A Proof of the Soliton Resolution Conjecture for the Focusing Nonlinear Schrödinger Equation

by

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Abstract

We give a proof of the long-time asymptotic behavior of the focusing nonlinear Schrödinger equation for generic initial condition in which we have simple discrete spectral data and an absence of spectral singularities. The proof relies upon the theory of Riemann-Hilbert problems and the $\bar{\partial}$ method for nonlinear steepest descent. To leading order, the solution will appear as a multi-soliton solution as $t \to \infty$.

1. Introduction

We begin by looking at the focusing nonlinear Schrödinger equation (NLS),

$$i\phi_t + \frac{1}{2}\phi_{xx} + |\phi|^2\phi = 0, \tag{1.1}$$

together with the initial condition

$$\phi(x, t = 0) = \phi_0(x), \quad \phi_0(x) \to 0 \text{ as } |x| \to \infty. \tag{1.2}$$

It is known that (1.1) has great significance to the field of optics, as it describes the propagation of optical solitons and envelope pulses in optical fibers, [9, 11, 14]. This specific form of the NLS models such phenomena in the absence of fiber loss, so that, in the present context, transmission distance of signals is not limited. The soliton resolution conjecture concerns the long-time asymptotic behavior of solutions to (1.1). It posits that for generic initial conditions, the solution will eventually appear as a collection of solitons with a decaying background radiation term. The purpose of this dissertation is to prove this conjecture. These results have been submitted for publication and the manuscript is included in the Appendix as part of this dissertation per university guidelines.

1.1. The Lax Pair

It was first shown in [18] that the solution to (1.1) can be found by the method of inverse scattering. Additionally, the theory developed in this section to describe the scattering and inverse scattering transforms can be found in [1].

We define the Lax pair for the NLS by

$$\mathcal{L} = -iz\sigma_3 + Q = \begin{bmatrix} -iz & \phi(x) \\ -\phi(x)^* & iz \end{bmatrix}, \tag{1.3}$$

$$\mathcal{B} = iz\mathcal{L} + \frac{1}{2}\sigma_3(Q^2 - Q_x) = \begin{bmatrix} z^2 - \frac{1}{2}|\phi|^2 & iz\phi + -\frac{1}{2}\phi_x \\ -iz\phi^* - \frac{1}{2}\phi_x^* & -z^2 + \frac{1}{2}|\phi|^2 \end{bmatrix}, \tag{1.4}$$

where

$$Q = \begin{bmatrix} 0 & \phi(x) \\ -\phi(x)^* & 0 \end{bmatrix},$$

and σ_3 is the Pauli Matrix defined by

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The compatibility condition for this Lax pair is derived from the simultaneous solution of the following pair of equations:

$$\partial_x \psi = \mathcal{L}\psi \tag{1.5}$$
$$i\partial_t \psi = \mathcal{B}\psi \tag{1.6}$$

Cross differentiating, we find that

$$i(\partial_x \psi)_t = i\mathcal{L}_t\psi + i\mathcal{L}\psi_t = i\mathcal{L}_t\psi + \mathcal{L}\mathcal{B}\psi \tag{1.7}$$
$$(i\partial_t \psi)_x = \mathcal{B}_x\psi + \mathcal{B}\psi_x = \mathcal{B}_x\psi + \mathcal{B}\mathcal{L}\psi. \tag{1.8}$$

Thus the requirement of equivalency leads to the condition

$$i\mathcal{L}_t - \mathcal{B}_x + [\mathcal{L}, \mathcal{B}] = \begin{bmatrix} 0 & i\phi_t + \frac{1}{2}\phi_{xx} + \phi|\phi|^2 \\ -i\phi_t^* + \frac{1}{2}\phi_{xx}^* + \phi^*|\phi|^2 & 0 \end{bmatrix} = 0 \tag{1.9}$$

The forward scattering procedure begins as follows: Given initial data $\phi_0(x)$, we seek a normalized solution to the eigenvalue problem (1.3). Specifically, we seek a unique matrix-valued function satisfying

$$\begin{cases} \partial_x \psi = \mathcal{L}\psi \\ m \equiv \psi \begin{bmatrix} e^{ixz} & 0 \\ 0 & e^{-ixz} \end{bmatrix} \to I \quad \text{as } x \to +\infty \\ m(x,z) \text{ bounded as } x \to -\infty \end{cases} \tag{1.10}$$

for $\mathfrak{Im}(z) = 0$. These solutions will contain analytic extensions off of the real line to either $\mathbb{C}^+$ or $\mathbb{C}^-$, depending on the column. This will be specified below. In fact, the third condition of (1.10) will only be relevant in the case where $\mathfrak{Im}(z) \neq 0$. Let J_+ denote the matrix solution to (1.10) that have the following normalization.

$$J_+ = \left[j_+^{(1)}; j_+^{(2)}\right] \to \begin{bmatrix} e^{-ixz} & 0 \\ 0 & e^{ixz} \end{bmatrix} \quad x \to \infty. \tag{1.11}$$

Similarly, let J_- denote the matrix solution to the eigenvalue problem

$$\begin{cases} \partial_x \psi = P\psi \\ m \equiv \psi \begin{bmatrix} e^{ixz} & 0 \\ 0 & e^{-ixz} \end{bmatrix} \to I \quad \text{as } x \to -\infty \\ m(x,z) \text{ bounded as } x \to \infty \end{cases} \tag{1.12}$$

with the normalization

$$J_- = \left[j_-^{(1)}; j_-^{(2)}\right] \to \begin{bmatrix} e^{-ixz} & 0 \\ 0 & e^{ixz} \end{bmatrix} \quad x \to -\infty. \tag{1.13}$$

The matrix solutions $J_\pm$ are referred to as Jost solutions in the literature. To verify the existence of these solutions will show details for the column solution $j_+^{(1)}$. Let us denote this particular solution by

$$j_+^{(1)} = \begin{bmatrix} e^{-ixz}u(x,z) \\ e^{ixz}v(x,z) \end{bmatrix}$$

The asymptotic behavior of $j_+^{(1)}$ will require that

$$\lim_{x\to\infty} u(x,z) = 1, \text{ and } \lim_{x\to\infty} v(x,z) = 0. \tag{1.14}$$

Applying (1.3) to this solution gives the differential equations

$$\begin{aligned} u_x &= e^{2ixz}\phi(x)v(x,z) \\ v_x &= -e^{-2ixz}\phi^*(x)u(x,z) \end{aligned}$$

These equations are equivalent to the system of integral equations

$$u(x,z) = 1 - \int_x^\infty e^{2itz}\phi(t)v(t,z)dt \tag{1.15}$$

$$v(x,z) = \int_x^\infty e^{-2itz}\phi^*(t)u(t,z)dt \tag{1.16}$$

Upon substituting (1.16) into (1.15), we arrive at the integral equation

$$\begin{aligned} u(x,z) &= 1 - \int_x^\infty e^{2itz}\phi(t)\left(\int_t^\infty e^{-2isz}\phi^*(s)u(s,z)ds\right)dt \qquad (1.17) \\ &= 1 - \int_x^\infty \int_x^s e^{2i(t-s)z}\phi(t)\phi^*(s)u(s,z)dtds \\ &= 1 - \int_x^\infty K(x,s,z)u(s,z)ds \end{aligned}$$

where

$$K(x,s,z) = \int_x^s e^{2i(t-s)z}\phi(t)\phi^*(s)dt \tag{1.18}$$

where we note that since $t \le s$ on the region of integration and $\mathfrak{Im}(z) = 0$, we have the following bound

$$\begin{aligned} |K(x,s,z)| &\le |\phi(s)| \int_x^s |e^{2i(t-s)z}\phi(t)|dt \qquad (1.19) \\ &\le |\phi^*(s)| \cdot \|\phi\|_1 \end{aligned}$$

Next we use Picard iteration to show the existence of a solution to (1.17). Let $u_0 \equiv 1$, we shall use induction to show that

$$|u_n(x,z) - u_{n-1}(x,z)| \leq \frac{C}{(n-1)!}\|\phi\|_1^{n-1}\left(\int_x^\infty |\phi| d\tilde{s}\right)^{n-1} \tag{1.20}$$

where

$$u_n = 1 + \int_x^\infty K(x,s,z)u_{n-1}(s,z)ds. \tag{1.21}$$

Let $n = 1$, so that

$$\begin{aligned}
|u_1 - u_0| &= \left|1 - \int_x^\infty K(x,s,z)u_0(s,z)ds - u_0\right| \\
&\leq \left|\int_x^\infty K(x,s,z)ds\right| \\
&\leq \int_x^\infty |K(x,s,z)|ds \\
&\leq \int_x^\infty |\phi^*(s)| \cdot \|\phi\|_1 ds \\
&\leq \|\phi\|_1^2 = C
\end{aligned}$$

Assuming (1.20) holds for u_n, we now prove that the u_{n+1} case holds as well:

$$\begin{aligned}
|u_{n+1} - u_n| &= \left|\int_x^\infty K(x,s,z)(u_n(s,z) - u_{n-1}(s,z))ds\right| \\
&\leq \frac{C}{(n-1)!}\|\phi\|_1^{n-1}\int_x^\infty |K(x,s,z)|\left(\int_s^\infty |\phi| d\tilde{s}\right)^{n-1} ds \\
&\leq \frac{C}{(n-1)!}\|\phi\|_1^{n-1}\int_x^\infty |\phi^*(s)| \cdot \|\phi\|_1\left(\int_s^\infty |\phi| d\tilde{s}\right)^{n-1} ds \\
&= \frac{C}{(n-1)!}\|\phi\|_1^{n}\int_x^\infty |\phi(s)|\left(\int_s^\infty |\phi| d\tilde{s}\right)^{n-1} ds \\
&= -\frac{C}{n!}\|\phi\|_1^{n}\int_x^\infty \frac{d}{ds}\left(\int_s^\infty |\phi| d\tilde{s}\right)^{n} ds \\
&= \frac{C}{n!}\|\phi\|_1^{n}\left(\int_x^\infty |\phi| d\tilde{s}\right)^{n}.
\end{aligned}$$

Thus we conclude that

$$|u_n - u_0| \leq \sum_{k=0}^{n-1}\frac{C}{k!}\|\phi\|_1^{k}\left(\int_x^\infty |\phi| d\tilde{s}\right)^{k} \tag{1.22}$$

and

$$\lim_{n\to\infty}|u_n - u_0| = |u-1| \le \sum_{k=0}^{\infty}\frac{C}{k!}\|\phi\|_1^k\left(\int_x^\infty|\phi|d\tilde{s}\right)^k = Ce^{\|\phi\|_1^2}. \tag{1.23}$$

On the right hand side of (1.23) we have a well-defined series, thus our series converges absolutely and our solution exists. We can also uniformly bound $|u-1|$ for any value of x by

$$|u-1| \le Ce^{\|\phi\|_1^2} \tag{1.24}$$

so that when we once again look at (1.15), we see that

$$\begin{aligned}|u(x,z)| &= \left|1-\int_x^\infty K(x,s,z)u(s,z)ds\right| \\ &\le 1+\int_x^\infty |K(x,s,z)||u(s,z)-1|ds + \int_x^\infty |K(x,s,z)|ds \\ &\le 1+\int_{-\infty}^\infty \chi_{[x,\infty)}(s)|\phi^*(s)|\cdot\|\phi\|_1 Ce^{\|\phi\|_1^2}ds + \int_{-\infty}^\infty \chi_{[x,\infty)}(s)|\phi^*(s)|\cdot\|\phi\|_1 ds \\ &\le 1+\|\phi\|_1 Ce^{\|\phi\|_1^2}\int_{-\infty}^\infty|\phi^*(s)|ds + \|\phi\|_1\int_{-\infty}^\infty|\phi^*(s)|ds \\ &< \infty,\end{aligned}$$

where $\chi_{[x,\infty)}(s)$ denotes the indicator function for the set $[x,\infty)$. Thus by the Dominated Convergence Theorem, we can take limiting values of u as $x\to\pm\infty$ so that

$$\begin{aligned}\lim_{x\to\infty}u(x,z) &= 1 \\ \lim_{x\to-\infty}u(x,z) &= 1+\int_{-\infty}^\infty K(-\infty,s,z)u(s,z)ds.\end{aligned}$$

We now use this result to show the existence of $v(x,z)$. Since (1.16) defines v as an integral of $e^{-2isz}\phi^*(s)u(s,z)$, a function that is absolutely integrable, we see that v is absolutely integrable. Now since u is uniformly bounded, we can apply The Dominated Convergence Theorem once again to show that v has well-defined values as $x\to\pm\infty$.

$$\begin{aligned}\lim_{x\to\infty}v(x,z) &= 0 \\ \lim_{x\to-\infty}v(x,z) &= \int_{-\infty}^\infty e^{-2isz}\phi^*(s)u(s,z)ds\end{aligned}$$

The above case was for $z\in\mathbb{R}$ and we now focus on the case where z lies in the lower-half complex plane. This particular extension of this Jost solution will be useful to us later. Let $z=\alpha+i\beta$, $\beta<0$, and let

$$\tilde{j}_+^{(1)} = \begin{bmatrix} e^{-ixz}\tilde{u}(x,z) \\ e^{-ixz}\tilde{v}(x,z)\end{bmatrix}$$

be a proposed solution to (1.3) with z as above and the boundary conditions

$$\lim_{x\to\infty} \tilde{u}(x,z) = 1, \text{ and } \lim_{x\to\infty} \tilde{v}(x,z) = 0. \tag{1.25}$$

After substitution into (1.3) we arrive at the differential equations

$$\tilde{u}_x = \phi(x)\tilde{v}(x,z) \tag{1.26}$$
$$\tilde{v}_x - 2iz\tilde{v} = -\phi^*(x)\tilde{u}(x,z). \tag{1.27}$$

With the use of an integrating factor and application of the Fundamental Theorem of Calculus we find that,

$$\tilde{v}(x,z) = \int_x^\infty e^{-2i(t-x)z}\phi^*(t)\tilde{u}(t,z)dt. \tag{1.28}$$

Similarly, applying (1.25) to $\tilde{u}$, we get the integral equation

$$\begin{aligned} \tilde{u}(x,z) &= 1 - \int_x^\infty \phi(t)\tilde{v}(t,z)dt \qquad (1.29)\\ &= 1 - \int_x^\infty \phi(t)\left(\int_t^\infty e^{-2i(s-t)z}\phi^*(s)\tilde{u}(s,z)ds\right)dt \\ &= 1 - \int_x^\infty \int_x^s e^{-2i(s-t)z}\phi(t)\phi^*(s)\tilde{u}(s,z)dtds \\ &= 1 - \int_x^\infty \tilde{K}(x,s,z)\tilde{u}(s,z)ds, \end{aligned}$$

where

$$\tilde{K}(x,s,z) = \int_x^s e^{-2i(s-t)z}\phi(t)\phi^*(s)dt. \tag{1.30}$$

Since $t < s$ on the domain of integration and $\beta < 0$, we have $2(s-t)\beta < 0$, and we have the bound

$$\begin{aligned} |\tilde{K}(x,s,z)| &\leq |\phi^*(s)| \int_x^s |e^{-2i(s-t)z}\phi(t)|dt \qquad (1.31)\\ &\leq |\phi^*(s)| \int_x^s e^{2(s-t)\beta}|\phi(t)|dt \qquad (1.32)\\ &\leq |\phi^*(s)| \cdot \|\phi\|_1. \qquad (1.33) \end{aligned}$$

The same arguments used to show Picard iteration and apply the Dominated Convergence Theorem above will also apply in the case where $\beta < 0$. Additionally, from the definitions (1.28) and (1.29), we can take limiting values as $\mathfrak{Re}(z) \to -\infty$ since the integrands will be absolutely bounded. Additionally, we also conclude that $\tilde{u}$ is analytic for $z \in \mathbb{C}^-$.

By the same iteration method, we can show the existence of all four Jost solutions given by (1.11) and (1.13).

These four Jost solutions have the following properties:

1. $j_+^{(1)}(x,z)$ is continuous for $\mathfrak{Im}(z) \leq 0$ and there exists an analytic extension for $\mathfrak{Im}(z) < 0$. For all $x \in \mathbb{R}$

$$\lim_{\mathfrak{Im}(z)\to-\infty} e^{ixz} j_+^{(1)}(x,z) \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.34}$$

2. $j_+^{(2)}(x,z)$ is continuous for $\mathfrak{Im}(z) \geq 0$ and there exists an analytic extension for $\mathfrak{Im}(z) > 0$. For all $x \in \mathbb{R}$

$$\lim_{\mathfrak{Im}(z)\to\infty} e^{-ixz} j_+^{(2)}(x,z) \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1.35}$$

3. $j_-^{(1)}(x,z)$ is continuous for $\mathfrak{Im}(z) \geq 0$ and there exists an analytic extension for $\mathfrak{Im}(z) > 0$. For all $x \in \mathbb{R}$

$$\lim_{\mathfrak{Im}(z)\to\infty} e^{ixz} j_-^{(1)}(x,z) \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.36}$$

4. $j_-^{(2)}(x,z)$ is continuous for $\mathfrak{Im}(z) \leq 0$ and there exists an analytic extension for $\mathfrak{Im}(z) < 0$. For all $x \in \mathbb{R}$

$$\lim_{\mathfrak{Im}(z)\to-\infty} e^{-ixz} j_-^{(2)}(x,z) \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1.37}$$

We note at this time that we will use these Jost solutions to build matrix functions that will satisfy (1.11) and (1.13), but a scaling will be required.

Since the trace of $\mathcal{L}$ is zero then the Wronskian of J_+ will be independent of x. Therefore, since $j_+^{(1)}$ and $j_+^{(2)}$ are the columns of J_+, by taking the determinant of J_+ as $x \to \infty$ we find that the Wronskian is 1 and the two columns of J_+ are linearly independent. Therefore, the other two solutions from J_- must be a linear combination of the solutions from J_+ and we may write the scattering relationship for $z \in \mathbb{R}$ as

$$J_-(x,z) = J_+(x,z)S(z), \tag{1.38}$$

where we refer to $S(z)$ as the scattering matrix.

1.2. The Scattering Matrix

In this section we will determine properties of the scattering matrix, S given in (1.38) for $z \in \mathbb{R}$. We first note that since $j_+^{(1)}$ is a solution to (1.10), we have that

$$\begin{bmatrix} j_{+1}^{(1)} \\ -j_{+2}^{(1)} \end{bmatrix}_x = \begin{bmatrix} -izj_{+1}^{(1)} + \phi j_{+2}^{(2)} \\ -\phi^* j_{+1}^{(1)} + izj_{+2}^{(2)}. \end{bmatrix}$$

If we now take the complex conjugate of the equation and multiply the second entries by -1 we see that

$$\begin{bmatrix} j_{+1}^{(1)} \\ -j_{+2}^{(1)} \end{bmatrix}_x^* = \begin{bmatrix} iz(j_{+1}^{(1)})^* + \phi^*(j_{+2}^{(2)})^* \\ \phi(j_{+1}^{(1)})^* - iz(-j_{+2}^{(2)})^* \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} j_{+1}^{(1)} \\ -j_{+2}^{(1)} \end{bmatrix}_x^* = \mathcal{L} \begin{bmatrix} j_{+1}^{(1)} \\ -j_{+2}^{(1)} \end{bmatrix}^*.$$

Thus we see that

$$\begin{bmatrix} j_{+1}^{(1)} \\ -j_{+2}^{(1)} \end{bmatrix}_x^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} j_+^{(1)*}$$

is also a solution to (1.10). By taking a limit we see that

$$\lim_{x\to\infty} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} j_+^{(1)*} = \begin{bmatrix} 0 \\ e^{ixz} \end{bmatrix},$$

and by the linear independence of the Jost solutions and the asymptotic behavior given by (1.11), we conclude that

$$j_+^{(2)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} j_+^{(1)*}.$$

In the same way, we can derive the relationship

$$j_-^{(2)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} j_-^{(1)*}$$

and summarize these relationships by

$$\begin{aligned} J_-^* &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} J_- \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ J_+^* &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} J_+ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \tag{1.39}$$

If we now return to (1.38), solve for S and take the complex conjugate, we see that

$$\begin{aligned} S^* &= J_+^{-1*} J_-^* \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} S \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \end{aligned}$$

and conclude that S must be of the form

$$S = \begin{bmatrix} a(z) & -b^*(z) \\ b(z) & a^*(z) \end{bmatrix}. \tag{1.40}$$

With this form of S we have

$$j_-^{(1)} = a(z)j_+^{(1)} + b(z)j_+^{(2)} \tag{1.41}$$

and we can derive an equation for a. If we form a matrix, $\left[j_-^{(1)}; j_+^{(2)}\right]$ (which will have an analytic extension to $\mathbb{C}^+$) and then take a determinant we see that

$$\begin{aligned} det\left(\left[j_-^{(1)}; j_+^{(2)}\right]\right) &= det\left(\left[a(z)j_+^{(1)} + b(z)j_+^{(2)}; j_+^{(2)}\right]\right) \\ &= a(z)det\left(\left[j_+^{(1)}; j_+^{(2)}\right]\right) \\ &= a(z). \end{aligned} \tag{1.42}$$

We now can conclude that $a(z)$ has an analytic extension from $\mathbb{R}$ to $\mathbb{C}^+$. Furthermore, for any value of $z \in \mathbb{C}^+$ such that $a(z) = 0$ we can deduce that $j_-^{(1)}$ must be a multiple of $j_+^{(2)}$, and hence we have a solution to (1.5) that decays as $x \to \pm\infty$. We call these values of z for which $a(z) = 0$ the eigenvalues of the scattering problem. Furthermore, we can deduce the asymptotic behavior of $a(z)$ from (1.42), (1.35), and (1.36) to conclude that

$$\lim_{\mathfrak{Im}(z)\to\infty} a(z) = 1. \tag{1.43}$$

1.3. Time-dependence of the Scattering Matrix

So far we have developed the theory of Jost solutions merely based upon the criteria that $J_\pm$ satisfy (1.5), and have seen that $S = S(z)$. To arrive at a solution ϕ to (1.1), we must have that $J_\pm = J_\pm(x, z, t)$ also satisfy (1.6), and therefore S should also depend on t. We will require that

$$\lim_{x\to\pm\infty} J_\pm(x, z, t)e^{ixz\sigma_3} = I, \tag{1.44}$$

and must derive conditions for S and $J_\pm(x, z, t)$ so that these Jost matrices will solve both (1.5) and (1.6). We already know that the time-independent solutions $J_\pm(x, z)$ satisfy

$$(J_\pm)_x = \mathcal{L}J_\pm, \tag{1.45}$$

so if we differentiate with respect to t we see that

$$\partial_t (J_\pm)_x = \partial_t\mathcal{L}J_\pm + \mathcal{L}\partial_t J_\pm. \tag{1.46}$$

We now use the left hand side of (1.9), we find that

$$(i\partial_t J_\pm - \mathcal{B}J_\pm)_x = \mathcal{L}\left(i\partial_t J_\pm - \mathcal{B}J_\pm\right) \tag{1.47}$$

and see that $i\partial_t J_\pm - \mathcal{B}J_\pm$ is also a time-independent solution to (1.5). However, we know that $J_\pm$ are linearly independent solutions to (1.5) so there must exist a matrix $C(z,t)$ such that

$$i\partial_t J_\pm - \mathcal{B}J_\pm = J_\pm C(z,t). \tag{1.48}$$

As $J_\pm$ is invertible, we can solve for $C(z,t)$ so that

$$C(z,t) = J_\pm^{-1}\left(i\partial_t J_\pm - \mathcal{B}J_\pm\right), \tag{1.49}$$

since C is independent of x we can take the limits of the right-hand side as $x \to \pm\infty$ so that

$$\begin{aligned} C(z,t) &= \lim_{x\to\pm\infty} J_\pm^{-1}\left(i\partial_t J_\pm - \mathcal{B}J_\pm\right) \\ &= e^{\pm ixz\sigma_3}\left(0 - z^2\sigma_3 e^{\mp ixz\sigma_3}\right) \\ &= -z^2\sigma_3. \end{aligned}$$

We can now differentiate (1.38) with respect to t and multiply by i to get

$$i\partial_t J_- = i\partial_t J_+ S + iJ_+\partial_t S \tag{1.50}$$

and if we use the substitution $i\partial_t J_\pm - \mathcal{B}J_\pm = J_\pm\left(-z^2\sigma_3\right)$, we can conclude that

$$\begin{gathered} \partial_t S = iz^2[S,\sigma_3] \\ \begin{bmatrix} \partial_t a & -\partial_t b^* \\ \partial_t b & \partial_t a^* \end{bmatrix} = \begin{bmatrix} 0 & 2iz^2 b^* \\ 2iz^2 b & 0 \end{bmatrix}. \end{gathered}$$

Hence, $a(z)$ is independent of t and $b(z,t) = b(z,0)e^{2iz^2t}$, and therefore $S(x,z,t)$ is of the form

$$S(z,t) = e^{-iz^2t\sigma_3}S(z,0)e^{iz^2t\sigma_3} = \begin{bmatrix} a(z) & -b^*(z)e^{-2iz^2t} \\ b(z)e^{2iz^2t} & a^*(z) \end{bmatrix}. \tag{1.51}$$

1.4. The Inverse Scattering Transform

Using the Jost solutions (1.11) and (1.13) we shall construct a matrix M defined by

$$M = \begin{cases} \begin{bmatrix} \frac{1}{a(z)}j_-^{(1)}(x,z) & j_+^{(2)}(x,z) \end{bmatrix} e^{ixz\sigma_3}, & \text{for } \mathfrak{Im}(z) > 0 \\ \begin{bmatrix} j_+^{(1)}(x,z) & \frac{1}{a^*(z^*)}j_-^{(2)}(x,z) \end{bmatrix} e^{ixz\sigma_3}, & \text{for } \mathfrak{Im}(z) < 0 \end{cases}. \tag{1.52}$$

The Jost solutions are paired in such a way so that M extends analytically to $\mathbb{C}^+$ and $\mathbb{C}^-$. Consequently, we may have poles that appear in M in either half-plane. From (1.39), we know that

$$j_+^{(1)*}(x,z^*) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} j_+^{(2)}(x,z)$$
$$j_-^{(2)*}(x,z^*) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} j_-^{(1)}(x,z)$$

and so for $z \in \mathbb{C}^+$ we derive the symmetry

$$\begin{aligned} M^*(x,z^*) &= \begin{bmatrix} e^{-ixz} j_+^{(1)*}(x,z^*) & \frac{e^{ixz}}{a^{**}(z^{**})} j_-^{(2)*}(x,z^*) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{e^{ixz}}{a(z)} j_-^{(1)}(x,z) & e^{-ixz} j_+^{(2)}(x,z) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}, \end{aligned}$$

or, equivalently,

$$M(x,z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} M^*(x,z^*) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.53}$$

As M is not continuous across $\mathbb{R}$ we wish to relate the definitions of M in $\mathbb{C}^+$ and $\mathbb{C}^-$ via (1.38). For $z \in \mathbb{R}$ let

$$M_\pm(x,z,t) \equiv \lim_{\epsilon \to 0^+} M(x, z + i\epsilon, t) \tag{1.54}$$

denote the boundary values of M. Then we see that

$$\begin{aligned} M_+ &= \begin{bmatrix} \frac{1}{a(z)} j_-^{(1)}(x,z) & j_+^{(2)}(x,z) \end{bmatrix} e^{ixz\sigma_3} \\ &= \begin{bmatrix} \frac{1}{a(z)} (a(z) j_+^{(1)}(x,z) + b(z) e^{2iz^2 t} j_+^{(2)}(x,z)) & j_+^{(2)}(x,z) \end{bmatrix} e^{ixz\sigma_3} \\ &= J_+ \begin{bmatrix} 1 & 0 \\ \frac{b(z)}{a(z)} e^{2iz^2 t} & 1 \end{bmatrix} e^{ixz\sigma_3} \end{aligned}$$

and

$$\begin{aligned} M_- &= \begin{bmatrix} j_+^{(1)}(x,z) & \frac{1}{a^*(z)} j_-^{(2)}(x,z) \end{bmatrix} e^{ixz\sigma_3} \\ &= \begin{bmatrix} j_+^{(1)}(x,z) & \frac{1}{a^*(z)} (-b^* e^{-2iz^2 t}(z) j_+^{(1)}(x,z) + a^*(z) j_+^{(2)}(x,z)) \end{bmatrix} e^{ixz\sigma_3} \\ &= J_+ \begin{bmatrix} 1 & -\frac{b^*(z)}{a^*(z)} e^{-2iz^2 t} \\ 0 & 1 \end{bmatrix} e^{ixz\sigma_3}. \end{aligned}$$

Thus we have the jump relation

$$M_+ = M_- V(z) = M_- \begin{bmatrix} 1+|r(z)|^2 & e^{-2i(xz+z^2t)}r^*(z) \\ e^{2i(xz+z^2t)}r(z) & 1 \end{bmatrix} \tag{1.55}$$

where we refer to $V(z)$ as the jump matrix and $r(z) = \dfrac{b(z)}{a(z)}$ is the reflection coefficient. We define the ratio $t(z) = \dfrac{1}{a(z)}$ as the transmission coefficient and from (1.38) we see that

$$det(S) = |a(z)|^2 + |b(z)|^2 = 1, \tag{1.56}$$

and consequently

$$1 + |r(z)|^2 = |t(z)|^2. \tag{1.57}$$

It can be shown (see [6]) that for initial data in the Sobolev space $H^{1,1}(\mathbb{R})$ defined by

$$H^{1,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : xf, f' \in L^2(\mathbb{R})\}, \tag{1.58}$$

$r(t)$ will also be in $H^{1,1}(\mathbb{R})$ for all $t \geq 0$.

