Triples in Finite Groups and a Conjecture of Guralnick and Tiep

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Notation

A_n The alternating group on a set of n elements

S_n The symmetric group on a set of n elements

$\mathrm{GL}_n^\epsilon(q)$ General linear group if $\epsilon = +$ and General unitary group if $\epsilon = -$.

$\mathrm{SL}_n^\epsilon(q)$ Special linear group if $\epsilon = +$ and Special unitary group if $\epsilon = -$.

$\mathrm{L}_n^\epsilon(q)$ Projective special linear group if $\epsilon = +$ and Projective special unitary group if $\epsilon = -$.

$\mathrm{S}_{2n}(q)$ Projective symplectic group

Ω_n^ϵ Omega group, i.e., derived subgroup of special orthogonal group

$\mathrm{O}_n(q)$ Simple orthogonal group ($\mathrm{O}_{2n+1}(q) = \Omega_{2n+1}(q)$ and $\mathrm{O}_{2n}^\epsilon(q) = \mathrm{P}\Omega_{2n}^\epsilon(q)$.)

$E_6^\epsilon(q)$ Simple group of type E_6 if $\epsilon = +$ and simple group of type ${}^2E_6$ is $\epsilon = -$.

$\pi(m)$, $n \in \mathbb{Z}$ The set of prime divisors of m.

$\zeta(q, n)$ The set of primitive prime divisors of $q^n - 1$.

$\Phi_n(q)$ The n-th cyclotomic polynomial

$X \prec G$ X is a section of G.

$\mathrm{Irr}(G)$ The set of irreducible characters of G

$\mathrm{Irr}_s(G)$ The set of semisimple characters of G

Abstract

In this thesis, we will see a way to use representation theory and the theory of linear algebraic groups to characterize certain family of finite groups. In Chapter 1, we see the history of preceding work. In particular, J. G. Thompson's classification of minimal finite simple nonsolvable groups and characterization of solvable groups will be given. In Chapter 2, we will describe some background knowledge underlying this project and notation that will be widely used in this thesis.

In Chapter 3, the main theorem originally conjectured by Guralnick and Tiep will be stated together with the base theorem which is a reduced version of main theorem to the case where we have a quasisimple group. Main theorem explains a way to characterize the finite groups with a composition factor of order divisible by two distinct primes p and q as the finite groups containing nontrivial 2-element x, p-element y, q-element z such that $xyz = 1$. In this thesis we more focus on the proof of showing a finite group G with a composition factor of order divisible by two distinct prime p and q contains nontrivial 2-element x, p-element y, q-element z such that $xyz = 1$.

In Chapter 4, we will prove a set of lemmas and proposition which will be used as key tools in the proof of the base theorem. In Chapters 5 to 7, we will establish the base theorem in the cases where a quasisimple group G has its simple quotient isomorphic to alternating groups or sporadic groups (Chapter 5), classical groups (Chapter 6), and exceptional groups (Chapter 7).

In Chapter 8, we show that any finite group G admitting nontrivial 2-element x, p-element y, q-element z such that $xyz = 1$ for two distinct odd primes p and q admits a composition factor of order divisible by pq. Also, we show that the question if a finite group G with a composition factor of order divisible by two distinct prime p and q contains nontrivial 2-element x, p-element y, q-element z such that $xyz = 1$ can be reduced to the base theorem.

Chapter 1
Introduction

This thesis focuses on the proof of a characterization of finite groups containing a composition factor of order divisible by pq for distinct odd prime divisors p and q. Before stating the main theorem, we describe preceding results on characterizing of solvable groups and p-solvable groups.

1.1 Characterization of Finite Solvable Groups

J. G. Thompson classified the minimal nonsolvable finite simple groups, nonsolvable finite simple groups all of which proper subgroups are solvable.

