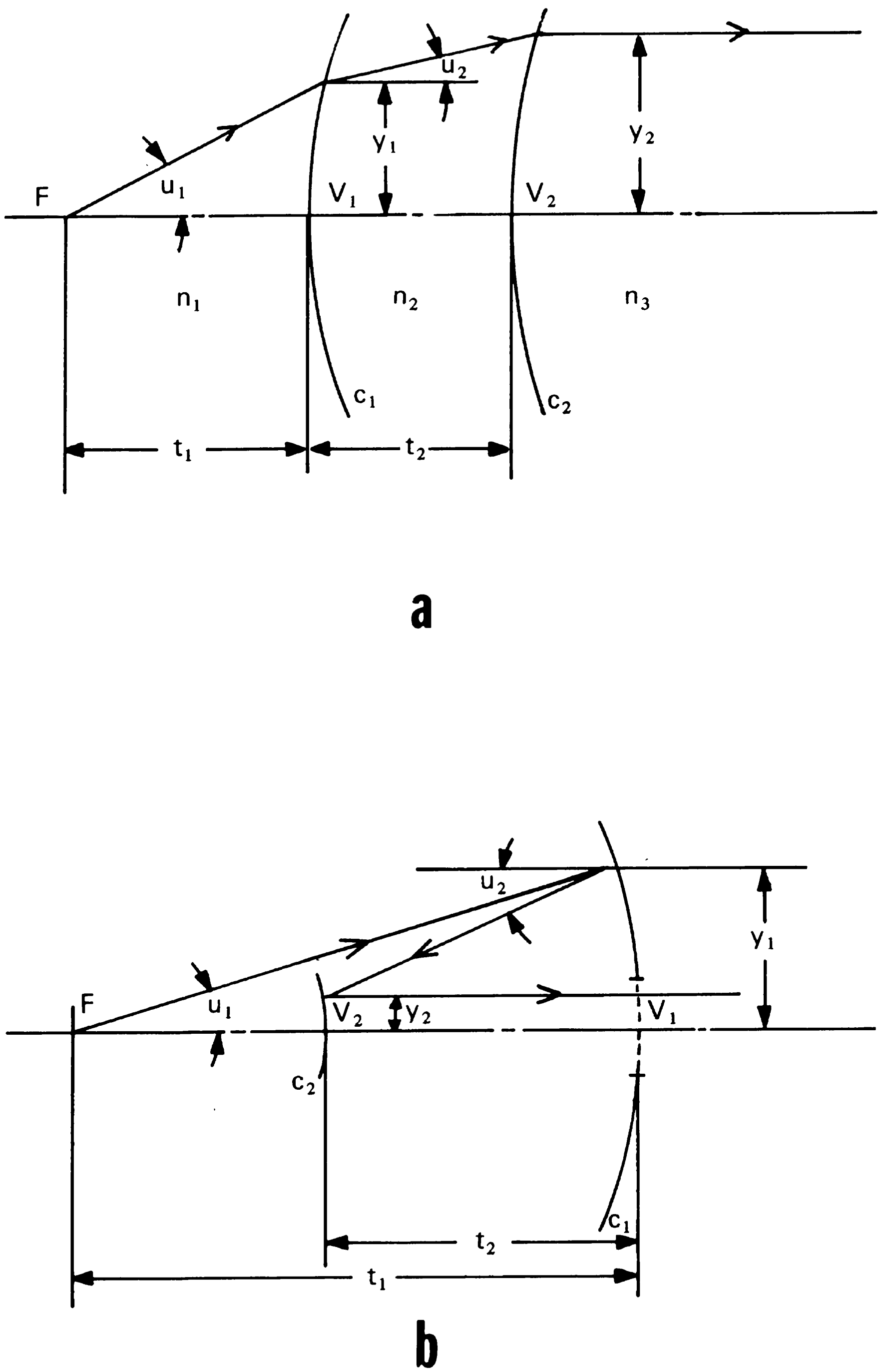


Fig. 1. Two-surface optical systems with image plane located at infinity. (a) Refracting system, (b) reflecting system.

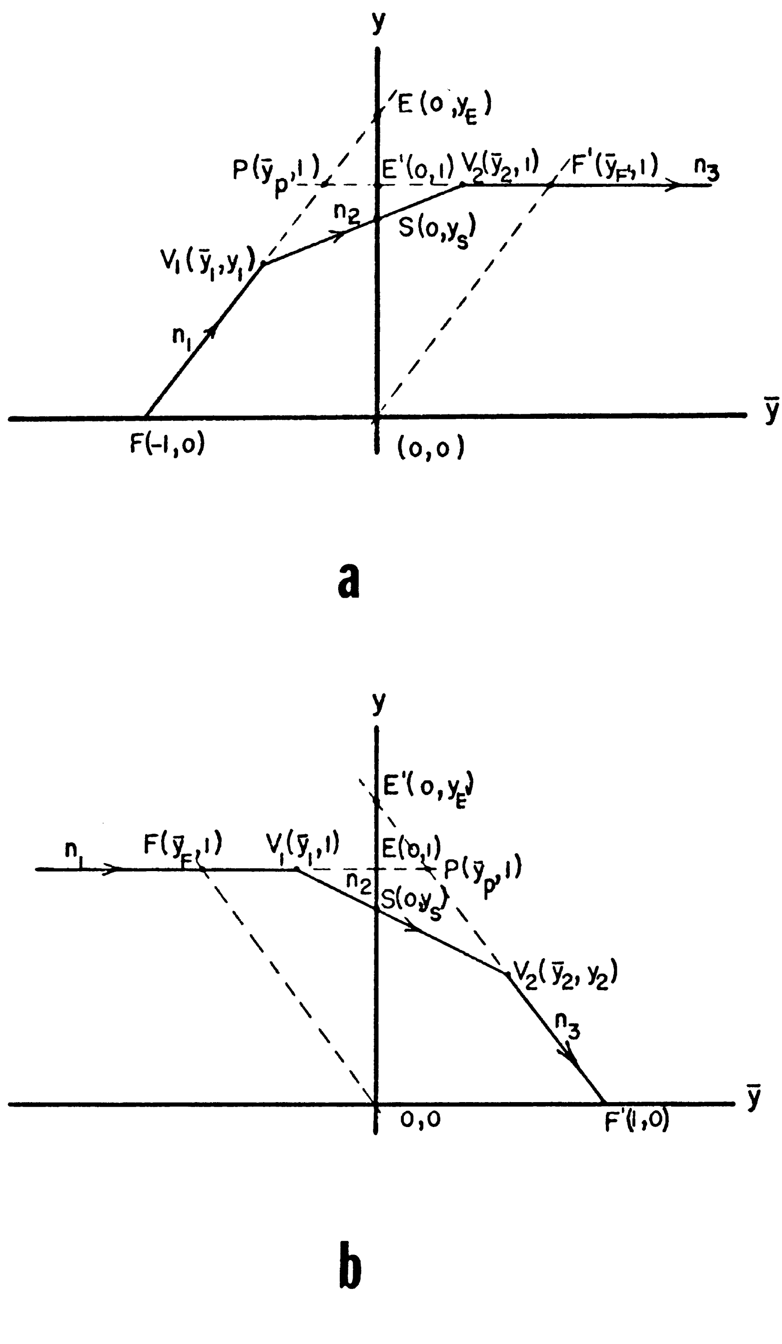


Fig. 2. Normalized $y,\bar{y}$ diagrams for a two-surface optical system with one conjugate point at infinity. (a) Image point at infinity, (b) object point at infinity.

The other auxiliary quantity associated with the paraxial marginal ray is defined for the jth surface by

$$S_j = \frac{-\,n_j y_j (u_{j+1} + i_j)(n_{j+1} - n_j)}{2\mathcal{H} n_{j+1}} \quad . \tag{18}$$

The same quantity defined for the paraxial principal ray at the jth surface is

$$\bar{S}_j = \frac{-\,n_j \bar{y}_j (\bar{u}_{j+1} + \bar{i}_j)(n_{j+1} - n_j)}{2\mathcal{H}\, n_{j+1}} \quad . \tag{19}$$

When expressed in terms of the $\Omega,\bar{\Omega}$ parameters, Eqs. (18) and (19) become

$$S_j = \tfrac{1}{2}(y_j/n_{j+1}^2)(\Omega_{j+1}\, n_j^2 - \Omega_j n_{j+1}^2) \tag{20}$$

$$\bar{S}_j = \tfrac{1}{2}(\bar{y}_j/n_{j+1}^2)(\bar{\Omega}_{j+1}\, n_j^2 - \bar{\Omega}_j n_{j+1}^2). \tag{21}$$

Third-Order Aberration Coefficients

The Seidel aberrations of an optical system may be computed from paraxial ray data, which provide information to calculate the third-order aberration coefficients. These Seidel aberration coefficients represent the algebraic sum of the third-order surface contributions throughout the optical system. The third-order aberration coefficients for an optical system of k surfaces are given by the following:

Spherical aberration $$B = \sum_{j=1}^{k} S_j i_j^2 \tag{22}$$

Coma $$F = \sum_{j=1}^{k} S_j i_j \bar{i}_j \tag{23}$$

Astigmatism $$C = \sum_{j=1}^{k} S_j \bar{i}_j^2 \tag{24}$$

Distortion $$E = \sum_{j=1}^{k} [\bar{S}_j i_j \bar{i}_j + \mathcal{H}(\bar{u}_j^2 - \bar{u}_{j+1}^2)] \tag{25}$$

Petzval curvature $$P = \sum_{j=1}^{k} c_j (n_j - n_{j+1})/n_j n_{j+1} \; . \tag{26}$$

The Module

The two-surface systems to be analyzed are those that have zero third-order spherical aberration relative to a pair of conjugate planes, one of which is at infinity. Such two-surface systems are called modules.

When Eq. (22) is expanded in terms of the normalized $y,\bar{y}$ and the $\Omega,\bar{\Omega}$ parameters of the two-surface system with image plane at infinity and the resulting expression is set equal to zero, we obtain

$$(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)^2(n_3 - n_2)^2 y_1 - n_1^2 n_3^2 \Omega_2^3 (n_2 - n_1)^2 = 0. \tag{27}$$

Solving for y_1 gives

$$y_1 = \frac{n_1^2 n_3^2 (n_2 - n_1)^2 \Omega_2^3}{(\Omega_2 n_1^2 - n_2^2)(n_2 - \Omega_2 n_1)^2 (n_3 - n_2)^2} . \tag{28}$$

Equation (28) gives the normalized value of the marginal ray height at the first surface of the module in terms of the Ω_2 parameter and the indices of refraction. This functional relationship is the condition that must be satisfied by such a two-surface system to be a module.

If Eq. (27) is expanded and the terms are rearranged, it is easily recognized to be a polynomial that is cubic in Ω_2. Stavroudis (1969b) solved this cubic expression in terms of the front focal distance, a parameter related to the focal length of the system, and another parameter that he called q. Applying the general method of solving cubic polynomials, attributed to Cardan (1545), he solved this cubic by introducing a new, nonoptical parameter k. The solution has the form

$$\Omega_2 = \frac{3n_2^2/n_1}{n_1 + 2n_2 - (n_2 - n_1)\left[\left(\frac{k+1}{k-1}\right)^{1/3} w^r + \left(\frac{k-1}{k+1}\right)^{1/3} w^{-r}\right]} \tag{29}$$

$$(r = 0,1,2),$$

where $w = \exp(2\pi i/3)$, a complex cube root of unity. The value of y_1 obtained in terms of the parameter k is

$$y_1 = \frac{27 n_2^2 n_3^2 (k^2 - 1)}{4n_1 (n_3 - n_2)^2 (n_2 - n_1)} . \tag{30}$$

Only those values of k that yield real values of Ω_2 and y_1 are of interest. From Eq. (30) real values of y_1 occur only if k is either real or pure imaginary. Stavroudis showed that for real values of k the solution to the cubic can be real only when r is equal to zero. Hence for real k

$$\Omega_2 = \frac{3n_2^2/n_1}{n_1 + 2n_2 - (n_2 - n_1)\left[\left(\frac{k+1}{k-1}\right)^{1/3} + \left(\frac{k-1}{k+1}\right)^{1/3}\right]} \, . \tag{31}$$

For pure imaginary values of k, Stavroudis introduced another free parameter θ, defined by $k = i \tan\theta$, which enables us to write Eqs. (29) and (30) in the following form:

$$\Omega_2 = \frac{3n_2^2/n_1}{n_1 + 2n_2 + 2(n_2 - n_1)\cos[2(\theta + \pi r)/3]} \qquad (r = 0,1,2) \tag{32}$$

$$y_1 = \frac{-27n_2^2 n_3^2 \sec^2\theta}{4n_1(n_3 - n_2)^2(n_2 - n_1)} \, . \tag{33}$$

Equations (32) and (33) form a set of three one-parameter families of solutions that is designated as the imaginary case. Either set of solutions (30) and (31) or (32) and (33) defines a family of modules having the property that third-order aberration is zero.

Canonical Optical Parameters

Stavroudis (1969b), in his analysis of modules, transformed the conventional optical parameters he used into a more convenient form, which he called *canonical optical parameters*. He defined the canonical optical parameters (written using my notation) as

$$\begin{aligned} C_1 &= (n_2 - n_1)c_1 f \\ C_2 &= (n_3 - n_2)c_2 f \\ T_1 &= t_1/n_1 f \\ T_2 &= t_1/n_2 f \end{aligned} \tag{34}$$

where $f = [n_2 t_1 + n_1 t_2 - (n_2 - n_1)c_1 t_1 t_2]/n_1 n_2$, c_1 and c_2 are the curvatures of the module's first and second surfaces, t_1 is the negative value of the front focal distance, and t_2 is the axial thickness of the module. He also defined $Q_r = q_r/n_1$. Comparing the above canonical optical parameters and the equations obtained with the

$y,\bar{y}$ diagram parameters, we note that the expression for Q_r and that for Ω_2 in Eq. (29) are identical, as are T_1 and y_1 in Eq. (30). Therefore, the canonical optical parameter T_1 is identical to y_1, and the parameter Q is identical to Ω_2.

The equivalence of the remaining canonical optical parameters and equations with the $y,\bar{y}$ diagram parameters used in this analysis of modules is easily established when the relationship between the parameter f and the Lagrange invariant is obtained. To show the relationship existing between f and $\mathcal{H}$, T_1 is equated to y_1, which gives

$$y_1 = t_1/n_1 f.$$

From the $y,\bar{y}$ diagram, $t_1 = n_1 y_1/\mathcal{H}$. From these two relationships we obtain $f = 1/\mathcal{H}$. Therefore, the parameter f defined and used by Stavroudis (1969b) is the reciprocal of the Lagrange invariant.

From the relationship between the parameter f and the Lagrange invariant, the equivalence between the remaining canonical optical parameters and the $y,\bar{y}$ diagram parameters is now established. They are as follows:

$$\begin{aligned} C_1 &= \Phi_1 \\ C_2 &= \Phi_2 \\ T_2 &= \tau_2 \mathcal{H}. \end{aligned} \tag{35}$$

In terms of the $y,\bar{y}$ and the $\Omega,\bar{\Omega}$ parameters, Eqs. (35) could be written as

$$\begin{aligned} C_1 &= \bar{\Omega}_2 - \Omega_2 \bar{\Omega}_1 = (1 - \Omega_2)/y_1 \\ C_2 &= \Omega_2 \\ T_2 &= (y_1 \bar{y}_2 - \bar{y}_1) = (1 - y_1)/\Omega_2. \end{aligned} \tag{36}$$

Critical Values of Free Parameters

Parameter k.—The first critical value, k_0, is the value of the free parameter k when $\Omega_2 = 1$. For Ω_2 equal to unity, Eq. (31) becomes

$$\left(\frac{k+1}{k-1}\right)^{1/3} + \left(\frac{k-1}{k+1}\right)^{1/3} = -(n_1 + 3n_2)/n_1, \tag{37}$$

which when solved for k gives

$$k_0 = \frac{(3n_2 + 2n_1)(3n_2 - n_1)^{½}}{3(3)^{½}\, n_2(n_1 + n_2)^{½}} = \sqrt{1 - \frac{4n_1^{\,3}}{27\, n_2(n_1 + n_2)}}\ . \tag{38}$$

This value of k results in zero values for the parameters c_1, C_1, and Φ_1.

The second critical value, k_∞, of the parameter k is its value when Ω_2 is infinite. For this value of Ω_2, Eq. (31) is reduced to

$$\left(\frac{k+1}{k-1}\right)^{1/3} + \left(\frac{k-1}{k+1}\right)^{1/3} = \frac{n_1 + 2n_2}{n_2 - n_1} , \tag{39}$$

which when solved for k gives

$$k_\infty = \frac{(n_2 + 2n_1)(4n_2 - n_1)^{1/2}}{3(3)^{1/2}\, n_2 (n_1)^{1/2}} = \sqrt{1 + \frac{4(n_2 - n_1)^3}{27 n_2{}^2 n_1}} . \tag{40}$$

At $k = k_\infty$, the values of the parameters c_1, C_1, Φ_1, t_2, T_2, and $Ж\tau_2$ vanish.

The third critical value, k^*, is the value of k when y_1 is unity. Substituting $y_1 = 1$ in Eq. (30) and solving for k, we obtain

$$k^* = \sqrt{1 + \frac{4n_1 (n_3 - n_2)^2 (n_2 - n_1)}{27 n_2^2 n_3^2}} . \tag{41}$$

This value of k makes t_2, T_2, and $\tau_2 Ж$ become zero.

Table I gives the corresponding values of the canonical optical parameters and their equivalent $y,\bar{y}$ diagram parameters for the three critical values of k including the values for $k = 1$, 0, and ∞.

TABLE I

Values of canonical optical parameters and equivalent $y,\bar{y}$ diagram parameters corresponding to the critical values of k.

	0	1	∞	k_0	k_∞	k^*
Ω_2 Φ_2 C_2 Q	$\frac{3n_2^2}{n_1(4n_2 - n_1)}$	0	$\frac{n_2^2}{n_1^2}$	1	∞	–
y_1 T_1	$\frac{-27n_2^2 n_3^2}{4n_1(n_3 - n_2)^2(n_2 - n_1)}$	0	∞	$\frac{-n_1^2 n_3^2}{(n_2^2 - n_1^2)(n_3 - n_2)^2}$	$\frac{n_3^2(n_2 - n_1)^2}{n_1^2(n_3 - n_2)^2}$	1
Φ_1 C_1	$\frac{4(n_3 - n_2)^2(n_2 - n_1)^2(3n_2 - n_1)}{27n_2^2 n_3^2(4n_2 - n_1)}$	∞	0	0	$-\infty$	–
$Ж\tau_2$ T_2	$\frac{n_1(4n_2 - n_1)}{3n_2^2} \cdot \left[1 + \frac{27n_2^2 n_3^2}{4n_1(n_3 - n_2)^2(n_2 - n_1)}\right]$	∞	$-\infty$	$1 + \frac{n_1^2 n_3^2}{(n_2^2 - n_1^2)(n_3 - n_2)^2}$	0	0

Parameter θ.—Corresponding to the critical values obtained for the real case, we have the following critical values of θ for the pure imaginary case:

$$
\begin{aligned}
\theta_0 &= -\pi r + (3/2)\text{arc cos}\left(\frac{n_1 + 3n_2}{2n_1}\right) \\
\theta_\infty &= -\pi r + (3/2)\text{arc cos}\left[\frac{n_1 + 2n_2}{2(n_1 - n_2)}\right] \\
\theta^* &= -\pi r + \text{arc cos}\left[\frac{3n_1 n_3}{2(n_3 - n_2)}\sqrt{\frac{3}{n_1(n_1 - n_2)}}\right] .
\end{aligned}
\tag{42}
$$

Table II gives the corresponding values of the canonical optical parameters and their equivalent $y,\bar{y}$ diagram parameters for the three critical values of θ.