We now turn to the eigenvalues of the scattering transform. These are the values $z_n \in \mathbb{C}^+$ for which $a(z_n) = 0$, which we will assume are simple. Additionally, we will assume that none of these eigenvalues are in $\mathbb{R}$. We shall refer to this set by $\mathcal{Z} = \{z_n\}_{n=1}^N \subset \mathbb{C}^+$. From (1.42), and since boundary conditions prevent $j_-^{(1)}$ and $j_+^{(2)}$ from being identically equal to zero, we must have

$$j_-^{(1)}(x, z_n) = \lambda_n(t) j_-^{(1)}(x, z_n). \tag{1.59}$$

for each eigenvalue z_n, where $\lambda_n(t) \neq 0$. If we wish to determine the residues of M at these poles we see that

$$\begin{aligned} \operatorname*{Res}_{z_n} M &= \lim_{z \to z_n} \begin{bmatrix} \frac{1}{a(z)} j_-^{(1)} e^{ixz} & j_+^{(2)} e^{-ixz} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\lambda_n j_+^{(2)}(x,z_n) e^{ixz_n}}{a'(z_n)} & 0 \end{bmatrix} \\ &= \lim_{z \to z_n} M \begin{bmatrix} 0 & 0 \\ \frac{\lambda_n e^{2ixz}}{a'(z)} & 0 \end{bmatrix}. \end{aligned}$$

In the previous statement we have relied on the fact that $a'(z) \neq 0$, and hence that the zeros of $a(z)$ are simple. Furthermore, since $a^*(z_n^*) = 0$ we will also have eigenvalues $z_n^* \in \mathbb{C}^-$ with residues given by

$$\operatorname*{Res}_{z_n^*} M = \lim_{z \to z_n^*} M \begin{bmatrix} 0 & \frac{-\lambda_n^* e^{-2ixz}}{a'(z)^*} \\ & 0 \end{bmatrix}.$$

To determine the time-dependence of the λ_n we will refer to (1.59) and (1.48) and deduce that

$$\lambda(t) = \frac{\lambda_n(0)}{a'(z_n)} e^{2i(xz_n + z_n^2 t)} = c_n e^{2i(xz_n + z_n^2 t)}, \tag{1.60}$$

where $c_n = \frac{\lambda_n(0)}{a'(z_n)}$. Finally, to recover ϕ from the matrix M we note that M must satisfy

$$M_x = -iz[\sigma_3, M] + QM. \tag{1.61}$$

If we take the Laurent expansion of M given by $M = I + z^{-1}M_1 + \mathcal{O}(z^{-2})$, plug into (1.61), and equate terms that are $\mathcal{O}(1)$, then we see that

$$Q = i[\sigma_3, M_1] \tag{1.62}$$

or, specifically that

$$\begin{bmatrix} 0 & \phi \\ -\phi^* & 0 \end{bmatrix} = 2i \begin{bmatrix} 0 & (M_1)_{12} \\ -(M_1)_{21} & 0 \end{bmatrix}. \tag{1.63}$$

Thus we can recover the solution to the NLS from M by means of the relation

$$\phi(x,t) = \lim_{z \to \infty} 2iz(M_1(x,t))_{12}. \tag{1.64}$$

We now detail the Riemann-Hilbert Problem describing the matrix M and note that a solution will exist and will be unique under the assumptions that $r(z) \in H^{1,1}(\mathbb{R})$ and that the discrete scattering data will be simple in that $a(z)$ will only have simple zeros at each z_n. We will keep these assumptions for the rest of this dissertation. To justify the assumption on r, we note that in [21] it was shown that the scattering transform is a map from

$$H^{k,j} = \{f \in L^2(\mathbb{R}) : x^j f, \partial_x^k f \in L^2(\mathbb{R})\}$$

to $H^{k,j}$. To find a reasonable solution we must look at weak solutions to (1.1) wherein we define ϕ to be a weak solution in $H^{k,j}$ if the mapping $t \to \phi(t)$ given by

$$\phi(t) = e^{it\Delta}\phi_0 + i\int_0^t e^{i(t-s)\Delta} 2|\phi(s)|^2 \phi(s) ds, \tag{1.65}$$

is continuous from $t \geq 0$ to $H^{k,j}$. The notation Δ in (1.65) denotes the operation of the heat kernel $K(x,t)$ defined by

$$K(x,t) = \frac{1}{\sqrt{i\pi t}} e^{\frac{ix^2}{2t}} \tag{1.66}$$

so that we may define the convolution $e^{it\Delta} f = K(x,t) \star f = \dfrac{1}{\sqrt{i\pi t}} \displaystyle\int e^{\frac{i(x-x')^2}{2t}} f(x')dx'$. These weak solutions to (1.1) exist and are unique, therefore if we look at the space

$H^{1,1}(\mathbb{R})$, then this will be the largest space for which ϕ will exist and for which the associated reflection coefficient will also be in $H^{1,1}(\mathbb{R})$. Furthermore, we have that the function determined by the inverse scattering of r via (1.64) will be our function ϕ. Detailed results for the existence and uniqueness of this solution may be found in [6, 11, 20].

Riemann-Hilbert Problem 1.4. *Let $\mathcal{D}(t) = \left\{ r(z), \{z_n, c_n\}_{n=1}^N \right\}$ be the scattering data for the function $\phi(x,t)$. Then to solve the inverse scattering problem $\mathcal{S}^{-1} : \mathcal{D}(t) \to \phi(x,t)$, we seek a meromorphic matrix function M defined on $\{z \in \mathbb{C} \backslash (\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*)\}$ such that*

1. *$M(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.*
2. *For each $z \in \mathbb{R}$, M takes continuous boundary values $M_\pm(z) := \lim_{\epsilon \to 0^+} M(z \pm i\epsilon)$ which satisfy the jump relation $M_+(z) = M_-(z)V(z)$. The jump matrix $V(z)$ is given by*
$$V(z) = \begin{bmatrix} 1 + |r(z)|^2 & r^*(z)e^{-2it\theta(z)} \\ r(z)e^{2it\theta(z)} & 1 \end{bmatrix}, \tag{1.67}$$
where
$$\theta(x,z,t) = z^2 - 2\xi z = (z-\xi)^2 - \xi^2, \quad \xi = -x/(2t). \tag{1.68}$$
3. *$M(z)$ has simple poles at each z_n and z_n^* with residue conditions*
$$\begin{aligned} \operatorname*{Res}_{z_n} M &= \lim_{z \to z_n} M \begin{bmatrix} 0 & 0 \\ c_n e^{2it\theta} & 0 \end{bmatrix}, \\ \operatorname*{Res}_{z_n^*} M &= \lim_{z \to z_n^*} M \begin{bmatrix} 0 & -c_n^* e^{-2it\theta} \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{1.69}$$

2. Long-Time Asymptotics for the Focusing NLS

2.1. Introduction

The long-time asymptotic behavior of (1.1) will be the focus of the remaining chapters and has been previously studied in [3, 4, 5, 6, 7, 19]. For a generic initial condition ϕ_0 from an appropriate function space, the solution of (1.1) for t can generate solitary waves referred to as solitons. In the forward scattering data, these solutions correspond to the eigenvalues z_n for which $a(z_n) = 0$. This however is not the only option for such generic data, as there are solutions that will show no solitons. The soliton resolution conjecture is a statement that for generic, decaying initial data, for large time the solution to (1.1) will resolve towards a sum of solitons and some background radiation. This background radiation corresponds to the effect of the reflection coefficient from the scattering data on the long-time behavior of the solution. This dissertation will prove this conjecture for the defocusing NLS equation.

In order to determine the long-time asymptotic behavior, the process is to first transform the initial data into its scattering data and then apply the time-dependence of the scattering transform. This will generate a Riemann-Hilbert Problem which, by a series of transformations, will generate a solution that is the product of matrices with known asymptotic behavior which will then be extracted via (1.64) to determine the behavior of ϕ.

We first need to make clear what is meant by generic initial data. We impose the condition that ϕ_0 belong to the Sobolev space $H^{1,1}(\mathbb{R})$ defined by

$$H^{1,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : xf, f' \in L^2(\mathbb{R})\}. \tag{2.1}$$

Furthermore, we wish to only include initial data which will generate a finite number of simple discrete spectra and a reflection coefficient defined on $\mathbb{R}$ which does not contain singularities.

2.2. The First Transformation

If M is in fact the solution to RHP 1.4, then by the stated boundary condition $M \to I$ as $x \to \infty$ for a fixed value of t. We wish to establish the asymptotic behavior of M along an arbitrary characteristic direction $x = vt$ to a sufficient level of precision that we can extract the asymptotic behavior of the solution to (1.1) for large t via (1.64). If we let $\xi = -\dfrac{x}{2t}$ be fixed, then we are essentially fixing a trajectory along which we will be observing the large t behavior. We observe that for certain values of z_n the residue conditions (1.69) will be unbounded in time. The aim of the first transformation is to create a new RHP that will have residue conditions that are bounded in time.

Let us partition the discrete spectra $\{z_n\}$ so that

$$
\begin{aligned}
D_\xi^- &= \{n \in \{0,1,\dots,N\} \,:\, \mathfrak{Re}(z_n) < \xi\}, \\
D_\xi^+ &= \{n \in \{0,1,\dots,N\} \,:\, \mathfrak{Re}(z_n) \geq \xi\}.
\end{aligned} \tag{2.2}
$$

Thus, if $n \in D_\xi^-$, then $\mathfrak{Im}(\theta(x,z_n,t)) = 2(\mathfrak{Re}(z_n) - \xi)\mathfrak{Im}(z_n) < 0$ and the residue of M at z_n will grow with t. In the same way, if $n \in D_\xi^+$ then the residue at z_n will remain bounded as $t \to \infty$. As our first step towards transforming M let us define the function

$$
\begin{aligned}
T(z) = T(z,\xi) &= \prod_{n \in D_\xi^-} \left(\frac{z - z_n^*}{z - z_n} \right) \exp\left(i \int_{-\infty}^{\xi} \frac{\kappa(s)}{s - z} ds \right), \\
\kappa(s) &= -\frac{1}{2\pi} \log(1 + |r(s)|^2).
\end{aligned} \tag{2.3}
$$

Proposition 1. *The function $T(z)$ as defined by (2.3) has the following properties:*

1. *$T(z)$ is meromorphic on the set $\mathbb{C} \setminus (-\infty, \xi)$ with simples poles at $\{z_n : n \in D_\xi^-\}$ and simple zeros at $\{z_n^* : n \in D_\xi^-\}$*

2. *For $z \in (-\infty, \xi)$ the boundary values $T_\pm(z) := \lim_{\epsilon \to 0^+} T(z \pm i\epsilon)$ satisfy the jump condition*
$$
T_+(z)/T_-(z) = 1 + |r(z)|^2, \quad z \in (-\infty, \xi). \tag{2.4}
$$

3. *As $|z| \to \infty$ with $|\arg(z)| \neq \pi$,*
$$
T(z) = 1 + \frac{i}{z} \left(2 \sum_{n \in D_\xi^-} \mathfrak{Im}(z_n) - \frac{1}{2\pi} \int_{-\infty}^{\xi} \log(1 + |r(s)|^2) ds \right) + \mathcal{O}(z^{-2}).
$$

4. *As $z \to \xi$ along any ray $\xi + e^{i\phi}u$ with $u \in \mathbb{R}^+$ and $|\arg \phi| < \pi$*
$$
\left| T(z,\xi) - T_0(\xi)(z - \xi)^{i\kappa(\xi)} \right| \leq C \|r\|_{H^1(\mathbb{R})} |z - \xi|^{1/2} \tag{2.5}
$$
where $T_0(\xi)$ is defined by
$$
T_0(\xi) = \prod_{n \in D_\xi^-} \left(\frac{\xi - z_n^*}{\xi - z_n} \right) e^{i\beta(\xi,\xi)} = \exp\left[i \left(\beta(\xi,\xi) - 2 \sum_{n \in D_\xi^-} \arg(\xi - z_n) \right) \right],
$$
and
$$
\beta(z,\xi) = -\kappa(\xi) \log(z - \xi + 1) + \int_{-\infty}^{\xi} \frac{\kappa(s) - \chi(s)\kappa(\xi)}{s - z} ds
$$

where $\chi(s)$ is the usual characteristic function of the interval $(\xi-1,\xi)$ and the logarithm has its principal branch on $(-\infty,\xi-1]$.

Proof. Part 1. follows from the fact that the exponential part of T is analytic for $z\in\mathbb{C}\setminus(-\infty,\xi)$. For Part 2. we use the Sokhotski-Plemelj formula, which states that if

$$f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(t)}{s-z}ds, \tag{2.6}$$

where φ satisfies the Hölder condition on Γ

$$|\varphi(t)-\varphi(s)|\le\delta|t-s|^\lambda \tag{2.7}$$

with $\delta>0$, $0<\lambda\le 1$, then for $z\in\Gamma$ (not an endpoint) we have the equation $f_+-f_-=\varphi$. Thus, if we let $y_\pm=\log(T_\pm)$, then $y_+-y_-=\log(1+|r(s)|^2)$ and the result follows by exponentiation of y. Part 3. follows if we use the expansions

$$\frac{z-z_n^*}{z-z_n}=1+\frac{i}{z}(2\mathfrak{Im}(z_n))+\mathcal{O}(z^{-2})$$

and

$$\exp\left(i\int_{-\infty}^{\xi}\frac{\kappa(s)}{s-z}ds\right)=1-\frac{i}{z}\int_{-\infty}^{\xi}\kappa(s)ds+\mathcal{O}(z^{-2}).$$

Finally, for Part 4. we use the expression

$$T(z,\xi)=\prod_{n\in D_\xi^-}\left(\frac{z-z_n^*}{z-z_n}\right)(z-\xi)^{i\kappa(\xi)}\exp\left(i\beta(z,\xi)\right), \tag{2.8}$$

and see that

$$\left|(z-\xi)^{i\kappa(\xi)}\right|\le\left|\exp\left(i\log(\xi+ue^{i\phi}-\xi)\right|^{\kappa(\xi)}\le e^{-\pi\kappa(\xi)}=\sqrt{1+|r(\xi)|^2} \tag{2.9}$$

Then we use Lemma 23.3 of [2] to conclude that

$$|\beta(z,\xi)-\beta(\xi,\xi)|\le C\|r\|_{H^1(\mathbb{R})}|z-\xi|^{1/2}. \tag{2.10}$$

□

We now define a new function $M^{(1)}$ based on the transformation

$$M^{(1)}(z)=M(z)T(z)^{-\sigma_3}. \tag{2.11}$$

The function $M^{(1)}$ defined by (2.11) now satisfies a new Riemann-Hilbert problem, where we make note that the jump matrices defined by (2.12) are factored in such a way as to isolate terms that will decay in regions of the complex plane as z is moved off of the real line.

Riemann-Hilbert Problem 2.2. *Let $M^{(1)}$ be a meromorphic matrix function defined on $\{z \in \mathbb{C}\backslash(\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*)\}$ such that:*

1. $M^{(1)}(z) = I + \mathcal{O}(z^{-1})$ *as* $z \to \infty$.

2. *For each* $z \in \mathbb{R}$, *the boundary values* $M_\pm^{(1)}(z)$ *satisfy the jump relation* $M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z)$ *where*

$$V^{(1)}(z) = \begin{cases} \begin{bmatrix} 1 & r^*(z)T(z)^2e^{-2it\theta} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r(z)T(z)^{-2}e^{2it\theta} & 1 \end{bmatrix} & z \in (\xi, \infty) \\ \begin{bmatrix} 1 & 0 \\ \frac{r(z)T_-(z)^{-2}}{1+|r(z)|^2}e^{2it\theta} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{r^*(z)T_+(z)^2}{1+|r(z)|^2}e^{-2it\theta} \\ 0 & 1 \end{bmatrix} & z \in (-\infty, \xi) \end{cases} \tag{2.12}$$

3. $M^{(1)}(z)$ *has simple poles at each* $z_n \in \mathcal{Z}$ *and* $z_n^* \in \mathcal{Z}^*$ *with residue conditions*

$$\begin{aligned} \operatorname*{Res}_{z_n} M^{(1)} &= \begin{cases} \lim_{z \to z_n} M^{(1)} \begin{bmatrix} 0 & c_n^{-1}(1/T)'(z_n)^{-2}e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & n \in D_\xi^- \\ \lim_{z \to z_n} M^{(1)} \begin{bmatrix} 0 & 0 \\ c_nT(z_n)^{-2}e^{2it\theta} & 0 \end{bmatrix} & n \in D_\xi^+ \end{cases} \\ \operatorname*{Res}_{z_n^*} M^{(1)} &= \begin{cases} \lim_{z \to z_n^*} M^{(1)} \begin{bmatrix} 0 & 0 \\ -(c_n^*)^{-1}T'(z_n^*)^{-2}e^{2it\theta} & 0 \end{bmatrix} & n \in D_\xi^- \\ \lim_{z \to z_n^*} M^{(1)} \begin{bmatrix} 0 & -c_n^*T(z_n^*)^2e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & n \in D_\xi^+ \end{cases} \end{aligned} \tag{2.13}$$

If we now look at the residue conditions for $M^{(1)}$ in (2.13) we see that all of the exponential terms will decay or remain bounded as $t \to \infty$.

2.3. The $\bar{\partial}$-Riemann-Hilbert Problem

The next step is to introduce a new transformation that will remove the jump across the real line. This will be done by defining extensions of $r(z)$ off of $\mathbb{R}$. The primary effect of this process will be to create new jumps which will correspond to jump matrices that have entries which are asymptotically close to I as $t \to \infty$. However, the transformation will result in matrices that involve non-analytic entries and consequently we will have to introduce a $\bar{\partial}$ component to the new RHP. This does provide a great advantage. In [6] a method of steepest-descent was developed to determine the long-time asymptotic behavior of the defocusing NLS. In order to move the jump condition off of the real line via transformation, the authors had to approximate r

using rational functions. These approximations were used to create a new RHP and it was necessary to prove that the solution to the new problem was asymptotically close to the solution of the original RHP involving only a jump across R. The use of non-analytic deformations and $\bar{\partial}$ problems presented here allows us to avoid the more complicated approximation issues that occur in using the steepest-descent method to solve RHP 2.2. This procedure of asymptotic analysis of Riemann-Hilbert problems by using $\bar{\partial}$ methods was developed in [12, 13].

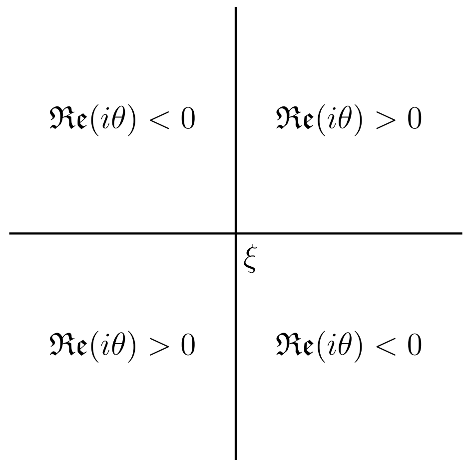

Figure 2.1: The signature table for the real part of $i\theta$.

First, we must detail the new contours that will define our new jump conditions. From the signature table given in Figure 2.1, we note that the real part of $i\theta$ will have a different sign in each of the four regions determined by ξ and the real line. Our goal is to define a new problem so that the new jumps and contours will decay in time based upon the region. Let our contours be denoted by

$$\Sigma_k = \xi + e^{i(2k-1)\pi/4}u, \quad u \in \mathbb{R}^+, \quad k = 1, 2, 3, 4, \tag{2.14}$$

oriented with increasing real part. These contours create six open sectors in $\mathbb{C}$, separated by $\mathbb{R}$ and the Σ_k, $k = 1, \ldots, 4$ which we shall denote by Ω_k, $k = 1, \ldots, 6$ as in Figure 2.2. Additionally, let

$$\mu = \operatorname{dist}(\mathcal{Z}, \mathbb{R}) \tag{2.15}$$

be the minimal distance from the discrete spectrum to the real axis, and

$$\rho = \min \left\{ \min_{\substack{n,k \in \mathcal{Z} \\ n \neq k}} |z_n - z_k|_\infty, \ \mu \right\} \tag{2.16}$$

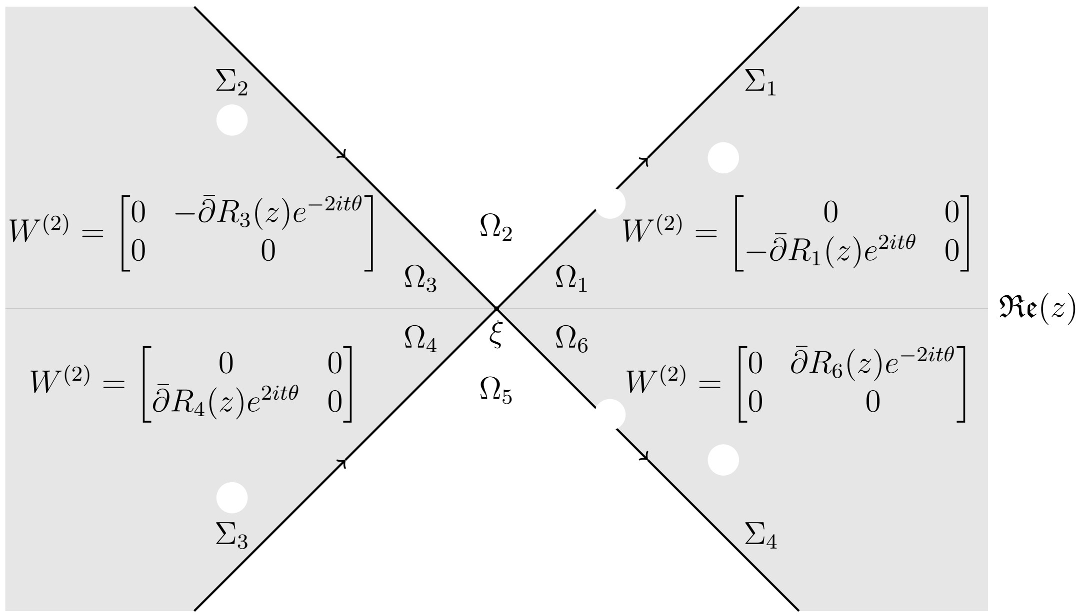

Figure 2.2: The contours Σ_k and regions Ω_k $k = 1, \ldots, 6$ defining the $\bar{\partial}$-relationship for the matrix $M^{(2)}$. The support of $\bar{\partial}M^{(2)}$, is shaded in gray.

the minimum of μ and the smallest ∞-norm distance between points of discrete spectra. Let the indicator $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C}, [0,1])$ be supported near the discrete spectrum as follows:

$$\chi_{\mathcal{Z}}(z) = \begin{cases} 1 & \operatorname{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) < \mu/3 \\ 0 & \operatorname{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) > 2\mu/3 \end{cases}. \tag{2.17}$$

With $\chi_{\mathcal{Z}}$ defined this way, we will be able to create extensions of $r(z)$ that are zero near discrete spectra, thus preserving the current residue conditions. We now state properties of our extensions as follows:

Proposition 2. *There exist functions $R_j : \overline{\Omega}_j \to \mathbb{C}$, $j = 1, 3, 4, 6$, with boundary* *values satisfying*

$$R_1(z) = \begin{cases} r(z)T(z)^{-2} & z \in (\xi, \infty) \\ r(\xi)T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_1 \end{cases}$$

$$R_3(z) = \begin{cases} \frac{r(z)^*}{1+|r(z)|^2} T_+(z)^2 & z \in (-\infty, \xi) \\ \frac{r(\xi)^*}{1+|r(\xi)|^2} T_0(\xi)^2 (z-\xi)^{2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_2 \end{cases}$$

$$R_4(z) = \begin{cases} \frac{r(z)}{1+|r(z)|^2} T_-(z)^{-2} & z \in (-\infty, \xi) \\ \frac{r(\xi)}{1+|r(\xi)|^2} T_0(\xi)^{-2} (z-\xi)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_3 \end{cases}$$

$$R_6(z) = \begin{cases} r(z)^*T(z)^2 & z \in (\xi, \infty) \\ r(\xi)^*T_0(\xi)^2(z-\xi)^{2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_4 \end{cases}$$

such that for a fixed constant $c_1 = c_1(\psi_0)$*, and an indicator function* $\chi_{\mathcal{Z}}$ *satisfying* (2.17) *we have*

$$\begin{gathered} \left|\bar{\partial} R_j(z)\right| \le c_1 \chi_{\mathcal{Z}}(z) + c_1 \left|r'\left(\mathfrak{Re}(z)\right)\right| + c_1 |z-\xi|^{-1/2}, \\ \bar{\partial} R_j(z) = 0 \qquad \text{if } dist(z, \mathcal{Z} \cup \mathcal{Z}^*) \le \mu/3. \end{gathered} \tag{2.18}$$

Additionally, the extension can be made so as to preserve the symmetry $R(z^*)^* = R(z)$*.*

The existence of these extensions is can be verified from the following definitions: Let

$$\begin{aligned} f_1(z) &= r(\xi)T^2(z)T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)} && z \in \overline{\Omega}_1 \\ f_3(z) &= \frac{r(\xi)^*}{1+|r(\xi)|^2} T(z)^2 T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)} && z \in \overline{\Omega}_3, \end{aligned}$$

then define, for $z \in \overline{\Omega}_j$, $j = 1, 3$, the extensions

$$\begin{aligned} R_1(z) &= \left[f_1(z) + \left(r(\mathfrak{Re}(z)) - f_1(z)\right)\cos(2\phi)\right] T(z)^{-2}(1-\chi_{\mathcal{Z}}(z)), \\ R_3(z) &= \left[f_3(z) + \left(\frac{r(\mathfrak{Re}(z))^*}{1+|r(\mathfrak{Re}(z))|^2} - f_3(z)\right)\cos(2\phi)\right] T(z)^2(1-\chi_{\mathcal{Z}}(z)). \end{aligned}$$

R_4 and R_6 can be defined using $z \in \mathbb{C} \backslash (-\infty, \xi]$, $T(z^*)^* = 1/T(z)$ and choosing $\chi_{\mathcal{Z}}(z)$ to respect Schwartz symmetry, then let $R_4 = R_3(z^*)^*$ and $R_6(z) = R_1(z^*)^*$. Additionally, we see that if $\text{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) \le \mu/3$, then $(1-\chi_{\mathcal{Z}}(z)) = 0$ and consequently,

$\bar{\partial}R_j(z) = 0$. We now can define a new matrix function $M^{(2)}$ as

$$M^{(2)}(z) = \begin{cases} M^{(1)}(z)\begin{bmatrix} 1 & 0 \\ -R_1(z)e^{2it\theta} & 1 \end{bmatrix} & z \in \Omega_1 \\ M^{(1)}(z)\begin{bmatrix} 1 & -R_3(z)e^{-2it\theta} \\ 0 & 1 \end{bmatrix} & z \in \Omega_3 \\ M^{(1)}(z)\begin{bmatrix} 1 & 0 \\ R_4(z)e^{2it\theta} & 1 \end{bmatrix} & z \in \Omega_4 \\ M^{(1)}(z)\begin{bmatrix} 1 & R_6(z)e^{-2it\theta} \\ 0 & 1 \end{bmatrix} & z \in \Omega_6 \\ M^{(1)}(z) & z \in \Omega_2 \cup \Omega_5 \end{cases} \tag{2.19}$$

and this function will satisfy a new $\bar{\partial}$-RHP.