Theorem 1.1. *[Tho68, Corollary 1] Every minimal nonsolvable finite simple group is isomorphic to one of the following minimal simple groups:*
(i) $\mathrm{L}_2(2^p)$, *p any prime.*
(ii) $\mathrm{L}_2(3^p)$, *p any odd prime.*
(iii) $\mathrm{L}_2(p)$, $p \geq 7$ *odd prime such that* $5 | p^2+1$.
(iv) ${}^2B_2(2^p)$, *p any odd prime.*
(v) $\mathrm{L}_3(3)$.

Based on the classification of minimal nonsolvable finite simple groups, J. G. Thompson gave a criterion to characterize the finite solvable groups by the existence of certain elements.

Theorem 1.1a. *[Tho68, Corollary 3] A finite group G is solvable if and only if it does not contain nontrivial elements $x, y, z \in G$ of pairwise coprime orders such that $xyz = 1_G$.*

G. Kaplan and D. Levy ([KL10]) showed that this solvability criterion can be improved as follows:

Theorem 1.1b. *A finite group G is solvable if and only if it does not admit nontrivial elements $x, y, z \in G$ where x is a 2-element, y a p-element for some odd prime p, and z an element of order coprime to $2p$ such that $xyz = 1_G$.*

R. M. Guralnick and P. H. Tiep sharpened the solvability criteria even further. They introduced the notion of a (p,q,r)-*triple* in a finite group G. For a finite group G, a (p,q,r)-*triple* in G is a triple (x,y,z) of nontrivial elements in G where x is a p-element, y a q-element, z an r-element such that $xyz = 1$.

Theorem 1.1c. *[GT15, Theorem 1.2] A finite group G is solvable if and only if it does not admit a (p,q,r)-triple for distinct prime divisors p,q,r of $|G|$.*

They even showed that it suffices to assume that $p = 2$ and $q \in \{3,5\}$.

1.2 Characterization of Finite p-Solvable Groups

R. M. Guralnick and P. H. Tiep in [GT15] used the approach that they used in their characterization of finite solvable groups to give a characterization of finite p-solvable groups for p an odd prime. Recall that a finite group G is p-solvable if the non-abelian composition factors of G are of order prime to p.

They first classified minimal finite non-p-solvable simple groups and used it to characterize the finite p-solvable groups using $(2,p,q)$-triples for a prime $q \neq p$, $q\big|\,|G|$.

Theorem 1.2. *[GT15, Theorem 5.4] Let S be a finite non-abelian simple group and $p \geq 5$ with $p\big||S|$. If every proper subgroup of S is p-solvable, then either $S = \mathrm{L}_2(p)$, A_p, or one of the following holds:*
(i) $S = \mathrm{L}_2(q)$ with $p|(q^2-1)$.
(ii) $S = \mathrm{L}_n(q)$, $n \geq 3$ is odd, and $p|(q^n-1)$ but $p \nmid \prod_{i=1}^{n-1}(q^i-1)$.
(iii) $S = \mathrm{U}_n(q)$, $n \geq 3$ is odd, and $p|(q^n-(-1)^n)$ but $p \nmid \prod_{i=1}^{n-1}(q^i-(-1)^i)$.
(iv) $S = {}^2B_2(q)$.
(v) $S = {}^2G_2(q)$, and $p|(q^2-q+1)$ but q^2-1.
(vi) $S = {}^2F_4(q)$, $q \geq 8$, and $p|(q^4-q^2+1)$.
(vii) $S = {}^3D_4(q)$, and $p|(q^4-q^2+1)$ but $q \nmid (q^6-1)$.
(viii) $S = E_8(q)$, and $p|(q^{30}-1)$ but $p \nmid \prod_{i=8,14,18,20,24}(q^i-1)$.
(ix) (S,p) is one of $(M_{23}, 23)$, $(J_1, 7$ or $19)$, $(Ly, 37$ or $67)$, $(J_4, 29$ or $43)$, $(Fi_{24}', 29)$, $(B, 47)$, $(M, 41$ or 59 or $71)$.