TABLE II

Values of canonical optical parameters and equivalent $y,\bar{y}$ diagram parameters corresponding to the critical values of θ.

	θ_0	θ_∞	θ^*
Ω_2, Φ_2, C_2, Q	1	∞	—
y_1, T_1	$\dfrac{-n_1^2 n_3^2}{(n_2^2 - n_1^2)(n_3 - n_2)^2}$	$\dfrac{n_3^2(n_2 - n_1)^2}{n_1^2(n_3 - n_2)^2}$	1
Φ_1, C_1	0	$-\infty$	—
τ_2, T_2	$1 + \dfrac{n_1^2 n_3^2}{(n_2^2 - n_1^2)(n_3 - n_2)^2}$	0	0

Parameter Bounds.—Values of Ω_2 and y_1 as functions of k are defined by Eqs. (30) and (31), and their corresponding values as functions of θ are defined by Eqs. (32) and (33). Because the canonical optical parameters and the equivalent $y,\bar{y}$ diagram parameters could be expressed in terms of Ω_2 and y_1, functional relationships exist between the parameter k or θ and the canonical optical parameters or equivalent $y,\bar{y}$ diagram parameters. Bounds on the free parameters k and θ are equivalent to bounds placed on the canonical optical parameters or the $y,\bar{y}$ diagram parameters.

The critical values of the free parameters k and θ are functions of the indices of refraction n_1, n_2, and n_3. Therefore, the order in which the critical values occur depends on the relative values of these refractive indices. Stavroudis (1969b) made an analysis of all possible orderings of the critical values and the corresponding conditions on the relative values of the set of three indices of refraction for real case modules.

ADDITIONAL PROPERTIES OF MODULES

This chapter presents further analysis of the remaining third-order aberrations of the module. Conditions for modules to simultaneously eliminate third-order spherical and other Seidel aberrations are derived. Limitations in the choice of values of the free parameters and the $y,\bar{y}$ diagram parameters are discussed for modules free of additional third-order aberrations. Numerical examples of modules are given in the appendix. All equations derived in this chapter conform with the assumptions made in the preceding chapter.

Modules with Zero Coma

The condition for a two-surface system with image plane at infinity to be free from coma can be obtained if Eq. (23) is expanded and set equal to zero. When the resulting equation is solved for y_1 and equated to Eq. (28), the condition for zero third-order spherical aberration, we obtain

$$(\bar{\Omega}_2 n_3 - n_2)(n_2 - \Omega_2 n_1) - n_3 \Omega_2 (\bar{\Omega}_1 n_2 - \bar{\Omega}_2 n_1) = 0. \tag{43}$$

This is the condition that must be satisfied by the $\Omega,\bar{\Omega}$ parameters in order for the module to eliminate coma.

Stavroudis (1969b) has shown that the two-surface module can be free from coma only when $k = 0$. Hence, for zero coma, $\bar{\Omega}_2$ in Eq. (31) becomes a function of only the refractive indices. That is,

$$\Omega_2 = 3n_2^2/n_1(4n_2 - n_1), \tag{44}$$

and the height of the marginal ray at the first surface is

$$y_1 = \frac{-27 n_2^2 n_3^2}{4 n_1 (n_3 - n_2)^2 (n_2 - n_1)} . \tag{45}$$

When the relationship $y_1 = (1-\Omega_2)/(\bar{\Omega}_2 - \Omega_2\bar{\Omega}_1)$ is solved for $\bar{\Omega}_2$ and Eqs. (44) and (45) are substituted into the resulting equation, we obtain

$$\bar{\Omega}_2 = \bar{\Omega}_1 + \xi \tag{46}$$

where ξ is a function of the indices n_1, n_2, and n_3.

If Eq. (43) is solved for $\bar{\Omega}_2$ and Eq. (44) is substituted into the resulting expression, we obtain

$$\bar{\Omega}_2 = n_2[n_1(n_2 - n_1) + 3n_2n_3\bar{\Omega}_1]/n_1n_3(4n_2 - n_1). \tag{47}$$

Solving Eqs. (46) and (47) simultaneously will give the values of $\bar{\Omega}_1$ and $\bar{\Omega}_2$ as functions of the refractive indices.

The location of the aperture stop $S(0,y_s)$ in the $y,\bar{y}$ diagram is defined by the point of intersection of the line associated with n_2, which is given by

$$\Omega_2\bar{y} - \bar{\Omega}_2 y + 1 = 0 \tag{48}$$

and the y axis. Hence, $y_s = 1/\bar{\Omega}_2$. Therefore, the value of $\bar{\Omega}_2$ obtained by solving simultaneously Eqs. (46) and (47) locates the stop position of the module with zero coma, and the obtained value of $\bar{\Omega}_1$ determines the height of the entrance pupil since $y_E = 1/\bar{\Omega}_1$.

Modules with Zero Astigmatism

The condition for a two-surface system with image plane located at infinity, to be free from astigmatism is obtained when Eq. (24) is set equal to zero. If this expression is solved for y_1 and is equated to Eq. (28), the condition for zero spherical aberration, the result provides the condition for modules to be free from astigmatism

$$(\bar{\Omega}_2n_3 - n_2)^2(n_2 - \Omega_2n_1)^2 - (\bar{\Omega}_1n_2 - \bar{\Omega}_2n_1)^2n_3^2\Omega_2^2 = 0. \tag{49}$$

We observe that one of the factors of Eq. (49) is identical to the expression for zero coma. Hence if the module satisfies the condition given by Eq. (43), then astigmatism is also zero, which occurs only when $k = 0$.

The second factor of Eq. (49) will also eliminate astigmatism if it vanishes. This leads to the equation

$$(\bar{\Omega}_2n_3 - n_2)(n_2 - \Omega_2n_1) + n_3\Omega_2(\bar{\Omega}_1n_2 - \bar{\Omega}_2n_1) = 0. \tag{50}$$

Since $\Omega_2 = (\Omega_2\bar{\Omega}_1y_1 + 1 - \Omega_2)/\Omega_2y_1$, Eq. (50) when solved for $\bar{\Omega}_2$ becomes

$$\bar{\Omega}_2 = \frac{n_2[y_1(n_2 - n_1\Omega_2) - n_3(\Omega_2 - 1)]}{2n_3y_1(n_2 - n_1\Omega_2)} . \tag{51}$$

Because Ω_2 and y_1 are functions of the free parameter k, the solution of Eq. (51) gives the value of $\bar{\Omega}_2$ for zero astigmatism indirectly in terms of k. Equation (51) also defines the location of the aperture stop in the $y,\bar{y}$ diagram for the module to be free from astigmatism since y_s is the reciprocal of $\bar{\Omega}_2$. Once the value of $\bar{\Omega}_2$ is obtained for zero astigmatism, the height of the entrance pupil could also be determined.

Both Coma and Astigmatism Equal to Zero

Modules can be free simultaneously of coma and astigmatism only if either the condition for zero coma is satisfied or the first factor of the condition for zero astigmatism vanishes. However, Eq. (43) vanishes only for concentric systems, as shown by Stavroudis (1969b) and Powell (1970). Therefore, modules with zero coma and astigmatism simultaneously must be concentric two-surface systems.

The $y,\bar{y}$ diagram parameters of modules with both coma and astigmatism equal to zero are defined by Eqs. (44), (45), (46), and (47). The parameters are all functions of the indices of refraction of the three media. The aperture stop location of the module is defined by the value of $\bar{\Omega}_2$, obtained by solving simultaneously Eqs. (46) and (47). Its entrance pupil height is equal to the reciprocal of $\bar{\Omega}_1$.

Modules with Zero Petzval Curvature

The condition for a two-surface system with image plane at infinity to have zero Petzval contribution is obtained if Eq. (26) is set equal to zero. In terms of the $\Omega,\bar{\Omega}$ parameters this condition may be written as

$$n_3(\bar{\Omega}_2 - \Omega_2\bar{\Omega}_1) + n_1\Omega_2 = 0, \tag{52}$$

which can also be expressed as

$$n_3(1 - \Omega_2) + n_1 y_1 \Omega_2 = 0. \tag{53}$$

When solved for y_1, Eq. (53) becomes

$$y_1 = -n_3(1 - \Omega_2)/n_1\Omega_2. \tag{54}$$

Equation (54) gives the value of y_1 as a function of Ω_2 and the indices of refraction of the object and image spaces for a two-surface system with zero Petzval curvature.

Equating Eqs. (54) and (28) and rearranging terms results in a quartic equation given by

$$A\Omega_2^4 + B\Omega_2^3 + C\Omega_2^2 + D\Omega_2 + E = 0, \tag{55}$$

where

$$
\begin{aligned}
A &= n_1^3(n_2^2 - n_1 n_3)(n_3 - n_1) \\
B &= n_1^2(n_3 - n_2)^2(n_1 + n_2)^2 \\
C &= -2n_1 n_2(n_3 - n_2)^2(n_1^2 + n_1 n_2 + n_2^2) \\
D &= n_2^2(n_3 - n_2)^2(n_1 + n_2)^2 \\
E &= -n_2^4(n_3 - n_2)^2 .
\end{aligned}
$$

This quartic gives the condition for a module to be free from Petzval curvature.

Real Zeros of the Quartic

The fundamental theorem of algebra states that a polynomial such as Eq. (55) has at least one root that could be either real or complex. We also know that such a polynomial has at most four zeros. Because we are interested only in the real values of Ω_2, let us analyze the nature of the real zeros of the given quartic. To do this we make use of an important theorem in algebra known as the Descartes rule of signs. Every standard text in algebra or theory of equations such as Dickson (1939, pp. 76-80) discusses this theorem.

From the equations of the coefficients of the quartic, we note that the relative values of the three indices of refraction determine the variations in sign of the successive terms of the given polynomial. We also note that the quartic can have zero roots only when $n_3 = n_2$, which is an impossible case for modules or any other two-surface system.

For a given set of three positive refractive indices n_1, n_2, and n_3, we observe that only the first term of the quartic could possibly change in sign. The remaining terms have the same algebraic signs regardless of the relative values of the three indices of refraction. Let Eq. (55) be denoted by $G(\Omega_2) = 0$. The change in the number of variations in sign of the terms in the given polynomial could be grouped in the following three cases:

Case I. (a) $n_2^2 - n_1 n_3 > 0, \quad n_3 - n_1 > 0$

(b) $n_2^2 - n_1 n_3 < 0, \quad n_3 - n_1 < 0$

Case II. (a) $n_2^2 - n_1 n_3 < 0, \quad n_3 - n_1 > 0$

(b) $n_2^2 - n_1 n_3 > 0, \quad n_3 - n_1 < 0$

Case III. (a) $n_2^2 - n_1 n_3 = 0$

(b) $n_3 - n_1 = 0.$

An analysis of the above three cases might prove useful in the selection of glasses for modules with zero Petzval sum.

Case I. Case I(a) implies that $n_2 > n_1$ and Case I(b) implies $n_1 > n_2$. For both, the number of variations in sign in $G(\Omega_2)$ is three; hence, $G(\Omega_2) = 0$ has at most three positive real roots or at least one. The given quartic has at most only one real negative root since the number of variations in sign in $G(-\Omega_2)$ is one. Therefore, for this case there are at most three positive values of Ω_2 and at most one real negative value that makes the module eliminate Petzval contribution.

Applying Sturm's theorem (Dickson, 1939, pp. 81-88), which is more powerful than Descartes' in revealing the number of real roots of $G(\Omega_2) = 0$, we find that the quartic yields, for this particular case, exactly two real zeros. One root is positive and the other is negative.

Case II. The number of variations in sign in $G(\Omega_2)$ for this case is four; hence $G(\Omega_2) = 0$ has at most four positive real zeros. The given quartic could also have two positive real roots or none at all. Because there is no variation in sign in $G(-\Omega_2)$, the quartic cannot have any real negative root. This implies that Ω_2 can have only positive real values for this particular case. Case II(a) implies that $n_3 > n_2$. Case II(b) implies $n_2 > n_3$. Sturm's theorem gives us exactly two positive real roots for this particular case.

Case III. For this case, the quartic reduces to a cubic. The variations in sign in the resultant cubic are three, which implies that $G(\Omega_2) = 0$ can have at most three positive real roots or at least one. There is no variation in sign in $G(-\Omega_2)$; therefore, the given polynomial can never have any real negative zero. Case III(b) applies to the case of modules in air or any medium common to the object and image spaces. It was verified by Sturm's theorem that the cubic can have only one real root, which is always positive.

The three possible cases revealed that the quartic Eq. (55) cannot have four real roots but at most two and at least one. Hence, for a given set of three indices of refraction n_1, n_2, n_3, there are either two or one refracting module that could have zero Petzval contribution.

Two-Mirror Case

The quartic Eq. (55) degenerates into a simple quadratic equation for the case of a two-mirror module in air. The resulting equation is

$$2\Omega_2^2 - 1 = 0, \tag{56}$$

which when solved gives values of $\pm\sqrt{2}/2$ for Ω_2. These two real values imply the existence of two modules in air consisting of spherical reflecting surfaces with zero Petzval curvature. Equation (31), applied to a two-mirror module in air, could be written as

$$\left(\frac{k+1}{k-1}\right)^{1/3} + \left(\frac{k-1}{k+1}\right)^{1/3} = (\Omega_2 + 3)/2\Omega_2, \tag{57}$$

which when solved gives $k = \pm 1.221032$ for $\Omega_2 = \sqrt{2}/2$ and gives $k = \pm i1.364291$ for $\Omega_2 = -\sqrt{2}/2$. These pure imaginary values of k correspond to $\theta = \pi r \pm 0.9383$ rad, where $r = 0,1,2$. Therefore, zero Petzval sum in a two-mirror module in air limits us to the above two values of the Ω_2 parameter, which implicitly impose a limit in the choice of values of the free parameters k and θ. Only the four calculated values of the free parameters satisfy the condition for this additional third-order property of the two-mirror module in air. Figures 3 and 4 show the configuration of the two types of reflecting module systems with zero Petzval contribution.

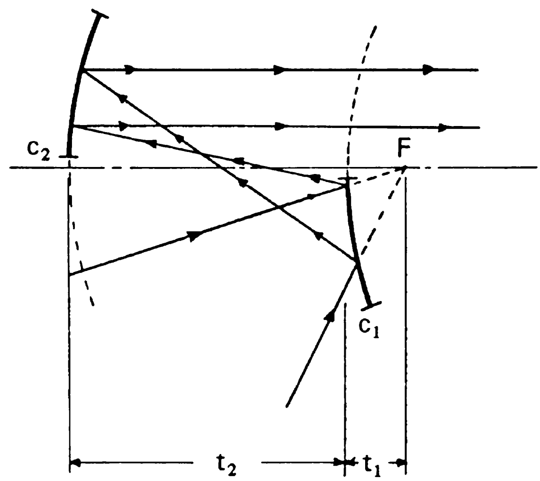


Fig. 3. Real-case two-mirror module system in air with zero Petzval curvature.

$k = \pm 1.221032$
$c_1 = c_2 = 0.353554$
$t_1 = -0.414213$
$t_2 = -2.0$
$f = 1$

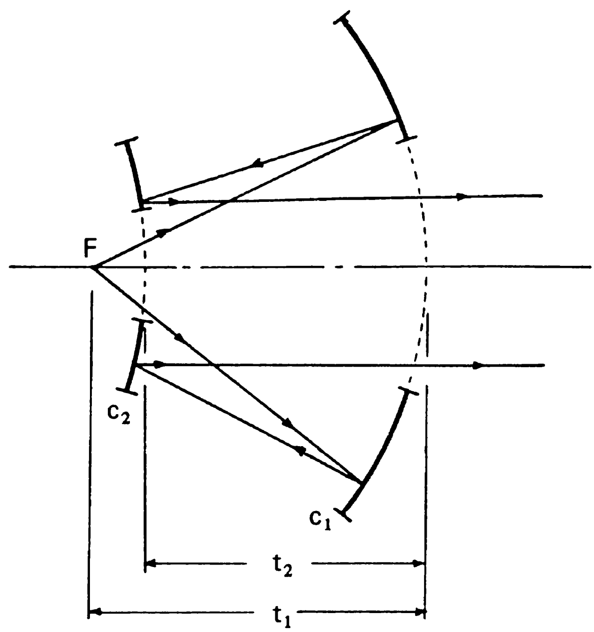


Fig. 4. Imaginary-case two-mirror module system in air with zero Petzval curvature.

$k = \pm i1.364291$
$c_1 = c_2 = -0.353554$
$t_1 = 2.41421$
$t_2 = -2.0$
$f = 1$

Both Coma and Petzval Sum Equal to Zero

The height of the marginal ray at the first surface of the module with zero coma is given by Eq. (45), and the marginal ray height at the same surface for a two-surface system that has zero Petzval curvature is given by Eq. (54). Equating Eqs. (45) and (54) results in an equation that provides the condition for a module to simultaneously have zero coma and Petzval sum. The resultant equation when solved for Ω_2 gives

$$\Omega_2 = \frac{4n_1(n_3 - n_2)^2(n_2 - n_1)}{27n_2^2 n_3 + 4n_1(n_3 - n_2)^2(n_2 - n_1)} \; . \tag{58}$$

Therefore, for a set of three indices of refraction, there is only one module that can simultaneously have zero coma and zero Petzval curvature.

Both Astigmatism and Petzval Sum Equal to Zero

The marginal ray height at the first surface of the module with zero astigmatism is obtained if Eq. (51) is solved for y_1, resulting in

$$y_1 = \frac{-n_2 n_3(\Omega_2 - 1)}{(n_2 - n_1\Omega_2)(2n_3\bar{\Omega}_2 - n_2)} \; . \tag{59}$$

If Eq. (59) is set equal to Eq. (54) and the resultant expression is solved for $\bar{\Omega}_2$, we obtain

$$\bar{\Omega}_2 = \frac{n_2(2n_1\Omega_2 - n_2)}{2n_3(n_1\Omega_2 - n_2)} \; . \tag{60}$$

Equation (60) provides the condition for a module with zero astigmatism, in which the Petzval contribution is also zero. Equation (60) also defines the location of the aperture stop in the $y,\bar{y}$ diagram as a function of the Ω_2 parameter and the three indices of refraction.

Zero Coma, Astigmatism, and Petzval Sum

The condition for modules to eliminate both coma and astigmatism is the vanishing of the expressions in Eq. (43). This implies that a module with zero coma because it satisfies Eq. (43) also has zero astigmatism. Hence, the condition for a module to have simultaneously zero coma and zero Petzval contribution, which is given by Eq. (58), is identical to the requirements for modules to be free simultaneously of coma, astigmatism, and Petzval curvature because Eq. (58) was derived with the assumption that Eq. (43) vanishes.

If Eq. (58) is substituted into Eq. (43) and the resulting equation is solved for $\bar{\Omega}_2$, we obtain

$$\bar{\Omega}_2 = n_2/n_3 + \psi(n_3\bar{\Omega}_1 - n_1), \tag{61}$$

where

$$\psi = \frac{4n_1(n_3 - n_2)^2(n_2 - n_1)}{n_3[27n_2^2n_3 + 4n_1(n_3 - n_2)^2(n_2 - n_1)]} .$$

Equation (61) gives the functional relationship between $\bar{\Omega}_2$ and $\bar{\Omega}_1$. It also implies the relative locations of the stop and the entrance pupil of the module to be free simultaneously of coma, astigmatism, and Petzval curvature.