$\bar{\partial}$-Riemann-Hilbert Problem 2.3. *Find a function $M^{(2)}$ defined on $\mathbb{C}\backslash(\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$, where $\Sigma^{(2)} = \bigcup_{j=1}^{4} \Sigma_j$, with the following properties:*

1. *$M^{(2)}$ has continuous first partial derivatives in $\mathbb{C}\backslash(\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$*
2. *$M^{(2)}(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.*
3. *For $z \in \Sigma^{(2)}$, we have the jump relation $M_+^{(2)}(z) = M_-^{(2)}(z)V^{(2)}(z)$, where*

$$V^{(2)}(z) = I + (1 - \chi_{\mathcal{Z}}(z))\delta V^{(2)}(z),$$

$$\delta V^{(2)}(z) = \begin{cases} \begin{bmatrix} 0 & 0 \\ r(\xi)T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0 \end{bmatrix} & z \in \Sigma_1 \\ \begin{bmatrix} 0 & \frac{r(\xi)^*T_0(\xi)^2}{1+|r(\xi)|^2}(z-\xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & z \in \Sigma_2 \\ \begin{bmatrix} 0 & 0 \\ \frac{r(\xi)T_0^{-2}(\xi)}{1+|r(\xi)|^2}(z-\xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0 \end{bmatrix} & z \in \Sigma_3 \\ \begin{bmatrix} 0 & r(\xi)^*T_0(\xi)^2(z-\xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & z \in \Sigma_4 \end{cases} \tag{2.20}$$

4. *For $z \in \mathbb{C}$ we have*

$$\bar{\partial}M^{(2)}(z) = M^{(2)}(z)W^{(2)}(z) \tag{2.21}$$

where

$$W^{(2)}(z) = \begin{cases} \begin{bmatrix} 0 & 0 \\ -\bar{\partial}R_1(z)e^{2it\theta} & 0 \end{bmatrix} & z \in \Omega_1 \\ \begin{bmatrix} 0 & -\bar{\partial}R_3(z)e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & z \in \Omega_3 \\ \begin{bmatrix} 0 & 0 \\ \bar{\partial}R_4(z)e^{2it\theta} & 0 \end{bmatrix} & z \in \Omega_4 \\ \begin{bmatrix} 0 & \bar{\partial}R_6(z)e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & z \in \Omega_6 \\ \boldsymbol{0} & z \in \Omega_2 \cup \Omega_5 \end{cases} \tag{2.22}$$

5. *$M^{(2)}(z)$ has simple poles at each $z_n \in \mathcal{Z}$ and $z_n^* \in \mathcal{Z}^*$ with residue conditions*

$$\begin{aligned} \operatorname*{Res}_{z_n} M^{(2)} &= \begin{cases} \lim\limits_{z \to z_n} M^{(2)} \begin{bmatrix} 0 & c_n^{-1}(1/T)'(z_n)^{-2}e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & n \in D_\xi^- \\ \lim\limits_{z \to z_n} M^{(2)} \begin{bmatrix} 0 & 0 \\ c_n T(z_n)^{-2}e^{2it\theta} & 0 \end{bmatrix} & n \in D_\xi^+ \end{cases} \\ \operatorname*{Res}_{z_n^*} M^{(2)} &= \begin{cases} \lim\limits_{z \to z_n^*} M^{(2)} \begin{bmatrix} 0 & 0 \\ -(c_n^*)^{-1}T'(z_n^*)^{-2}e^{2it\theta} & 0 \end{bmatrix} & n \in D_\xi^- \\ \lim\limits_{z \to z_n^*} M^{(2)} \begin{bmatrix} 0 & -c_n^* T(z_n^*)^2 e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & n \in D_\xi^+ \end{cases} \end{aligned} \tag{2.23}$$

Here we note that the residue conditions of $\bar{\partial}$-Riemann-Hilbert Problem 2.3 are unchanged since $R_i \equiv 0$ near each pole and by (2.19), $M^{(2)} = M^{(1)}$.

2.4. Removing the Riemann-Hilbert Component of the $\bar{\partial}$-Problem

In this section, we will build a pure $\bar{\partial}$-problem from $\bar{\partial}$-Riemann-Hilbert Problem 2.3, so that we can focus on the Riemann-Hilbert component separately. This pure $\bar{\partial}$-problem will not have any residue or jump conditions. Let us call the solution to the Riemann-Hilbert part of this problem $M^{(2)}_{RHP}$. That is to say, let $M^{(2)}_{RHP}$ be the solution to $\bar{\partial}$-Riemann-Hilbert Problem 2.3 with $W^{(2)} = 0$ in (2.21). In building a solution $M^{(2)}_{RHP}$, we must deal both with the residues conditions and the jump condition detailed in $\bar{\partial}$-Riemann-Hilbert Problem 2.3, Part 5. and Part 3. respectively. This solution will be of the form

$$M^{(2)}_{RHP}(z) = \begin{cases} E(z)M^{(out)}(z) & |z-\xi| > \mu/2 \\ E(z)M^{(\xi)}(z) & |z-\xi| < \mu/2. \end{cases} \tag{2.24}$$

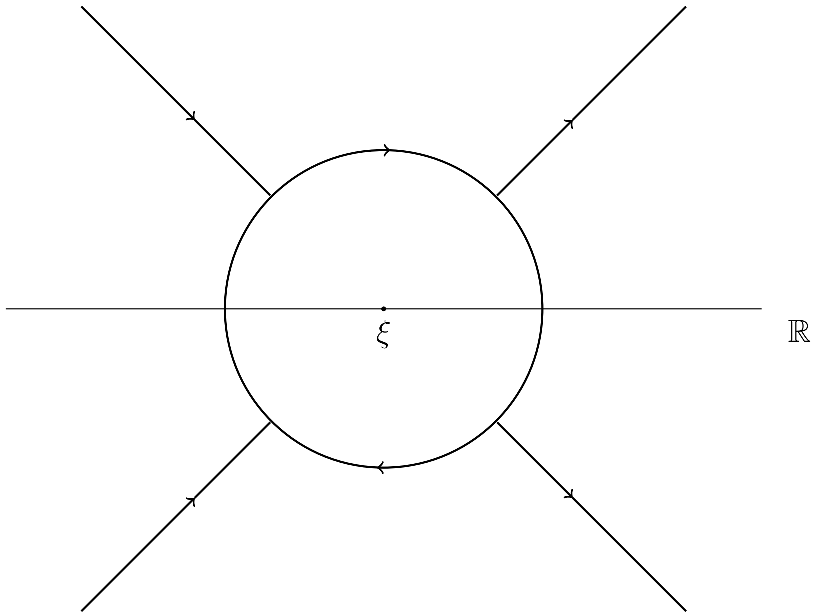

Figure 2.3: The contour $\Sigma^{(E)} = \partial\mathcal{U}_\xi \cup (\Sigma^{(2)} \backslash \partial\mathcal{U}_\xi)$, where the radius of region $\partial\mathcal{U}_\xi$ is $\mu/2$.

Here the intent is to create matrix valued functions $M^{(out)}(z)$, $M^{(\xi)}(z)$, and $E(z)$ in such a way that when we multiply $M^{(2)}$ by the inverse of $M^{(2)}_{RHP}$, we will have a new matrix that will be defined by a pure $\bar{\partial}$-problem. $M^{(out)}(z)$ will be formed in a way that will remove the residue conditions. $M^{(\xi)}(z)$ will be built out of a RHP that only has a jump condition across $\Sigma^{(2)}$. Finally, $E(z)$ will the solution of a small norm Riemann-Hilbert problem that will, in a way, piece together the functions $M^{(out)}(z)$and $M^{(\xi)}(z)$ without making a significant contribution to the overall asymptotic behavior of the problem. We will prove $E(z)$ exists and bound it asymptotically so that we may determine the contribution that this new transformation will make to the long-time asymptotic behavior of our original function M. Let us now define the region $\mathcal{U}_\xi$ such that

$$\mathcal{U}_\xi = \{z \,:\, |z - \xi| < \mu/2\}. \tag{2.25}$$

The boundary of this region, $\partial\mathcal{U}_\xi$ will determine a new contour to be used in our new RHP, and we shall assign a clockwise orientation to $\partial\mathcal{U}_\xi$. Let us denote this new contour (see Figure 2.3) by

$$\Sigma^{(E)} = \partial\mathcal{U}_\xi \cup (\Sigma^{(2)} \backslash \partial\mathcal{U}_\xi). \tag{2.26}$$

2.4.1. The N-Soliton Outer Problem

Outside of $\mathcal{U}_\xi$ we note that $V^{(2)}$ satisfies the bound

$$\|V^{(2)} - I\|_{L^\infty(\Sigma^{(2)}\setminus\partial\mathcal{U}_\xi)} = \mathcal{O}(\rho^{-2}e^{-\sqrt{2}t|z-\xi|^2}), \tag{2.27}$$

where ρ is given by (2.16), so that for large t, the jump matrix is close to identity and we seek a solution to a RHP of purely discrete spectra.

The N-Soliton Problem 2.4.1. *There exists a meromorphic function $M^{(out)}$ that satisfies*

- $M^{(out)}(z) = I + \mathcal{O}(z^{-1})$ *as* $z \to \infty$.
- $M^{(out)}$ *has simple poles at each* $z_k \in \mathcal{Z}$ *and* $z_k^* \in \mathcal{Z}^*$ *satisfying the residue conditions*

$$\begin{aligned}
\operatorname*{Res}_{z_n} M^{(out)} &= \begin{cases} \lim\limits_{z\to z_n} M^{(out)} \begin{bmatrix} 0 & c_n^{-1}(1/T)'(z_n)^{-2}e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & n \in D_\xi^- \\ \lim\limits_{z\to z_n} M^{(out)} \begin{bmatrix} 0 & 0 \\ c_n T(z_n)^{-2}e^{2it\theta} & 0 \end{bmatrix} & n \in D_\xi^+ \end{cases} \\
\operatorname*{Res}_{z_n^*} M^{(out)} &= \begin{cases} \lim\limits_{z\to z_n^*} M^{(out)} \begin{bmatrix} 0 & 0 \\ -(c_n^*)^{-1}T'(z_n^*)^{-2}e^{2it\theta} & 0 \end{bmatrix} & n \in D_\xi^- \\ \lim\limits_{z\to z_n^*} M^{(out)} \begin{bmatrix} 0 & -c_n^* T(z_n^*)^2 e^{-2it\theta} \\ 0 & 0 \end{bmatrix} & n \in D_\xi^+ \end{cases}
\end{aligned} \tag{2.28}$$

In order to find this solution, we simply find the solution to the purely N-Soliton problem for the defocusing NLS that corresponds to the adjusted discrete scattering data $\sigma_d^{(out)} := \{z_k, \widetilde{c}_k(\xi)\}_{k=1}^N$ where

$$\widetilde{c}_k(\xi) = c_k \exp\left(\frac{i}{\pi}\int_{-\infty}^{\xi} \frac{\log(1+|r(s)|^2)}{s - z_k} ds\right), \tag{2.29}$$

and $\{z_k, c_k\}_{k=1}^N$ is the original discrete scattering data from RHP (1.4). We will denote the solution to N-Soliton problem (2.4.1) by $\phi(x,t;\sigma_d^{(out)}) = \phi_{out}$, and note that proof of the existence of this solution can be found in the appendices of the paper found in the Appendix. The solution to this problem will be of the form

$$M^{(out)}(z) = I + \frac{1}{2iz}\begin{bmatrix} m_{11}^1 & \phi_{out} \\ \phi_{out}^* & m_{22}^1 \end{bmatrix} + \mathcal{O}(z^{-2}), \tag{2.30}$$

where $m_{11}^1 = -\int_x^\infty |\phi_{out}|^2 ds + \sum\limits_{k\in D_\xi^-} 4\mathfrak{Im}(z_n)$ and $m_{22}^1 = -m_{11}^1$.

2.4.2. The Inner Problem and the Parabolic Cylinder Problem

For the inner problem where $z \in \mathcal{U}_\xi$, we seek a function that solves a RHP having jumps identical to those defined by (2.20). Such a problem exists and is referred to as the Parabolic Cylinder Problem. Details of this derivation can be found in [10] and in the paper included in the Appendix.

The Parabolic Cylinder Problem 2.4.2. *For a parameter r_0 such that $|r_0| < 1$ there exists a matrix function $P(\zeta)$ that satisfies:*

1. *P is analytic for $\zeta \in \mathbb{C} \setminus \Sigma^{(2)}$.*

2. *$P(\zeta) = I + \zeta^{-1} P_1^\infty + \mathcal{O}(\zeta^{-2})$ as $\zeta \to \infty$.*

3. *For $\zeta \in \Sigma^{(2)}$, we have the jump relation $P_+(\zeta) = P_-(\zeta) V^{(P)}(\zeta)$, where*

$$V^{(P)}(\zeta) = \begin{cases} \begin{bmatrix} 1 & 0 \\ r_0 \zeta^{-2i\kappa} e^{i\zeta^2/2} & 1 \end{bmatrix} & \zeta \in \Sigma_1 \\ \begin{bmatrix} 1 & \frac{r_0^*}{1+|r_0|^2} \zeta^{2i\kappa} e^{-i\zeta^2/2} \\ 0 & 1 \end{bmatrix} & \zeta \in \Sigma_2 \\ \begin{bmatrix} 1 & 0 \\ \frac{r_0}{1+|r_0|^2} \zeta^{-2i\kappa} e^{i\zeta^2/2} & 1 \end{bmatrix} & \zeta \in \Sigma_3 \\ \begin{bmatrix} 1 & r_0^* \zeta^{2i\kappa} e^{-i\zeta^2/2} \\ 0 & 1 \end{bmatrix} & \zeta \in \Sigma_4. \end{cases} \tag{2.31}$$

Through matrix manipulation, the entries of the solution to the RHP can be related to the solution of the parabolic cylinder equation

$$\frac{d^2}{d\zeta^2} D_a(\zeta) + \left(\frac{1}{2} - \frac{\zeta^2}{4} + a \right) D_a(\zeta) = 0 \tag{2.32}$$

where a is a fixed complex number as can be seen in the Appendix. The information that we seek from this problem is in the entries of the matrix P_1^∞ which are given by

$$P_1^\infty = \begin{bmatrix} 0 & -i\beta_{12} \\ i\beta_{21} & 0 \end{bmatrix}, \tag{2.33}$$

where

$$\beta_{12}(r_0) = \frac{(2\pi)^{1/2} e^{i\pi/4} e^{-\pi\kappa/2}}{r_0 \Gamma(-i\kappa)}, \qquad \beta_{21}(r_0) = \frac{\kappa r_0 \Gamma(-i\kappa)}{(2\pi)^{1/2} e^{i\pi/4} e^{-\pi\kappa/2}} = \frac{\kappa}{\beta_{12}}. \tag{2.34}$$

If we let
$$\zeta = \zeta(z) = 2\sqrt{t}(z - \xi) \tag{2.35}$$
and
$$r_0 = r(\xi)T_0(\xi)^{-2}e^{2i(\kappa(\xi)\log(2\sqrt{t})-t\xi^2)}, \tag{2.36}$$
then we have exactly the Parabolic Cylinder problem since $|r_0| = |r(\xi)|$. Now since $M^{(out)}$ is analytic and bounded inside of $\mathcal{U}_\xi$, then if we define
$$M^{(\xi)}(z) = M^{(out)}(z)P(\zeta(z), r_0) \qquad z \in \mathcal{U}_\xi, \tag{2.37}$$
then $M^{(\xi)}$ will satisfy the jump condition (2.20) in this region.

2.4.3. The Riemann-Hilbert Problem for $E(z)$

Before we can move on to a pure $\bar{\partial}$-problem, we must first ensure that if $M^{(out)}(z)$ and $M^{(\xi)}(z)$ are defined as above, then the function $E(z)$ determined from (2.24) exists and is well-defined. This function will have a jump condition across (2.26) where the circular portion $\partial\mathcal{U}_\xi$ is given a clockwise orientation. It follows from (2.24) that E will satisfy the following Riemann-Hilbert Problem.

The Small-Norm Riemann-Hilbert Problem 2.4.3. *There exists a holomorphic function E defined on $\mathbb{C}\backslash\Sigma^{(E)}$ that has the properties:*

1. $E(z) = I + \mathcal{O}(z^{-1})$ *as* $z \to \infty$.
2. *For each* $z \in \Sigma^{(E)}$ *the boundary values* $E_\pm(z)$ *will satisfy the jump condition* $E_+(z) = E_-(z)V^{(E)}(z)$ *where*

$$V^{(E)}(z) = \begin{cases} M^{(out)}(z)V^{(2)}(z)M^{(out)}(z)^{-1} & z \in \Sigma^{(2)}\backslash\mathcal{U}_\xi \\ M^{(out)}(z)P(\zeta(z), r_\xi)M^{(out)}(z)^{-1} & z \in \partial\mathcal{U}_\xi. \end{cases} \tag{2.38}$$

The existence of a solution to this problem follows from (2.27) and from the fact that for $z \in \partial\mathcal{U}_\xi$ and from relations (5.14) and (5.15) in the Appendix. Specifically, $M^{(out)}$ will be bounded so that $|V^{(E)} - I| = |M^{(out)}||P(\zeta(z), r_\xi) - I||M^{(out)}(z)^{-1}| = \mathcal{O}(\zeta^{-1}) = \mathcal{O}(t^{-1/2})$. Then by the established theory for the existence of small norm Riemann-Hilbert problems (see [3],[6] Section 2), we can deduce that E does exist.

The intuition for why this is true is as follows. We consider a problem similar to RHP (2.4.3) where we replace the jump matrix with the identity matrix I. In such a case, we are looking for a solution that will be an entire function with asymptotic behavior $I + \mathcal{O}(z^{-1})$ as $z \to \infty$. By Liouville's theorem we can conclude that the solution must by I. Returning to RHP (2.4.3), we are looking at a problem that has a jump condition that is asymptotically close to I as $t \to \infty$, so the underlying principal of small norm Riemann-Hilbert problems is that the solution E should be asymptotically close to I for large t.

From the small norm existence theory we conclude not only that E exists but also that E is a small perturbation of the identity matrix. E will have the form

$$E(z) = I + z^{-1}E_1 + \mathcal{O}(z^{-2}) \tag{2.39}$$

where we will need the long-time asymptotic behavior of E_1 for our original problem. From definition (2.24) and (2.4.3) we can determine this long-time behavior to be

$$E_1 = \frac{t^{-1/2}}{2} M^{(out)}(\xi) \begin{bmatrix} 0 & -i\beta_{12}(r_\xi) \\ i\beta_{21}(r_\xi) & 0 \end{bmatrix} M^{(out)}(\xi)^{-1} + \mathcal{O}(t^{-1}). \tag{2.40}$$

2.4.4. The Pure $\bar{\partial}$-Problem

If we define now

$$M^{(3)}(z) := M^{(2)}(z)M^{(2)}_{RHP}(z)^{-1}, \tag{2.41}$$

then $M^{(3)}$ will satisfy the pure $\bar{\partial}$-problem below.

The $\bar{\partial}$-Problem 2.4.4. *Find a function $M^{(3)}$ with the following properties.*

1. *$M^{(3)}$ has continuous first partial derivatives in $\mathbb{C}$.*
2. *$M^{(3)}(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.*
3. *For $z \in \mathbb{C}$, we have*

$$\bar{\partial}M^{(3)}(z) = M^{(3)}(z)W^{(3)} \tag{2.42}$$

where we define $W^{(3)} = M^{(2)}_{RHP}(z)W^{(2)}(z)M^{(2)}_{RHP}(z)^{-1}$.

This results from the fact that, by construction, $M^{(2)}_{RHP}(z)$ is meromorphic and so

$$\begin{aligned} \bar{\partial}M^{(3)}(z) &= \bar{\partial}M^{(2)}(z)M^{(2)}_{RHP}(z)^{-1} + M^{(2)}(z)\bar{\partial}M^{(2)}_{RHP}(z)^{-1} \\ &= M^{(2)}W^{(2)}(z)M^{(2)}_{RHP}(z)^{-1} + 0 \\ &= M^{(3)}M^{(2)}_{RHP}(z)W^{(2)}(z)M^{(2)}_{RHP}(z)^{-1}. \end{aligned}$$

2.5. Long-Time Asymptotics of the $\bar{\partial}$-Problem

The primary benefit of $\bar{\partial}$-Problem 2.4.4 is that it can be reformulated as the integral equation

$$M^{(3)}(z) = I + \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial}M^{(3)}(s)}{s-z}dA(s) = I + \frac{1}{\pi}\int_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}}{s-z}dA(s), \tag{2.43}$$

where $s = u + iv$. Alternatively, we can write (2.43) in operator notation using the solid Cauchy operator

$$\mathbb{S}[f] = \frac{1}{\pi}\int_{\mathbb{C}} \frac{fW^{(3)}}{s-z} dA(s), \tag{2.44}$$

so that

$$(I - \mathbb{S})[M^{(3)}(z)] = I. \tag{2.45}$$

Equation (2.43) results from the facts that the Cauchy kernel $k(z) = p.v.\frac{1}{z}$ is a fundamental solution of the $\bar{\partial}$-operator and $\bar{\partial}u = f$ if and only if $u = \mathbb{S}[f]$ (see [12, 15]). We show in the Appendix that $\mathbb{S}$ is bounded such that

$$\|\mathbb{S}\|_{L^\infty \to L^\infty} \leq Ct^{-1/4}, \tag{2.46}$$

where we have relied on the fact that $M^{(2)}_{RHP}$ is uniformly bounded in the plane. Specifically, let us assume without loss of generality that $f \in L^\infty(\Omega_1)$ is supported inside Ω_1. Then from (2.22) we see that

$$\begin{aligned} |\mathbb{S}[f]| &\leq \int\int_{\Omega_1} \frac{|fM^{(2)}_{RHP}(z)W^{(2)}(z)M^{(2)}_{RHP}(z)^{-1}|}{|s-z|} dA(s) \\ &\leq \|f\|_\infty \|M^{(2)}_{RHP}(z)\|_\infty \|M^{(2)}_{RHP}(z)^{-1}\|_\infty \int\int_{\Omega_1} \frac{|\bar{\partial}R_1 e^{2it\theta}|}{|s-z|} dA(s), \end{aligned}$$

and if we apply (2.18) we find that

$$|\mathbb{S}[f]| \leq C\left(I_1 + I_2 + I_3\right),$$

where

$$\begin{aligned} I_1 &= \int\int_{\Omega_1} \frac{|\chi_{\mathcal{Z}}(z)|e^{-4tv(u-\xi)}}{|s-z|} dA(s), \\ I_2 &= \int\int_{\Omega_1} \frac{|r'(u)|e^{-4tv(u-\xi)}}{|s-z|} dA(s), \\ \text{and } I_3 &= \int\int_{\Omega_1} \frac{|z-\xi|^{-1/2}e^{-4tv(u-\xi)}}{|s-z|} dA(s). \end{aligned}$$

We then carry out analysis of these three integrals to determine the decay rate in t as detailed in Appendix C of the manuscript in the Appendix. The result of (2.46) is that we may choose t large enough so that (2.45) may be inverted by Neumann series. From the asymptotic behavior of $M^{(3)}$, the solution will be of the form $M^{(3)} = I + z^{-1}M_1^{(3)} + \mathcal{O}(z^{-2})$. Specifically, we write that

$$M^{(3)}(z) = I - \frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{M^{(3)}(s)W^{(3)}}{z} - \frac{sM^{(3)}(s)W^{(3)}}{z(s-z)} \right) dA(s), \tag{2.47}$$

so that if we again use the uniform boundedness of $M^{(2)}_{RHP}$ and similar analysis of the remaining 2-D integral, then we see that $M_1^{(3)}$ will satisfy the bound

$$|M_1^{(3)}| \leq ct^{-3/4}, \tag{2.48}$$

where c is a constant.

2.6. Results for the Long-Time Asymptotics for NLS

We now wish to reverse all of the transformations for which we now have long-time asymptotic behavior and determine the asymptotics for (1.4). In the region Ω_2, M will have the form

$$M = M^{(3)}(z)E(z)M^{(out)}(z)T^{\sigma_3}.$$

We now take the large z expansions of these matrices and find that the coefficient of the z^{-1} term in the Laurent expansion of M is given by

$$M_1 = M_1^{(3)} + E_1 + M_1^{(out)} + T_1. \tag{2.49}$$

As we will only need the off diagonal entry to determine ϕ from (1.64), T will not make a contribution. Furthermore, let $\mathcal{I} = [a, b]$ be a real interval and we make the following definitions:

$$\begin{gathered} \mathcal{Z}(\mathcal{I}) = \{z_n \in \mathcal{Z} \,:\, \mathfrak{Re}(z_n) \in \mathcal{I}\} \quad \text{and} \quad N(\mathcal{I}) = |\mathcal{Z}(\mathcal{I})| \\ D_\xi^-(\mathcal{I}) = \{n \in \{0, 1, \ldots, N\} \,:\, a \leq \mathfrak{Re}(z_n) < \xi\} \\ \widehat{c}_n(\mathcal{I}) = c_n \prod_{\substack{z_j \in \mathcal{Z}(\mathcal{I}) \\ \mathfrak{Re}(z_j) < a}} \left(\frac{z_n - z_j}{z_n - z_j^*}\right)^2 \exp\left(\frac{i}{\pi}\int_{-\infty}^{\xi} \frac{\log(1+|r(s)|^2)}{s - z_n} ds\right). \end{gathered} \tag{2.50}$$

Let us now denote the discrete scattering data associated with (2.50) by $\widehat{\sigma}_d^{(out)} = \{(z_n, \widehat{c}_n(\mathcal{I}) \,:\, z_n \in \mathcal{Z}(\mathcal{I})\}$ and the corresponding $N(\mathcal{I})$-Soliton solution for this data by $\widehat{\phi}(x, t; \widehat{\sigma}_d^{(out)}) = \widehat{\phi}_{out}$. This effectively gives us a pure soliton solution that will only have the solitons associated with the set $\widehat{\sigma}_d^{(out)}$. We have the asymptotic relationship

$$\left|\phi_{out} - \widehat{\phi}_{out}\right| = \mathcal{O}(e^{-4\rho t}) \tag{2.51}$$

as $t \to \infty$ along all characteristics $x = x_0 + vt$ inside the truncated cone

$$x_1 + v_1 t \leq x \leq x_2 + v_2 t, \qquad t \geq 0$$

where we determine the interval $\mathcal{I}$ by the relation $\mathcal{I} = [-v_2/2, -v_1/2]$ (see Figure 2.4). Essentially, this means that inside a certain viewing window determined by the cone, all of the solitons in ϕ_{out} will line up with all of the solitons of $\widehat{\phi}_{out}$ as t grows large. We now state the final result for the long-time asymptotics for (1.1).

Long-time Asymptotic Behavior for Focusing NLS. *Let $\phi(x,t)$ be the solution of the focusing nonlinear Schrödinger Equation* (1.1) *with initial data $\phi(x,t=0)=\phi_0(x)\in H^{1,1}(\mathbb{R})$ and associated scattering data $\{r,\{z_n,c_n\}_{k=1}^N\}$, such that $r\in H^{1,1}(\mathbb{R})$, all z_n are simple and ϕ_0 does not generate any spectral singularities. Fix $x_1,x_2,v_1,v_2\in\mathbb{R}$ with $v_1\leq v_2$. Let $\mathcal{I}=[-v_2/2,-v_1/2]$, and define $\xi=-x/(2t)$. Then as $t\to\infty$ inside the cone*

$$x_1+v_1t\leq x\leq x_2+v_2t,\qquad t\to\infty$$

we have the asymptotic behavior

$$\phi(x,t)=\widehat{\phi}(x,t;\widehat{\sigma}_d^{(out)})+t^{-1/2}f(x,t)+\mathcal{O}(t^{-3/4}),$$

where $\widehat{\phi}(x,t;\widehat{\sigma}_d^{(out)})$ is the $N(\mathcal{I})$ soliton corresponding to the modified discrete scattering data given by $\widehat{\sigma}_d^{(out)}=\{(z_n,\widehat{c}_n(\mathcal{I}):z_n\in\mathcal{Z}(\mathcal{I})\}$, with $\mathcal{Z}(\mathcal{I})$ and $\widehat{c}_n(\mathcal{I})$ given by (2.50)*. Furthermore, let*

$$f(x,t)=m_{11}(\xi)^2\alpha_1(\xi)e^{ix^2/(2t)-i\kappa(\xi)\log(4t)}+m_{12}(\xi)^2\alpha_2(\xi)e^{-ix^2/(2t)+i\kappa(\xi)\log(4t)},$$

where

$$|\alpha_1(\xi)|^2=|\alpha_2(\xi)|^2=|\kappa(\xi)|,\quad \arg\alpha_2(s)=-\arg\alpha_1(s),$$

$$\arg\alpha_1(s)=2\int_{-\infty}^{\xi}\frac{\kappa(s)-\chi(s)\kappa(\xi)}{s-z}ds-4\sum_{k\in D_\xi^-}\arg(\xi-z_k)+\frac{\pi}{4}+\arg\Gamma(i\kappa(s))-\arg r(s),$$

and the coefficients $m_{11}(\xi)$ and $m_{12}(\xi)$ are the entries in the first row of the solution of (2.4.1) *with discrete spectral data $\widehat{\sigma}_d^{(out)}$ and $D_\xi^-(\mathcal{I})$ evaluated at $z=\xi$.*

We now discuss some previous results that have been attained in the area of long-time asymptotics of integrable partial differential equations. In [17], the author provides a survey of results for long-time asymptotic behavior of NLS equations. At the time of publication the author noted that the literature was "far from proving" any results towards soliton resolution of any interesting equations. The author goes on to describe results which focused on initial data that were small perturbations of N-soliton solutions as detailed in [16]. In this case, the soliton resolution was confirmed for the equation

$$i\frac{\partial\psi}{\partial t}=-\Delta x-F(|\psi|^2)\psi, \tag{2.52}$$

where $x\in\mathbb{R}^n$ for $n\geq 3$.

In [8] a similar soliton resolution conjecture was proven for the Korteweg-de Vries equation,

$$u_t-6uu_x+u_{xxx}=0 \tag{2.53}$$

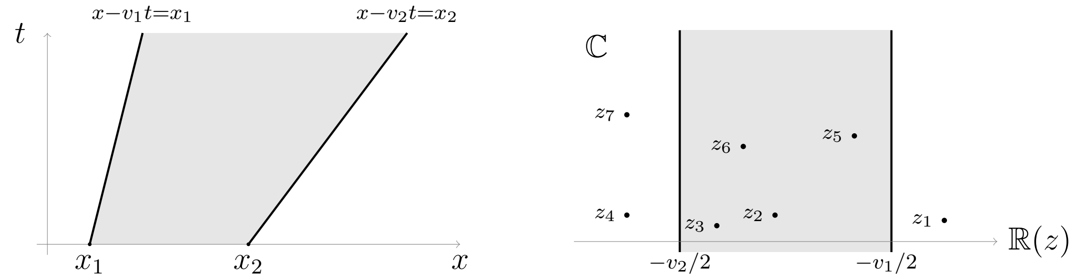

Figure 2.4: On the left we see the cone $x_1 + v_1 t \le x \le x_2 + v_2 t$ as $t \to \infty$. Given initial data ϕ_0 with scattering data $\{r, \{z_n, c_n\}_{k=1}^N\}$, the z_n values appear in the right figure in the complex plane where the cone has been mapped to the region $-v_2/2 \le \mathbb{R}(z) \le -v_1/2)$. To construct the reduced soliton solution corresponding to the cone, we retain only the z_n within the shaded region and adjust the c_n by the relation (2.51).

with initial condition

$$u(x, t = 0) = u_0(x).$$

In this result, $u_0(x)$ is assumed to be sufficiently smooth and to decay rapidly enough as $|x| \to \infty$ so that the scattering data produced will be well behaved. Under such conditions one may choose constants $v > 0$ and $M \ge 0$ so that

$$\sup_{x \ge -M + vt} |u(x,t) - u_d(x,t)| = \mathcal{O}(\sigma(t)) \quad t \to \infty, \tag{2.54}$$

where $u_d(x,t)$ is the N-soliton solution of the discrete scattering data produced by u_0 and $\sigma(t) \to 0$ as $t \to \infty$ at a rate that depends on the reflection coefficient r. As this result was proven in 1983, it relies on analysis of the Gelfand-Levitan equation rather than analysis of a Riemann-Hilbert problem.

Finally, we note that results for the long-time asymptotics of NLS do exist in the absence of discrete scattering spectra. The defocusing NLS is defined by

$$iq_t + q_{xx} - 2|q|^2 q = 0, \tag{2.55}$$

with initial condition $q(x, 0) = q_0$. Under the condition that q_0 vanishes as $|x| \to \infty$, solutions to (2.55) will produce no solitons and the scattering data associated with q_0 will contain only continuous spectra given by the reflection coefficient r. In [6], the authors applied the nonlinear steepest descent method to determine that for $q_0 \in H^{0,1}$ the long-time asymptotic behavior of q is given by

$$q(x,t) = t^{-1/2} \alpha(z_0) e^{i\frac{x^2}{4t} - i\nu(z_0)\log 2t} + \mathcal{O}(t^{-(1/2+\kappa)}) \tag{2.56}$$

where κ is a fixed constant such that $0 < \kappa < \frac{1}{4}$ and

$$\nu(z) = -\frac{1}{2\pi}\log(1 - |r(z)|^2), \qquad |\alpha(z)|^2 = \frac{\nu(z)}{2}, \tag{2.57}$$

and

$$\arg \alpha(z) = \frac{1}{\pi} \int_{-\infty}^{z} \log(z-s) d \left(\log(1-|r(s)|^2) \right) + \frac{\pi}{4} + \arg \Gamma(i\nu(z)) - \arg r(z). \quad (2.58)$$

More recently in [7], $\bar{\partial}$ methods were applied to (2.55) and the error given in (2.56) was improved to $\mathcal{O}(t^{-3/4})$. We note that the for the results of the main theorem presented in this dissertation, away from any solitons (or in the absence of discrete spectra) we will have that $\widehat{\phi}(x, t; \widehat{\sigma}_d^{(out)})$ and m_{12} will be exponentially small in t. Additionally, m_{11} will be exponentially close to 1 as $t \to \infty$ so that the long-time asymptotic behavior of (1.1) will reduce to

$$\phi(x,t) = t^{-1/2} \alpha_1(\xi) e^{ix^2/(2t) - i\kappa(\xi) \log(4t)} + \mathcal{O}(t^{-3/4}), \quad (2.59)$$

which is similar to the behavior seen in the defocusing NLS.

Appendix

This appendix includes a copy of the paper ”Long Time Asymptotic Behavior of the Focusing Nonlinear Schrodinger Equation” by M. Borghese, R. Jenkins, and K. D. T.-R. McLaughlin, which can be found at arxiv.org, reference arXiv:1604.07436v1. A version of this paper may appear as a journal publication at a future time.

LONG TIME ASYMPTOTIC BEHAVIOR OF THE FOCUSING NONLINEAR SCHRODINGER EQUATION

MICHAEL BORGHESE, ROBERT JENKINS, AND KENNETH D. T.-R. MCLAUGHLIN

Abstract. We study the Cauchy problem for the focusing nonlinear Schrödinger (NLS) equation. Using the $\overline{\partial}$ generalization of the nonlinear steepest descent method we compute the long time asymptotic expansion of the solution $\psi(x,t)$ in any fixed space-time cone $x_1 + v_1 t \le x \le x_2 + v_2 t$ with $v_1 \le v_2$ up to an (optimal) residual error of order $\mathcal{O}\left(t^{-3/4}\right)$. In each (x,t) cone the leading order term in this expansion is a multi-soliton whose parameters are modulated by soliton-soliton and soliton-radiation interactions as one moves through the cone. Our results only require that the initial data possess one $L^2(\mathbb{R})$ moment and (weak) derivative and that it not generate any spectral singularities (embedded eigenvalues).

1. Introduction

In this paper we study the long time asymptotic behavior of the focusing nonlinear Schrödinger (fNLS) equation on $\mathbb{R} \times \mathbb{R}_+$:

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \qquad \psi(x,0) = \psi_0(x). \tag{1.1}$$

The long time behavior of the *defocusing* NLS equation—equation (1.1) with the sign of cubic nonlinearity reversed—has been thoroughly studied [17, 7, 6, 8, 9, 10]. In the defocusing case, one finds that as $t \to \infty$,

$$\psi(x,t) = t^{-1/2}\alpha(z_0)e^{ix^2/(2t) - i\nu(z_0)\log(4t)} + \mathcal{E}(x,t) \tag{1.2}$$

where

$$\nu(z) = -\frac{1}{2\pi}\log(1 - |r(z)|^2), \quad |\alpha(z)|^2 = \nu(z)^2,$$

and

$$\arg\alpha(z) = \frac{1}{\pi}\int_{-\infty}^{z}\log(z-s)d(\log(1-|r(s)|^2)) + \frac{\pi}{4} + \arg\Gamma(i\nu(z)) - \arg r(z).$$

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Here $z_0 = -x/(2t)$, Γ is the gamma function, and r is the so called reflection coefficient for the potential $\psi_0(x)$ described below. Estimates for the size of the error term $\mathcal{E}(x,t)$ depend on smoothness and decay assumptions on ψ_0. The leading term without estimates was first obtained in [17]. Using the nonlinear steepest descent method [5], it was shown in [8, 6] that if ψ_0 had a high degree of smoothness and decay that $\mathcal{E}(x,t) = \mathcal{O}\left(t^{-1}\log t\right)$. This was later improved [9] to $\mathcal{E}(x,t) = \mathcal{O}\left(t^{-(1/2+\kappa)}\right)$ for any $0 < \kappa < 1/4$ under the much weaker assumption that ψ_0 belonged to the weighted Sobolev space

$$H^{1,1} = \left\{ f \in L^2(\mathbb{R}) \,:\, xf,\ f' \in L^2(R) \right\}.$$

Recently, McLaughlin and Miller [15, 16], developed a method of asymptotic analysis of Riemann-Hilbert problems based on $\overline{\partial}$ problems, rather than the asymptotic analysis of singular integrals on contours. This was successfully adapted to study defocusing NLS both for finite mass initial data [10] and finite density initial data [4]; the later of which supports soliton solutions. The advantages of this method are two fold: 1) it avoids delicate estimates involving L^p estimates of Cauchy projection operators (central to the work in [9]), and 2) it improves error estimates without additional restrictions on the initial data. The result in [10], which can be shown to be sharp, is that for $\psi_0 \in H^{1,1}$, the error $\mathcal{E}(x,t) = \mathcal{O}\left(t^{-3/4}\right)$.

In this work we apply these $\overline{\partial}$-techniques to the inverse scattering transform (IST) for NLS to obtain the long-time asymptotic behavior of solutions to (1.1). The long-time behavior of solutions of focusing NLS are necessarily more detailed than in the defocusing case due to the presence of solitons which correspond to discrete spectra of the non self-adjoint ZS-AKNS (Dirac) scattering operator associated with focusing NLS (cf. (2.1a) below). Given initial data $\psi_0 \in L^2(\mathbb{R})$ the ZS-AKNS operator for (1.1) allows for (complex conjugate pairs of) discrete spectra anywhere in $\mathbb{C}\backslash\mathbb{R}$. In the defocusing case the ZS-AKNS operator is self-adjoint and the discrete spectrum is empty for finite mass ($L^2(\mathbb{R})$) initial data; discrete spectra are possible for the finite density type data studied in [4], but they are restricted to lie in a fixed interval of the real axis set by the initial data. The description of the minimal scattering data for the forward/inverse scattering transform is necessarily more complicated in the focusing case.

Let us briefly consider the minimal scattering data for (1.1), more details are given in Section 2 and the references therein. Associated with any $z_k \in \mathbb{C}^+$ of simple discrete spectrum is a nonzero complex number c_k called a norming constant. The real axis is the continuous spectrum of the ZS-AKNS operator along which we define a *reflection coefficient* $r : \mathbb{R} \to \mathbb{C}$. In the focusing case, the reflection coefficient r may take any value in $\mathbb{C}$; it is also possible that r may posses singularities along the real line—such points are called *spectral singularities*. When spectral singularities exist it is possible for their to be a (countably) infinite discrete spectrum which must accumulate at a spectral singularity; if no spectral singularities exist, the discrete spectrum is finite. For initial data ψ_0 which produces only simple discrete spectra and has no spectral singularities, the minimal scattering data for focusing NLS is the collection $\mathcal{D} = \{r(z), \{(z_k, c_k)\}_{k=1}^N\}$. This is the classical scattering map $\mathcal{S} : \psi_0 \mapsto \mathcal{D}$ for NLS. As described in [2, 3] such initial data is generic. In the general, non-generic, case where spectral singularities or higher order spectra may exist the classical scattering map is replaced by $\mathcal{S} : \psi_0 \mapsto v$ where v is a certain matrix defined along a contour Γ consisting of the real axis and a closed circle around infinity as described in [20].

In either case the amazing fact of integrability is that the scattering map $\mathcal{S}$ linearizes the time evolution; for a potential ψ_0 evolving according to (1.1) the scattering data evolution is trivial: $\mathcal{D}(t) = \{r(z)e^{2iz^2t}, \{(z_k, c_k e^{2iz_k^2 t})\}_{k=1}^N\}$ (or $v(t) = e^{-iz^2t\sigma_3} v e^{iz^2t\sigma_3}$ in the general case). It is often remarked in the literature that the scattering map $\mathcal{S}$ is a kind of nonlinear Fourier transform, and indeed it preserves regularity and smoothness in the same way; as shown in [20] the scattering map is a bijective (in fact bi-Lipschitz) map from $H^{j,k}(\mathbb{R})$ to $H^{k,j}(\Gamma)$ for any $j > 0$ and $k \geq 1$ (in the classical setting without spectral singularities this reduces to the reflection coefficient $r \in H^{k,j}(\mathbb{R})$). However, it is a trivial calculation that in order for the time evolving scattering data to persist in the weighted Sobolev space $H^{k,j}$ one must have $j \geq k$. It follows that the largest space $H^{j,k}$ from which the IST for (1.1) is well defined in $H^{1,1}$, and this is precisely the space in which we will work.

Spectral data $\{r \equiv 0, \{(z_k, c_k)\}_{k=1}^N\}$ for which the reflection coefficient vanishes identically correspond to soliton solutions of (1.1). If the spectrum consist of a single point, $\sigma_d = \{(\xi + i\eta, c)\}$ the corresponding solution of (1.1) is the one-soliton

$$\begin{gathered}\psi_{\mathrm{sol}}(x,t) = \psi_{\mathrm{sol}}(x,t;\{(\xi+i\eta,c)\}) = 2\eta\,\mathrm{sech}(2\eta(x+2\xi t - x_0))e^{-2i(\xi x + (\xi^2-\eta^2)t)}e^{-i\phi_0},\\ \|\psi_{\mathrm{sol}}(\cdot,t)\|_{L^2(\mathbb{R})}^2 = 4\eta\end{gathered} \tag{1.3}$$

where the phase shift x_0 and constant ϕ_0 are

$$x_0 = \frac{1}{2\eta}\log\left|\frac{c}{2\eta}\right|, \qquad \phi_0 = \frac{\pi}{2} + \arg(c).$$

This solution is a localized pulse with speed $v = -2\xi$ and maximum amplitude 2η. When $N > 1$ the solution of (1.1) with scattering data $\{r \equiv 0,\ \sigma_d = \{(z_k, c_k)\}_{k=1}^N\}$, which we label $\psi_{\mathrm{sol}}(x,t;\sigma_d)$, is called an N-soliton solution (corresponding to the discrete scattering data σ_d). The long time behavior of the N-soliton is a straightforward exercise in linear algebra and goes back to [18]. Generically, the solution breaks apart into N independent one-solitons; each traveling at distinct speed $v_k = -2\,\mathrm{Re}\, z_k$. When the spectra do not have distinct real parts the long-time behavior is more complicated; we give a streamlined review of this in Appendix B. Likewise, in the absence of solitons the defocusing methods mentioned above go through with only superficial changes of certain signs. The interesting question is how, in the generic case, the soliton and reflection coefficient terms interact to affect the long time limit. Formula (1.4) used in Theorem 1.1 characterizes this interaction in the general setting and (1.5) shows explicitly how these interactions affect the asymptotic phase shifts of individual solitons.

1.1. **Main Results and Remarks.** Our main result describes the asymptotic behavior of the solution (1.1) as $t \to \infty$, for generic initial data $\psi_0 \in H^{1,1}(\mathbb{R})$. In order to state our

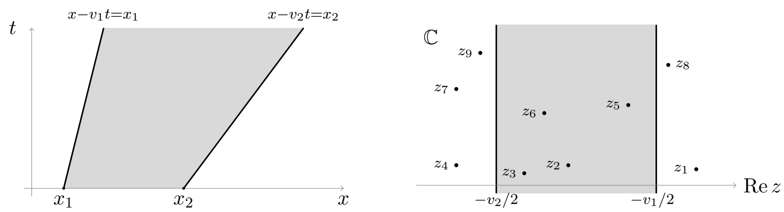

Figure 1. *Given initial data ψ_0 with scattering data $\{r, \{(z_k, c_k)\}_{k=1}^N\}$, the asymptotic behavior of $\psi(x,t)$, the solution of* (1.1)*, in the space-time cone $x_1 + v_1 t \le x \le x_2 + v_2 t$ as $t \to \infty$, is described to leading order by the $N(\mathcal{I})$-soliton $\psi_{\mathrm{sol}}(x, t; \widehat{\sigma}_d)$ corresponding to the discrete spectral values in $\mathcal{Z}(\mathcal{I})$ and connection coefficients $\widehat{c}_k$ modified by the self-interaction between solitons and with the reflection coefficient as described in Theorem 1.1. In the example here, the original data has nine spectral values, but inside the shaded (x,t) cone the solution is described by a 4-soliton with spectrum $\mathcal{Z}(\mathcal{I}) = \{z_2, z_3, z_5, z_6\}$.*

results we define the following quantities derived from given scattering data $\{r, \{(z_k, c_k)\}_{k=1}^N\}$. Let $\mathcal{Z}$ denote the projection of the discrete spectral data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N$ onto its first coordinate $\mathcal{Z} = \{z_k\}_{k=1}^N \subset \mathbb{C}^+$; define

$$\kappa(s) = -\frac{1}{2\pi} \log(1 + |r(s)|^2),$$

and for any real number ξ let

$$\Delta_\xi^- = \{k \in \{0, 1, \dots, N\} \,:\, \operatorname{Re} z_k < \xi\}.$$

Given any real interval $\mathcal{I} = [a, b]$ let

$$\begin{gathered}
\mathcal{Z}(\mathcal{I}) = \{z_k \in \mathcal{Z} \,:\, \operatorname{Re} z_k \in \mathcal{I}\} \quad \text{and} \quad N(\mathcal{I}) = |\mathcal{Z}(\mathcal{I})| \\
\Delta_\xi^-(\mathcal{I}) = \{k \in \{0, 1, \dots, N\} \,:\, a \le \operatorname{Re} z_k < \xi\} \\
\widehat{c}_k(\mathcal{I}) = c_k \prod_{\substack{z_j \in \mathcal{Z} \\ \operatorname{Re} z_j < a}} \left(\frac{z_k - z_j}{z_k - z_j^*} \right)^2 \exp\left(\frac{i}{\pi} \int_{-\infty}^{\xi} \log(1 + |r(s)|^2) \frac{ds}{s - z_k} \right)
\end{gathered} \tag{1.4}$$

Theorem 1.1. *Let $\psi(x,t)$ be the solution of* (1.1) *corresponding to initial data $\psi(x, t = 0) = \psi_0(x) \in H^{1,1}(\mathbb{R})$ and suppose that ψ_0 does not generate any spectral singularities. Let $\{r, \{z_k, c_k\}_{k=1}^N\}$ denote the spectral data generated from ψ_0. Fix $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $v_1 \le v_2$. Let $\mathcal{I} = [-v_2/2, -v_1/2]$, and let $\xi = -x/(2t)$. Then as $t \to \infty$ inside the truncated cone*

$$x_1 + v_1 t \le x \le x_2 + v_2 t, \qquad t \to \infty$$

we have

$$\psi(x,t) = \psi_{\text{sol}}(x,t;\widehat{\sigma}_d) + t^{-1/2} f(x,t) + \mathcal{O}\left(t^{-3/4}\right).$$

Here, $\psi_{\text{sol}}(x,t;\widehat{\sigma}_d)$ *is the* $N(\mathcal{I})$ *soliton corresponding to the modified discrete scattering data (see Figure 1) given by* $\widehat{\sigma}_d = \{(z_k, \widehat{c}_k(\mathcal{I})) \,:\, z_k \in \mathcal{Z}(\mathcal{I})\}$*, with* $\mathcal{Z}(\mathcal{I})$ *and* $\widehat{c}_k(\mathcal{I})$ *as defined by* (1.4)*, and*

$$f(x,t) = m_{11}(\xi)^2 \alpha_1(\xi) e^{ix^2/(2t) - i\kappa(\xi)\log(4t)} + m_{12}(\xi)^2 \alpha_2(\xi) e^{-ix^2/(2t) + i\kappa(\xi)\log(4t)},$$

with

$$|\alpha_1(\xi)|^2 = |\alpha_2(\xi)|^2 = |\kappa(\xi)|, \quad \arg\alpha_2(s) = -\arg\alpha_1(s),$$

and

$$\arg\alpha_1(s) = 2\int_{-\infty}^{\xi} \frac{\kappa(s) - \chi(s)\kappa(\xi)}{s - z} ds - 4\sum_{k\in\Delta_\xi^-} \arg(\xi - z_k) + \frac{\pi}{4} + \arg\Gamma(i\kappa(s)) - \arg r(s).$$

The coefficients $m_{11}(\xi)$ *and* $m_{12}(\xi)$ *are the entries in the first row of the solution of RHP B.2 with discrete spectral data* $\widehat{\sigma}_d$ *and* $\Delta = \Delta_\xi^-(\mathcal{I})$ *evaluated at* $z = \xi$*.*

Our result is essentially optimal. For initial data in the weakest possible space in which the IST can be formulated, we derive an asymptotic description up to a residual $\mathcal{O}\left(t^{-3/4}\right)$ error; this is the same order that arises in the Fourier analysis of the free Schrödinger equation $i\psi_t + \frac{1}{2}\psi_{xx} = 0$. We avoid the consideration of spectral singularities only to limit the length of the paper. Even subject to spectral singularities, our results should still hold in any (x,t) cone $x_1 + v_1 t < x < x_2 + v_2 t$, such that the spectral interval $\mathcal{I}$ does not contain any spectral singularities.