Modules with Zero Distortion

The two-surface system with image plane at infinity will eliminate distortion if Eq. (25) vanishes.

For a module to eliminate distortion we have the necessary condition

$$\begin{aligned} &n_1^2n_3^4\Omega_2^3(\bar{\Omega}_2n_1^2 - \bar{\Omega}_1n_2^2)(\bar{\Omega}_1n_2 - \bar{\Omega}_2n_1)(\bar{\Omega}_1 - \bar{\Omega}_2) \\ &\quad + (1 - \Omega_2)(\Omega_2n_1^2 - n_2^2)(n_2 - \Omega_2n_1)\Big\{n_3^2(n_2^2 - \bar{\Omega}_2n_3^2)(\bar{\Omega}_2 - 1)n_1^2\bar{\Omega}_2 \\ &\quad + \text{[illegible]}(n_3 - n_2)^2[n_3^2(\bar{\Omega}_1^2n_2^2 - \bar{\Omega}_2^2n_1^2) + n_1^2(\bar{\Omega}_2^2n_3^2 - n_2^2)]\Big\} = 0. \end{aligned} \tag{62}$$

We observe that the above condition involves all the unknown $\Omega,\bar{\Omega}$ parameters of the module and is difficult to satisfy in practice.

IMAGINARY-CASE MODULES

In this chapter we analyze the properties of imaginary-case design modules with particular attention to the critical values of the free parameters. The results of this study when combined with similar results obtained for real-case modules will enable us to devise methods of conjoining modules to form the desired first-order layout of optical systems.

Change of Parameter and New Parameter Domain

The transformation $k = i\tan\theta$, used to obtain Eqs. (32) and (33), implies that $\theta = \arctan(-ik)$. To avoid ambiguity and considering that θ is a free parameter, we require θ to be single-valued. Because the arctangent function is multivalued, we confine the above transformation to a branch such that θ has the range $-\pi/2 < \theta \leqslant \pi/2$ and the quantity $(-ik)$ has the domain $-\infty < -ik \leqslant \infty$. Let $\phi = 2(\theta + \pi r)/3$ so that Eqs. (32) and (33) may now be rewritten in the form

$$\Omega_2 = \frac{3n_2^2/n_1}{n_1 + 2n_2 + 2(n_2 - n_1)\cos\phi} \tag{63}$$

$$y_1 = \frac{-27n_2^2 n_3^2 \sec^2(3\phi/2)}{4n_1(n_3 - n_2)^2(n_2 - n_1)} \; . \tag{64}$$

Under such a change of the free parameter, the relationship between k and the new parameter ϕ becomes $k = i\tan(3\phi/2)$. The new parameter ϕ runs from zero to 2π rad, and its domain of definition may be divided into the following three subdomains, each corresponding to a root of the cubic:

$$r = 0: \quad -\pi/3 < \phi \leqslant \pi/3 \tag{65}$$

$$r = 1: \quad \pi/3 < \phi \leqslant \pi \tag{66}$$

$$r = 2: \quad \pi < \phi \leqslant 5\pi/3. \tag{67}$$

The branch cuts of ϕ are located at $\phi = \pi/3$, π, and $5\pi/3$. For each subdomain of ϕ, the quantity $-ik$ has the range $-\infty < -ik \leqslant \infty$. In the following discussion we will refer to each subdomain of the parameter ϕ by specifying the value of r.

Critical Values of ϕ

It has been shown that the value of ϕ in Eqs. (63) and (64) yields real values of y_1 and Ω_2, which in turn yield the remaining optical parameters c_1, c_2, and t_2. Because all four optical parameters could be expressed in terms of ϕ, functional relationships exist between ϕ and the optical parameters. Owing to the change in parameterization, we have the following critical values of ϕ for the pure imaginary case of k corresponding to those given in Eqs. (42):

$$\phi_0 = \arccos[(n_1 + 3n_2)/2n_1] \tag{68}$$

$$\phi_\infty = \arccos[(n_1 + 2n_2)/2(n_1 - n_2)] \tag{69}$$

$$\phi^* = (1/3)\arccos\left[\frac{27n_2^2 n_3^2}{2n_1(n_3 - n_2)^2(n_1 - n_2)} - 1\right]. \tag{70}$$

The critical value ϕ_0 is the value of ϕ for which $\Omega_2 = 1$, resulting in c_1 equal to zero; ϕ_∞ is the value of ϕ when Ω_2 becomes infinite; and $\phi = \phi^*$ when $y_1 = 1$. At $\phi = \phi_\infty$, the value of t_2 is zero and c_1 and c_2 are infinite. At $\phi = \phi^*$, the module's axial thickness, t_2, is zero.

In addition to ϕ_0, ϕ_∞, and ϕ^*, we note that the values of the parameter ϕ at the branch cuts are also critical. At each of these values of ϕ the optical parameters t_1 and t_2 become infinite and c_1 approaches zero.

Critical Values for Refracting Systems

We observe that the critical values of ϕ that are located at the branch cuts are fixed and independent of the indices of refraction of the module. It is also obvious that they all exist for both refracting and reflecting module systems. In this section we study the remaining three critical values of ϕ.

It should be noted that the three critical values ϕ_0, ϕ_∞, and ϕ^* depend only on the three indices of refraction n_1, n_2, and n_3. Although these critical values of ϕ exist in the strict mathematical sense, we will analyze their existence in the practical or applied sense by restricting the values of the refractive indices to those of commonly available optical materials, particularly optical glasses. The order in which the critical values of ϕ occur, vital in establishing bounds on the parameter ϕ, ultimately depends on the relative values of the three indices of refraction.

In the domain of definition of the parameter ϕ, the cosine function has the range $-1 \leqslant \cos\phi \leqslant 1$. In order for the first critical value ϕ_0 to exist, the value of $\cos\phi_0$ should belong to the closed interval $[-1,1]$. This implies that

$$-1 \leqslant (n_1 + 3n_2)/2n_1 \leqslant 1. \tag{71}$$

The inequality (71) when simplified may be written in the form $-1 \leqslant n_2/n_1 \leqslant 1/3$. For refractive indices bounded away from zero, this implies

$$-1 \leqslant \frac{n_2}{n_1} < 0 \quad \text{and} \quad 0 < \frac{n_2}{n_1} \leqslant 1/3, \tag{72}$$

from which we obtain the following relationships between the refractive indices n_1 and n_2 in order for ϕ_0 to exist:

$$-n_1 \leqslant n_2 < 0 \tag{73}$$

$$0 < 3n_2 \leqslant n_1. \tag{74}$$

It is of interest to note that the inequality (73) can be satisfied only by reflecting module systems, the discussion of which is deferred until the next section. Although the condition given by inequality (74) might be satisfied by some optical materials in the infrared region, it cannot be satisfied by the indices of refraction of common optical glasses over the visual wavelength. Hence, for most practical applications of the results in this study of modules, we conclude that the critical value ϕ_0 does not exist for the case of refracting systems.

Similarly, if ϕ_∞ is to exist in the domain of ϕ, the value of $\cos\phi_\infty$ must satisfy

$$-1 \leqslant (n_1 + 2n_2)/2(n_1 - n_2) \leqslant 1, \tag{75}$$

which reduces to

$$0 \leqslant 3n_1/(n_1 - n_2) \leqslant 4. \tag{76}$$

We observe that inequality (76) requires that $n_1 - n_2 > 0$ and, if written in the form $0 \leqslant 3n_1 \leqslant 4(n_1 - n_2)$, implies

$$4n_2 \leqslant n_1, \tag{77}$$

the condition that must be satisfied for the critical value ϕ_∞ to exist. We note that this condition cannot be met by refracting module systems but could be met by reflecting modules. Hence, ϕ_∞ exists only for the case of reflecting systems.

The third critical value ϕ^* exists if the three indices of refraction n_1, n_2, and n_3 satisfy

$$0 \leqslant [27n_2^2 n_3^2/n_1(n_3 - n_2)^2(n_1 - n_2)] \leqslant 4. \tag{78}$$

Again, we observe that the condition for the existence of ϕ^* requires that $n_1 > n_2$. It is easily shown that, for reflecting module systems where $n_1 = n_3$ and $n_2 = -n_1$, the value of the expression inside the [...] in the inequality (78) is within the closed interval [0,4]. Hence, ϕ^* exists for the case of reflecting systems.

For the case of refracting module systems we have to analyze further the condition given by inequality (78). From this condition, we obtain the relationship

$$27n_2^2/n_1(n_1 - n_2) \leqslant 4(n_3 - n_2)^2/n_3^2. \tag{79}$$

Let $\mu_1 = n_2/n_1$ and $\mu_2 = n_2/n_3$. Then the inequality (79) may be written in the form

$$\mu_2 \leqslant 1 - 3\sqrt{3}\mu_1/2\sqrt{1 - \mu_1}\ . \tag{80}$$

When the inequality sign in (80) is replaced by strict equality, μ_2 is a function of μ_1. This curve is of degree three and is asymptotic with the line $\mu_1 = 1$. A segment of this curve for $0 < \mu_1 < 1$ is shown in Fig. 5. This is the only region of the curve that interests us because the requirement $n_1 - n_2 > 0$ in the inequality (78) implies that $0 < \mu_1 < 1$ for refracting module systems. Within this interval of μ_1, μ_2 can have

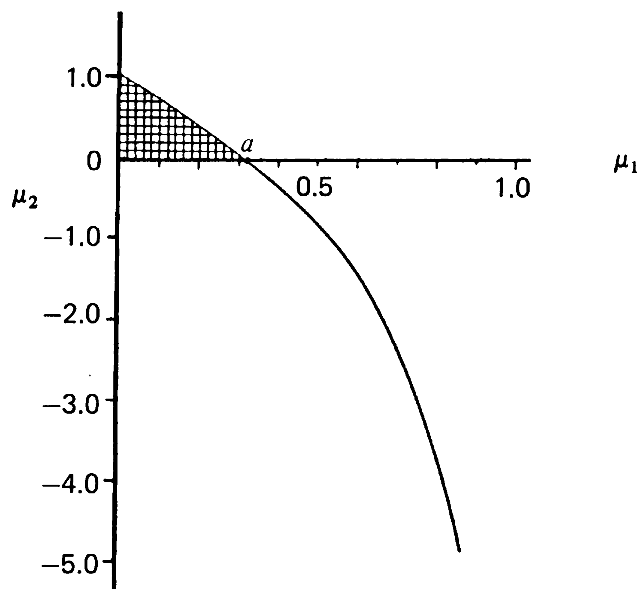


Fig. 5. A segment of the graph of μ_2 as a function of μ_1.

any value in the region under the curve. It is interesting to note that μ_2 decreases monotonically as μ_1 approaches 1. Let a be the value of μ_1 for which $\mu_2 = 0$. Within the interval $0 < \mu_1 < 1$, μ_2 is positive (i.e., $0 < \mu_2 < 1$) only for $0 < \mu_1 < a$. To determine a, the point where the curve crosses the μ_1 axis, we set the expression on the right side of the inequality (80) equal to zero. Clearing the radical yields the quadratic equation

$$27\mu_1^2 + 4\mu_1 - 4 = 0. \tag{81}$$

The roots of this quadratic equation are $2(-1 \pm 2\sqrt{7})/27$. Hence, $a = 2(-1 + 2\sqrt{7})/27$, which is approximately 0.318. The conditions that must be satisfied by the indices of refraction for the case of refracting module systems are

$$0 < \mu_1 < 0.318 \tag{82}$$

$$0 < \mu_2 < 1. \tag{83}$$

This region is shown shaded in Fig. 5. Inequality (82) implies that $n_2 < 0.318n_1$, and inequality (83) implies that $n_2 < n_3$. We observe that the condition given by inequality (82) cannot be satisfied by the indices of optical glasses since in order for n_2 to have a value of unity, which is minimum for the refracting case, n_1 should be greater than 3.145. We therefore conclude that the critical value ϕ^* of the parameter ϕ does not exist for refracting module systems.

Critical Values for Reflecting Systems

We have shown that, for most practical applications of modules, the critical values ϕ_0, ϕ_∞, and ϕ^* do not exist for refracting systems but do exist for reflecting systems. In this section we calculate these critical values of ϕ for any two-mirror module system.

For reflecting systems, where $n_2/n_1 = -1$, the first critical value given by Eq. (68) becomes $\phi_0 = \arccos(-1)$, which implies that $\phi_0 = \pi$. At this value of ϕ the optical parameters t_1 and t_2 become infinite while c_1 approaches zero.

The second critical value, ϕ_∞, given by Eq. (69), yields $\phi_\infty = \arccos(-1/4)$. Within the domain of definition of the parameter ϕ, ϕ_∞ is two-valued ($\phi_\infty = \pm 0.58\pi$), the values being within the $r = 1$ and the $r = 2$ subdomains, respectively.

For a two-mirror module system, Eq. (70) becomes $3\phi^* = \arccos(11/16)$, which implies that $\cos 3\phi^* = 11/16$. Using the trigonometric identity $\cos 3x = 4\cos^3 x - 3\cos x$, this may be written in the form

$$\cos^3\phi^* - (3/4)\cos\phi^* - 11/64 = 0. \tag{84}$$

Equation (84) is a reduced cubic of the form $y^3 + py + q = 0$, whose discriminant is given by $\Delta = -4p^3 - 27q^2$. The discriminant of Eq. (84) is positive, which implies that this cubic has three distinct real roots. Using Descartes' rule of signs, we note that Eq. (84) has one positive root and two negative real roots. Solution of the cubic yields $\cos\phi^* = -0.713525, 0.963525, -0.2500$, corresponding to $\phi^* = \pm 0.753\pi$, $\pm 0.086\pi$, $\pm 0.58\pi$, respectively. We note that the critical values $\phi^* = \pm 0.58\pi$ are identical to those obtained for ϕ_∞. The locations of the critical values ϕ_0, ϕ_∞, and ϕ^* in the polar plane for any reflecting module system are shown in Fig. 6. Values of q and the optical parameters at the critical values of ϕ for a two-mirror module system in air are given in Table III.

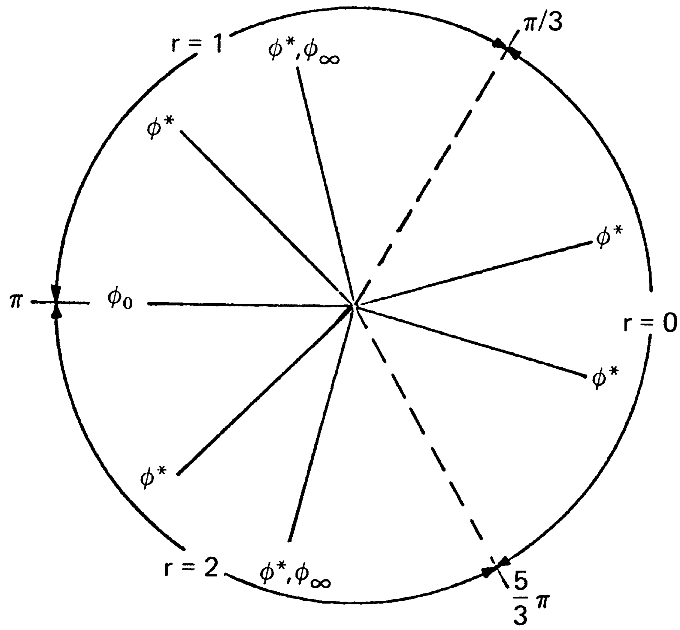


Fig. 6. Locations of the critical values and subdomains of ϕ in the polar plane.

TABLE III

Values of q and the optical parameters corresponding to the critical values of ϕ for two-mirror modules in air.

	0	$\pm\pi/3$	$\pm\pi/2$	$\pm 2\pi/3$	$\phi_0=\pi$	$\phi_\infty,\phi^*=0.58\pi$	$\phi^*=\pm 0.086\pi$	$\phi^*=\pm 0.753\pi$
q	$-3/5$	-1	-3	3	1	$-\infty$	-0.618034	1.618036
t_0	$27f/32$	∞	$27f/16$	$27f/32$	∞	f	f	f
c_1	$-128/135f$	0	$-32/27f$	$32/27f$	0	$-\infty$	$-0.809017/f$	$0.309018/f$
t_1	$25f/96$	∞	$-11f/48$	$-5f/96$	∞	0	0	0
c_2	$-3/10f$	$-1/2f$	$-3/2f$	$3/2f$	$1/2f$	$-\infty$	$-0.309017/f$	$0.809018/f$

Behavior of q, the Optical Parameters, and Pupil Plane Positions

We have shown that the critical values ϕ_0, ϕ_∞, and ϕ^*, in general, do not exist for refracting systems. Classification of modules possessing pure imaginary free parameters based on the critical values, as in the real case of k, is therefore not feasible. In this section we discuss the behavior of the nonoptical parameter q as well as of the optical parameters and the pupil plane locations. This study may lead to a method of classifying modules in the imaginary case in order to determine whether they can be coupled so that they are constructible and their adjacent foci and pupil planes coincide. By constructible we mean a module whose thickness is positive and whose curvatures are spherical surfaces that it is practicable to construct.

We observe that the nonoptical parameter q, which is equal to $n_1\Omega_2$, is always positive for refracting modules, while for reflecting modules q may be positive or negative depending on the value of ϕ. In neither case can the parameter q be zero. The parameter q is infinite only when $\phi = \phi_\infty$, which for all practical purposes occurs only for reflecting systems.

The optical parameter t_1 can also never be zero for any value of ϕ. We have also mentioned that t_1 becomes infinite at the branch cuts of ϕ for both refracting and reflecting cases. In addition, t_1 may be positive or negative depending on whether f is positive or negative and on the relative values of n_1 and n_2.

The curvature c_2 may be either positive or negative but can never be zero since q is always nonzero. For positive values of f, c_2 follows the sign of q and can be infinite in value only when $\phi = \phi_\infty$, which occurs only for reflecting systems.