Remark 1.1. Spectral singularities may exist for data in any weighted Sobolev space $H^{j,k}$; there are even examples [19, Example 3.3.16] of Schwarz class data for which spectral singularities occur. However, if the initial data decays exponentially, i.e., for some $c > 0$, $\int_{\mathbb{R}} e^{c|x|}|\psi_0(x)|^2 dx < \infty$ then it is easily shown that spectral singularities cannot occur.

In Theorem 1.1 we give the asymptotic description in cones in order to accommodate many situations at once. In particular by considering small cones instead of fixed frames which of reference we are able to account for uncertainties in the computation (or measurement) of the spectral data and thus speed of the resulting solitons. We believe that such a description should also be useful to study non-integrable perturbations of focusing NLS where the discrete spectra would no longer be stationary.

If one has additional knowledge of the spectral data, then the formulae above can be simplified greatly in fixed frames of reference $x - vt = \mathcal{O}(1)$. In a frame of reference different than any soliton speed, i.e., if we have $|\xi - \operatorname{Re} z_k| \geq c > 0$ for all $k = 1, \ldots, N$, then $\psi_{\text{sol}}(x,t)$, $m_{11}(\xi) - 1$, and $m_{12}(\xi)$ are each exponentially small in t so the asymptotic description reduces to

$$\psi(x,t) = t^{-1/2}\alpha_1(\xi) e^{ix^2/(2t) - i\kappa(\xi)\log(4t)} + \mathcal{O}\left(t^{-3/4}\right).$$

This is the analog of the defocusing result (1.2). Next, we consider the frame of reference of a distinct 1-soliton, that is, suppose that $z_k = \xi_k + i\eta_k \in \mathcal{Z}$ is a discrete spectral value of the initial data ψ_0 whose real part is distinct from that of all other spectral values (except its complex conjugate) and let c_k be its associated norming constant. Then as $t \to \infty$ with $x + 2\operatorname{Re}(\xi_k)t = \mathcal{O}(1)$ the asymptotic solution reduces to

$$\begin{gathered} \psi(x,t) = \psi_{\mathrm{sol}}(x,t;(z_k,\widehat{c}_k)) + \mathcal{O}\left(t^{-1/2}\right) \\ \psi_{\mathrm{sol}}(x,t;(z_k,\widehat{c}_k)) = 2\eta_k \operatorname{sech}\left(2\eta_k(x+2\xi_k t - x_0)\right) e^{-2i(\xi_k x + (\xi_k^2-\eta_k^2)t)} e^{-i\phi_0} \end{gathered} \tag{1.5a}$$

where

$$\begin{aligned} x_0 &= \frac{1}{2\eta_k}\log\left|\frac{c_k}{2\eta_k}\right| + \eta_k^{-1}\sum_{\operatorname{Re} z_j<\xi_k}\log\left|\frac{z_k-z_j}{z_k-z_j^*}\right| - \frac{1}{2\pi}\int_{-\infty}^{-x/(2t)}\log(1+|r(s)|^2)\frac{ds}{(s-\xi_k)^2+\eta_k^2} \\ \phi_0 &= \frac{\pi}{2} + \arg c_k + 2\sum_{\operatorname{Re} z_j<\xi_k}\arg\left(\frac{z_k-z_j}{z_k-z_j^*}\right) + \frac{1}{\pi}\int_{-\infty}^{-x/(2t)}\log(1+|r(s)|^2)\frac{s-\xi_k}{(s-\xi_k)^2+\eta_k^2}ds \end{aligned} \tag{1.5b}$$

describe the asymptotic phase shifts. The last two terms in each expression above describe the asymptotic effect of the soliton-soliton interaction and the interaction of the soliton with the radiative component of the solution respectively.

If all of the solitons have distinct real parts, then the solution separates asymptotically in the sense that uniformly for $x \in \mathbb{R}$ as $t \to \infty$,

$$\psi(x,t) = \sum_{k=1}^{N}\psi_{\mathrm{sol}}(x,t;(z_k,\widehat{c}_k)) + \mathcal{O}\left(t^{-1/2}\right),$$

and the correction of order $t^{-1/2}$ can be explicitly computed using the results of Theorem 1.1.

Remark 1.2. Though we say that initial data ψ_0 whose spectra have distinct real parts are generic (in the sense that small perturbations of any non-generic initial datum will be generic) there are important classes of non-generic data. The so called Klaus-Shaw 'single lobe' potentials, $\psi_0(x) = A(x)e^{ikx+i\phi_0}$ with $k, \phi_0 \in \mathbb{R}$ and $A(x)$ a bounded piecewise smooth function which is nondecreasing to the left of some x_0 and nonincreasing to the right of x_0, are such that all of the discrete spectra have the same real part. Such potentials have been extensively studied in the semi-classical limit where the number of spectra is asymptotically large.

Organization of the rest of the paper. In Section 2 we describe the forward scattering transform step of the IST in greater detail collecting the necessary results for our later work and provide references for their proofs. The section ends with the characterization of the inverse scattering transform in terms of a Riemann-Hilbert problem RHP 2.1. Section 3 begins the Riemann-Hilbert analysis by describing the initial conjugation of RHP 2.1 to better condition the problem for asymptotic analysis in a given frame of reference. Section 4 introduces the $\overline{\partial}$ analysis to define extensions of the jump matrix for the non-linear steepest descent method. In Section 5 we construct a global model solution which captures the leading order asymptotic behavior of the solution. Removing this component of the solution results in a small-norm $\overline{\partial}$ problem which is analyzed in Section 6 culminating in a proof of our main result Theorem 1.1.

2. Results of scattering theory for focusing NLS

The focusing NLS equation can be integrated [1, 18] using the ZS-AKNS operator associated with Lax pair for NLS:

$$(\partial_x - \mathcal{L})\Phi = 0, \qquad \mathcal{L} = -iz\sigma_3 + \Psi, \tag{2.1a}$$

$$(i\partial_t - \mathcal{B})\Phi = 0, \qquad \mathcal{B} = iz\mathcal{L} + \frac{1}{2}\sigma_3(\Psi^2 - \Psi_x), \tag{2.1b}$$

where

$$\Psi = \Psi(x,t) = \begin{pmatrix} 0 & \psi(x,t) \\ -\psi(x,t)^* & 0 \end{pmatrix},$$

and σ_3 is the third Pauli matrix $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The existence of a simultaneous solution of this overdetermined system of equations requires that the potential $\Psi(x,t)$ satisfy the zero-curvature equation,

$$i\mathcal{L}_t - \mathcal{B}_x + [\,\mathcal{L}, \mathcal{B}\,] = i\Psi_t + \frac{1}{2}\sigma_3\Psi_{xx} - \sigma_3\Psi^2 = 0, \tag{2.2}$$

which is just a restatement of (1.1).

In the forward scattering step given initial data $\psi_0(x)$ one constructs solutions $\Phi(x,z)$ of (2.1a) with $z \in \mathbb{R}$; in particular one constructs the two *Jost solutions* $\Phi^{(\pm)}(x;z) = m^{(\pm)}(x,z)e^{-izx\sigma_3}$, which satisfy

$$i\partial_x m = -iz[\sigma_3, m] + \Psi m, \qquad \lim_{x\to\pm\infty} m^{\pm}(x,z) = I. \tag{2.3}$$

These solutions can be expressed as Volterra type integrals

$$m^{(\pm)}(z) = I + \int_{\pm\infty}^{x} e^{iz(x-y)\sigma_3}\Psi(y)m^{(\pm)}(y)e^{-iz(x-y)\sigma_3}dy$$

By iteration one shows that these equations have bounded continuous solutions in both x and z whenever $\psi_0 \in L^1(\mathbb{R})$.

As the differential equation (2.1a) is traceless, the determinant of any solution Φ is independent of x and it follows that $\det \Phi^{(\pm)} = m^{(\pm)} \equiv 1$; for complex z solutions must also possess the symmetry $m(x,z^*) = \sigma_2 m(x,z)^*\sigma_2$. It follows that for $z \in \mathbb{R}$ both $m^{(+)}$ and

$m^{(-)}$ define a fundamental solution set for (2.3) and so there exists a continuous matrix function $S(z)$, *the scattering matrix*, satisfying

$$\begin{aligned} &\Phi^{(-)}(x;z) = \Phi^{(+)}(x;z)S(z), \qquad z \in \mathbb{R}, \\ &S(z) = \begin{pmatrix} a(z) & -b(z)^* \\ b(z) & a(z)^* \end{pmatrix}, \qquad \det S(z) = |a(z)|^2 + |b(z)|^2 = 1 \end{aligned} \tag{2.4}$$

the coefficients $a(z)$ and $b(z)$ can be expressed as

$$\begin{aligned} a(z) &= \det\left[m_1^{(-)},\, m_2^{(+)}\right] = 1 + \int_R \psi(y)^* m_{12}^{(+)}(y)dy = 1 + \int_R \psi(y) m_{21}^{(-)}(y)dy, \\ b(z) &= \det\left[m_1^{(+)},\, m_1^{(-)}\right] = -\int_R \psi(y)^* e^{-2izy} m_{11}^{(+)}(y)dy = -\int_R \psi(y)^* e^{-2izy} m_{11}^{(-)}(y)dy \end{aligned} \tag{2.5}$$

where

$$m^{(\pm)} = \left(m_1^{(\pm)}\,,\, m_2^{(\pm)}\right) = \begin{pmatrix} m_{11}^{(\pm)} & m_{12}^{(\pm)} \\ m_{21}^{(\pm)} & m_{22}^{(\pm)} \end{pmatrix}.$$

The following results are standard, proofs and details can be found in the literature, see for example [2, 3, 9].

Let $m_j^{(\pm)}$ denote the j^{th} column of $m^{(\pm)}$ and e_j denote the j^{th} column of the identity matrix:

- $m_1^{(-)}(x,z)$, $m_2^{(+)}(x,z)$ and $a(z)$ extend analytically to $z \in \mathbb{C}^+$ with continuous boundary values on $\mathbb{R}$. As $z \to \infty$ in $\mathbb{C}^+$, $m_1^{(-)}(x,z) \to e_1$, $m_2^{(+)}(x,z) \to e_2$ and $a(z) \to 1$. Analogous statements hold for the other pair of columns for $z \in \mathbb{C}^-$. Generally, $b(z)$ is defined only for $z \in \mathbb{R}$.

- At any $z_k \in \mathbb{C}^+$ for which $a(z_k) = 0$, the solutions $\Phi_1^{(-)}(x,z_k)$ and $\Phi_2^{(+)}(x,z_k)$ are linearly dependent. Specifically, a *norming constant* c_k exists such that:

  $$\Phi_1^{(-)}(x,z_k) = c_k \Phi_2^{(+)}(x,z_k).$$

  As these solution decays exponentially as $x \to \mp\infty$ respectively, this indicates that z_k is an L^2 eigenvalue of (2.1a) with eigenfunction $\Phi_1^{(-)}(x;z_k)$. The symmetry $a(z^*) = a(z)^*$ implies these eigenvalues come in conjugate pairs.

- The *reflection coefficient* $r : \mathbb{R} \to \mathbb{C}$ and *transmission coefficient* $\tau : \mathbb{C}^+ \to \mathbb{C}$ are defined by

  $$r(z) = \frac{b(z)}{a(z)} \qquad \tau(z) = \frac{1}{a(z)} \tag{2.6}$$

  and it follows from (2.4) that $1 + |r(z)|^2 = |\tau(z)|^2$ for each $z \in \mathbb{R}$.

- The properties of the scattering coefficients are similar to those of the Fourier transform. Given initial data Ψ_0 in the weighted Sobolev space
$$H^{j,k}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \partial_x^j f,\, |x|^k f \in L^2(\mathbb{R}) \right\}$$
the scattering coefficients $a(z) - 1 \in H^{k,1}$ and $b(z) \in H^{k,j}$. It follows that, in the absence of spectral singularities (real zeros of $a(z)$), the map $\mathcal{R} : \psi_0 \mapsto r$ is a map from $H^{j,k}$ to $H^{k,j}$ (c.f. [9]).

The collection of data $\mathcal{D} = \{r(z), \{z_k, c_k\}_{k=1}^N\}$ are called the scattering data for $\psi_0(x)$ and the map $\mathcal{S} : \psi_0 \mapsto \mathcal{D}$ is called the (forward) scattering map. The essential fact of integrability, is that if the potential $\psi_0(x)$ evolves according to (1.1) then the evolution of the scattering data $\mathcal{D}$ is trivial

$$\mathcal{D}(t) = \left\{ r(z,t), \{z_k(t), c_k(t)\}_{k=1}^N \right\} = \left\{ r(z) e^{2itz^2}, \{z_k, c_k e^{2itz_k^2}\}_{k=1}^N \right\}. \tag{2.7}$$

The inverse scattering map $\mathcal{S}^{-1} : \mathcal{D}(t) \mapsto \psi(x,t)$ seeks to recover the solution of (1.1) from its scattering data. This is done as follows: from the (now time evolving) Jost function $\Phi^{(\pm)}(x,t;z) = m^{(\pm)}(x,t;z) e^{-izx\sigma_3}$ one constructs the function

$$M(z) = M(z;x,t) := \begin{cases} \left[\dfrac{m_1^{(-)}(x,t;z)}{a(z)}, m_2^{(+)}(x,t;z) \right] & : \quad z \in \mathbb{C}^+ \\ \sigma_2 M(z^*;x,t)^* \sigma_2 & : \quad z \in \mathbb{C}^-. \end{cases} \tag{2.8}$$

Assuming that the data ψ_0 is generic in the sense that $a(z)$ has only simple zeros in $\mathbb{C}^+$ and no spectral singularities, the matrix M defined above is the solution of the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 2.1 *Find a meromorphic function $M : \mathbb{C} \backslash (\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*) \to SL_2(\mathbb{C})$ with the following properties*

1. *$M(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.*
2. *For each $z \in \mathbb{R}$, M takes continuous boundary values $M_\pm(z) := \lim_{\epsilon \to 0^+} M(z \pm \mathrm{i}\epsilon)$ which satisfy the jump relation $M_+(z) = M_-(z) V(z)$ where*
$$V(z) = \begin{pmatrix} 1 + |r(z)|^2 & r^*(z) e^{-2it\theta(z)} \\ r(z) e^{2it\theta(z)} & 1 \end{pmatrix}, \tag{2.9}$$
where
$$\theta = \theta(z;x,t) = z^2 - 2\xi z = (z-\xi)^2 - \xi^2, \quad \xi = -x/(2t). \tag{2.10}$$
3. *$M(z)$ has simple poles at each $z_k \in \mathcal{Z}$ and $z_k^* \in \mathcal{Z}^*$ at which*
$$\begin{aligned} \operatorname*{Res}_{z_k} M &= \lim_{z \to z_k} M \begin{pmatrix} 0 & 0 \\ c_k e^{2it\theta} & 0 \end{pmatrix}, \\ \operatorname*{Res}_{z_k^*} M &= \lim_{z \to z_k^*} M \begin{pmatrix} 0 & -c_k^* e^{-2it\theta} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.11}$$

It's a simple consequence of Liouville's theorem that if a solution exists it is unique. Expanding this solution as $z \to \infty$, $M = I + z^{-1} M^{(1)}(x,t) + o\left(z^{-1}\right)$ and inserting this into (2.3) one finds that

$$M = I + \frac{1}{2iz} \begin{bmatrix} -\int_x^\infty |\psi(s,t)|^2 ds & \psi(x,t) \\ \psi(x,t)^* & \int_x^\infty |\psi(s,t)|^2 ds \end{bmatrix} + o\left(z^{-1}\right), \tag{2.12}$$

and it follows that the solution of (1.1) is given by

$$\psi(x,t) = \lim_{z \to \infty} 2iz M_{12}(z;x,t). \tag{2.13}$$

For non-generic potentials various parts of the above characterization must be altered. There can exist points $z \in \mathbb{R}$ for which $a(z) = 0$, in which case $m_\pm(x,z)$ fail to exist; these are called *spectral singularities*. The number of discrete spectra of (2.1a) may be infinite, due to the first property of the solution m, the discrete spectra must accumulate at a spectral singularity along the real axis. In the absence of spectral singularities the discrete spectrum is finite. Smoothness and decay of the initial data does not preclude the existence of spectral singularities; in [19] an explicit example is given of a Schwarz-class potential which generates an infinite discrete spectrum accumulating at $z = 0$. Finally, even in the case of a finite spectrum, poles may coalesce resulting in higher order singularities at certain points of the discrete spectrum; in this case the pole conditions (2.11) must be altered. For simplicity we will consider here only the generic setting. Special cases of a single spectral singularity and of an infinite number of solitons have been partially described in [14, 13].

3. Conjugation

The function $M(z;x,t)$ defined by (2.8) which solves solvs RHP 2.1, is normalized such that it has identity asymptotics as $x \to +\infty$ with t fixed. It is not unreasonable to assume that the RHP should be well conditioned as $t \to \infty$ along a characteristic $x = vt$ where $v \gg 1$. However, along an arbitrary characteristic there is no reason to expect that M will remain near identity. In this section we describe a transformation $M \mapsto M^{(1)}$ which renormalizes the RHP such that it is well behaved as $t \to \infty$ along an arbitrary characteristic.

Let $\xi = -x/(2t)$. Define the partition of $\{0, 1, \dots, N\} = \Delta_\xi^- \cup \Delta_\xi^+$ by

$$\begin{aligned} \Delta_\xi^- &= \{k \in \{0,1,\dots,N\} \,:\, \operatorname{Re} z_k < \xi\}, \\ \Delta_\xi^+ &= \{k \in \{0,1,\dots,N\} \,:\, \operatorname{Re} z_k \geq \xi\}. \end{aligned} \tag{3.1}$$

This partition splits the the residues c_k in (2.11) into two sets: As $t \to \infty$ with $x \geq -2\xi t$, it follows from (2.10) that for each $k \in \Delta_\xi^-$, $\operatorname{Im}(\theta(z_k)) < 0$ and thus the residue of $M(z)$ at z_k in (2.11) grows without bound as $t \to \infty$, similarly, for z_k with $k \in \Delta_\xi^+$, the resides are bounded or near zero.

The first step in our analysis is to introduce a transformation which renormalizes the Riemann-Hilbert problem such that it is well conditioned for $t \to \infty$ with ξ fixed. In order to arrive at a problem which is well normalized, we introduce the function

$$\begin{aligned} T(z) = T(z,\xi) &= \prod_{k \in \Delta_\xi^-} \left(\frac{z - z_k^*}{z - z_k} \right) \exp\left(i \int_{-\infty}^{\xi} \frac{\kappa(s)}{s - z} ds \right), \\ \kappa(s) &= -\frac{1}{2\pi} \log(1 + |r(s)|^2). \end{aligned} \tag{3.2}$$

A standard result of the forward scattering theory [11] is the following trace formula for the transmission coefficient

$$\frac{1}{a(z)} = \prod_{k=1}^{N} \left(\frac{z - z_k^*}{z - z_k} \right) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \log(1 + |r(s)|^2) \frac{ds}{s - z} \right) \tag{3.3}$$

from which we see that our function $T(z,\xi)$ is a partial transmission coefficient which approaches the total transmission $1/a(z)$ as $\xi \to \infty$.

Proposition 3.1. *The function $T(z)$ defined by* (3.2) *has the following properties:*

a. T is meromorphic in $\mathbb{C} \backslash (-\infty, \xi]$. For each $k \in \Delta_\xi^-$, $T(z)$ has a simple pole at z_k and a simple zero at z_k^*.

b. For $z \in \mathbb{C} \backslash (-\infty, \xi]$, $T(z^*)^* = 1/T(z)$.

c. For $z \in (-\infty, \xi)$, the boundary values $T_\pm$ satisfy

$$T_+(z)/T_-(z) = 1 + |r(z)|^2, \quad z \in (-\infty, \xi). \tag{3.4}$$

d. As $|z| \to \infty$ with $|\arg(z)| \neq \pi$,

$$T(z) = 1 + \frac{i}{z} \left[2 \sum_{k \in \Delta_\xi^-} \operatorname{Im} z_k - \frac{1}{2\pi} \int_{-\infty}^{\xi} \log(1 + |r(s)|^2) ds \right] + \mathcal{O}\left(z^{-2}\right).$$

e. As $z \to \xi$ along any ray $\xi + e^{i\phi}\mathbb{R}_+$ with $|\arg \phi| < \pi$

$$\left| T(z,\xi) - T_0(\xi)(z - \xi)^{i\kappa(\xi)} \right| \leq C \|r\|_{H^1(\mathbb{R})} |z - \xi|^{1/2} \tag{3.5}$$

where $T_0(\xi)$ is the complex unit

$$T_0(\xi) = \prod_{k \in \Delta_\xi^-} \left(\frac{\xi - z_k^*}{\xi - z_k} \right) e^{i\beta(\xi,\xi)} = \exp\left[i \left(\beta(\xi,\xi) - 2 \sum_{k \in \Delta_\xi^-} \arg(\xi - z_k) \right) \right],$$

$$\beta(z,\xi) = -\kappa(\xi) \log(z - \xi + 1) + \int_{-\infty}^{\xi} \frac{\kappa(s) - \chi(s)\kappa(\xi)}{s - z} ds,$$

and $\chi(s)$ is the characteristic function of the interval $(\xi - 1, \xi)$ and the logarithm is principally branched along $(-\infty, \xi - 1]$.

Proof. Parts *a.–c.* are elementary consequences of the definition (3.2) and the Sokhotski-Plemelj formula. For part *d.* one geometrically expands the product term and the factor $(s-z)^{-1}$ for large z, and use the fact that $\|\kappa\|_{L^1(\mathbb{R})} \leq \|r\|_{L^2(\mathbb{R})}$ to bound the remainder in the integral term for z bounded away from the contour of integration. For part *e.* we write

$$\begin{aligned} T(z,\xi) &= \prod_{k\in\Delta_\xi^-} \left(\frac{z-z_k^*}{z-z_k}\right) \exp\left(i\int_{\xi-1}^{\xi} \frac{\kappa(\xi)}{s-z} ds + i\int_{-\infty}^{\xi} \frac{\kappa(s)-\chi(s)\kappa(\xi)}{s-z} ds\right) \\ &= \prod_{k\in\Delta_\xi^-} \left(\frac{z-z_k^*}{z-z_k}\right) (z-\xi)^{i\kappa(\xi)} \exp\left(i\beta(z,\xi)\right). \end{aligned}$$

The result then follows from the facts that $\left|(z-\xi)^{i\kappa(\xi)}\right| \leq e^{-\pi\kappa(\xi)} = \sqrt{1+|r(\xi)|^2}$ and using Lemma 23.3 of [3]

$$|\beta(z,\xi) - \beta(\xi,\xi)| \leq C\|r\|_{H^1(\mathbb{R})}|z-\xi|^{1/2}.$$

□

We define a new unknown function $M^{(1)}$ using our partial transmission coefficient

$$M^{(1)}(z) = M(z)T(z)^{-\sigma_3} \tag{3.6}$$

Proposition 3.2. *The function $M^{(1)}$ defined by* (3.6) *satisfies the following Riemann-Hilbert problem*

Riemann-Hilbert Problem 3.1 *Find a meromorphic function $M^{(1)} : \mathbb{C}\backslash\mathbb{R} \to SL_2(\mathbb{C})$ with the following properties*

1. *$M^{(1)}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.*

2. *For each $z \in \mathbb{R}$, the boundary values $M_\pm^{(1)}(z)$ satisfy the jump relation $M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z)$ where*

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & r^*(z)T(z)^2e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)T(z)^{-2}e^{2it\theta} & 1 \end{pmatrix} & z \in (\xi,\infty) \\ \begin{pmatrix} 1 & 0 \\ \frac{r(z)T_-(z)^{-2}}{1+|r(z)|^2}e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{r^*(z)T_+(z)^2}{1+|r(z)|^2}e^{-2it\theta} \\ 0 & 1 \end{pmatrix} & z \in (-\infty,\xi) \end{cases} \tag{3.7}$$

3. $M^{(1)}(z)$ has simple poles at each $z_k \in \mathcal{Z}$ and $z_k^* \in \mathcal{Z}^*$ at which

$$\begin{aligned}\operatorname*{Res}_{z_k} M^{(1)} &= \begin{cases} \lim\limits_{z\to z_k} M^{(1)} \begin{pmatrix} 0 & c_k^{-1}(1/T)'(z_k)^{-2}e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim\limits_{z\to z_k} M^{(1)} \begin{pmatrix} 0 & 0 \\ c_k T(z_k)^{-2} e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \\ \operatorname*{Res}_{z_k^*} M^{(1)} &= \begin{cases} \lim\limits_{z\to z_k^*} M^{(1)} \begin{pmatrix} 0 & 0 \\ -(c_k^*)^{-1} T'(z_k^*)^{-2} e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim\limits_{z\to z_k^*} M^{(1)} \begin{pmatrix} 0 & -c_k^* T(z_k^*)^2 e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases}\end{aligned} \tag{3.8}$$

Proof. That $M^{(1)}$ is unimodular, analytic in $\mathbb{C}\backslash(\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*)$, and approaches identity as $z \to \infty$ follows directly from it's definition, Proposition 3.1 and the properties of M. The jump (3.7) follows directly from using the factorizations of V, (2.9), given by

$$V^{(1)}(z) = \begin{cases} T(z)^{\sigma_3} \begin{pmatrix} 1 & r(z)^* e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)e^{2it\theta} & 1 \end{pmatrix} T(z)^{-\sigma_3} & z > \xi \\ T_-(z)^{\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{r(z)e^{2it\theta}}{1+|r(z)|^2} & 1 \end{pmatrix} \left(\frac{T_+(z)}{T_-(z)}\right)^{\sigma_3} \begin{pmatrix} 1 & \frac{r(z)^* e^{-2it\theta}}{1+|r(z)|^2} \\ 0 & 1 \end{pmatrix} T_+(z)^{-\sigma_3} & z < \xi \end{cases}$$

to the right and left of $z = \xi$ on the real line respectively and making use of the jump relation (3.4) satisfied by $T(z)$ on $(-\infty, \xi)$. Concerning the residues, since $T(z)$ is analytic at each z_k, z_k^* with $k \in \Delta_\xi^+$, the residue conditions at these poles are an immediate consequence of (3.2). For $k \in \Delta_\xi^-$, $T(z)$ has a zero at z_k^* and a pole at z_k, so that $M_1^{(1)} = M_1(z)T(z)^{-1}$ has a removable singularity at z_k, but acquires a pole at z_k^*. For $M_2^{(1)} = M_2(z)T(z)$ the situation is reversed; it has a pole at z_k and a removable singularity at z_k^*. At z_k we have

$$\begin{aligned} M_1^{(1)}(z_k) &= \lim_{z\to z_k} M_1(z)T(z)^{-1} = \operatorname*{Res}_{z_k} M_1(z) \cdot (1/T)'(z_k) \\ &= c_k e^{2it\theta_k} M_2(z_k)(1/T)'(z_k), \\ \operatorname*{Res}_{z_k} M_2^{(1)}(z) &= \operatorname*{Res}_{z=z_k} M_2(z)T(z) = M_2(z_k)\left[(1/T)'(z_k)\right]^{-1} \\ &= c_k^{-1}\left[(1/T)'(z_k)\right]^{-2} e^{-2it\theta} M_1^{(1)}(z_k), \end{aligned}$$

from which the first formula in (3.8) clearly follows. The computation of the residue at z_k^* for $k \in \Delta_\xi^-$ is similar. □

4. Introducing $\overline{\partial}$ extensions of jump factorization

The next step in our analysis is to introduce factorizations of the jump matrix whose factors admit continuous–but not necessarily analytic–extensions off the real axis. Using these extensions we define a new unknown that deforms the oscillatory jump along the real axis onto new contours along which the jumps are decaying. The price we pay for this non-analytic transformation is that the new unknown has nonzero $\overline{\partial}$ derivatives inside the regions in which the extensions are introduced and satisfies a hybrid $\overline{\partial}$/Riemann-Hilbert problem.

Define the contours

$$\Sigma_k = \xi + e^{i(2k-1)\pi/4}\,\mathbb{R}_+, \quad k = 1,2,3,4, \tag{4.1}$$

oriented with increasing real part and denote the six open sectors in $\mathbb{C}$ — separated by $\mathbb{R}$ and the collection of Σ_k, $k = 1,\dots,4$ — by Ω_k, $k = 1,\dots,6$ starting with the sector Ω_1 between $[\xi,\infty)$ and Σ_1 and numbered consecutively continuing counterclockwise, see Figure 2. Additionally, let

$$\mu = \operatorname{dist}(\mathcal{Z},\mathbb{R}) \qquad \rho = \min\left\{ \min_{\substack{j,k\in\mathcal{Z}\\ j\neq k}} |z_j - z_k|_\infty,\ \mu \right\} \tag{4.2}$$

be the minimal distance from the discrete spectrum to the real axis (positive by assumption) and the lesser of μ and the minimal ∞-norm distance between points of discrete spectra respectively. Let $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C},[0,1])$ be supported near the discrete spectrum such that

$$\chi_{\mathcal{Z}}(z) = \begin{cases} 1 & \operatorname{dist}(z,\mathcal{Z}\cup\mathcal{Z}^*) < \mu/3 \\ 0 & \operatorname{dist}(z,\mathcal{Z}\cup\mathcal{Z}^*) > 2\mu/3 \end{cases} \tag{4.3}$$

Standard practice in the analysis of RHPs dictates that we should extend the first and last terms in each factorization in (3.7) to the right and left sides of the contour respectively. We define these extensions of the off-diagonal entries in (3.7) in the following lemma.

Lemma 4.1. *It is possible to define functions $R_j : \overline{\Omega}_j \to \mathbb{C}$, $j = 1,3,4,6$, with boundary values satsifying*

$$R_1(z) = \begin{cases} r(z)T(z)^{-2} & z \in (\xi,\infty) \\ r(\xi)T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_1 \end{cases}$$

$$R_3(z) = \begin{cases} \dfrac{r(z)^*}{1+|r(z)|^2}T_+(z)^2 & z \in (-\infty,\xi) \\ \dfrac{r(\xi)^*}{1+|r(\xi)|^2}T_0(\xi)^2(z-\xi)^{2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_2 \end{cases}$$

$$R_4(z) = \begin{cases} \dfrac{r(z)}{1+|r(z)|^2}T_-(z)^{-2} & z \in (-\infty,\xi) \\ \dfrac{r(\xi)}{1+|r(\xi)|^2}T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_3 \end{cases}$$

$$R_6(z) = \begin{cases} r(z)^*T(z)^2 & z \in (\xi,\infty) \\ r(\xi)^*T_0(\xi)^2(z-\xi)^{2i\kappa(\xi)}(1-\chi_{\mathcal{Z}}(z)) & z \in \Sigma_4 \end{cases}$$

such that for a fixed constant $c_1 = c_1(\psi_0)$, and a fixed cutoff function $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C}, [0,1])$ satisfying (4.3) *we have*

$$\begin{aligned} \left|\overline{\partial} R_j(z)\right| &\le c_1 \chi_{\mathcal{Z}}(z) + c_1 \left|r'(\operatorname{Re} z)\right| + c_1 |z-\xi|^{-1/2}, \\ \overline{\partial} R_j(z) &= 0 \quad \text{if } \operatorname{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) \le \mu/3. \end{aligned} \tag{4.4}$$

Moreover, if we set $R : \mathbb{C} \to \mathbb{C}$ by $R(z)\big|_{z \in \Omega_j} = R_j(z)$, (with $R_2(z) = R_5(z) = 0$), the extension can be made to preserve the symmetry $R(z^*)^* = R(z)$.

Proof. Using the constant $T_0(\xi)$ defined in Prop. 3.1, define the functions

$$\begin{aligned} f_1(z) &= r(\xi) T^2(z) T_0(\xi)^{-2} (z-\xi)^{-2i\kappa(\xi)} & z \in \overline{\Omega}_1 \\ f_3(z) &= \frac{r(\xi)^*}{1+|r(\xi)|^2} T(z)^2 T_0(\xi)^{-2} (z-\xi)^{-2i\kappa(\xi)} & z \in \overline{\Omega}_3, \end{aligned}$$

Define, for $z \in \overline{\Omega}_j$, $j = 1, 3$, the extensions

$$\begin{aligned} R_1(z) &= \left[f_1(z) + (r(\operatorname{Re} z) - f_1(z)) \cos(2\phi)\right] T(z)^{-2} (1 - \chi_{\mathcal{Z}}(z)), \\ R_3(z) &= \left[f_3(z) + \left(\frac{r(\operatorname{Re} z)^*}{1+|r(\operatorname{Re} z)|^2} - f_3(z)\right)\cos(2\phi)\right] T(z)^2 (1-\chi_{\mathcal{Z}}(z)). \end{aligned}$$

The extensions R_4 and R_6 are defined using part b. of Prop. 3.1 and choosing $\chi_{\mathcal{Z}}(z)$ to respect Schwartz symmetry; we define $R_4 = R_3(z^*)^*$ and $R_6(z) = R_1(z^*)^*$ which preserves the Schwartz reflection symmetry.

We give the rest of the details for R_1 only. The other cases are easily inferred. Clearly, $R_1(z)$ satisfies the boundary conditions of the lemma as $\cos(2\phi)$ vanishes on Σ_1 and $\chi_{\mathcal{Z}}(z)$ is zero on the real axis. Since $\overline{\partial} = (\partial_x + i\partial_y)/2 = e^{i\phi}(\partial_\rho + i\rho^{-1}\partial_\phi)/2$, we have

$$\begin{aligned} \overline{\partial} R_1(z) = &- \left[f_1(z) + (r(\operatorname{Re} z) - f_1(z)) \cos 2\phi\right] T(z)^{-2} \overline{\partial} \chi_{\mathcal{Z}}(z) \\ &+ \left[\frac{1}{2} r'(\operatorname{Re} z)\cos(2\phi) - i e^{i\phi} \frac{r(\operatorname{Re}(z)) - f_1(z)}{|z-\xi|} \sin(2\phi)\right] T(z)^{-2}(1-\chi_{\mathcal{Z}}(z)) \end{aligned}$$

We arrive at (4.4) by observing that $r(z)$ is bounded (since we are assuming there are no imbedded eigenvalues) and as both $1 - \chi_{\mathcal{Z}}(z)$ and $\chi'_{\mathcal{Z}}(z)$ are supported away from the discrete spectrum, the poles and zeros of $T(z)$ do not affect the bound. This gives the first two terms in the bound. For the last term we write

$$|r(\operatorname{Re} z) - f_1(z)| \le |r(\operatorname{Re} z) - r(\xi)| + |r(\xi) - f_1(z)|$$

and use Cauchy-Scwartz to bound each term as follows:

$$|r(\operatorname{Re} z) - r(\xi)| \le \left| \int_\xi^{\operatorname{Re} z} r'(s) ds \right| \le \|r\|_{H^1(\mathbb{R})} |z-\xi|^{1/2}$$

and

$$|r(\xi) - f_1(z)| \le |r(\xi)|(1+|r(\xi)|^2) \left| T(z,\xi)^2 - T_0(\xi)^2 (z-\xi)^{2i\kappa(\xi)|} \right| \le C_\xi \|r\|_{H^1(\mathbb{R})} |z-\xi|^{1/2}.$$

The last estimate uses (3.5) and the fact that $T(z,\xi)$ and $(z-\xi)^{i\kappa(\xi)}$ are bounded functions in a neighborhood of $z=\xi$. The bound (4.4) for $z\in\Omega_1$ follows immediately. □

We use the extension in Lemma 4.1 and the factorized jump matrices in (3.7) to define a new unknown function

$$M^{(2)}(z)=\begin{cases} M^{(1)}(z)\begin{pmatrix} 1 & 0 \\ -R_1(z)e^{2it\theta} & 1\end{pmatrix} & z\in\Omega_1 \\ M^{(1)}(z)\begin{pmatrix} 1 & -R_3(z)e^{-2it\theta} \\ 0 & 1\end{pmatrix} & z\in\Omega_3 \\ M^{(1)}(z)\begin{pmatrix} 1 & 0 \\ R_4(z)e^{2it\theta} & 1\end{pmatrix} & z\in\Omega_4 \\ M^{(1)}(z)\begin{pmatrix} 1 & R_6(z)e^{-2it\theta} \\ 0 & 1\end{pmatrix} & z\in\Omega_6 \\ M^{(1)}(z) & z\in\Omega_2\cup\Omega_5 \end{cases} \tag{4.5}$$

Let $\Sigma^{(2)}=\bigcup_{j=1}^4\Sigma_k$. It is an immediate consequence of Lemma 4.1 and RHP 3.1 that $M^{(2)}$ satisfies the following $\overline{\partial}$-Riemann-Hilbert problem.

$\overline{\partial}$-Riemann-Hilbert Problem 4.1 *Find a function $M^{(2)}:\mathbb{C}\backslash(\Sigma^{(2)}\cup\mathcal{Z}\cup\mathcal{Z}^*)\to SL_2(\mathbb{C})$ with the following properties.*

1. *$M^{(2)}$ has continuous first partial derivatives in $\mathbb{C}\backslash(\Sigma^{(2)}\cup\mathcal{Z}\cup\mathcal{Z}^*)$.*
2. *$M^{(2)}(z)=I+\mathcal{O}\left(z^{-1}\right)$ as $z\to\infty$.*
3. *For $z\in\Sigma^{(2)}$, the boundary values satisfy the jump relation $M_+^{(2)}(z)=M_-^{(2)}(z)V^{(2)}(z)$, where*

$$V^{(2)}(z)=I+(1-\chi_{\mathcal{Z}}(z))\delta V^{(2)},$$

$$\delta V^{(2)}(z)=\begin{cases} \begin{pmatrix} 0 & 0 \\ r(\xi)T_0(\xi)^{-2}(z-\xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0\end{pmatrix} & z\in\Sigma_1 \\ \begin{pmatrix} 0 & \frac{r(\xi)^*T_0(\xi)^2}{1+|r(\xi)|^2}(z-\xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0\end{pmatrix} & z\in\Sigma_2 \\ \begin{pmatrix} 0 & 0 \\ \frac{r(\xi)T_0^{-2}(\xi)}{1+|r(\xi)|^2}(z-\xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0\end{pmatrix} & z\in\Sigma_3 \\ \begin{pmatrix} 0 & r(\xi)^*T_0(\xi)^2(z-\xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0\end{pmatrix} & z\in\Sigma_4 \end{cases} \tag{4.6}$$

4. *For $z\in\mathbb{C}$ we have*

$$\overline{\partial}M^{(2)}(z)=M^{(2)}(z)W^{(2)}(z)$$

where

$$W^{(2)}(z) = \begin{cases} \begin{pmatrix} 0 & 0 \\ -\overline{\partial} R_1(z) e^{2it\theta} & 0 \end{pmatrix} & z \in \Omega_1 \\ \begin{pmatrix} 0 & -\overline{\partial} R_3(z) e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Omega_3 \\ \begin{pmatrix} 0 & 0 \\ \overline{\partial} R_4(z) e^{2it\theta} & 0 \end{pmatrix} & z \in \Omega_4 \\ \begin{pmatrix} 0 & \overline{\partial} R_6(z) e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & z \in \Omega_6 \\ \mathbf{0} & \textit{elsewhere} \end{cases} \tag{4.7}$$

5. $M^{(2)}(z)$ *has simple poles at each* $z_k \in \mathcal{Z}$ *and* $z_k^* \in \mathcal{Z}^*$ *at which*

$$\begin{aligned} \operatorname*{Res}_{z_k} M^{(2)} &= \begin{cases} \lim_{z \to z_k} M^{(2)} \begin{pmatrix} 0 & c_k^{-1} (1/T)'(z_k)^{-2} e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim_{z \to z_k} M^{(2)} \begin{pmatrix} 0 & 0 \\ c_k T(z_k)^{-2} e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \\ \operatorname*{Res}_{z_k^*} M^{(2)} &= \begin{cases} \lim_{z \to z_k^*} M^{(2)} \begin{pmatrix} 0 & 0 \\ -(c_k^*)^{-1} T'(z_k^*)^{-2} e^{2it\theta} & 0 \end{pmatrix} & k \in \Delta_\xi^- \\ \lim_{z \to z_k^*} M^{(2)} \begin{pmatrix} 0 & -c_k^* T(z_k^*)^2 e^{-2it\theta} \\ 0 & 0 \end{pmatrix} & k \in \Delta_\xi^+ \end{cases} \end{aligned} \tag{4.8}$$

Remark 4.1. In the $\overline{\partial}$-RHP for $M^{(2)}$ above, it is useful to recall how the extensions $R_j(z)$ are defined in Lemma 4.1, particularly the second condition in (4.4). Though (4.7) may seem to suggest that $M^{(2)}$ is non-analytic near the points of discrete spectra, the $\overline{\partial}$-derivative vanishes in small neighborhoods of each point of discrete spectra so that $M^{(2)}$ is analytic in each neighborhood.

The $\overline{\partial}$-Riemann-Hilbert problem 4.1 is now ideally conditioned for large t asymptotic analysis. It has jump matrices which approach identity point-wise, all residues corresponding to solitons whose speeds differ from the characteristic defined by ξ are exponentially small, and Lemma 4.1 controls the $\overline{\partial}$ derivatives in a manageable way. The final two sections construct the solution $M^{(2)}$ as follows,

(1) The $\overline{\partial}$ component of $\overline{\partial}$-RHP 4.1 is ignored, and we prove the existence of a solution of the resulting pure Riemann-Hilbert problem and compute its asymptotic expansion.

(2) Conjugating off the solution of the first step, we arrive at a pure $\overline{\partial}$ problem which we show has a solution and bound its size.

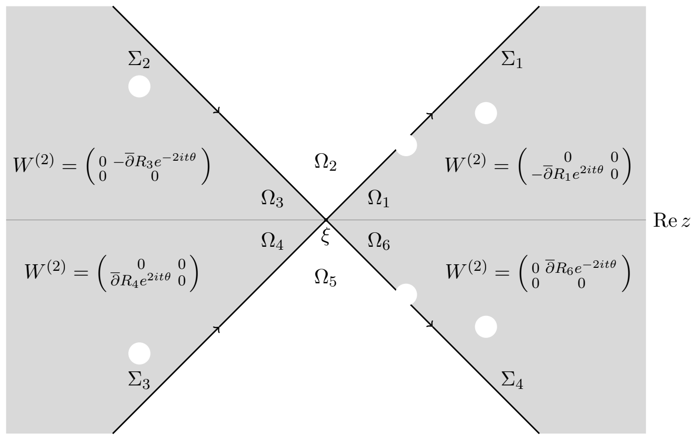

Figure 2. *The contours Σ_k and regions Ω_k $k = 1, \ldots, 6$ defining the $\bar{\partial}$-relationship for the matrix $M^{(2)}$. The support of the $\bar{\partial}$-derivatives, $\overline{\partial} M^{(2)} = M^{(2)} W^{(2)}$, is shaded in gray.*

Unwinding the series of transformations that led from the M to $M^{(2)}$ we recover the solution RHP 2.1 and then from (2.13) we recover a long time asymptotic expansion of the solution $q(x,t)$ of NLS for our class of initial data.

5. Removing the Riemann-Hilbert component of the solution

In this section we build a solution $M^{(2)}_{\text{RHP}}$ to the Riemann-Hilbert problem that results from the $\overline{\partial}$-RHP for $M^{(2)}$ by dropping the $\overline{\partial}$ component. Specifically,

$$\text{Let } M^{(2)}_{\text{RHP}} \text{ be the solution of the Riemann-Hilbert problem resulting from setting } W^{(2)} \equiv 0 \text{ in } \overline{\partial}\text{-RHP 4.1.} \tag{5.1}$$

In this section we will prove that the solution $M^{(2)}_{\text{RHP}}$ exists and construct its asymptotic expansion for large t. Before we embark upon this adventure, we first show that if $M^{(2)}_{\text{RHP}}$ exists, it reduces $\overline{\partial}$-RHP 4.1 to a pure $\overline{\partial}$ problem.

Proposition 5.1. *Suppose that $M^{(2)}_{\text{RHP}}$ is a solution of the Riemann-Hilbert problem described in* (5.1)*, then the ratio*

$$M^{(3)}(z) := M^{(2)}(z) M^{(2)}_{\text{RHP}}(z)^{-1} \tag{5.2}$$

is a continuously differentiable function satisfying the following $\overline{\partial}$-problem.

$\overline{\partial}$ **Problem 5.1** *Find a function $M^{(3)} : \mathbb{C} \to SL_2(\mathbb{C})$ with the following properties.*

1. *$M^{(3)}$ has continuous first partial derivatives in $\mathbb{C}$.*
2. *$M^{(3)}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.*
3. *For $z \in \mathbb{C}$, we have*

$$\overline{\partial} M^{(3)}(z) = M^{(3)}(z) W^{(3)} \tag{5.3}$$

where $W^{(3)} := M^{(2)}_{\text{RHP}}(z) W^{(2)}(z) M^{(2)}_{\text{RHP}}(z)^{-1}$ and $W^{(2)}$ is as defined in (4.7).

Proof. Both $M^{(2)}$ and $M^{(2)}_{\text{RHP}}$ are unimodular and approach identity as z tends to infinity. It follows from (5.2) that $M^{(3)}$ inherits these properties as well as continuous differentiability in $\mathbb{C} \backslash \Sigma^{(2)}$. Since both $M^{(2)}$ and $M^{(2)}_{\text{RHP}}$ satisfy the same jump relation (4.6), we have

$$\begin{aligned} {M^{(3)}}_{-}^{-1} M^{(3)}_{+} &= M^{(2)}_{\text{RHP}-}(z) M^{(2)}_{-}(z)^{-1} M^{(2)}_{+}(z) M^{(2)}_{\text{RHP}+}(z)^{-1} \\ &= M^{(2)}_{\text{RHP}-}(z) V^{(2)}(z) \left(M^{(2)}_{\text{RHP}-}(z) V^{(2)}(z) \right)^{-1} = I, \end{aligned}$$

from which it follows that $M^{(3)}$ and its first partials extend continuously to $\Sigma^{(2)}$.

Both $M^{(2)}$ and $M^{(2)}_{\text{RHP}}$ are analytic in some deleted neighborhood of each point of discrete spectra z_k and satisfy the residue relation (4.8). Let N_k denote the constant (in z) nilpotent matrix which appears in the left side of (4.8), then we have the Laurent expansions

$$\begin{aligned} M^{(2)}(z) &= C_0 \left[\frac{N_k}{z - z_k} + I \right] + \mathcal{O}\left(z - z_k\right), \\ M^{(2)}_{\text{RHP}}(z)^{-1}(z) &= \left[\frac{-N_k}{z - z_k} + I \right] \widehat{C}_0 + \mathcal{O}\left(z - z_k\right), \end{aligned} \tag{5.4}$$

where C_0 and $\widehat{C}_0$ are the constant terms in the Laurent expansions of $M^{(2)}(z)$ and $M^{(2)}_{\text{RHP}}(z)^{-1}$ respectively. This implies that

$$M^{(2)}(z) M^{(2)}_{\text{RHP}}(z)^{-1}(z) = \mathcal{O}(1), \tag{5.5}$$

i.e., $M^{(3)}$ has only removable singularities at each z_k. The last property follows immediately from the definition of $M^{(3)}$, exploiting the fact that $M^{(2)}_{\text{RHP}}$ has no $\overline{\partial}$ component:

$$\overline{\partial} M^{(3)}(z) = \overline{\partial} M^{(2)}(z) M^{(2)}_{\text{RHP}}(z)^{-1} = M^{(2)} W^{(2)}(z) M^{(2)}_{\text{RHP}}(z)^{-1} = M^{(3)} W^{(3)}(z).$$

□

5.1. **Constructing the model problems.** We will construct the solution $M^{(2)}_{\text{RHP}}$ by seeking a solution of the form

$$M^{(2)}_{\text{RHP}}(z) = \begin{cases} E(z) M^{(\text{out})}(z) & |z - \xi| > \mu/2 \\ E(z) M^{(\xi)}(z) & |z - \xi| < \mu/2 \end{cases} \tag{5.6}$$

where $M^{(\mathrm{out})}$ and $M^{(\xi)}$ are models which we construct below, and the error $E(z)$, the solution of a small norm Riemann-Hilbert problem, we will prove exists and bound it asymptotically.

5.1.1. *The outer model: an N-soliton potential.* The matrix $M^{(2)}_{\mathrm{RHP}}$ is meromorphic away from the contour $\Sigma^{(2)}$ on which its boundary values satisfy the jump relation (4.6). However, at any distance from the saddle point $z = \xi$, the jump is uniformly near identity. Specifically, let $\mathcal{U}_\xi$ denote the open neighborhood

$$\mathcal{U}_\xi = \{z \,:\, |z-\xi| < \mu/2\}, \tag{5.7}$$

on which $M^{(2)}_{\mathrm{RHP}}$ is pole free. Using the spectral bounds (4.2) and (4.6) we have

$$\|V^{(2)} - I\|_{L^\infty(\Sigma^{(2)})} = \mathcal{O}\left(\rho^{-2} e^{-\sqrt{2}t|z-\xi|^2}\right), \tag{5.8}$$

which is exponentially small in $\Sigma^{(2)} \backslash \mathcal{U}_\xi$, since $|z-\xi| \geq \mu/2$ outside $\mathcal{U}_\xi$. This estimate justifies constructing a model solution outside $\mathcal{U}_\xi$ which ignores the jumps completely.

Proposition 5.2. *Let $M^{(out)} : \mathbb{C} \to SL_2(\mathbb{C})$ be a meromorphic function such that*

- *$M^{(out)}$ is a meromorphic function from $\mathbb{C} \to SL_2(\mathbb{C})$*
- *$M^{(out)}(z) = I + \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$.*
- *$M^{(out)}$ has simple poles at each $z_k \in \mathcal{Z}$ and $z_k^* \in \mathcal{Z}^*$ satisfying the residue relations in (4.8) with $M^{(out)}$ replacing $M^{(2)}$.*

These conditions uniquely determine $M^{(out)}$. Moreover,

$$\lim_{z\to\infty} 2iz M^{(out)}_{12}(z;x,t) = \psi(x,t;\sigma_d^{out})$$

where $\psi(x,t;\sigma_d^{(out)})$ is the N-soliton solution of (1.1) corresponding to the discrete scattering data $\sigma_d^{(out)} := \{z_k, \widetilde{c}_k(\xi)\}_{k=1}^N$ where

$$\widetilde{c}_k(\xi) = c_k \exp\left(\frac{i}{\pi}\int_{-\infty}^{\xi} \log(1+|r(s)|^2)\frac{ds}{s-z_k}\right).$$

Proof. This is a simple consequence of the results in Appendix B. The properties required of $M^{(\mathrm{out})}$ are equivalent to RHP B.2 with $\Delta = \Delta_\xi^-$ and $\sigma_d = \sigma_d^{(\mathrm{out})}$. The uniqueness of solution and asymptotic behavior are consequences of Prop. B.1 and (B.7). □

5.1.2. *Local model near the saddle point $z = \xi$.* For $z \in \mathcal{U}_\xi$ the bound (5.8) gives a pointwise, but not uniform estimate on the decay of the jump $V^{(2)}$ to identity. In order to arrive at a uniformly small jump Riemann-Hilbert problem for the function E, implicitly defined by (5.6) we introduce a different local model $M^{(\xi)}$ which exactly matches the jumps of

$M^{(2)}_{\text{RHP}}$ on $\Sigma^{(2)} \cap \mathcal{U}_\xi$. In order to motivate the model let $\zeta = \zeta(z)$ denote the rescaled local variable

$$\zeta = \zeta(z) = 2\sqrt{t}(z-\xi) \quad \Rightarrow \quad \zeta^2/2 = 2t(z-\xi)^2 \tag{5.9}$$

which maps $\mathcal{U}_\xi$ to an expanding neighborhood of $\zeta = 0$. Additionally, let

$$r_\xi := r(\xi) T_0(\xi)^{-2} e^{2i(\kappa(\xi)\log(2\sqrt{t}) - t\xi^2)}. \tag{5.10}$$

Then, since $1 - \chi_{\mathcal{Z}}(z) \equiv 1$ for $z \in \mathcal{U}_\xi$, the jumps of $M^{(2)}_{\text{RHP}}$ in $\mathcal{U}_\xi$ can be expressed as

$$V^{(2)}(z)\Big|_{z\in\mathcal{U}_\xi} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_\xi\, \zeta(z)^{-2i\kappa(\xi)} e^{i\zeta(z)^2/2} & 1 \end{pmatrix} & z \in \Sigma_1 \\ \begin{pmatrix} 1 & \frac{r_\xi^*}{1+|r_\xi|^2} \zeta(z)^{2i\kappa(\xi)} e^{-i\zeta(z)^2/2} \\ 0 & 1 \end{pmatrix} & z \in \Sigma_2 \\ \begin{pmatrix} 1 & 0 \\ \frac{r_\xi}{1+|r_\xi|^2} \zeta(z)^{-2i\kappa(\xi)} e^{i\zeta(z)^2/2} & 1 \end{pmatrix} & z \in \Sigma_3 \\ \begin{pmatrix} 1 & r_\xi^*\, \zeta(z)^{2i\kappa(\xi)} e^{-i\zeta(z)^2/2} \\ 0 & 1 \end{pmatrix} & z \in \Sigma_4, \end{cases} \tag{5.11}$$

which are *exactly* the jumps of the parabolic cylinder model problem (A.3) described in Appendix A. Then using (A.4) we define the local model $M^{(\xi)}$ in (5.6) by

$$M^{(\xi)}(z) = M^{(\text{out})}(z) M^{(\text{PC})}(\zeta(z), r_\xi), \qquad z \in \mathcal{U}_\xi, \tag{5.12}$$

which satisfies the jump $V^{(2)}$ of $M^{(2)}_{\text{RHP}}$ as $M^{(\text{out})}$ is an analytic and bounded function in $\mathcal{U}_\xi$.