The module thickness t_2 may be positive or negative depending on the relative values of $n_1 f$ and t_1 for the case of refracting module systems. This optical parameter can be infinite only when t_1 is infinite or when q is zero. Since q can never be zero, t_2 becomes infinite identically with t_1, which occurs at $\phi = \pm\pi/3, \pi$, the branch cuts of ϕ. The optical parameter t_2 becomes zero either when q is infinite (when $\phi = \phi_\infty$) or when $t_1 = n_1 f$ (when $\phi = \phi^*$). Since ϕ_∞ and ϕ^* exist only for reflecting systems, t_2 may be zero only for this particular case. A necessary condition in terms of the parameter ϕ for which t_2 vanishes may be obtained if t_2 is expressed as a function of ϕ and the resulting expression is set equal to zero. This yields a quartic of the form

$$a_1 \cos^4\phi + b_1 \cos^3\phi + c_1 \cos^2\phi + d_1 \cos\phi + e_1 = 0, \tag{85}$$

where

$$\begin{aligned}
a_1 &= 16n_1(n_3 - n_2)^2(n_2 - n_1)^2 \\
b_1 &= 8n_1(n_3 - n_2)^2(n_2 - n_1)(2n_2 + n_1) \\
c_1 &= -12n_1(n_3 - n_2)^2(n_2 - n_1)^2 \\
d_1 &= (n_2 - n_1)[54n_2^2n_3^2 - 2n_1(n_3 - n_2)^2(4n_2 + 5n_1)] \\
e_1 &= (2n_2 + n_1)[2n_1(n_3 - n_2)^2(n_2 - n_1) + 27n_2^2n_3^2].
\end{aligned}$$

Our objective is to isolate the real roots of this quartic and, for a given set of three indices of refraction, to obtain the domain of the parameter ϕ where t_2 is finite and nonzero in value. This requirement is necessary to ensure the constructibility of the module obtained for any choice of ϕ. Equation (85) may be factored into

$$[\cos\phi + (n_1 + 2n_2)/2(n_2 - n_1)]\, G(\cos\phi) = 0, \tag{86}$$

where

$$\begin{aligned} G(\cos\phi) = {} & 8n_1(n_3 - n_2)^2(n_2 - n_1)\cos^3\phi - 6n_1(n_3 - n_2)^2(n_2 - n_1)\cos\phi \\ & + 27n_2^2n_3^2 + 2n_1(n_3 - n_2)^2(n_2 - n_1). \end{aligned} \tag{87}$$

If we divide Eq. (87) by the coefficient of the $\cos^3\phi$ term, we obtain

$$G(\cos\phi) = \cos^3\phi - (3/4)\cos\phi + (1/4)\left[\frac{27n_2^2n_3^2}{2n_1(n_3 - n_2)^2(n_2 - n_1)} + 1\right], \tag{88}$$

which is a reduced cubic of the form $y^3 + py + q$. One of the general properties of equations is that every polynomial of an odd degree has at least one real root of a sign opposite to that of its last term. Hence Eq. (88) may have a positive or a negative real root depending on the relative values of n_2 and n_1. To analyze further the nature of the roots of this cubic we have to calculate its discriminant $\Delta = -4p^3 - 27q^2$. The discriminant of this cubic may be written in the form

$$\Delta = \frac{-729n_2^2n_3^2}{16n_1(n_3 - n_2)^2(n_2 - n_1)}\left[\frac{27n_2^2n_3^2}{4n_1(n_3 - n_2)^2(n_2 - n_1)} + 1\right]. \tag{89}$$

We note that Δ is negative only if $n_2 > n_1$. This implies that Eq. (88) will have one real root and two complex conjugate imaginary roots. We also note that the discriminant can be zero only if the quantity in [...] of Eq. (89) vanishes. This implies that

$$27n_2^2n_3^2 = 4n_1(n_3 - n_2)^2(n_1 - n_2), \tag{90}$$

which can hold only if $n_2 < n_1$. If Eq. (90) is satisfied by a set of three indices of refraction, the cubic Eq. (88) will have at least two equal real roots.

We also observe that Δ can be positive when $n_2 < n_1$ and the quantity in [...] of Eq. (89) is greater than zero. The conditions $\Delta \geqslant 0$ may be cast together into a single inequality

$$27n_2{}^2/n_1(n_1 - n_2) \leqslant 4(n_3 - n_2)^2/n_3{}^2 , \tag{91}$$

which is identical to the inequality (79), the condition for the existence of the critical value ϕ^*. Since we showed that ϕ^* can never exist for refracting modules in the visual region, we conclude that for refractive indices of optical materials in the visual wavelengths the discriminant Eq. (89) can be neither positive nor zero. Hence the cubic Eq. (88) can have neither three distinct real zeros nor at least two equal real roots.

We shall now consider only the case where $n_2 > n_1$, which gives rise to a negative discriminant. To isolate the real roots of the cubic we will apply a general theorem in algebra that states that if a real polynomial $f(x)$ for $x = a$ and $x = b$ takes values $f(a)$ and $f(b)$ of opposite signs so that, for instance, $f(a) < 0, f(b) > 0$, then there is at least one real root of the equation $f(x) = 0$ in the interval (a,b). This theorem when applied to Eq. (88) yields positive values for $G(-1)$, $G(0)$, $G(1)$, and $G(+\infty)$. The cubic has a negative value for $G(-\infty)$. We observe that there is a permanence in sign of $G(\cos\phi)$ in the interval $(-1,\infty)$ and a variation in $(-\infty,-1)$. Because the discriminant of the cubic is negative, we conclude that the single real root of this cubic lies in $(-\infty, -1)$. But since $-1 \leqslant \cos\phi \leqslant 1$ and there is no real root in the closed interval $[-1,1]$, the refracting module's thickness t_2 can never be zero for any choice of value of the parameter ϕ. This supports our statement that t_2 may be zero only for the case of reflecting systems.

A more powerful method to effect the separation of real roots of polynomials is based on Sturm's theorem (Dickson, 1939), which allows one to find the exact number of real roots contained between two given numbers for an equation without multiple roots. To check the results we have just obtained, let us apply this theorem to the cubic, Eq. (88).

Let $x = \cos\phi$. Starting with $G(x)$, it is possible, and in many ways, to form a sequence of polynomials called *Sturm's functions*

$$G(x),\ G_1(x),\ G_2(x),\ \ldots,\ G_n(x) \tag{92}$$

in a given interval (a,b), where $a < b$ and neither is a root of $G(x) = 0$. This sequence of polynomials is called a Sturm's series relative to (a,b). The number of real roots of $G(x) = 0$ between a and b is equal to the excess of the number of variations of signs in the sequence (92) for $x = a$ over the number of variations of signs for $x = b$. The terms that vanish are to be dropped out before counting the variations of signs. The process of obtaining the Sturm's functions is a refinement of the greatest common divisor process of Euclid. The method of constructing the Sturm's series is discussed in detail in most standard books on the theory of equations (i.e., Dickson, 1939, pp. 81-88, and Uspensky, 1948, pp. 138-150).

To apply Sturm's theorem to the cubic, Eq. (88), we form the sequence (92), whose terms are the following Sturm's functions:

$$
\begin{aligned}
G(x) &= x^3 - (3/4)x + (1/4)\left[\frac{27n_2{}^2n_3{}^2}{2n_1(n_3 - n_2)^2(n_2 - n_1)} + 1\right] \\
G_1(x) &= x^2 - 1/4 \\
G_2(x) &= x - (1/2)\left[\frac{27n_2{}^2n_3{}^2}{2n_1(n_3 - n_2)^2(n_2 - n_1)} + 1\right] \\
G_3(x) &= -1.
\end{aligned}
\tag{93}
$$

We note that in this sequence $G_n(x) = G_3(x) \neq 0$. This implies that $G(x) = 0$ has no repeated root. Let $V(x)$ denote the variations in signs of the numbers in the sequence $G(x)$, $G_1(x)$, $G_2(x)$, $G_3(x)$ when x is a particular real number that is not a root of $G(x) = 0$. We find that $V(+\infty) = V(1) = V(0) = V(-1) = 1$ and $V(-\infty) = 2$. These imply that the given cubic has only one real root and it lies in the interval $(-\infty,-1)$, which proves our previous result. For reflecting modules, Eq. (85) becomes

$$256\cos^4\phi + 64\cos^3\phi - 192\cos^2\phi - 92\cos\phi - 11 = 0, \tag{94}$$

which when solved gives $\cos\phi = -0.25, -0.25, 0.963525, -0.713525$. We note that these roots of Eq. (94) correspond to the critical values ϕ_∞ and ϕ^* of reflecting module systems, which agrees with our previous statement.

The first curvature c_1 of the module may be infinite in value either when q is infinite or when t_1 is zero. Since t_1 can never be zero, c_1 becomes infinite only when q is infinite, which occurs when $\phi = \phi_\infty$, and occurs only for the case of reflecting systems. We note that c_1 may have a value of zero when t_1 is infinite, which occurs only for values of ϕ at the branch cuts. Expressing c_1 as a function of ϕ and setting the resulting equation equal to zero, we obtain the following necessary condition in order for c_1 to vanish:

$$
\begin{aligned}
&8n_1\cos^4\phi - 4(n_1 + 3n_2)\cos^3\phi - 6n_1\cos^2\phi \\
&\qquad + (9n_2 + 5n_1)\cos\phi - (n_1 + 3n_2) = 0.
\end{aligned}
\tag{95}
$$

Equation (95) may be factored into

$$(\cos\phi + 1)(\cos\phi - 1/2)^2\left(\cos\phi - \frac{n_1 + 3n_2}{2n_1}\right) = 0. \tag{96}$$

The first factor implies $\phi = \pi$ and the second implies $\phi = \pm\pi/3$. The last factor implies $\phi = \phi_0$. Since we have shown that the critical value ϕ_0 can never exist for indices of refraction of optical materials in the visible region, we conclude that the curvature c_1 can be zero only for values of ϕ at the branch cuts. Because t_1 and t_2 become infinite at the branch cuts, there is no constructible imaginary-case refracting module that is plane on one surface, as c_2 is always nonzero. Hence for constructible modules the choice of values of ϕ should be bounded away from its values at the branch cuts.

For the reflecting case, Eq. (95) becomes

$$4\cos^4\phi + 4\cos^3\phi - 3\cos^2\phi - 2\cos\phi + 1 = 0, \tag{97}$$

which may be factored into

$$(\cos\phi + 1)^2(2\cos\phi - 1)^2 = 0. \tag{98}$$

The first factor implies $\phi = \pi$ and the second implies $\phi = \pm\pi/3$. These results agree with our observation that c_1 is zero at the branch cuts.

Stavroudis (1969b) has defined pupil planes at which the module eliminates third-order astigmatism. The locations of the pupil planes are given by

$$\bar{t} = \frac{2n_3 f t_1 q(n_2 - q)}{fn_3(n_1 - q)(2q - n_2) + n_2 t_1(n_2 - q)} \tag{99}$$

$$\bar{t}' = \frac{-n_1 f[t_1(n_2 - 2n_3)(n_2 - q) + n_2 n_3 f(n_1 - q)]}{2t_1 q(n_2 - q)} \tag{100}$$

where $\bar{t}$ is the distance from the object point to the entrance pupil and $\bar{t}'$ is the distance from the second surface to the exit pupil plane. Note that $\bar{t}$ will be zero if the numerator of Eq. (99) equals zero or if the denominator becomes infinite. When Eq. (99) is expressed as a function of ϕ and set equal to zero, we obtain $\phi = \pm\pi/3$. Beforehand we know that, at $\phi = \pm\pi/3$, t_1 is infinite and $q = n_2$. This implies that Eq. (99) becomes indeterminate at this value of ϕ. In the limit, as t_1 approaches infinity and q approaches n_2, $\bar{t}$ has nonzero value. From this result we conclude that, although for $\bar{t}$ to be zero the numerator of Eq. (99) must vanish or the denominator must become infinite, these conditions are not sufficient.

If we set the denominator of Eq. (99) equal to zero, we obtain a condition for $\bar{t}$ to have a value of infinity. In terms of the parameter ϕ, this condition results in a quintic polynomial of the form

$$a_2\cos^5\phi + b_2\cos^4\phi + c_2\cos^3\phi + d_2\cos^2\phi + e_2\cos\phi + f_2 = 0, \tag{101}$$

where

$$\begin{aligned}
a_2 &= 16n_1(n_2 - n_1) \\
b_2 &= -8(3n_2^2 + 2n_1n_2 - 2n_1^2) \\
c_2 &= 8(n_1^2 - n_1n_2 + 6n_2^2) \\
d_2 &= [54n_2^3n_3/(n_3 - n_2)^2] + 2(9n_2^2 + 8n_1n_2 - 8n_1^2)
\end{aligned}$$

$$e_2 = \frac{27n_2^3n_3(n_2 + 2n_1)}{(n_3 - n_2)^2(n_2 - n_1)} + 7(n_1 - 3n_2)(n_1 + 2n_2)$$

$$f_2 = (n_1 + 3n_2)(4n_2 - n_1) - \frac{27n_2^3n_3(2n_2 + n_1)}{2(n_3 - n_2)^2(n_2 - n_1)} .$$

Equation (101) may be factored into

$$(\cos\phi - \tfrac{1}{2})(A_2\cos^4\phi + B_2\cos^3\phi + C_2\cos^2\phi + D_2\cos\phi + E_2) = 0, \qquad (102)$$

where

$$\begin{aligned}
A_2 &= 16n_1(n_2 - n_1) \\
B_2 &= 8(n_1^2 - n_1n_2 - 3n_2^2) \\
C_2 &= 12(n_1^2 - n_1n_2 + 3n_2^2) \\
D_2 &= [54n_2^3n_3/(n_3 - n_2)^2] + 2(18n_2^2 + 5n_1n_2 - 5n_1^2) \\
E_2 &= [27n_2^3n_3(2n_2 + n_1)/(n_3 - n_2)^2(n_2 - n_1)] - 2(4n_2 - n_1)(3n_2 + n_1).
\end{aligned}$$

To isolate the real roots of the second factor of Eq. (102), which is a quartic, we will make use of a theorem due to Budan and Fourier (MacDuffee, 1954, pp. 59-60), which is much easier to apply than that of Sturm. The Budan-Fourier theorem states that, if a and b are real numbers, $a < b$, neither of them being a root of $f(x) = 0$ (an equation of degree n with real coefficients), and if $V(a)$ denotes the number of variations of signs in the sequence

$$f(x), f'(x), f''(x), f'''(x), \ \ldots \ , f^{(n)}(x) \qquad (103)$$

for $x = a$ after vanishing terms have been deleted, then $V(a) - V(b)$ is either the number of real roots of $f(x) = 0$ between a and b or exceeds the number of those roots by an even integer. A root of multiplicity m is here counted as m roots. In particular, in case $V(a) - V(b)$ is 0 or 1, it is the exact number of real roots between

a and b. Although this method is much simpler to apply, it is less powerful than that of Sturm. Descartes' rule of signs is an immediate consequence of this theorem. An elementary proof of the Budan-Fourier theorem is given by Conkwright (1943).

Let $x = \cos\phi$. Denote the quartic factor of Eq. (102) by $H(x)$. The terms corresponding to those in the sequence (103) may be obtained by taking the derivatives of $H(x)$:

$$\begin{aligned}
H(x) &= A_2x^4 + B_2x^3 + C_2x^2 + D_2x + E_2 \\
H'(x) &= 4A_2x^3 + 3B_2x^2 + 2C_2x + D_2 \\
H''(x) &= 12A_2x^2 + 6B_2x + 2C_2 \\
H'''(x) &= 24A_2x + 6B_2 \\
H''''(x) &= 24A_2.
\end{aligned} \tag{104}$$

To analyze the variations of signs in the sequence $H(x)$, $H'(x)$, $H''(x)$, $H'''(x)$, $H''''(x)$ we consider two cases. The first case is when $n_2 - n_1 > 0$ and the second is when $n_2 - n_1 < 0$. Since we are considering these two cases for refracting modules, the indices of refraction are restricted to positive values.

$\underline{n_2 - n_1 > 0.}$ For this case we find that $V(+\infty) = 0$, $V(1) = 2$, $V(0) = 2$, and $V(-\infty) = 4$. We observed that $V(-1)$ may be either 4 or 2 in value because we can have either $H'(-1) < 0$ or $H'(-1) \geqslant 0$. The sign of $H'(-1)$ is the same as the sign of the inequality

$$n_2{}^3 n_3 \gtreqless (2n_2 - n_1)(n_1 + n_2)(n_3 - n_2)^2. \tag{105}$$

If we divide (105) by $n_1{}^2 n_3{}^2$ on both sides and set $\mu_1 = n_2/n_1$ and $\mu_2 = n_2/n_3$, we obtain

$$\mu_1{}^2\mu_2 \gtreqless (2\mu_1 - 1)(1 + \mu_1)(1 - \mu_2)^2, \tag{106}$$

which after grouping and expansion of terms may be written as

$$\mu_2/(1 - \mu_2)^2 \gtreqless 2 - \left(\frac{1}{\mu_1{}^2} - \frac{1}{\mu_1}\right). \tag{107}$$

Let $z = 1/\mu_1 = n_1/n_2$. If we complete the squares of the quantities inside the parentheses on the right side of (107) we get, after transposition of terms,

$$(z - ½)^2 \gtreqless \frac{9}{4} - \frac{\mu_2}{(1 - \mu_2)^2}. \tag{108}$$

Solving for z, inequality (108) yields

$$z \gtreqless \tfrac{1}{2} + \frac{\sqrt{9\mu_2^2 - 22\mu_2 + 9}}{2(1 - \mu_2)} \quad . \tag{109}$$

When the sign of the inequality (109) is replaced by strict equality, z is a function of μ_2. This functional relationship is plotted in Fig. 7 for $0 < \mu_2 < 2$. The graph has discontinuity in the interval $0.5195 < \mu_2 < 1.92495$, the region where quantities under the radical sign in (109) are negative thus giving imaginary values of the square root. In the interval $0 < \mu_2 < 2$, $z = \tfrac{1}{2}$ at $\mu_2 = 0.5195$ and at $\mu_2 = 1.92495$, which occur when the sum of the quantities under the radical in (109) vanishes. Because $n_2 > n_1$, then $0 < z < 1$. For $0 < \mu_2 < 2$, we observe that z is broken into $0 < z < 0.5$ and $0.5 < z < 1$. These regions are shown by solid line segments in Fig. 7.

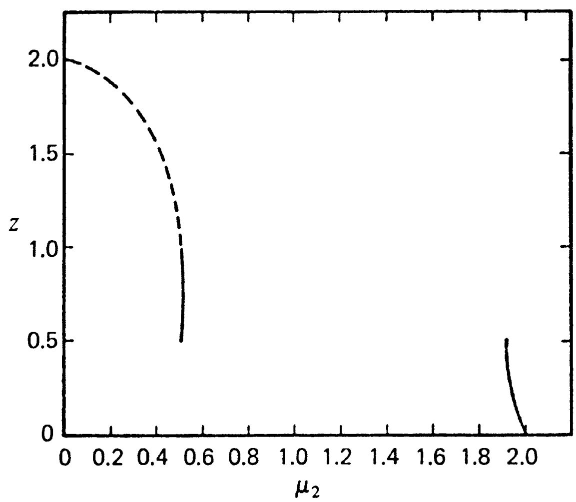


Fig. 7. Graph of z as a function of μ_2 for $0 < \mu_2 < 2$.

For $H'(-1) = 0$, Fig. 7 shows that, for $0.5 < z < 1$, μ_2 is in the interval $0.5 < \mu_2 < 0.5195$, and for $0 < z < 0.5$, μ_2 is in the interval $1.92495 < \mu_2 < 2$. Because $1 < n_3 < 2$, the only possible range of n_2 in the first interval of μ_2 is $1 < n_2 < 1.039$. But n_2 is greater than n_1, and because 1.0 is the minimum value that n_1 can have for refracting systems, $H'(-1) = 0$ may not be satisfied in the practical sense for $1 < n_2 < 1.039$. The second interval $1.92495 < \mu_2 < 2$ implies that $1.92495 n_3 < n_2 < 2n_3$. Because $1 < n_3 < 2$ and because n_2 must also be in the range $1 < n_2 < 2$, $H'(-1)$ can be zero only if $n_3 = 1$, and for this particular case the only possible range of n_2 is $1.92495 < n_2 < 2$. We note that in the visual region only a few glasses of the types La SF and SF could be used for the case $H'(-1) = 0$.

For $H'(-1) > 0$ we find that $0 < z < 1$ and $0.5 < \mu_2 < 2$. The interval $0.5 < \mu_2 < 2$ implies that $0.5n_3 < n_2 < 2n_3$. Because $1 < n_3 < 2$, n_2 has also the range $1 < n_2 < 2$. In this particular case, for every value of n_2 that is in the interval (1,2) such that $n_2 > n_1$, any value of n_3 in the interval [1,2) satisfies $H'(-1) > 0$.

For the case where $H'(-1) < 0$ and $0 < z < 1$, μ_2 may be either in the interval $0 < \mu_2 < 0.5195$ or $1.92495 < \mu_2 < 2$. These represent the region under the segments of the curve in Fig. 7. The first interval of μ_2 implies that $0 < n_2 < 0.5195n_3$. For $1 < n_3 < 2$, this range of n_3 is $0 < n_2 < 1.039$, which is not practicable to satisfy since $n_2 > n_1$. The other range $1.92495 < \mu_2 < 2$ can be satisfied only as in the $H'(-1) = 0$ case for $n_3 = 1$ where $1.92495 < n_2 < 2$.

These results for the case $n_2 - n_1 > 0$ reveal that $H'(-1) \leqslant 0$ can occur only when $n_3 = 1$ and $1.92495 < n_2 < 2$. If $H'(-1) < 0$, then $V(-1) = 4$, and since $V(0) = 2$, Budan-Fourier theorem tells us that either two real roots or none may lie in the interval (−1,0). When $H'(-1) \geqslant 0$, $V(-1) = 2$, and since $V(+1) = 2$, there is no real root in the interval [−1,1] where our parameter ϕ is defined. Although either two roots or none may lie in each of the intervals $(-\infty,-1)$ and $(1,\infty)$, these roots if they really exist will not be considered since they are outside the domain of the value of $\cos\phi$.

$\underline{n_2 - n_1 < 0.}$ We find that $V(+\infty) = 0$, $V(1) = 2$, $V(0) = 2$, and $V(-\infty) = 4$. We observed that the variations of signs in the sequence for $x = -1$ may be either 4 or 2. Since $V(-1) = 4$ for $H''(-1) < 0$ and $V(-1) = 2$ when $H''(-1) \geqslant 0$, we note that $H''(-1)$ follows the sign of the inequality

$$n_2{}^2 - n_1(n_1 - n_2) \gtreqless 0. \tag{110}$$

If we divide inequality (110) by $n_1{}^2$ and let $\mu_1 = n_2/n_1$, we get

$$\mu_1{}^2 + \mu_1 - 1 \gtreqless 0. \tag{111}$$

Because $n_2 < n_1$, then $0 < \mu_1 < 1$. Let $f(\mu_1) = \mu_1{}^2 + \mu_1 - 1$. The graph of $f(\mu_1)$ is a parabola, and this is shown in Fig. 8 for $0 < \mu_1 < 1$. The point where the curve crosses the μ_1 axis, designated by $b = 0.618034$, is the value of μ_1 if $H''(-1) = 0$. At b, $n_2 = 0.618034n_1$ where $1 < n_1 < 2$. For $H''(-1) < 0$, $0 < \mu_1 < 0.618034$, and for $H''(-1) > 0$, $0.618034 < \mu_1 < 1$. Because $1 < n_1 < 2$, the practical range of n_2 is $1 < n_2 < 1.236068$ if $H''(-1) < 0$ and $1.236068 < n_2 < 2$ if $H''(-1) > 0$. The Budan-Fourier theorem reveals that either two roots or none lies in the interval (−1,0) if $H''(-1) < 0$ whereas, if $H''(-1) \geqslant 0$, we are certain that there is no root in the closed interval [−1,1], beyond which we are not interested. Again, we observe that there are either two roots or none in $(-\infty, -1)$ if $H''(-1) < 0$ and in $(1, \infty)$ if $H''(-1) \gtreqless 0$, but $\cos\phi$ is not defined in these intervals.

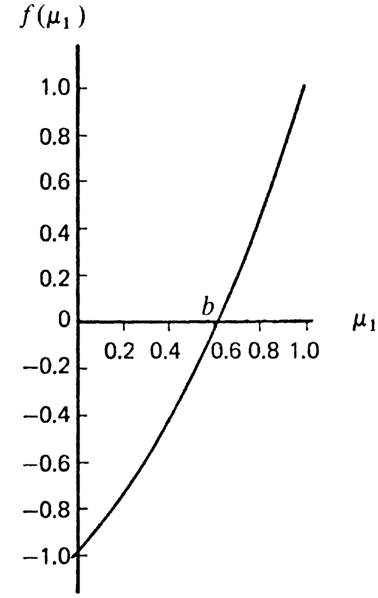


Fig. 8. Graph of $f(\mu_1)$ for $0 < \mu_1 < 1$.

We have shown that for the two cases $n_2 - n_1 > 0$ with $H'(-1) \geqslant 0$, and $n_2 - n_1 < 0$ with $H''(-1) \geqslant 0$, the quartic $H(x) = 0$ yields no real zeros in the interval $[-1,1]$, which implies that the entrance pupil plane distance $\overline{T}'$ can never be infinite in value for any choice of the parameter ϕ in these two particular cases. For the other two cases, where $n_2 - n_1 > 0$ with $H'(-1) < 0$ and $n_2 - n_1 < 0$ with $H''(-1) < 0$, we find that the Budan-Fourier theorem is uncertain as to the exact number of roots in the interval $(-1,0)$. To dissolve this uncertainty, other methods must be used to supplement the theorem.

For the reflecting case, Eq. (101) becomes

$$32\cos^5\phi - 8\cos^4\phi - 64\cos^3\phi + (55/2)\cos^2\phi + (197/8)\cos\phi - 187/16 = 0, \quad (112)$$

which when solved yields $\cos\phi = 0.81066, 0.963525, -0.713525, 0.5, -1.31066$. The first root of Eq. (112) implies $\phi = \pm 0.199\pi$ and the second and third roots are equal to the two sets of values of ϕ^*. The fourth implies $\phi = \pm\pi/3$, but at this value of ϕ the expression for $\overline{T}$ becomes indeterminate and yields a finite value when we take the limit of $\overline{t}$ as $t_1 \to \infty$ and $q \to n_2$. Hence, we conclude that the vanishing of the denominator of Eq. (99) is necessary for $\overline{t}$ to be infinite, but it is not a sufficient condition. The fifth root, whose absolute value is greater than 1, is extraneous.

Finally, we discuss the behavior of the exit pupil distance, $\bar{T}'$. We observe that a condition for $\bar{T}'$ to be infinite is when the denominator of Eq. (100) vanishes. If this denominator is set equal to zero and the resulting equation is solved for ϕ, we again obtain $\phi = \pm\pi/3$, which implies the insufficiency of that condition for $\bar{T}'$ to be infinite.

When Eq. (100) is expressed as a function of ϕ and set equal to zero, we obtain

$$
\begin{aligned}
&[2n_2 + n_1 + 2(n_2 - n_1)\cos\phi]^2 [27n_2^2 n_3(2\cos\phi - 1)(n_2 - 2n_3) \\
&\quad - 2(n_3 - n_2)^2(n_2 - n_1)(2n_1\cos\phi - n_1 - 3n_2) \\
&\qquad \times (1 + 4\cos^3\phi - 3\cos\phi)] = 0. \qquad (113)
\end{aligned}
$$

Equation (113) is the condition for $\bar{T}'$ to be zero. The first factor yields $\phi = \phi_\infty$ and the second is a quartic polynomial that may be written

$$a_3\cos^4\phi + b_3\cos^3\phi + c_3\cos^2\phi + d_3\cos\phi + e_3 = 0, \qquad (114)$$

where

$$
\begin{aligned}
a_3 &= 16n_1(n_3 - n_2)^2(n_2 - n_1) \\
b_3 &= -8(n_1 + 3n_2)(n_3 - n_2)^2(n_2 - n_1) \\
c_3 &= -12n_1(n_3 - n_2)^2(n_2 - n_1) \\
d_3 &= 2(n_3 - n_2)^2(n_2 - n_1)(5n_1 + 9n_2) - 54n_2^2 n_3(n_2 - 2n_3) \\
e_3 &= 27n_2^2 n_3(n_2 - 2n_3) - 2(n_3 - n_2)^2(n_2 - n_1)(n_1 + 3n_2).
\end{aligned}
$$

If Eq. (114) is divided by a_3, the coefficient of the $\cos^4\phi$ term, the resulting equation may be factored into

$$(\cos\phi - \tfrac{1}{2})F(\cos\phi) = 0 \qquad (115)$$

where

$$
\begin{aligned}
F(\cos\phi) = \cos^3\phi - \frac{3n_2}{2n_1}\cos^2\phi - \frac{3(n_2 + n_1)}{4n_1}\cos\phi \\
+ \frac{2(n_1 + 3n_2)(n_3 - n_2)^2(n_2 - n_1) + 27n_2^2 n_3(2n_3 - n_2)}{8n_1(n_3 - n_2)^2(n_2 - n_1)}. \qquad (116)
\end{aligned}
$$

The real roots of Eq. (116) may be isolated using the Budan-Fourier theorem. Let $x = \cos\phi$. We form the sequence $F(x)$, $F'(x)$, $F''(x)$, $F'''(x)$ made up of the terms

$$F(x) = x^3 - \frac{3n_2}{2n_1}x^2 - \frac{3(n_2+n_1)}{4n_1}x + \frac{2(n_1+3n_2)(n_3-n_1)^2(n_2-n_1)+27n_2^2n_3(2n_3-n_2)}{8n_1(n_3-n_2)^2(n_2-n_1)}$$

$$F'(x) = 3x^2 - \frac{3n_2}{n_1}x - \frac{3(n_2+n_1)}{4n_1}$$

$$F''(x) = 6x - \frac{3n_2}{n_1} \tag{117}$$

$$F'''(x) = 6.$$

The variations of signs in the sequence may be analyzed if the problem is divided into two cases: $n_2 - n_1 > 0$ and $n_2 - n_1 < 0$. Again we consider only positive values of refractive indices.

$\underline{n_2 - n_1 > 0}$. For this case we find the following variations of signs in the sequence $V(-\infty) = 3$, $V(-1) = 2$, $V(0) = 2$ and $V(\infty) = 0$. We observe that $V(1)$ may be either 1 or 2 since $F(1)$ may be greater or less than zero. The sign of $F(1)$ follows the sign of the inequality

$$27n_2^2n_3(2n_3 - n_2) \gtrless 4(3n_2 - n_1)(n_3 - n_2)^2(n_2 - n_1). \tag{118}$$

The Budan-Fourier theorem reveals that, for $F(1) > 0$, $F(x)$ has one negative root in the interval $(-\infty,-1)$ and either two roots or none in the interval $(1,\infty)$. Because we are concerned only with real roots in the closed interval $[-1,1]$ where $\cos\phi$ is defined, we note that in the case where $n_2 > n_1$ and $F(1) > 0$, the exit pupil plane distance, $\overline{t}'$, can never be zero for any value of ϕ. On the other hand, when $F(1) < 0$, the variation of signs in the sequence for $x = 1$ is $V(1) = 1$. Thus $F(x) = 0$ has three distinct real roots, which lie on each of the intervals $(-\infty, -1)$, $(0,1)$, and $(1,\infty)$. For $x = 1/2$, we find $V(1/2) = 2$. Hence the only real root with which we are concerned lies in the interval $(1/2,1)$, which implies that $\overline{t}'$ may be zero for the case $n_2 < n_1$ if $F(1) < 0$, and in this particular case the parameter ϕ is in the intervals $(0,\pi/3)$ and $(-\pi/3,0)$, which we recall belong to the subdomain $r = 0$. It is only in this subdomain of ϕ that $\overline{t}'$ can be zero. An example of this particular case is when $n_1 = 1.5731$, $n_2 = 1.9525$, and $n_3 = 1$. For this set of three indices of refraction where $F(1) < 0$, solution of $F(x) = 0$ gives $x = 0.90682548, 2.1432607, -1.1883099$. Only the first root, which implies $\phi = \arccos(0.90682548)$, is the value of the free parameter that makes $\overline{t}'$ equal to zero.

$n_2 - n_1 < 0$. The variations of signs in the sequence $F(x)$, $F'(x)$, $F''(x)$, $F'''(x)$ for this case give $V(-\infty) = 3$, $V(-1) = 3$, $V(0) = 1$, and $V(+\infty) = 0$. We note that $V(1) = 1$ even if $F'(1)$ may either be equal to, greater than, or less than zero. Because $V(1) - V(\infty) = 1$, one real root of $F(x) = 0$ lies in the interval $(1,\infty)$. The theorem also reveals that either two roots or none occur in the interval $(-1,0)$. Here again is a case where we find that the Budan-Fourier theorem must be supplemented by additional investigation in order to locate the exact number of real roots of $F(x) = 0$ in the interval $(-1,0)$ with certainty. If we know the sign of the discriminant of $F(x)$, which is a cubic, together with our result from the Budan-Fourier theorem, we might be able to draw conclusions on the true nature of the real roots in $(-1,0)$. It so happens that when the discriminant of $F(x)$ is calculated, it is too complicated to deduce whether it is positive, negative, or zero. When Sturm's theorem was applied to $F(x)$, which is very complicated but powerful, the examination of the Sturm's series $F(x)$, $F_1(x)$, $F_2(x)$, $F_3(x)$ revealed that there is no real root in the interval $(-1,0)$. Hence $F(x) = 0$ has no root in $[-1,1]$, which implies that for the case where $n_2 < n_1$, $\overline{t}'$ can never be zero for any choice of ϕ.

We have shown that $\overline{t}'$ can be zero only for the case where $n_2 - n_1 > 0$ with $F(1) < 0$ and that ϕ is in subdomain $r = 0$. The first factor of Eq. (113) that yields $\phi = \phi_\infty$ will not make $\overline{t}'$ vanish since we have shown that this critical value does not exist for refracting modules. The first factor of Eq. (115) implies $\phi = \pm\pi/3$, but $\overline{t}'$ approaches a nonzero value as ϕ approaches these values at the branch cuts. This shows that Eq. (113) is only a necessary condition for $\overline{t}'$ to be zero but not a sufficient one. For reflecting systems, Eq. (114) becomes

$$128\cos^4\phi + 128\cos^3\phi - 96\cos^2\phi - 226\cos\phi + 113 = 0. \qquad (119)$$

Solution of Eq. (119) yields real values of ϕ equal to $\pm 0.167\pi$. Therefore, at $\phi = \pm 0.167\pi$ and at ϕ_∞, $\overline{t}'$ is zero for reflecting module systems.

CONCLUSIONS

It has been shown that parameters derivable from the Delano $y,\bar{y}$ diagram form a convenient set of independent parameters that completely describe and define some of the general properties of design modules. Such representations were found to yield insights in the analysis of modules that cannot readily be obtained using better known methods. The only drawback of the Delano diagram is that the constructibility of modules is not at once visualized from these parameters without transforming to the conventional optical parameters of curvatures and axial separation. Such transformations are straightforward, however.

The canonical optical parameters defined by Eq. (34) were introduced by Stavroudis (1969b) as a convenient form of describing modules. These are comparable to the parameters associated with the $y,\bar{y}$ diagram when the parameter f is equated to $1/\mathscr{H}$, the reciprocal of the Lagrange invariant.

Critical values of the free nonoptical parameters, k and θ, were defined in terms of the $y,\bar{y}$ diagram parameters and the three indices of refraction n_1, n_2, and n_3. Values of the canonical optical parameters and equivalent $y,\bar{y}$ diagram parameters, which correspond to the critical values of k and θ, were given in Tables I and II.

Conditions for modules to eliminate simultaneously third-order spherical and other Seidel aberrations were obtained. The limitations or constraints in the choice of values of the free parameters and the $y,\bar{y}$ diagram parameters were analyzed. The proper location of the aperture stop defined by Eq. (51) eliminates third-order astigmatism. For a given set of three indices of refraction n_1, n_2, and n_3 the module was found to eliminate Petzval curvature if Ω_2 is chosen so that it satisfies the quartic Eq. (55). Application of Descartes' rule of signs and Sturm's theorem in this quartic showed that there are at most two possible values of Ω_2 and at least one that will provide values of the free parameter k for refracting modules with zero Petzval contribution. For a two-mirror module in air, the quartic degenerates into a quadratic, which implies the existence of two possible modules with zero Petzval sum. It was also shown that for a two-mirror module in air there are exactly two real and two pure imaginary values of the free parameter k that provide zero Petzval curvature. The configuration of such mirror systems was shown in Figs. 3 and 4.

It was shown that modules with zero coma and Petzval curvature may be defined algebraically by Eq. (58). Their constructibility depends on the relative values of the three refractive indices n_1, n_2, and n_3. In like manner, modules with zero astigmatism and Petzval sum, modules with zero coma, astigmatism, and Petzval sum, and modules with zero distortion have been defined. Conditions for such desirable combinations seem difficult to satisfy for the case of zero distortion, but the others may be feasible depending on the choice of parameters. Numerical examples of modules generated with the aid of the General Electric time-sharing computer yield promising data for possible applications of these results in the process of optical design.

It has been found that the critical values ϕ_0, ϕ_∞, and ϕ^* do not exist for refracting modules in the imaginary case when indices of refraction are confined to common optical materials, particularly optical glasses. This result is surprising considering that their counterparts do exist for modules in the real case. These critical values of ϕ are functions of only the three indices of refraction associated with the module. A main objective of this study was to determine whether modules in the pure imaginary case may be grouped or classified in accordance with their parameter domains, which would lead to criteria that would ensure their constructibility. These parameter domains are delimited by the critical values, which in turn depend on the refractive indices.

Because the critical values of ϕ do not exist for refracting modules in the imaginary case, a new basis for classifying these refracting modules is needed so that we can compare real-case and imaginary-case modules. The purpose of such a comparison would be to determine conditions under which modules could be coupled. A possible classification of refracting modules other than from the point of view of the critical values could be based on the zeros of the expressions for the optical parameters and pupil positions. These were analyzed for the polynomials given by Eqs. (85), (95), (101), and (114). This is the direction now being considered for future work on this subject.

For the case of reflecting module systems, we showed that the critical values of ϕ do exist and have fixed values. This result enables us to classify two-mirror module systems, both real and imaginary cases, into groups so that we can compare each class with each other class to determine whether they are compatible for coupling purposes. By coupling, we mean that not only the focal planes of adjacent module elements but also their pupil planes must coincide to ensure that the resulting optical system eliminates both third-order spherical aberration and third-order astigmatism. A detailed account of these classifications is beyond the scope of this paper.

The method of applying the results of this study of the properties of modules to the design of multi-element refracting systems is clear. Arrays of modules could be arranged so that the rear and front foci of successive modules coincide to yield systems with initially zero third-order spherical aberration. The parameters associated with the individual modules could be varied to optimize the design. Some examples of coupled modules are given in the Appendix.

Appendix. EXAMPLES OF MODULES

To illustrate what has been discussed about modules, two values of refractive indices, $n_1 = 1.7335$ and $n_2 = 1.51823$, are randomly picked from the glass catalog. For $n_3 = 1$, the calculated critical values of the free parameter k are as follows:

$$k_0 = 0.947121$$
$$k_\infty = 0.999815$$
$$k^* = 0.996774.$$

The above critical values of k were computed with the aid of the General Electric time-sharing service Mark I computer, using the program "MODULE" in BASIC language. The computer program, "MODULE," generates tables of conventional optical parameters for modules, real case, for the three input values of refractive indices. In addition to the critical values, tables calculated at five values between critical values were generated. An option for including additional values of the free parameters was provided in the program.

Another computer program, "MOD-BAR," which is a modification of the "MODULE" has been written to generate tables of the $y,\bar{y}$ diagram parameters, real case of k, for three input values of refractive indices. The data generated from the "MODULE" and the "MOD-BAR" using the same set of three input values of refractive indices and an f value ($1/\text{Ж}$) of 0.1 are shown plotted as functions of k in Figs. 9, 10, 11, and 12.