5.2. **The small norm Riemann-Hilbert problem for $E(z)$.** Using the functions $M^{(\text{out})}$ and $M^{(\xi)}$ defined by Prop 5.2 and (5.12) respectively, (5.6) implicitly defines an unknown $E(z)$ which is analytic in $\mathbb{C}\backslash\Sigma^{(E)}$,

$$\Sigma^{(E)} = \partial\mathcal{U}_\xi \cup (\Sigma^{(2)} \backslash \mathcal{U}_\xi),$$

where we orient $\partial\mathcal{U}_\xi$ clockwise. It is straightforward to show that $E(z)$ must satisfy the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 5.1 *Find a holomorphic function $E : \mathbb{C}\backslash\Sigma^{(E)} \to SL_2(\mathbb{C})$ with the following properties*

1. $E(z) = I + \mathcal{O}\left(z^{-1}\right)$ *as* $z \to \infty$.
2. *For each* $z \in \Sigma^{(E)}$ *the boundary values* $E_\pm(z)$ *satisfy* $E_+(z) = E_-(z) V^{(E)}(z)$ *where*

$$V^{(E)}(z) = \begin{cases} M^{(out)}(z)(z) V^{(2)}(z) M^{(out)}(z)^{-1} & z \in \Sigma^{(2)} \backslash \mathcal{U}_\xi \\ M^{(out)}(z) M^{(\text{PC})}(\zeta(z), r_\xi) M^{(out)}(z)^{-1} & z \in \partial\mathcal{U}_\xi \end{cases} \tag{5.13}$$

Starting from (5.13) and using (5.8) for $z \in \mathbb{C} \backslash \mathcal{U}_\xi$ and, using (5.9),(A.6) and the boundedness of $M^{(\mathrm{out})}$ for $z \in \mathcal{U}_\xi$, one finds that

$$|V_E(z) - I| = \begin{cases} \mathcal{O}\left(\rho^{-2} e^{-\sqrt{2}t|z-\xi|^2}\right) & z \in \Sigma^{(E)} \backslash \mathcal{U}_\xi \\ \mathcal{O}\left(t^{-1/2}\right) & z \in \partial \mathcal{U}_\xi, \end{cases} \tag{5.14}$$

and it follows that

$$\|V_E - I\|_{L^{k,p}(\Sigma^{(E)})} = \mathcal{O}\left(t^{-1/2}\right) \qquad p \in [1, \infty],\ k \geq 0. \tag{5.15}$$

This uniformly vanishing bound on $V_E - I$ establishes RHP 5.1 as a small-norm Riemann-Hilbert problem, for which there is a well known existence and uniqueness theorem [6, 9, 19]. In fact, we may write

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma^{(E)}} \frac{(I + \eta(s))(V_E(s) - I)}{s - z} ds \tag{5.16}$$

where $\eta \in L^2(\Sigma^{(E)})$ is the unique solution of

$$(\mathbf{1} - C_{V^{(E)}})\,\eta = C_{V^{(E)}} I. \tag{5.17}$$

Here $C_{V^{(E)}} : L^2(V^{(E)}) \to L^2(V^{(E)})$ is the singular integral operator defined by $C_{V^{(E)}} f = C_-(f(V_E - I))$ where C_- is the Cauchy projection operator

$$C_- f(z) \lim_{z \to \Sigma^{(E)}_-} \frac{1}{2\pi i} \int_{\Sigma^{(E)}} f(s) \frac{ds}{s - z}.$$

It's well known that $\|C_-\|_{L^2(\Gamma) \to L^2(\Gamma)}$ is bounded for a very large class of contours Γ including the class of finite unions of analytic curves with finite intersection which includes $\Sigma^{(E)}$. It then follows from (5.15) that

$$\|C_{V^{(E)}}\|_{L^2(\Sigma^{(E)}) \to L^2(\Sigma^{(E)})} = \mathcal{O}\left(t^{-1/2}\right), \tag{5.18}$$

which guarantees the existence of the resolvent operator $(\mathbf{1} - C_{V^{(E)}})^{-1}$ and thus of both η and E.

The existence of the solution $E(z)$ completes the definition of $M^{(2)}_{\mathrm{RHP}}(z)$ given by (5.6) which in turn solves (5.1) and thus also justifies the transformation (5.2) of Proposition 5.1 to an unknown $M^{(3)}$ which satisfies the pure $\overline{\partial}$-Problem 5.1.

In order to reconstruct the solution $\psi(x, t)$ of (1.1) we need the large z behavior of the solution of RHP 2.1. This will include the large z expansion of E which we give here. Geometrically expanding $(s - z)^{-1}$ for z large in (5.16), which is justified by the finiteness of moments in (5.15), we have

$$E(z) = I + z^{-1} E_1 + \mathcal{O}\left(z^{-2}\right) \tag{5.19}$$

where

$$E_1 = -\frac{1}{2\pi i} \int_{\Sigma^{(E)}} (I + \eta(s))(V^{(E)}(s) - I)\, ds. \tag{5.20}$$

Then using (5.17)-(5.18) and the bounds on $V_E - I$ in (5.14)-(5.15) we have

$$E_1 = -\frac{1}{2\pi i}\oint_{\partial\mathcal{U}_\xi} \left(V^E(s) - I\right) ds + \mathcal{O}\left(t^{-1}\right).$$

This last integral, using (5.13), (A.6), (A.5) and (5.9) can be asymptotically computed by residues yielding (recall that $\partial\mathcal{U}_\xi$ is clockwise oriented) to leading order

$$E_1 = \frac{t^{-1/2}}{2} M^{(\text{out})}(\xi) \begin{pmatrix} 0 & -i\beta_{12}(r_\xi) \\ i\beta_{21}(r_\xi) & 0 \end{pmatrix} M^{(\text{out})}(\xi)^{-1} + \mathcal{O}\left(t^{-1}\right). \tag{5.21}$$

6. Analysis of the remaining $\overline{\partial}$-problem

$\overline{\partial}$-Problem 5.1 is equivalent to the integral equation

$$M^{(3)}(z) = I + \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{\partial} M^{(3)}(s)}{s - z} dA(s) = I + \frac{1}{\pi}\int_{\mathbb{C}} \frac{M^{(3)}(s) W^{(3)}}{s - z} dA(s), \tag{6.1}$$

where $s = u + iv$.

Equation (6.1), can be written using operator notation as

$$(I - S)[M^{(3)}(z)] = I, \tag{6.2}$$

where S is the solid Cauchy operator

$$S[f] = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f W^{(3)}}{s - z} dA(s). \tag{6.3}$$

The goal at this point is to show that S is small in operator norm so that (6.2) may be inverted by Neumann series.

Proposition 6.1. *There exists a constant C such that for all $t > 0$, the operator* (6.3) *satisfies the inequality*

$$\|S\|_{L^\infty \to L^\infty} \le C t^{-1/4}. \tag{6.4}$$

Proof. We detail the case for matrix functions having support in the region Ω_1, the case for the other regions follows similarly. Let $A \in L^\infty(\Omega_1)$, then from (4.7) and (5.1) it follows that

$$\begin{aligned} |S[A]| &\le \iint_{\Omega_1} \frac{|A M^{(2)}_{\text{RHP}}(z) W^{(2)}(z) M^{(2)}_{\text{RHP}}(z)^{-1}|}{|s - z|} dA(s) \\ &\le \|A\|_\infty \|M^{(2)}_{\text{RHP}}(z)\|_\infty \|M^{(2)}_{\text{RHP}}(z)^{-1}\|_\infty \iint_{\Omega_1} \frac{|\bar{\partial} R_1 e^{2it\theta}|}{|s - z|} dA(s), \end{aligned}$$

where we note that $M^{(2)}_{\text{RHP}}(z) W^{(2)}(z) M^{(2)}_{\text{RHP}}(z)^{-1}$ is supported away from the poles z_k so that $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\text{supp}(R_1))}$.

From (4.4) we have the inequality

$$|S[A]| \leq C\left(I_1 + I_2 + I_3\right),$$

where

$$I_1 = \iint_{\Omega_1} \frac{|\chi_{\mathcal{Z}}(z)| e^{-4tv(u-\xi)}}{|s-z|} dA(s), \qquad I_2 = \iint_{\Omega_1} \frac{|r'(u)| e^{-4tv(u-\xi)}}{|s-z|} dA(s),$$

and

$$I_3 = \iint_{\Omega_1} \frac{|z-\xi|^{-1/2} e^{-4tv(u-\xi)}}{|s-z|} dA(s).$$

As detailed in C, we see that there exist constants c_1, c_2, and c_3 such that for all $t > 0$ we have the bounds

$$|I_1| \leq \frac{c_1}{t^{1/4}} \qquad |I_2| \leq \frac{c_2}{t^{1/4}} \qquad |I_3| \leq \frac{c_3}{t^{1/4}}.$$

and the result is proven.

□

For sufficiently large t it is possible to invert the operator (6.2) by Neumann series. Furthermore, to detail the long-time asymptotic behavior of $\psi(x,t)$ as mentioned in (2.13), it is necessary to determine the asymptotic behavior of the coefficient of the $\frac{1}{z}$ term in the Laurent expansion of $M^{(3)}$. An integral representation of this term is given by the expansion

$$M^{(3)}(z) = I + \frac{1}{\pi} \int_{\mathbb{C}} \frac{M^{(3)}(s) W^{(3)}}{s-z} dA(s) = I - \frac{1}{\pi} \int_{\mathbb{C}} \left(\frac{M^{(3)}(s) W^{(3)}}{z} - \frac{s M^{(3)}(s) W^{(3)}}{z(s-z)} \right) dA(s).$$

Therefore we seek the asymptotic behavior of

$$M_1^{(3)} = \int_{\mathbb{C}} M^{(3)}(s) W^{(3)} dA(s), \tag{6.5}$$

as in the following proposition.

Proposition 6.2. *For all $t > 0$ there exists a constant c such that*

$$|M_1^{(3)}| \leq c t^{-3/4}. \tag{6.6}$$

A proof of Proposition 6.2 is detailed in Appendix C.

7. Long Time Asymptotics for NLS

From equations (3.6), (4.5), (5.2), and (5.6) we see that in the region Ω_2 we have the relationship

$$M = M^{(3)}(z)E(z)M^{(\text{out})}(z)T^{\sigma_3}.$$

We now take the large z expansions of these matrices to see that

$$M = \left(I + \frac{M_1^{(3)}}{z} + \cdots\right)\left(I + \frac{E_1}{z} + \cdots\right)\left(I + \frac{M_1^{(\text{out})}}{z} + \cdots\right)\left(I + \frac{T_1}{z} + \cdots\right),$$

and consequently the coefficient of the z^{-1} in the Laurent expansion of M is given by

$$M_1 = M_1^{(3)} + E_1 + M_1^{(\text{out})} + T_1. \tag{7.1}$$

From equation (2.13), we see that to recover the solution to the NLS equation we seek the off-diagonal entry of M_1 and therefore find that T_1 will make no contribution due to its diagonal structure. The asymptotic behavior of $M_1^{(3)}$ is detailed in Section 6 and does not make the dominant contribution to the asymptotic behavior. The contribution from $M_1^{(\text{out})}$ will make a dominant contribution to the asymptotics wherever $t \to \infty$ along any characteristic $x = x_0 + vt$ inside the truncated cone

$$x_1 + v_1 t \leq x \leq x_2 + v_2 t, \qquad t \geq 0$$

as described in Proposition B.2 of Appendix B. Away from such a trajectory, the contribution will be exponentially small. As for the contribution from E_1, we use (5.21), (5.10), and the fact that $\det(M^{(\text{out})}) = 1$ to see that

$$2i(E_1)_{12} = t^{-1/2}M^{(\text{out})}(\xi)_{11}^2\beta_{12}(r_\xi) + t^{-1/2}M^{(\text{out})}(\xi)_{12}^2\beta_{21}(r_\xi) + \mathcal{O}\left(t^{-1}\right),$$

where $\beta_{21} = \dfrac{\kappa}{\beta_{12}}$ is detailed in (A.5). Away from the soliton trajectories mentioned above, $M^{(\text{out})}(\xi)_{12}$ is exponentially small and $M^{(\text{out})}(\xi)_{11}$ is exponentially close to 1 so that the dominant asymptotic behavior is given by $t^{-1/2}\beta_{12}(r_\xi)$ and the main result follows from (3.5) and the identity $|\Gamma(i\kappa)|^2 = |\Gamma(-i\kappa)|^2 = \pi(\kappa\sinh(\pi\kappa))^{-1}$.

Appendix A. The parabolic cylinder model problem

Let $\Sigma_{PC} = \bigcup_{j=1}^4 \Sigma_j$, where Σ_j denotes the complex contour

$$\Sigma_j = \left\{\zeta \in \mathbb{C} \mid \arg\zeta = \frac{2j-1}{4}\pi\right\}, \quad j = 1, \ldots, 4, \tag{A.1}$$

oriented with increasing real part. Let Ω_j, $j = 1, \ldots 6$ denote the six maximally connected open sectors in $\mathbb{C}\backslash(\Sigma_{PC} \cup \mathbb{R})$, where Ω_1 denotes the sector abutting the positive real axis from above, the rest labelled sequentially as one encircles the origin in a counterclockwise fashion. Finally, fix $r \in \mathbb{C}$ and let

$$\kappa = \kappa(r) := -\frac{1}{2\pi} \log(1 + |r|^2). \tag{A.2}$$

Then consider the following Riemann-Hilbert problem

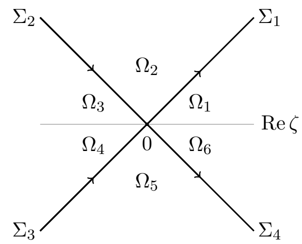

Figure 3. *The contours Σ_j and sectors Ω_j in the ζ-plane defining RHP A.1.*

Parabolic Cylinder Model Riemann-Hilbert Problem A.1 *Fix $r \in \mathbb{C}$, find an analytic function $M^{(PC)}(\cdot, r) : \mathbb{C} \backslash \Sigma^{(PC)} \to SL_2(\mathbb{C})$ such that*

1. $M^{(PC)}(\zeta, r) = I + \frac{M^{(PC)(1)}(r)}{\zeta} + \mathcal{O}\left(\zeta^{-2}\right)$ *uniformly as $\zeta \to \infty$.*

2. *For $\zeta \in \Sigma^{(PC)}$, the continuous boundary values $M_\pm^{(PC)}(\zeta, r)$ satisfying the jump relation $M_+^{(PC)}(\zeta, r) = M_-^{(PC)}(\zeta, r) V^{(PC)}(\zeta, r)$ where*

$$V^{(PC)}(\zeta, r) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r\zeta^{-2i\kappa} e^{i\zeta^2/2} & 1 \end{pmatrix} & \arg \zeta = \pi/4 \\ \begin{pmatrix} 1 & r^* \zeta^{2i\kappa} e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix} & \arg \zeta = -\pi/4 \\ \begin{pmatrix} 1 & \frac{r^*}{1+|r|^2} \zeta^{2i\kappa} e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix} & \arg \zeta = 3\pi/4 \\ \begin{pmatrix} 1 & 0 \\ \frac{r}{1+|r|^2} \zeta^{-2i\kappa} e^{i\zeta^2/2} & 1 \end{pmatrix} & \arg \zeta = -3\pi/4 \end{cases} \tag{A.3}$$

RHP A.1 has an explicit solution $M^{(PC)}(\zeta, r)$ which is expressed in terms of $D_a(\pm z)$, solutions of the parabolic cylinder equation, $\left(\frac{\partial^2}{\partial z^2} + \left(\frac{1}{2} - \frac{z^2}{2} + a\right)\right) D_a(z) = 0$, as follows:

$$M^{(PC)}(\zeta, r) = \Phi(\zeta, r) \mathcal{P}(\zeta, r) e^{\frac{i}{4}\zeta^2 \sigma_3} \zeta^{-i\kappa\sigma_3} \tag{A.4}$$

where

$$\mathcal{P}(\zeta,r)=\begin{cases}\begin{pmatrix}1 & 0\\ -r & 1\end{pmatrix} & \zeta\in\Omega_1\\ \begin{pmatrix}1 & \frac{-r^*}{1+|r|^2}\\ 0 & 1\end{pmatrix} & \zeta\in\Omega_3\\ \begin{pmatrix}1 & 0\\ \frac{r}{1+|r|^2} & 1\end{pmatrix} & \zeta\in\Omega_4\\ \begin{pmatrix}1 & r^*\\ 0 & 1\end{pmatrix} & \zeta\in\Omega_6\\ I & \zeta\in\Omega_2\cup\Omega_5\end{cases}$$

$$\Phi(\zeta,r)=\begin{cases}\begin{pmatrix}e^{-\frac{3\pi\kappa}{4}}D_{i\kappa}\left(e^{\frac{-3i\pi}{4}}\zeta\right) & -i\beta_{12}e^{\frac{\pi}{4}(\kappa-i)}D_{-i\kappa-1}\left(e^{-\frac{i\pi}{4}}\zeta\right)\\ i\beta_{21}e^{\frac{-3\pi}{4}(\kappa+i)}D_{i\kappa-1}\left(e^{\frac{-3i\pi}{4}}\zeta\right) & e^{\frac{\pi\kappa}{4}}D_{-i\kappa}\left(e^{\frac{-i\pi}{4}}\zeta\right)\end{pmatrix} & \zeta\in\mathbb{C}^+\\ \begin{pmatrix}e^{\frac{\pi\kappa}{4}}D_{i\kappa}\left(e^{\frac{i\pi}{4}}\zeta\right) & -i\beta_{12}e^{\frac{-3\pi}{4}(\kappa-i)}D_{-i\kappa-1}\left(e^{\frac{3i\pi}{4}}\zeta\right)\\ i\beta_{21}e^{\frac{\pi}{4}(\kappa+i)}D_{i\kappa-1}\left(e^{\frac{i\pi}{4}}\zeta\right) & e^{\frac{-3\pi\kappa}{4}}D_{-i\kappa}\left(e^{\frac{3i\pi}{4}}\zeta\right)\end{pmatrix} & \zeta\in\mathbb{C}^-\end{cases}$$

and β_{12} and β_{21} are the complex constants

$$\beta_{12}=\beta_{12}(r)=\frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\kappa/2}}{r\Gamma(-i\kappa)},\qquad \beta_{21}=\beta_{21}(r)=\frac{-\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\kappa/2}}{r^*\Gamma(i\kappa)}=\frac{\kappa}{\beta_{12}}. \tag{A.5}$$

A derivation of this result is given in [5], a direct verification of the solution in given in the appendix of [12]. The essential fact for our needs is the asymptotic behavior of the solution given in the above references, as is easily verified using the well known asymptotic behavior of $D_a(z)$,

$$M^{(PC)}(\zeta,r)=I+\frac{1}{\zeta}\begin{pmatrix}0 & -i\beta_{12}(r)\\ i\beta_{21}(r) & 0\end{pmatrix}+\mathcal{O}\left(\zeta^{-2}\right). \tag{A.6}$$

Appendix B. Meromorphic solutions of the NLS Riemann-Hilbert problem

Here we consider the solutions of the the Riemann-Hilbert problem associated with the NLS equation, RHP 2.1, for which the reflection coefficient $r(z)\equiv 0$. In this case the unknown function is analytic across the real axis and has isolated poles in the plane, i.e., the solution is meromorphic. The resulting, *reflectionless*, solutions of NLS, $\psi(x,t)$, derived from the solution of the Riemann-Hilbert problem, are (multi-)solitons. Here we give a simple proof of the existence and uniqueness of solutions of this problem and briefly discuss some well known results concerning the asymptotic behavior of these solutions as $t\to\infty$.

Given a finite set of discrete spectra and associated normalization constants, the reflectionless Riemann-Hilbert problem associated to NLS can be stated as follows.

Riemann-Hilbert Problem B.1 *Given discrete data* $\sigma_d=\{(z_k,c_k)\}_{k=1}^N\subset\mathbb{C}^+\times\mathbb{C}_*$ *find a meromorphic function* $m:\mathbb{C}\to SL_2(\mathbb{C})$ *with the following properties.*

1. $m(z)=I+\mathcal{O}(z)$ *as* $z\to\infty$

2. $m(z)$ is holomorphic in $\mathbb{C}\backslash(\mathcal{Z}\cup\mathcal{Z}^*)$ and has simple poles at each point $z_k \in \mathcal{Z}$ and $z_k^* \in \mathcal{Z}^*$ satisfying the residue conditions

$$\operatorname*{Res}_{z=z_k} m(z) = \lim_{z\to z_k} m(z) n_k \qquad \textit{and} \qquad \operatorname*{Res}_{z=z_k^*} m(z) = \lim_{z\to z_k^*} m(z)\sigma_2 n_k^* \sigma_2 \tag{B.1}$$

where n_k is the nilpotent matrix,

$$n_k = \begin{pmatrix} 0 & 0 \\ \gamma_k(x,t) & 0 \end{pmatrix} \qquad \gamma_k(x,t) := c_k \exp(2i(tz_k^2 + xz_k)).$$

It's a direct consequence of the symmetries in RHP B.1 (and more generally in RHP 2.1) that any solution of the problem must posses the symmetry $m(z) = \sigma_2 m(z^*)^* \sigma_2$. It follows that any solution of RHP B.1 must admit a partial fraction expansion of the form

$$m(z) = I + \sum_{k=1}^{N} \frac{1}{z-z_k} \begin{pmatrix} \alpha_k(x,t) & 0 \\ \beta_k(x,t) & 0 \end{pmatrix} + \frac{1}{z-z_k^*} \begin{pmatrix} 0 & -\beta_k(x,t)^* \\ 0 & \alpha_k(x,t)^* \end{pmatrix} \tag{B.2}$$

for coefficients $\alpha_k(x,t), \beta_k(x,t)$ to be determined.

Proposition B.1. *Given data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times \mathbb{C}_*$ such that $z_j \neq z_k$ for $j \neq k$, RHP B.1 has a unique solution.*

B.1. **Renormalizations of the reflectionless Riemann-Hilbert problem.**
The Riemann-Hilbert problem RHP B.1 which encodes the N-soliton solutions of (1.1) arises from a particular choice of normalization in the forward scattering step of the IST. Specifically, recalling that $J_1^-(x,t;z)$ and $J_2^+(x,t;z)$ denote the first and second columns respectively of the left and right normalized Jost functions $J^{\pm}(x,t;z)$ of the ZS-AKNS scattering problem, (2.1a), the matrix $m(z)$ in RHP B.1 is defined for $z \in \mathbb{C}^+$ as

$$m(z) = m(z;x,t) = \left[\frac{J_1^-(x,t;z)}{a(z)} \;\middle|\; J_2^+(x,t;z) \right] e^{i(tz^2+xz)\sigma_3}, \qquad a(z) = \prod_{k=1}^{N} \left(\frac{z-z_k}{z-z_k^*} \right), \tag{B.3}$$

where $1/a(z)$ is the transmission coefficient of the reflectionless initial data. This choice of normalization ensures that for any fixed t, $\lim_{x\to+\infty} m(z;x,t) = I$, but is not the only choice available to us.

Let $\Delta \subseteq \{1,2,\dots,N\}$ and $\nabla = \Delta^c = \{1,\dots,N\}\backslash\Delta$. Define

$$a_\Delta(z) = \prod_{k\in\Delta} \left(\frac{z-z_k}{z-z_k^*} \right) \qquad \text{and} \qquad a_\nabla(z) = \frac{a(z)}{a_\Delta(z)} = \prod_{k\in\nabla} \left(\frac{z-z_k}{z-z_k^*} \right). \tag{B.4}$$

The renormalization

$$m^\Delta(z) = m(z) a_\Delta(z)^{\sigma_3} = \left[\frac{J_1^-(x,t;z)}{a_\nabla(z)} \;\middle|\; \frac{J_2^+(x,t;z)}{a_\Delta(z)} \right] e^{i(tz^2+xz)\sigma_3}. \tag{B.5}$$

then splits the poles between the columns of $m^\Delta(z)$ according to the choice of Δ. It's a simple calculation to show that the renormalization m^Δ satisfies a modified discrete Riemann Hilbert problem.

Riemann-Hilbert Problem B.2 *Given discrete data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times \mathbb{C}_*$ and $\Delta \subseteq \{1, \ldots, N\}$ find a meromorphic function $m^\Delta : \mathbb{C} \to SL_2(\mathbb{C})$ with the following properties.*

1. $m^\Delta(z) = I + \mathcal{O}(z)$ *as* $z \to \infty$
2. $m^\Delta(z)$ *is holomorphic in* $\mathbb{C}\backslash(\mathcal{Z} \cup \mathcal{Z}^*)$ *and has simple poles at each point* $z_k \in \mathcal{Z}$ *and* $z_k^* \in \mathcal{Z}^*$ *satisfying the residue conditions*

$$\operatorname*{Res}_{z=z_k} m^\Delta(z) = \lim_{z\to z_k} m(z) n_k^\Delta \qquad \text{and} \qquad \operatorname*{Res}_{z=z_k^*} m^\Delta(z) = \lim_{z\to z_k^*} m(z)\sigma_2 (n_k^\Delta)^* \sigma_2 \tag{B.6}$$

where n_k is the nilpotent matrix,

$$n_k^\Delta = \begin{cases} \begin{pmatrix} 0 & 0 \\ \gamma_k(x,t) a_\Delta(z_k)^2 & 0 \end{pmatrix} & k \in \nabla \\ \begin{pmatrix} 0 & \gamma_k(x,t)^{-1} a_\Delta{}'(z_k)^{-2} \\ 0 & 0 \end{pmatrix} & k \in \Delta, \end{cases} \qquad \gamma_k(x,t) := c_k \exp(2i(t z_k^2 + x z_k)),$$

and a_Δ is as defined in (B.4).