Using the same set of three indices of refraction, the fourth degree Eq. (55) was solved with the aid of the General Electric Mark II time-sharing computer. A system subroutine called "ZORP***," in FORTRAN, was employed to determine the zeros of the polynomial. The quartic equation yields only two real roots of Ω_2 (2.0138932 and 0.55637065), which imply the existence of two modules with zero Petzval sum for the given set of three indices of refraction.

The program "MODULE" was also modified to generate tables of $\bar{\Omega}$ and $\bar{y}$ parameters for modules with zero astigmatism. Graphs showing the values of these parameters as functions of k, using the same set of three refractive indices and f-value, are plotted in Figs. 13 and 14.

Some examples of constructible real-case modules are shown in Figs. 15 and 16. The corresponding $y,\bar{y}$ diagrams of two typical real-case modules are illustrated in Fig. 17.

The program "IMMODR" was written to calculate tables of optical parameters for imaginary-case modules for three input values of refractive indices. Graphs showing the values of q, the optical parameters, and the pupil distances of two-mirror modules in air as functions of ϕ are shown in Figs. 18 to 21.

Figure 22 shows two examples of constructible two-mirror modules. All dimensions in these and succeeding illustrations are in centimeters. No attempt was made to correct the chromatic aberrations or any third-order aberrations besides spherical aberration and astigmatism.

In Fig. 23 we have examples of, first, two real-case modules defined by free parameters k_1 and k_2, coupled so that the pupil planes defined by Eq. (100) for the modules coincide. Relative to the object point O and image point I, both third-order spherical aberration and astigmatism are zero for this system. Two other configurations of two-module systems at finite conjugates are also shown.

Figure 24 is an example of three real-case modules with parameters k_1, k_2, and k_3 coupled so that pupil planes and focal planes of adjacent modules coincide. Relative to two conjugate planes, one at a finite distance and the other at infinity, both third-order spherical aberration and astigmatism are zero.

The cover illustration shows a seven-module system formed by combining a four-module afocal system generated by parameters k_1, k_2, k_3, and k_4 and a three-module system with parameters k_5, k_6, and k_7. The modules of this system were coupled so that the pupil planes and focal planes of adjacent modules coincide, thus guaranteeing that third-order spherical aberration and astigmatism are zero relative to the two conjugate planes of the resulting seven-module system. Optical data are given below.

focal length = 20.5 cm
semifield angle = 20°
f/number = 5.2
back focal distance = 18.7878 cm
stop position: 0.772896 cm from eighth surface

$n_1 = n_{15} = 1.0$	$c_1 = 0.091537$	$t_1 = 0.27087$	$k_1 = 1.00275$
$n_2 = 1.96052$ (LaSF-6)	$c_2 = 0.353252$	$t_2 = 2.96976$	$k_2 = 0.9875$
$n_3 = 1.9525$ (SF-59)	$c_3 = 0.219743$	$t_3 = 1.05006$	$k_3 = 0.985$
$n_4 = 1.0$	$c_4 = 0.212528$	$t_4 = 0.954597$	$k_4 = 0.9937525$
$n_5 = 1.4645$ (FK-3)	$c_5 = 0.359662$	$t_5 = 0.771267$	$k_5 = 0.9925$
$n_6 = 1.0$	$c_6 = 0.258797$	$t_6 = 1.14959$	$k_6 = 0.9825$
$n_7 = 1.9212$ (LaSF-7)	$c_7 = -0.0967274$	$t_7 = 0.601411$	$k_7 = 1.0021875$
$n_8 = 1.65332$ (KzFS-5)	$c_8 = 0.194559$	$t_8 = 1.414110$	
$n_9 = 1.0$	$c_9 = -0.17913$	$t_9 = 0.921304$	
$n_{10} = 1.62230$ (SSK-2)	$c_{10} = 0.0801456$	$t_{10} = 0.939737$	
$n_{11} = 1.9525$ (SF-59)	$c_{11} = -0.222083$	$t_{11} = 1.47854$	
$n_{12} = 1.0$	$c_{12} = -0.366516$	$t_{12} = 0.37597$	
$n_{13} = 1.4645$ (FK-3)	$c_{13} = -0.299313$	$t_{13} = 0.877542$	
$n_{14} = 1.9525$ (SF-59)	$c_{14} = -0.199782$		

These examples are given primarily to illustrate the potential of using modules in design problems. Further studies of optical design modules and modular methods in lens design are currently being undertaken. This report presents only the preliminary work done on the subject.

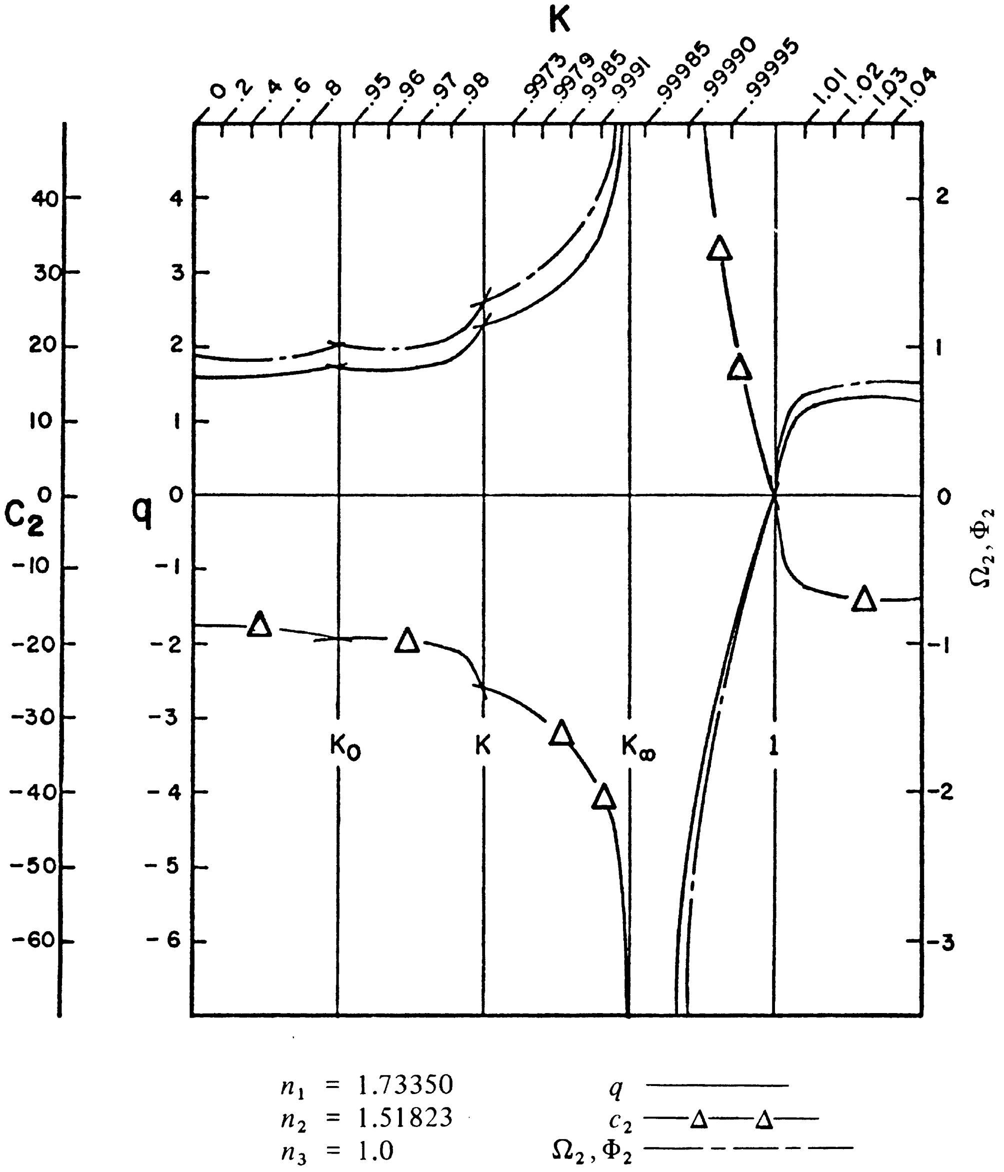


Fig. 9. Values of q, c_2, Ω_2, and Φ_2 as functions of k.

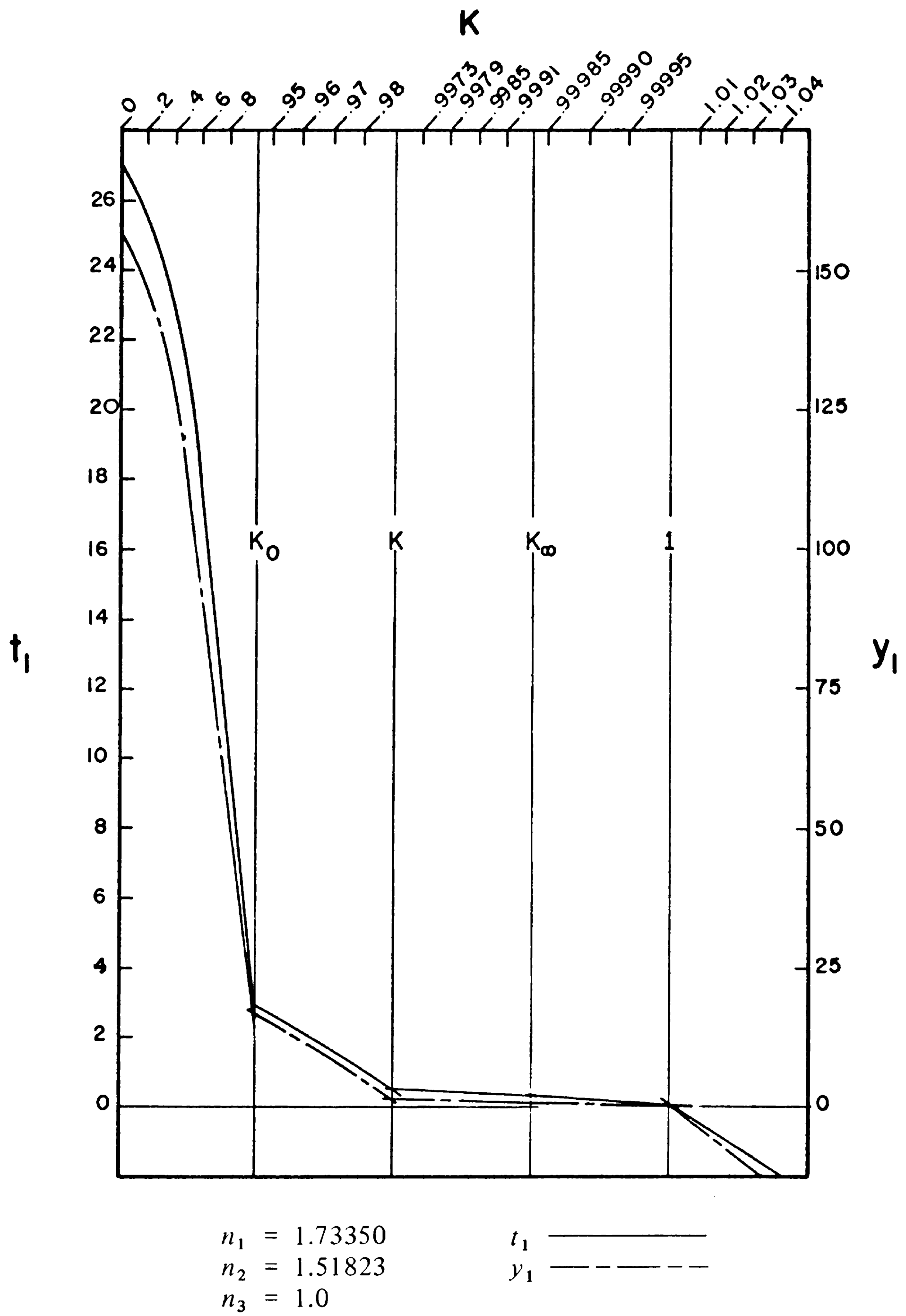
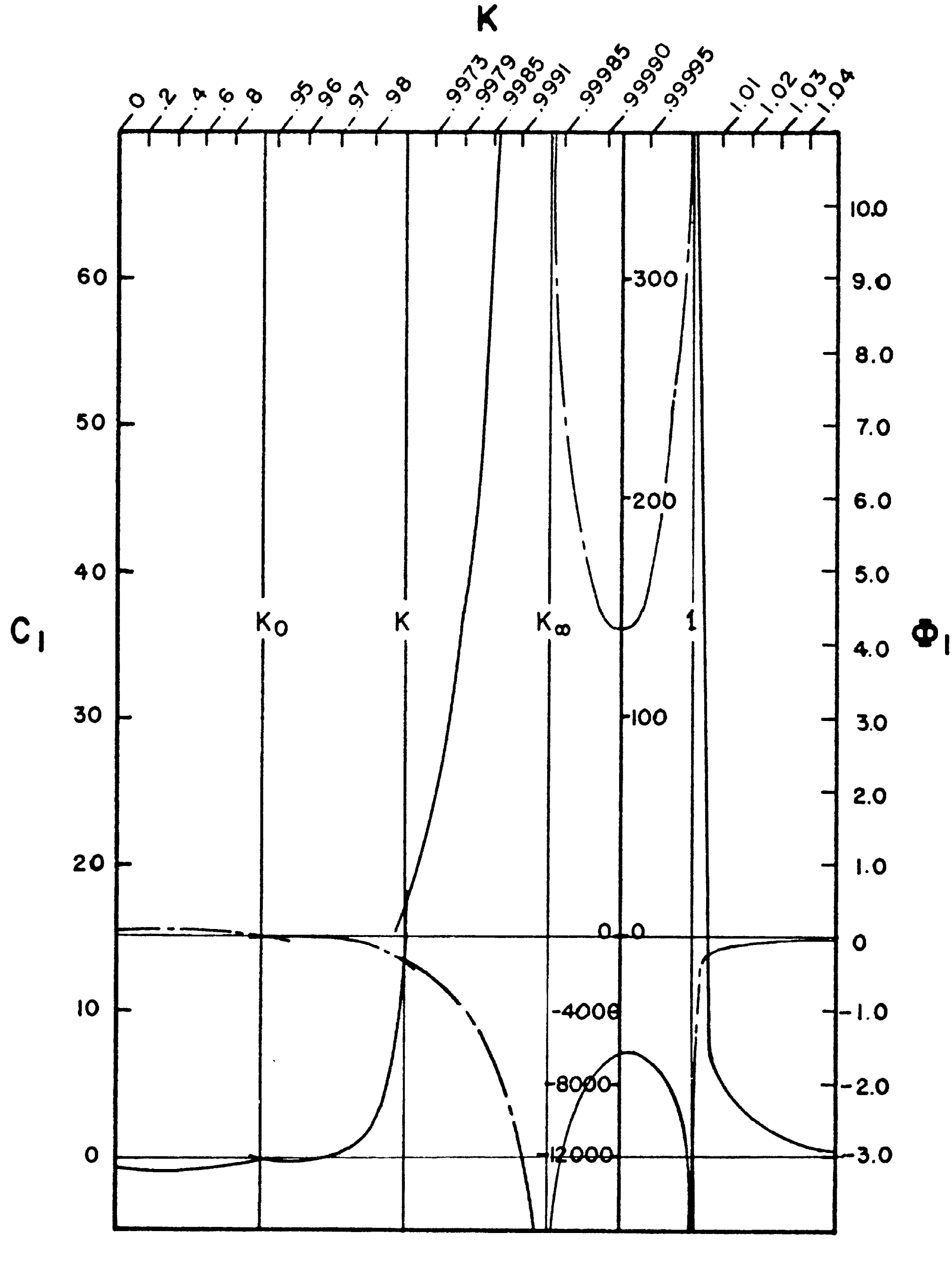


Fig. 10. Values of t_1 and y_1 as functions of k.



$n_1 = 1.73350$

$n_2 = 1.51823$

$n_3 = 1.0$

c_1 ————————

Φ_1 —— – – ——

Fig. 11. Values of c_1 and Φ_1 as functions of k.

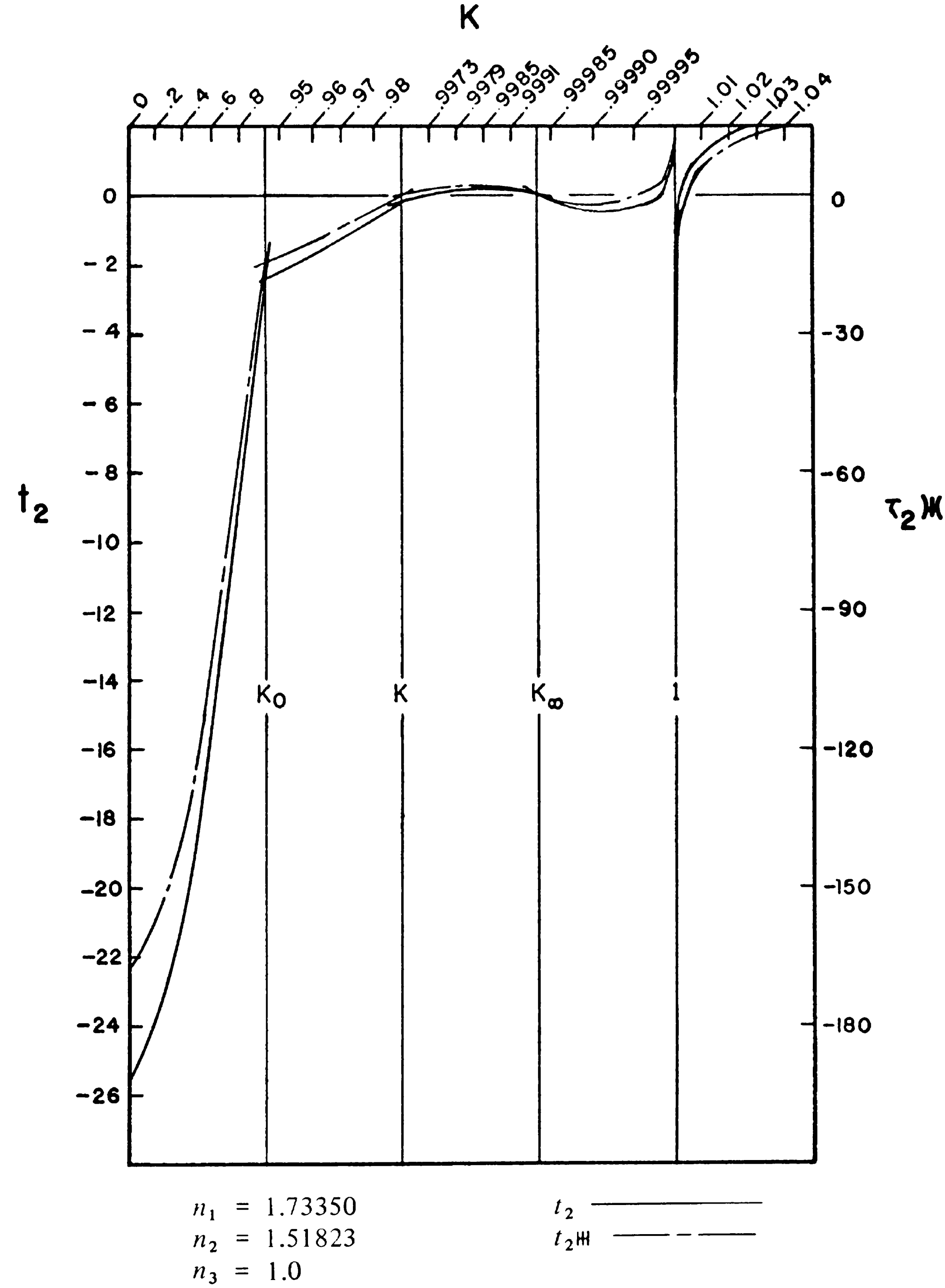
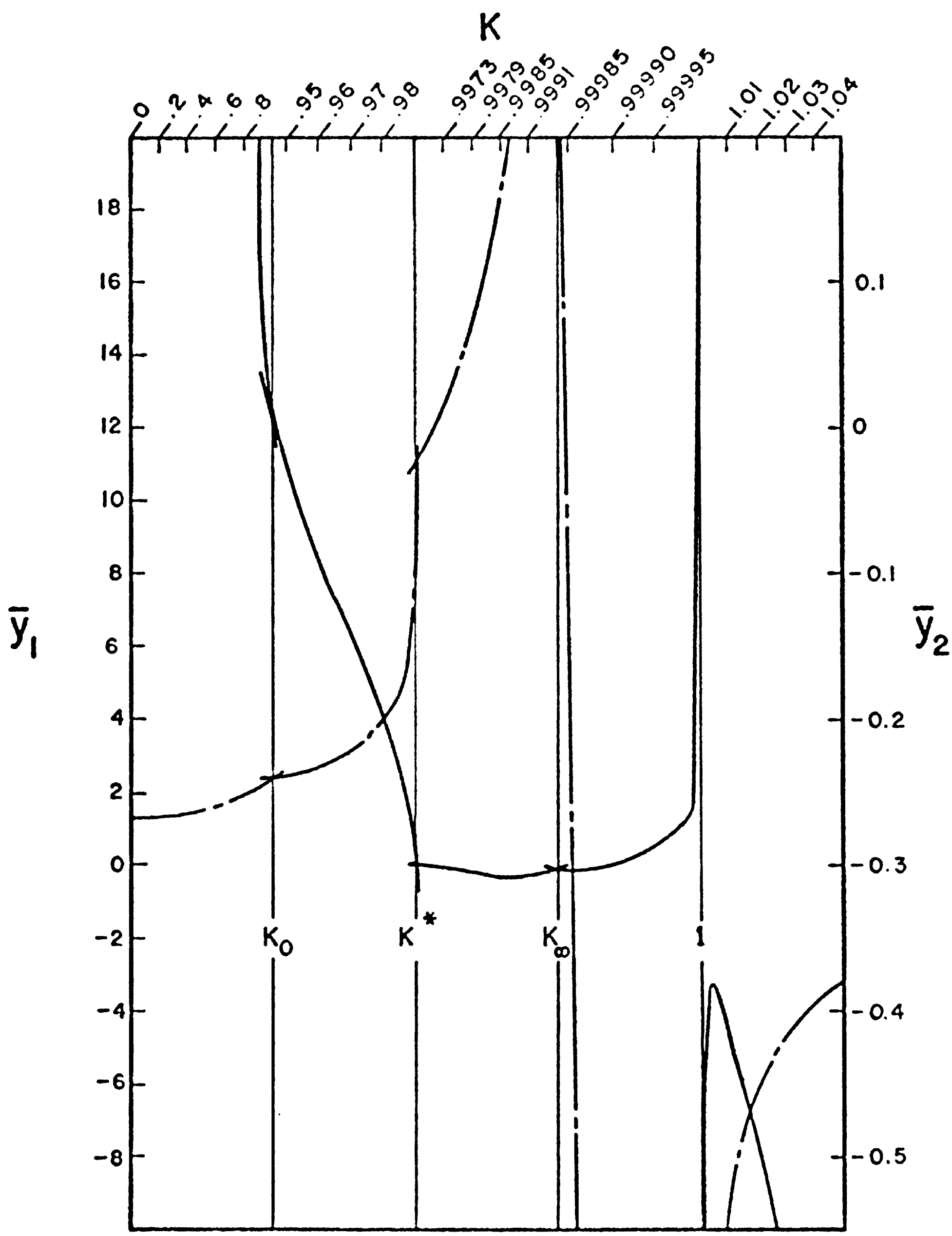


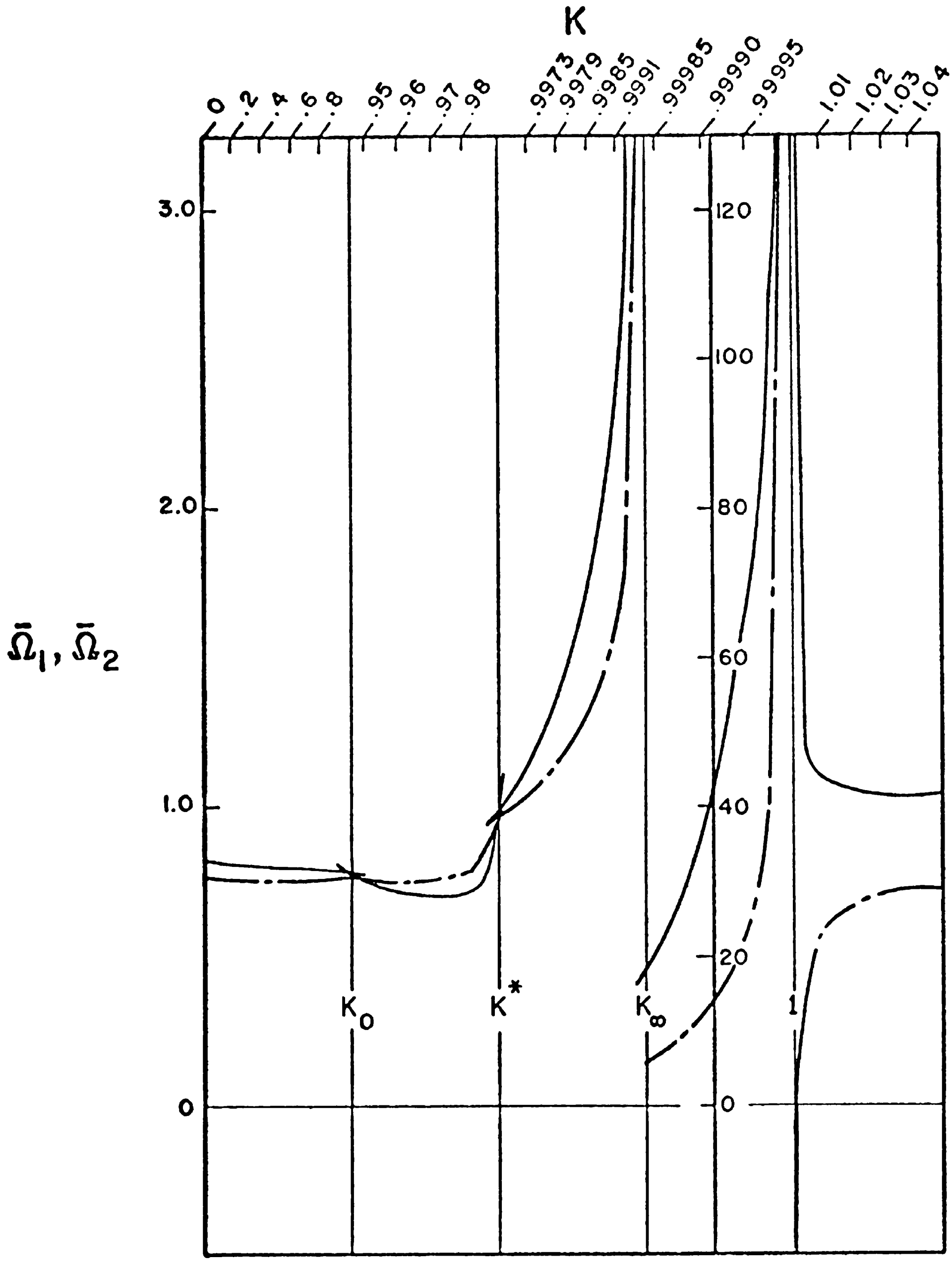
Fig. 12. Values of t_2 and τ_2ж as functions of k.



n_1 = 1.73350
n_2 = 1.51823
n_3 = 1.0

$\bar{y}_1$ ————————
$\bar{y}_2$ — — — — —

Fig. 13. Values of $\bar{y}_1$ and $\bar{y}_2$ as functions of k for modules with zero astigmatism.

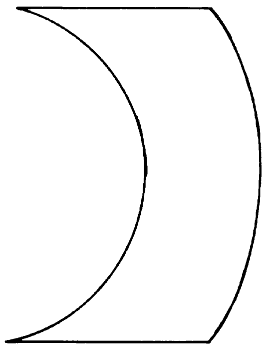


$n_1 = 1.73350$ $\bar{\Omega}_1$ ————————

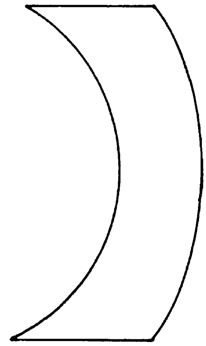
$n_2 = 1.51823$ $\bar{\Omega}_2$ — — — — — —

$n_3 = 1.0$

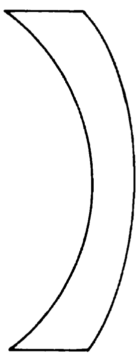
Fig. 14. Values of $\overline{\Omega}_1$ and $\overline{\Omega}_2$ as functions of k for modules with zero astigmatism.



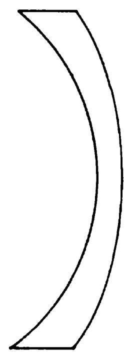
$c_1 = -0.443581$
$c_2 = -0.248473$
$t_1 = 12.775$
$t_2 = 1.55025$
$k = 1.008508$



$c_1 = -0.409394$
$c_2 = -0.245106$
$t_1 = 13.517$
$t_2 = 1.17703$
$k = 1.0090$



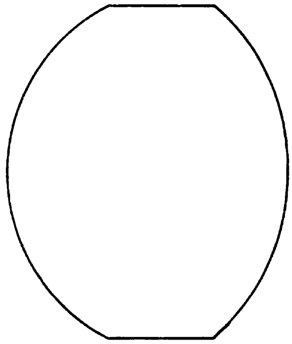
$c_1 = -0.362926$
$c_2 = -0.240297$
$t_1 = 14.7244$
$t_2 = 0.545859$
$k = 1.00980$



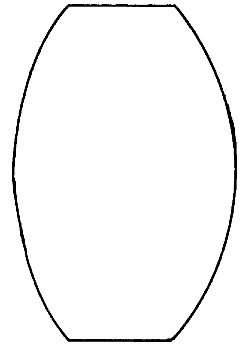
$c_1 = -0.347866$
$c_2 = -0.238673$
$t_1 = 15.1774$
$t_2 = 0.302244$
$k = 1.0101$

$n_1 = 1.5731 \quad n_2 = 1.9525 \quad n_3 = 1.0 \quad 1/Ж = 10$

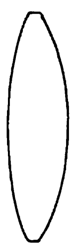
Fig. 15. Examples of modules ($k_0 < k_\infty < k^* > 1$).



$c_1 = 0.414699$
$c_2 = -0.324973$
$t_1 = 10.7023$
$t_2 = 3.81669$
$k = 0.9960$



$c_1 = 0.264356$
$c_2 = -0.294339$
$t_1 = 13.3712$
$t_2 = 2.93919$
$k = 0.9950$

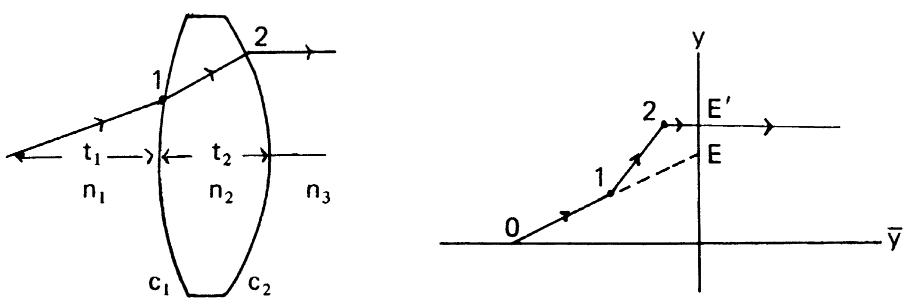


$c_1 = 0.146009$
$c_2 = -0.263775$
$t_1 = 18.0353$
$t_2 = 0.793953$
$k = 0.993250$

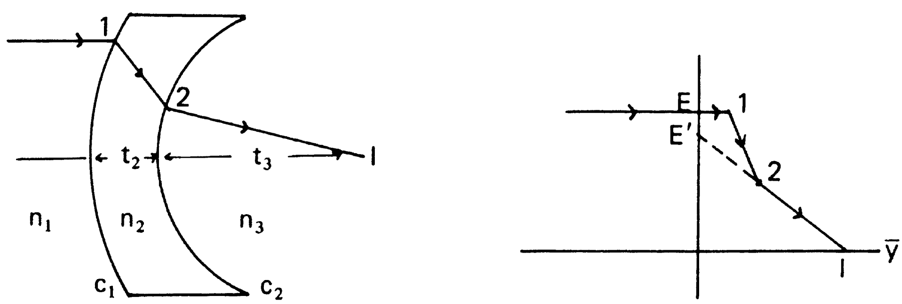
$c_1 = 0.135933$
$c_2 = -0.260681$
$t_1 = 18.7009$
$t_2 = 0.444406$
$k = 0.9930$

$n_1 = 1.9525 \quad n_2 = 1.5731 \quad n_3 = 1.0 \quad 1/M = 10$

Fig. 16. Examples of modules ($k_0 < k^* < k_\infty < 1$).



$n_1 = 1.9525$	$t_1 = 16.0374$	$c_1 = 0.184135$
$n_2 = 1.5731$	$t_2 = 1.78539$	$c_2 = -0.274616$
$n_3 = 1.0$		$k = 0.994$



$n_1 = 1.0$	$t_2 = 1.17703$	$c_1 = 0.245106$
$n_2 = 1.9525$	$t_3 = 13.517$	$c_2 = 0.409394$
$n_3 = 1.5731$		$k = 1.009$

Fig. 17. Constructible modules and their $y,\bar{y}$ diagrams.

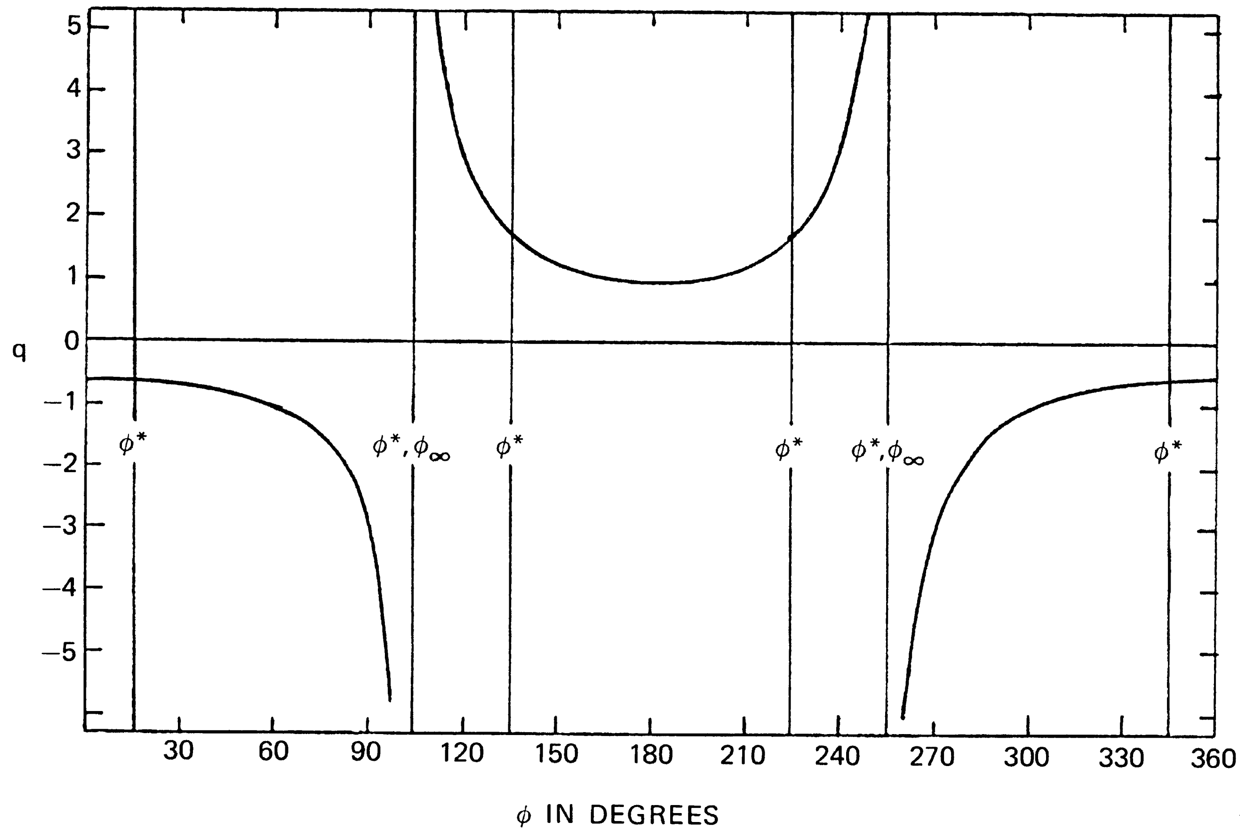


Fig. 18. Values of q as a function of ϕ for two-mirror modules in air with f equal to unity.

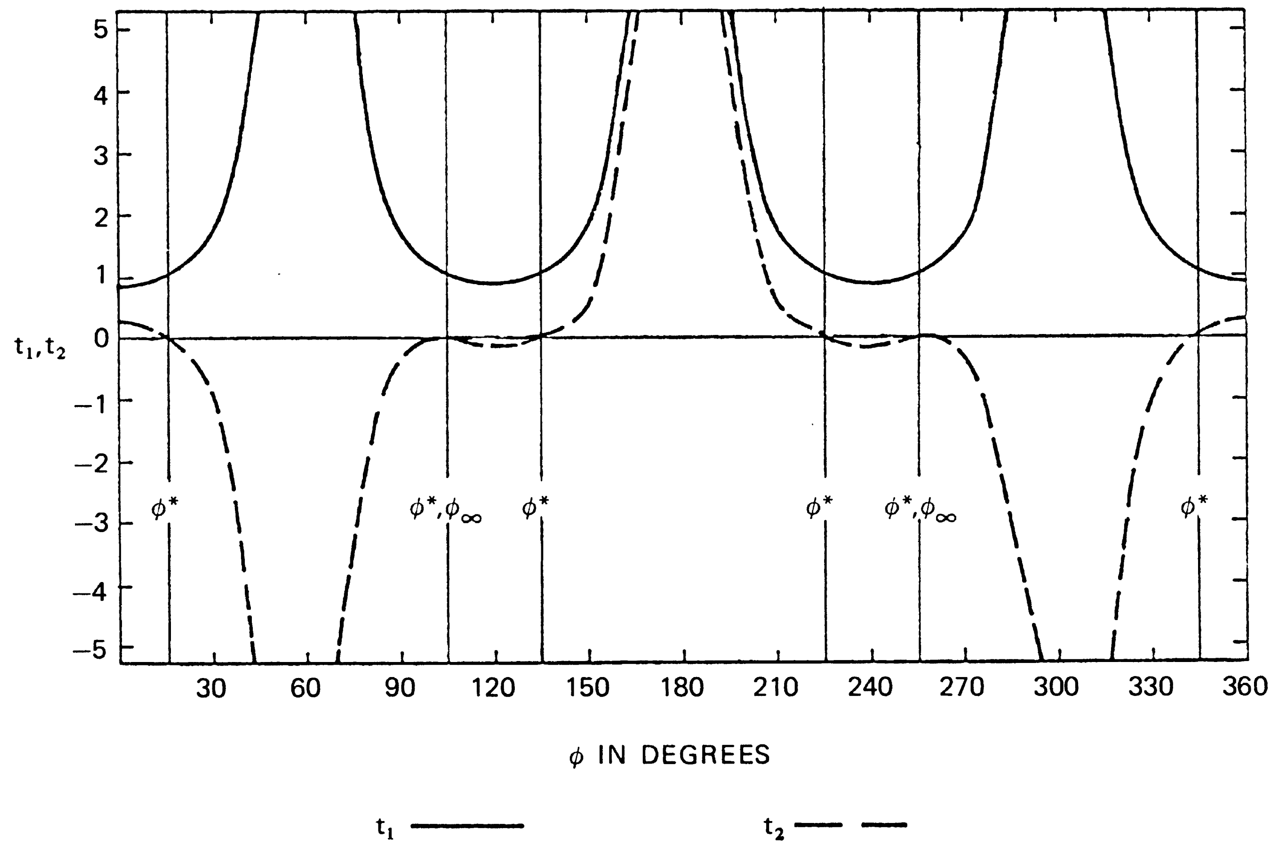


Fig. 19. Values of t_1 and t_2 as functions of ϕ for two-mirror modules in air with f equal to unity.