As $m^\Delta(z)$ is an explicit transformation of $m(z)$, it follows directly from Prop. B.1 that RHP B.2 has a unique solution whenever the poles $z_k \in \mathcal{Z}$ are distinct. Moreover, if $\psi_{\mathrm{sol}}(x,t) = \psi_{\mathrm{sol}}(x,t;\sigma_d)$ denotes the N-soliton solution of (1.1) encoded by RHP B.1, then using (2.13) and (B.5) we have

$$m^\Delta(z) = I + \frac{1}{2iz} \begin{bmatrix} -\int_x^\infty |\psi_{\mathrm{sol}}(s,t)|^2 ds + \sum_{k\in\Delta} 4 \operatorname{Im} z_k & \psi_{\mathrm{sol}}(x,t) \\ \psi_{\mathrm{sol}}^*(x,t) & \int_x^\infty |\psi_{\mathrm{sol}}(s,t)|^2 ds - \sum_{k\in\Delta} 4 \operatorname{Im} z_k \end{bmatrix} + \mathcal{O}\left(z^{-2}\right). \tag{B.7}$$

This shows that each normalization encodes ψ_{sol} in the same way. The advantage of the nonstandard normalizations is, as we will see below, that by choosing Δ correctly, other asymptotic limits in which $t \to \infty$ with $-x/2t = \xi$ bounded are under better asymptotic control. The new sums appearing on the diagonal entries above, when compared to (2.13), represent the squared L^2 mass of the solitons corresponding to each z_k, $k \in \Delta$.

B.2. **Long time behavior of soliton solutions.** If $N = 1$, then the scattering data consists of only a single point $\sigma_d = \{(\xi + i\eta, c_1)\}$. In this case, the algebraic system for $\alpha_1(x,t)$ and $\beta_1(x,t)$ implied by (B.1)-(B.2) is trivial. Using (2.13), the solution of (1.1), $\psi(x,t) = -2i\beta_1(x,t)^*$, is given by

$$\begin{aligned} \psi(x,t;\sigma_d) &= 2\eta \operatorname{sech}\left(2\eta(x + 2\xi t - x_0)\right) e^{-2i(\xi x + (\xi^2 - \eta^2)t)} e^{-i\phi_0}, \\ x_0 &= \frac{1}{2\eta} \log\left|\frac{c_1}{2\eta}\right|, \qquad \phi_0 = \frac{\pi}{2} + \arg(c_1), \end{aligned} \tag{B.8}$$

which is a localized traveling wave of maximum amplitude $2\operatorname{Im} z_0$ traveling at speed $-2\operatorname{Re} z_0$; the normalization constant c determines the initial location and constant phase shift of the solution.

For $N > 1$ exact formulas for the solution become ungainly, and we will not present them here. However, as is well known, the N-soliton solutions undergo elastic collisions and asymptotically separate as $t \to \infty$ into, generically, N single soliton solutions each traveling at speed $-2\operatorname{Re} z_k$, one for each point in the discrete spectra $\{z_k\}_{k=1}^N$ which define RHP B.1. The exception, of course, is the non-generic case in which two (or more) points of discrete spectra lie on a vertical line $\xi + i\mathbb{R}$. This can be made precise as follows; for any (possibly degenerate) interval $\mathcal{I} = [\xi_1, \xi_2]$ let

$$\mathcal{Z}(\mathcal{I}) = \{z_k \in \mathcal{Z} \,:\, \operatorname{Re} z_k \in \mathcal{I}\} \quad \text{and} \quad n(\mathcal{I}) = |\mathcal{Z}(\mathcal{I})| \tag{B.9a}$$

denote the set of point spectra in the vertical strip extending over $\mathcal{I}$ and its cardinality respectively; let

$$\rho = \rho(\mathcal{I}) = \min_{z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})} \operatorname{Im}(z_k) \operatorname{dist}(\operatorname{Re} z_k, \mathcal{I}) \tag{B.9b}$$

Proposition B.2. *Let $\psi(x,t;\sigma_d)$ denote the N-soliton solution of the NLS equation* (1.1) *corresponding to discrete scattering data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times \mathbb{C}_*$. Fix $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $v_1 \leq v_2$. Let $\mathcal{I} = [-v_2/2, -v_1/2]$. Then as $t \to \infty$ along any characteristic $x = x_0 + vt$ inside the truncated cone*

$$x_1 + v_1 t \leq x \leq x_2 + v_2 t, \qquad t \geq 0$$

we have

$$|\psi(x,t;\sigma_d) - \psi(x,t;\widehat{\sigma}_d)| = \mathcal{O}\left(e^{-4\rho t}\right) \tag{B.10}$$

where $\psi(x,t;\widehat{\sigma}_d)$ is the reduced $N(\mathcal{I})$-soliton solution of NLS given by scattering data $\widehat{\sigma}_d = \{(z_k, \widehat{c}_k) \,:\, z_k \in \mathcal{Z}(\mathcal{I})\}$ where

$$\widehat{c}_k = c_k \prod_{\substack{z_j \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I}) \\ \operatorname{Re} z_j < -v_2/2}} \left(\frac{z_k - z_j}{z_k - z_j^*} \right)^2 \tag{B.11}$$

Proof. Let $x = x_0 + vt$ be a characteristic inside the cone and let $\xi = -v/2$ so that $\xi \in \mathcal{I}$. Define $\Delta_\xi^\pm$ as in (3.1) and let

$$a_{\Delta_\xi^-}(z) = \prod_{k \in \Delta_\xi^-} \left(\frac{z - z_k}{z - z_k^*} \right).$$

Using $a_{\Delta_\xi^-}$ we renormalize the problem as in (B.5) by defining[1] $m^{\Delta_\xi^-}(z) = m(z) a_{\Delta_\xi^-}(z)^{\sigma_3}$. The new unknown $m^{\Delta_\xi^-}(z)$ then satisfies RHP B.2 with $\Delta = \Delta_\xi^-$. The important fact about

[1]This transformation can be thought of as a reflectionless version of the more general version (3.6) that appears in the analysis of the full problem.

this choice of normalization is that

$$|\gamma_k(x_0+vt,t)| = |c_k| \exp(-2x_0 \operatorname{Im}(z_k)) \exp(-4t \operatorname{Im}(z_k) \operatorname{Re}(z_k-\xi))$$

which shows that $|\gamma_k|$ grows with t only for those z_k with $k \in \Delta_\xi^-$. The effect of the renormalization $m^{\Delta_\xi^-}$ is to reciprocate these coefficients in the nilpotent matrices defining the residue conditions. Specifically, as $t \to \infty$ along the characteristic,

$$\|n_k^{\Delta_\xi^-}\| = \begin{cases} \mathcal{O}(1) & z_k \in \mathcal{Z}(\mathcal{I}) \\ \mathcal{O}(\exp(-4t\rho)) & z_k \in \mathcal{Z}\backslash\mathcal{Z}(\mathcal{I}). \end{cases}$$

This suggest that the poles in $\mathcal{Z}\backslash\mathcal{Z}(\mathcal{I})$ do not meaningfully contribute to the solution. Let $\widehat{m}^{\Delta_\xi^-}$ denote the reduced solution of RHP B.2 with poles only in $\mathcal{Z}(\mathcal{I})$ which results from ignoring the pole conditions at each $z_k \in \mathcal{Z}\backslash\mathcal{Z}(\mathcal{I})$ for $\widehat{m}^{\Delta_\xi^-}$. 'Un'-renormalizing the solution $\widehat{m}^{\Delta_\xi^-}$ to $\widehat{m}$ (so that all of the poles are in the first column of $\widehat{m}$) one sees that it is defined by the scattering data $\widehat{\sigma}_d = \{(z_k, \widehat{c}_k) \,:\, z_k \in \mathcal{Z}(\mathcal{I})\}$ where the $\widehat{c}_k$ are defined by (B.11).

The residue relations (B.6) satisfied by $m^{\Delta_\xi^-}(z)$ imply that it admits a partial fraction expansion of the form

$$m^{\Delta_\xi^-}(z) = I + \sum_{k\in\Delta_\xi^+} \frac{\begin{pmatrix} \alpha_k & 0 \\ \beta_k & 0 \end{pmatrix}}{z-z_k} + \frac{\begin{pmatrix} 0 & -\beta_k^* \\ 0 & \alpha_k^* \end{pmatrix}}{z-z_k^*} + \sum_{k\in\Delta_\xi^-} \frac{\begin{pmatrix} 0 & \beta_k \\ 0 & \alpha_k \end{pmatrix}}{z-z_k} + \frac{\begin{pmatrix} \alpha_k^* & 0 \\ -\beta_k^* & 0 \end{pmatrix}}{z-z_k^*} \tag{B.12}$$

whose coefficients satisfy the following system of $2N$ equations:
For each $j \in \Delta_\xi^+$:

$$\begin{aligned} &\alpha_j + \gamma_k(x,t) a_{\Delta_\xi^-}(z_j)^2 \left(\sum_{k\in\Delta_\xi^+} \frac{\beta_k^*}{z_j - z_k^*} - \sum_{k\in\Delta_\xi^-} \frac{\beta_k}{z_j - z_k} \right) = 0 \\ &\beta_j^* - \frac{\gamma_k(x,t)^*}{a_{\Delta_\xi^-}(z_j^*)^2} \left(\sum_{k\in\Delta_\xi^+} \frac{\alpha_k}{z_j^* - z_k} + \sum_{k\in\Delta_\xi^-} \frac{\alpha_k^*}{z_j^* - z_k^*} \right) = \frac{\gamma_k(x,t)^*}{a_{\Delta_\xi^-}(z_j^*)^2} \end{aligned} \tag{B.13a}$$

For each $j \in \Delta_\xi^-$:

$$\begin{aligned} &\alpha_j^* + \frac{\gamma_j(x,t)^{*-1}}{a'_{\Delta_\xi^-}(z_j)^2} \left(-\sum_{k\in\Delta_\xi^+} \frac{\beta_k^*}{z_j^* - z_k^*} + \sum_{k\in\Delta_\xi^-} \frac{\beta_k}{z_j^* - z_k} \right) = 0 \\ &\beta_j - \frac{\gamma_j(x,t)^{-1}}{a'_{\Delta_\xi^-}(z_j)^2} \left(\sum_{k\in\Delta_\xi^+} \frac{\alpha_k}{z_j - z_k} + \sum_{k\in\Delta_\xi^-} \frac{\alpha_k^*}{z_j - z_k^*} \right) = \frac{\gamma_j(x,t)^{-1}}{a'_{\Delta_\xi^-}(z_j)^2} \end{aligned} \tag{B.13b}$$

Letting $\epsilon := \exp(-4\rho t)$ and rearranging the variables so that the $2N(\mathcal{I})$ equations for the coefficients corresponding to the poles in $\mathcal{Z}(\mathcal{I})$ come first we can write the full system of $2N$ equations in the block matrix form:

$$\begin{bmatrix} \boldsymbol{I}+\widehat{\boldsymbol{A}} & \boldsymbol{A}_{12} \\ \epsilon \boldsymbol{A}_{21} & \boldsymbol{I}+\epsilon \boldsymbol{A}_{22} \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{b_1} \\ \epsilon \boldsymbol{b_2} \end{bmatrix} \tag{B.14}$$

where $\boldsymbol{x}$ is the vector of α_k and β_k's from (B.13); each of the coefficient blocks $\widehat{\boldsymbol{A}}$, $\boldsymbol{A}_{12}$, $\boldsymbol{A}_{21}$, $\boldsymbol{A}_{22}$ and the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are all $\mathcal{O}(1)$ and the upper left $N(\mathcal{I})\times N(\mathcal{I})$ block $\widehat{\boldsymbol{A}}$ and target vector $\boldsymbol{b}_1$ are precisely the data for the linear system corresponding to the the reduced $N(\mathcal{I})$-soliton problem $\widehat{m}^{\Delta_\xi^-}(z)$. The solvability of the soliton problem guaranteed by Prop B.1 implies that $\widehat{\boldsymbol{A}}$ is invertible and thus (B.14) is equivalent to the system

$$\left(\boldsymbol{I} + \epsilon \begin{bmatrix} (\boldsymbol{I}+\widehat{\boldsymbol{A}})^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{21} & -(\boldsymbol{I}+\widehat{\boldsymbol{A}})^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \right) \boldsymbol{x} = \begin{bmatrix} (\boldsymbol{I}+\widehat{\boldsymbol{A}})^{-1}(\boldsymbol{b_1} - \epsilon \boldsymbol{A}_{12}\boldsymbol{b_2}) \\ \epsilon \boldsymbol{b_2} \end{bmatrix}.$$

As all of the coefficients blocks are $\mathcal{O}(1)$ the system is near identity and can be expanded asymptotically in ϵ which gives

$$\boldsymbol{x} = \begin{bmatrix} (\boldsymbol{I}+\widehat{\boldsymbol{A}})^{-1}\boldsymbol{b}_1 \\ \boldsymbol{0} \end{bmatrix} + \mathcal{O}(\epsilon).$$

Which justifies the claim that the leading order behavior of $m^{\Delta_\xi^-}(z;x,t)$ in the prescribed wedge is given by $\widehat{m}^{\Delta_\xi^-}(z;x,t)$. The result (B.10) then follows from (B.7) and (B.12). □

Proof of Proposition B.1. Inserting the partial fraction expansion (B.2) into the residue conditions (B.1) leads to, after some renormalization, the following linear system of equations for $j = 1, \ldots, N$,

$$\widehat{\alpha}_j + \sum_{k=1}^{N} \frac{\gamma_j{}^{1/2}\gamma_k^{*1/2}}{z_j - z_k^*}\widehat{\beta}_k^* = 0, \qquad \widehat{\beta}_j^* - \sum_{k=1}^{N} \frac{\gamma_j^{*1/2}\gamma_k^{1/2}}{z_j^* - z_k}\widehat{\alpha}_k, = \gamma_j^{*1/2} \tag{B.15}$$

where we've defined the renormalized parameters

$$\widehat{\alpha}_j = \alpha_j/\gamma_j^{1/2}, \quad \text{and} \quad \widehat{\beta}_j^* = \beta_j^*/\gamma_j^{*1/2},$$

and for brevity we've suppress the (x,t) dependence of α_j, β_j, and γ_j. Letting $\widehat{\boldsymbol{\alpha}} = (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_N)^\intercal$, $\widehat{\boldsymbol{\beta}} = (\widehat{\beta}_1, \ldots, \widehat{\beta}_N)^\intercal$, $\boldsymbol{\gamma}^{1/2} = (\gamma_1^{1/2}, \ldots, \gamma_N^{1/2})^\intercal$, and $\boldsymbol{A}$ be the $N \times N$ matrix with entries

$$\boldsymbol{A}_{jk} = \frac{-i\gamma_j^{*1/2}\gamma_k^{1/2}}{(z_j^* - z_k)}, \qquad j,k = 1, \ldots, N$$

the system (B.15) is equivalent to the block matrix equation

$$\begin{bmatrix} \boldsymbol{I}_N & -i\,\boldsymbol{A}^* \\ -i\boldsymbol{A} & \boldsymbol{I}_N \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{\alpha}} \\ \widehat{\boldsymbol{\beta}}^* \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{\gamma}^{*1/2} \end{bmatrix}. \tag{B.16}$$

Note that $\boldsymbol{A}^*$ denotes only the complex, not hermitian, conjugate of $\boldsymbol{A}$. Equation (B.16) will have a unique solution if and only if

$$\det \begin{bmatrix} \boldsymbol{I}_N & -i\,\boldsymbol{A}^* \\ -i\boldsymbol{A} & \boldsymbol{I}_N \end{bmatrix} = \det\left(\boldsymbol{I}_N + \boldsymbol{A}\boldsymbol{A}^*\right) \neq 0.$$

Clearly, $\boldsymbol{A}$ is hermitian. Observing also that $\boldsymbol{A}$ has the inner product structure

$$\boldsymbol{A}_{jk} = \int_0^\infty \gamma_j^{*1/2} \gamma_k^{1/2} e^{i(z_k - z_j^*)s} ds = \left\langle {\gamma_j}^{1/2} e^{i z_j s},\ {\gamma_k}^{1/2} e^{i z_k s} \right\rangle$$

where the functions $f_j(s) = {\gamma_j}^{1/2} e^{i z_j s}$ are linearly independent in $L^2(\mathbb{R}_+)$ since $z_j \neq z_k$ by assumption. It follows that $\boldsymbol{A}$ is positive definite. Let $\boldsymbol{A}^{1/2}$ denote the unique positive definite square root of $\boldsymbol{A}$. Now the eigenvalues of $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^{1/2}\left(\boldsymbol{A}^{1/2}\boldsymbol{A}^*\right)$ are the same as those of $\boldsymbol{A}^{1/2}(\boldsymbol{A}^*)\boldsymbol{A}^{1/2}$ which is itself positive definite. If we denote these eigenvalues as $\{\mu_k\}_{k=1}^N \subset \mathbb{R}_+$ then it follows that

$$\det\left(\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^*\right) = \prod_{k=1}^N (1 + \mu_k) > 0.$$

This proves the proposition. □

Appendix C. Details of Calculations for the $\overline{\partial}$ problem

Proposition C.1. *There exist constants c_1, c_2, and c_3 such that for all $t > 0$ we have the bounds*

$$|I_1| \leq \frac{c_1}{t^{1/4}} \qquad |I_2| \leq \frac{c_2}{t^{1/4}} \qquad |I_3| \leq \frac{c_3}{t^{1/4}}.$$

Proof. The calclulations shown here follow those found in [10]. We will make use of the fact that

$$\begin{aligned} \left\| \frac{1}{|s-z|} \right\|_{L^2(v+\xi,\infty)} &= \left(\int_{v+\xi}^\infty \frac{1}{(u-\alpha)^2 + (v-\beta)^2} du \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} \frac{1}{(s)^2 + (v-\beta)^2} ds \right)^{1/2} \\ &= \left(\frac{\pi}{|v-\beta|} \right)^{1/2}, \end{aligned}$$

where we recall that $s = u + iv$ and $z = \alpha + i\beta$. Therefore, we see that

$$\begin{aligned}|I_1| &\le \int_0^\infty \int_{v+\xi}^\infty \frac{|\chi_{\mathcal{Z}}(z)|}{|s-z|} e^{-8tv(u-\xi)} du dv \\ &\le \int_0^\infty e^{-tv^2} \int_{v+\xi}^\infty \frac{|\chi_{\mathcal{Z}}(z)|}{|s-z|} du dv \\ &\le \int_0^\infty e^{-tv^2} \|\chi_{\mathcal{Z}}(z)\|_{L^2(v+\xi,\infty)} \cdot \left\| \frac{1}{|s-z|} \right\|_{L^2(v+\xi,\infty)} dv \\ &\le c_1 \int_0^\infty e^{-tv^2} \left(\frac{\pi}{|v-\beta|} \right)^{1/2} dv \\ &= c_1 \left(\int_0^\beta \frac{e^{-tv^2}}{\sqrt{\beta - v}} dv + \int_\beta^\infty \frac{e^{-tv^2}}{\sqrt{v-\beta}} dv \right).\end{aligned}$$

For the first integral we make the substitution $v = \beta w$ and remark that since $t > 0$, $\beta > 0$, and $w > 0$ we have the inequality $\sqrt{\beta} e^{-t\beta^2 w^2} = \frac{(t^{1/4}\beta w)^{1/2}}{t^{1/4} w^{1/2}} e^{-(t^{1/4}\beta w)^2} \le c t^{-1/4} w^{-1/2}$, so that

$$\int_0^\beta \frac{e^{-tv^2}}{\sqrt{\beta - v}} dv = \int_0^1 \sqrt{\beta} \frac{e^{-t\beta^2 w^2}}{\sqrt{1-w}} dw \le c t^{-1/4} \int_0^1 \frac{1}{\sqrt{w(1-w)}} dw \le C t^{-1/4}.$$

Furthermore, for the second integral we make the substitution $w = v - \beta$ to get

$$\int_\beta^\infty \frac{e^{-tv^2}}{\sqrt{v-\beta}} dv \le \int_0^\infty \frac{e^{-tw^2}}{\sqrt{w}} dw \le t^{-1/4} \int_0^\infty \frac{e^{-s^2}}{\sqrt{s}} ds \le C t^{-1/4}.$$

The bound for I_2 is similar to I_1, remarking that $r \in H^{1,1}(\mathbb{R})$ and thus,

$$\begin{aligned}|I_2| &\le \int_0^\infty e^{-tv^2} \int_{v+\xi}^\infty \frac{|r'(u)|}{|s-z|} du dv \\ &\le \int_0^\infty e^{-tv^2} \|r'(u)\|_{L^2(v+\xi,\infty)} \cdot \left\| \frac{1}{|s-z|} \right\|_{L^2(v+\xi,\infty)} dv \\ &\le \frac{c_2}{t^{1/4}}.\end{aligned}$$

To arrive at the third bound, we begin with the following estimates for $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$:

$$\begin{aligned}\left\|\frac{1}{\sqrt{|s-\xi|}}\right\|_{L^p(v+\xi,\infty)} &= \left(\int_{v+\xi}^{\infty}\left(\frac{1}{(u-\xi)^2+v^2}\right)^{p/4} du\right)^{1/p} \\ &= \left(\int_{v}^{\infty}\frac{1}{(u^2+v^2)^{p/4}}du\right)^{1/p} \\ &= (v^{1/p-1/2})\left(\int_{1}^{\infty}\frac{1}{(1+w^2)^{p/4}}dw\right)^{1/p} \\ &\leq cv^{1/p-1/2},\end{aligned} \tag{C.1}$$

$$\begin{aligned}\left\|\frac{1}{|s-z|}\right\|_{L^q(v+\xi,\infty)} &= \left(\int_{v+\xi}^{\infty}\frac{1}{((u-\alpha)^2+(v-\beta)^2)^{q/2}}du\right)^{1/q} \\ &\leq \left(\int_{\mathbb{R}}\frac{1}{(s^2+(v-\beta)^2)^{q/2}}ds\right)^{1/2} \\ &\leq c|v-\beta|^{1/q-1}.\end{aligned}$$

We now apply the above estimates to see that

$$\begin{aligned}|I_3| &\leq C\int_0^{\infty} e^{-tv^2}\left\|\frac{1}{\sqrt{|s-\xi|}}\right\|_{L^p(v+\xi,\infty)}\left\|\frac{1}{|s-z|}\right\|_{L^q(v+\xi,\infty)} dv \\ &\leq C\int_0^{\infty} e^{-tv^2}v^{1/p-1/2}|v-\beta|^{1/q-1}dv \\ &\leq C\left(\int_0^{\beta} e^{-tv^2}v^{1/p-1/2}(\beta-v)^{1/q-1}dv + \int_{\beta}^{\infty} e^{-tv^2}v^{1/p-1/2}(v-\beta)^{1/q-1}dv\right).\end{aligned}$$

For the first integral we again use the substitution $v = \beta w$ and the bound $\sqrt{\beta}e^{-t\beta^2w^2} \leq ct^{-1/4}w^{-1/2}$, so that

$$\begin{aligned}\int_0^{\beta} e^{-tv^2}v^{1/p-1/2}(\beta-v)^{1/q-1}dv &= \int_0^{1}\sqrt{\beta}e^{-t\beta^2w^2}w^{1/p-1/2}(1-w)^{1/q-1}dw \\ &\leq ct^{-1/4}\int_0^1 w^{1/p-1}(1-w)^{1/q-1}dw \\ &\leq Ct^{-1/4}.\end{aligned}$$

For the final integral, we use the substitution $v = w + \beta$ as above so that

$$\begin{aligned}\int_{\beta}^{\infty} e^{-tv^2} v^{1/p-1/2}(v-\beta)^{1/q-1} dv &= \int_0^{\infty} e^{-t(w+\beta)^2}(w+\beta)^{1/p-1/2} w^{1/q-1} dw \\ &\le \int_0^{\infty} e^{-tw^2} w^{-1/2} dw \\ &\le Ct^{-1/4},\end{aligned}$$

and the result is confirmed. □

Proposition C.2. *For all $t > 0$ there exists a constant c such that*

$$|M_1^{(3)}| \le ct^{-3/4}. \tag{C.2}$$

Proof. The proof given here follows calculations that can be found in [10]. Let A be supported in the region Ω_1 such that $A \in L^{\infty}(\Omega)$. Then, where

$$\begin{aligned}|M_1^{(3)}| &\le \frac{1}{\pi}\int\int_{\Omega_1} |AM_{\text{RHP}}^{(2)}(z)W^{(2)}(z)M_{\text{RHP}}^{(2)}(z)^{-1}| dA \\ &\le \frac{1}{\pi}\|A\|_{\infty}\|M_{\text{RHP}}^{(2)}(z)\|_{\infty}\|M_{\text{RHP}}^{(2)}(z)^{-1}\|_{\infty}\int\int_{\Omega_1} |\bar{\partial}R_1 e^{2it\theta}| dA \\ &\le C\left(\int\int_{\Omega_1} |\chi_{\mathcal{Z}}| e^{-tuv} dA + \int\int_{\Omega_1} |r'| e^{-tuv} dA + \int\int_{\Omega_1} |z-\xi|^{-1/2} e^{-tuv} dA\right) \\ &\le C(I_4 + I_5 + I_6)\end{aligned}$$

where again we note that $M_{\text{RHP}}^{(2)}(z)W^{(2)}(z)M_{\text{RHP}}^{(2)}(z)^{-1}$ is supported away from the poles z_k so that $\|\cdot\|_{\infty} = \|\cdot\|_{L^{\infty}(\text{supp}(R_1))}$.

To bound I_4 we use the Cauchy-Schwarz inequality on the inner integral as follows:

$$\begin{aligned}|I_4| &\le \int_0^{\infty} \|\chi_{\mathcal{Z}}(z)\|_{L^2(v+\xi,\infty)} \left(\int_{v+\xi}^{\infty} e^{-2tuv} du\right)^{1/2} dv \\ &\le ct^{-1/2}\int_0^{\infty} \frac{e^{-tv^2}}{\sqrt{v}} dv \le ct^{-3/4}\int_0^{\infty} \frac{e^{-w^2}}{\sqrt{w}} dw \le \frac{c}{t^{3/4}}.\end{aligned}$$

The bound for I_5 follows in the same manner as for I_4. Turning to I_6 we once again use Hölder's inequality for $2 < p < 4$ and the bound (C.1). Thus,

$$\begin{aligned}|I_6| &\le c\int_0^{\infty} v^{1/p-1/2}\left(\int_{v+\xi}^{\infty} e^{-qtuv} du\right)^{1/q} dv \\ &\le ct^{-1/q}\int_0^{\infty} v^{2/p-3/2} e^{-tv^2} dv \le ct^{-3/4}\int_0^{\infty} w^{2/p-3/2} e^{-w^2} dw \le \frac{c}{t^{3/4}},\end{aligned}$$

where we have used the substitution $w = t^{1/2}v$ and the fact that $-1 < \frac{2}{p} - \frac{3}{2} < -\frac{1}{2}$. □

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