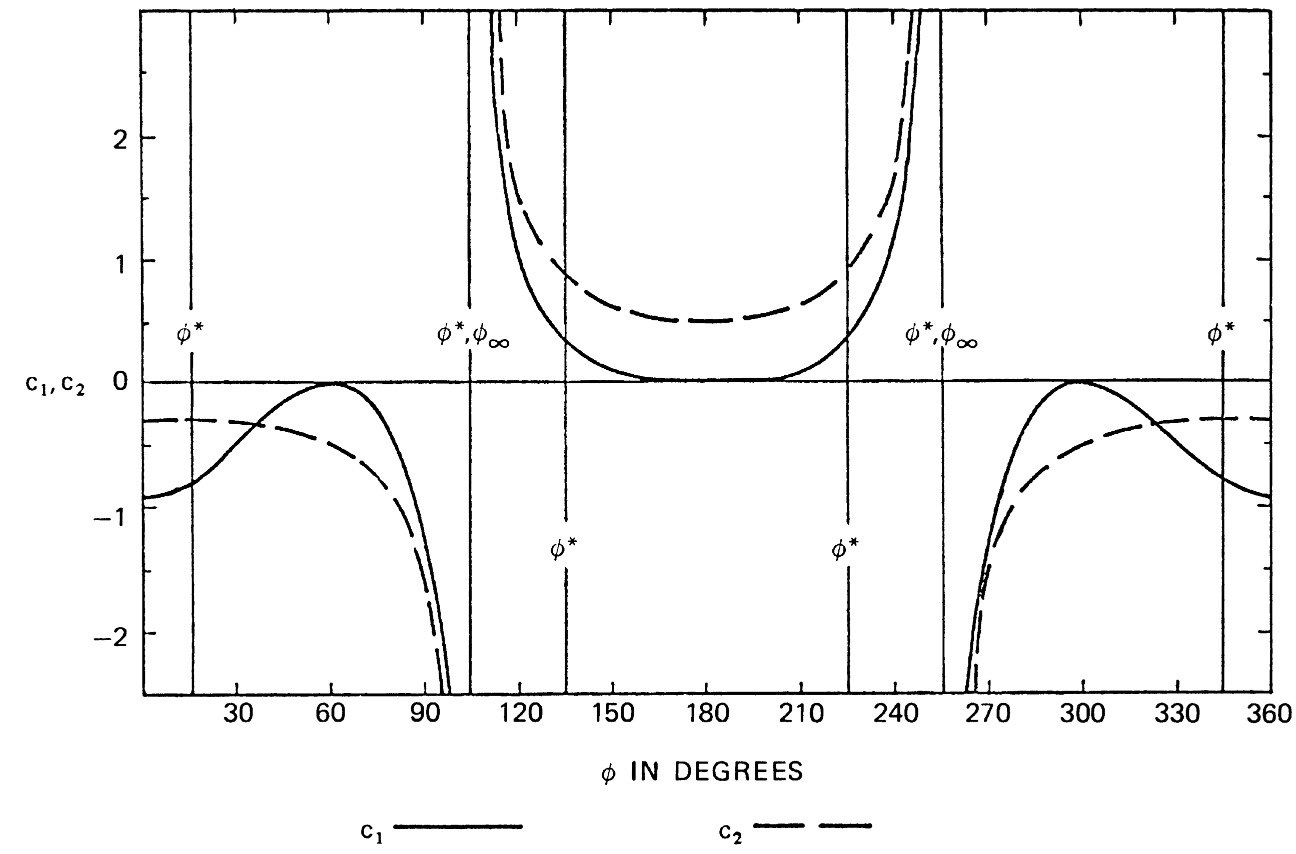


Fig. 20. Values of c_1 and c_2 as functions of ϕ for two-mirror modules in air with f equal to unity.

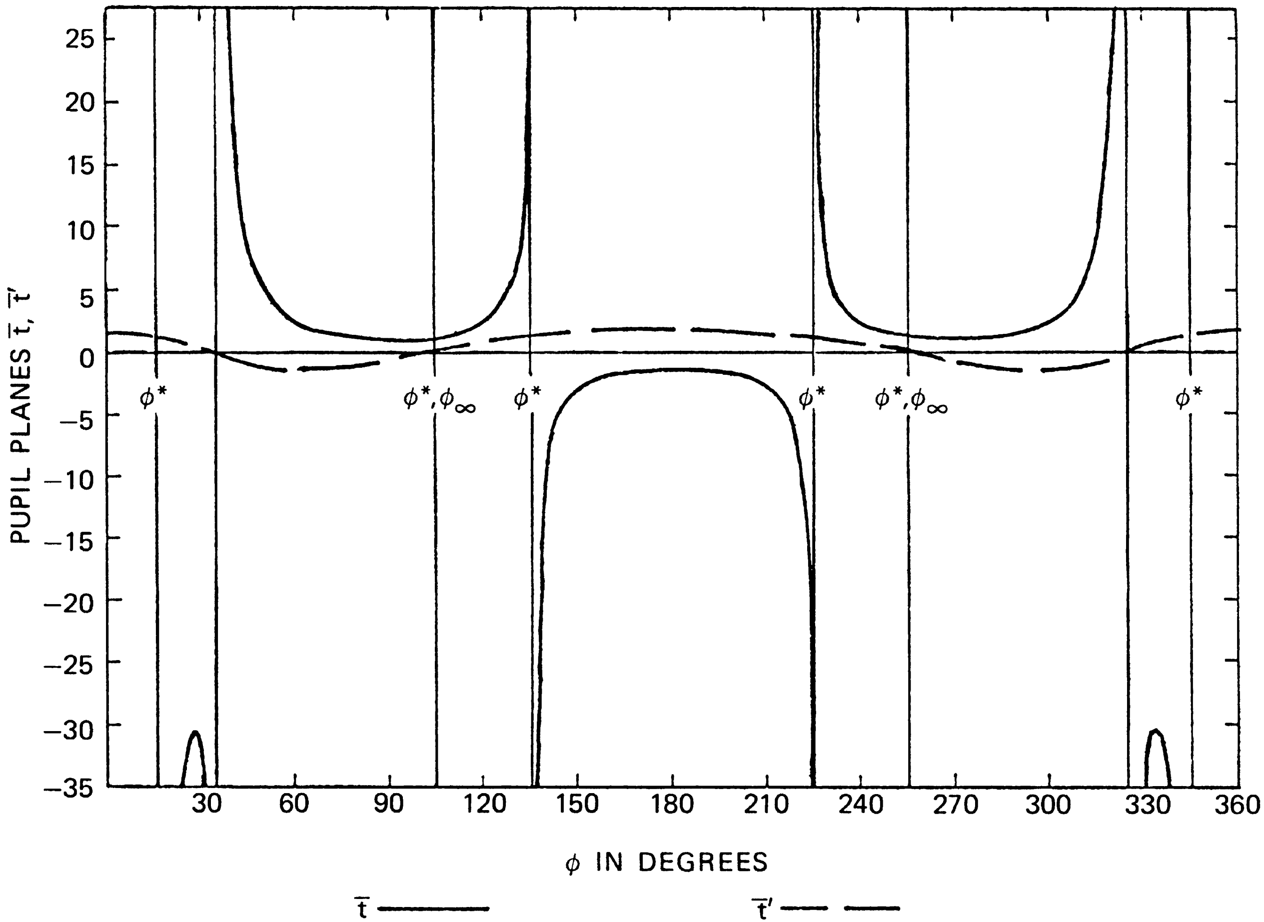
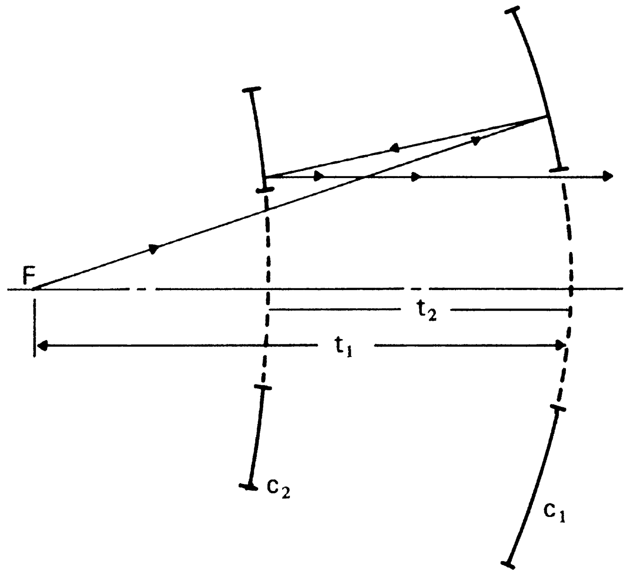
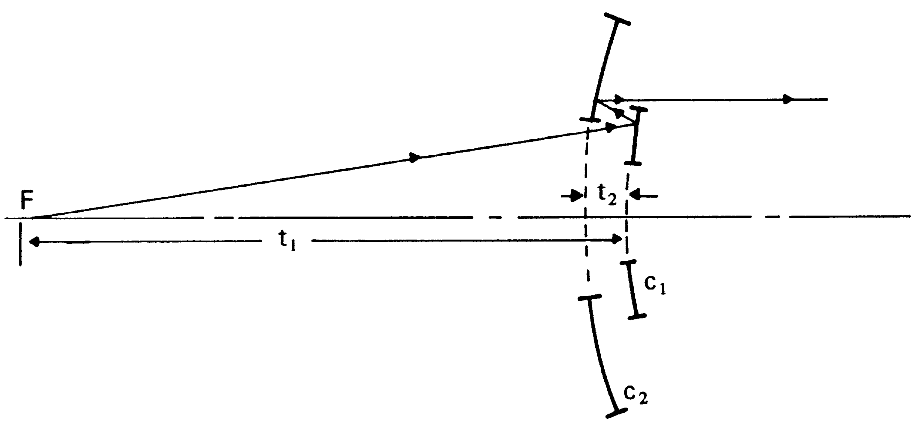


Fig. 21. Values of $\bar{t}$ and $\bar{t}'$ as functions of ϕ for two-mirror modules in air with f equal to unity.



ϕ = 29°
c_1 = −0.519742
c_2 = −0.333446

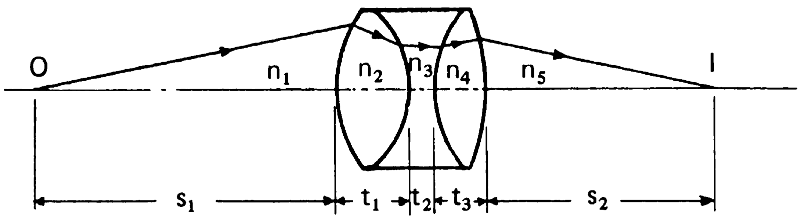
t_1 = 1.60358
t_2 = −0.905057
f = 1



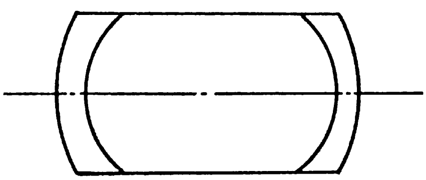
ϕ = 124°
c_1 = 0.0835611
c_2 = 0.121284

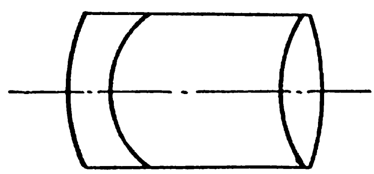
t_1 = 8.53071
t_2 = −0.605726
f = 10

Fig. 22. Typical examples of two-mirror module systems in air.



$n_1 = n_5 = 1.0$	$c_1 = 0.351083$	$s_1 = 11.4979$
$n_2 = 1.62230$	$c_2 = -0.457917$	$s_2 = 10.4927$
$n_3 = 1.9525$	$c_3 = 0.392119$	$k_1 = 1.0005758$
$n_4 = 1.75496$	$c_4 = -0.224021$	$k_2 = 1.000195$
$t_1 = 1.6071$	$t_2 = 0.662728$	$t_3 = 1.1162$

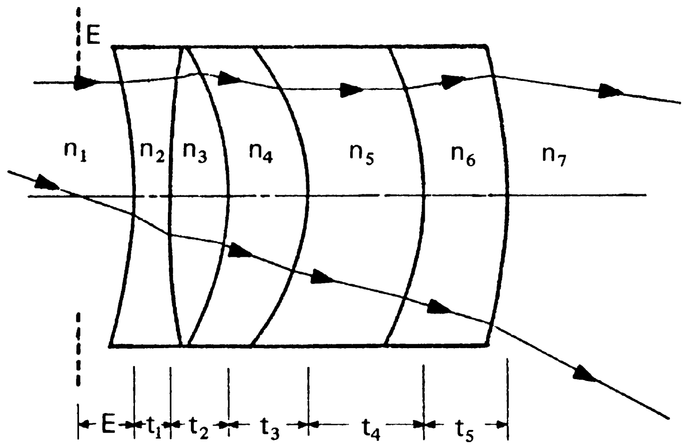




$n_1 = n_5 = 1.0$	$k_1 = 1.001135$
$n_2 = n_4 = 1.9525$	$k_2 = 1.00112$
$n_3 = 1.5731$	$s_1 = 10.5491$
$c_1 = 0.269896$	$s_2 = 10.4559$
$c_2 = 0.451214$	$t_1 = 0.626223$
$c_3 = -0.448943$	$t_2 = 5.67939$
$c_4 = -0.271434$	$t_3 = 0.522644$

$n_1 = n_5 = 1.0$	$k_1 = 1.00045$
$n_2 = 1.9525$	$k_2 = 1.00009325$
$n_3 = 1.6968$	$s_1 = 8.56744$
$n_4 = 1.5731$	$s_2 = 15.3428$
$c_1 = 0.252134$	$t_1 = 1.04765$
$c_2 = 0.516981$	$t_2 = 3.85895$
$c_3 = 0.339141$	$t_3 = 0.856998$
$c_4 = -0.185293$	

Fig. 23. Examples of two-module systems with finite object and image distances.



focal length = 10
semifield angle = 25°
f/number = 5
back focal distance = 10.1911

$n_1 = n_7 = 1.0$	$c_1 = -0.21259$
$n_2 = 1.5731$ (LF-1)	$c_2 = 0.0989137$
$n_3 = n_6 = 1.9525$ (SF-59)	$c_3 = -0.404631$
$n_4 = 1.4645$ (FK-3)	$c_4 = -0.539845$
$n_5 = 1.6968$ (LaK N-14)	$c_5 = -0.413695$
$t_1 = 0.335079$	$c_6 = -0.210235$
$t_2 = 0.490371$	$k_1 = 0.9925$
$t_3 = 0.747837$	$k_2 = 0.9993435$
$t_4 = 1.10755$	$k_3 = 1.00043525$
$t_5 = 0.661058$	$E = 0.516713$

Fig. 24. Example of a three-module system. Ray trace data for this system, generated using the program "TRIP-4," are shown on the opposite page.

```
                         MARGINAL          PRINCIPAL
            Y =          1                 0
            U =          0                 0.466308

SURFACE #   1
SPH =       -2.38584E-3  CØMA =            -4.65839E-3
AST =       -9.09559E-3  CURV =            -2.71531E-2
DST =       -4.72404E-2  PETZ =             0.077449
LØNG CØL =   1.81872E-3  LAT CØL =          3.55108E-3
            I =          -0.21259          -0.415085
            Y =           1                -0.240947
            U =          -0.077449          0.315087

SURFACE #   2
SPH =        2.38584E-3  CØMA =            -4.65839E-3
AST =        9.09559E-3  CURV =             1.19443E-2
DST =       -1.08158E-2  PETZ =            -1.22182E-2
LØNG CØL =  -4.44745E-3  LAT CØL =          8.68372E-3
            I =           0.17893          -0.349363
            Y =           1.02595          -0.346526
            U =          -4.26803E-2        0.247201

SURFACE #   3
SPH =        4.92943E-2  CØMA =             7.49737E-3
AST =        1.1403E-3   CURV =             1.72409E-2
DST =        4.60328E-3  PETZ =            -6.90556E-2
LØNG CØL =  -1.48973E-2  LAT CØL =         -2.2658E-3
            I =          -0.38092          -5.79357E-2
            Y =           1.04688          -0.467746
            U =           8.42498E-2        0.266506

SURFACE #   4
SPH =       -4.92943E-2  CØMA =             7.49737E-3
AST =       -1.1403E-3   CURV =            -1.29066E-2
DST =        3.7304E-3   PETZ =             5.04659E-2
LØNG CØL =   2.29498E-3  LAT CØL =         -3.49052E-4
            I =          -0.61539           9.35971E-2
            Y =           0.983876         -0.667049
            U =          -1.92993E-9        0.27932

SURFACE #   5
SPH =       -1.37375E-2  CØMA =             4.20593E-3
AST =       -1.2877E-3   CURV =            -8.73214E-3
DST =        5.30948E-3  PETZ =             3.19293E-2
LØNG CØL =   1.12423E-2  LAT CØL =         -3.44199E-3
            I =          -0.407024          0.124616
            Y =           0.983876         -0.97641
            U =          -0.053304          0.29564

SURFACE #   6
SPH =        1.37375E-2  CØMA =             4.20593E-3
AST =        1.2877E-3   CURV =             0.0252
DST =        1.27542E-2  PETZ =            -0.10256
LØNG CØL =  -7.67212E-3  LAT CØL =         -2.34892E-3
            I =          -0.160949         -4.92768E-2
            Y =           1.01911          -1.17184
            U =           0.1               0.342576

IMAGE PLANE
            Y =           1.62679E-8       -4.66308
            U =           0.1               0.342576

SUMS ØF ABERRATIØNS
SPH =       -6.22085E-10 CØMA =             1.40898E-2
AST =        8.73115E-11 CURV =             5.59331E-3
DST =       -3.16589E-2  PETZ =            -2.39898E-2
LØNG CØL =  -1.16609E-2  LAT CØL =          3.82904E-3
```

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