Theorem 1.2a. *[GT15, Theorem 5.7] Let G be a finite group and $p \big| |G|$ an odd prime. Then G is p-solvable if and only if G does not admit a $(2, p, q)$-triple for any odd prime $q \neq p$.*

1.3 Characterization of Finite Groups Containing a Composition Factor of Order Divisible by pq

R. M. Guralnick and P. H. Tiep in [GT15] conjectured the following characterization of finite groups containing a composition factor of order divisible by pq for two distinct odd primes p and q.

Theorem A (Main Theorem)**.** *Let p, q be distinct odd primes and let G be a finite group. The following statements are equivalent:*
(i) G contains a composition factor whose order is divisible by pq;
(ii) G contains a $(2, p, q)$-triple.

One can see Theorem A as a generalization of Theorem 1.1c and Theorem 1.2a. Theorem 1.1c can be rephrased as follows: a finite group G has a non-abelian composition factor if and only if there is a (p, q, r)-triple for some distinct odd prime divisors p, q, r of $|G|$. Similarly, Theorem 1.2a states that a finite group G has a non-abelian composition factor of order divisible by p if and only if there is a $(2, p, q)$-triple for some prime divisor q of $|G|$ such that $q \neq 2, p$.

Also we note that some further generalization from Theorem A is not possible. We cannot characterize the finite groups containing a composition factor of order divisible by three distinct primes p, q, r as the finite groups containing a (p, q, r)-triple. R. M. Guralnick and P. H. Tiep in [GT15] investigated the possibility of this generalization, and they found that there is a counterexample. However, one could further study under what condition the above characterization would work. For example, $p = 2$, which is the case in Theorem A.

Chapter 2
Preliminaries

2.1 Linear Algebraic Groups

In this section, we give definitions of various algebraic groups and related concepts that will be used throughout the thesis. Most of them can be found in [MT11, Chapter 8] with more details. A *linear algebraic group* $\mathcal{G}$ is an affine variety defined over k an algebraic closure of a finite field of characteristic p with a group structure where the multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and inverse $\mathcal{G} \to \mathcal{G}$ are morphisms of varieties. Let $\mathcal{G}$ be a linear algebraic group. If the maximal connected normal unipotent subgroup of $\mathcal{G}$ is trivial, then $\mathcal{G}$ is *reductive*, and if the maximal connected normal subgroup of $\mathcal{G}$ is trivial then $\mathcal{G}$ is *semisimple*. If a semisimple algebraic group $\mathcal{G}$ dos not contain a proper closed normal subgroup, then $\mathcal{G}$ is called a *simple algebraic group*.

If $\mathcal{G}$ is a connected reductive algebraic group, then its derived subgroup $[\mathcal{G},\mathcal{G}]$ is a semisimple algebraic group. In particular, $\mathcal{G} = [\mathcal{G},\mathcal{G}]Z(\mathcal{G})^\circ$ where $Z(\mathcal{G})^\circ$ is the connected component of the center of G.

Let $\mathcal{G}$ be a semisimple algebraic group. Then $\mathcal{G}$ is perfect, i.e., $\mathcal{G} = [\mathcal{G},\mathcal{G}]$, and $\mathcal{G} = \mathcal{G}_1 \cdot \cdots \cdot \mathcal{G}_n$ for simple algebraic groups $\mathcal{G}_i$s where each $\mathcal{G}_i$ is a closed connected normal subgroups of $\mathcal{G}$ such that $[\mathcal{G}_i,\mathcal{G}_j] = 1$ for any distinct i, j and $|\mathcal{G}_i \cap \prod_{j\neq i} \mathcal{G}_j| < \infty$.

2.1.1 Classification of Semisimple Algebraic Groups

To classify the semisimple algebraic groups, we first need to define a root datum of an algebraic group. Here we mainly follow the way it is defined in [Car93, Section 1.8] and [MT11, Section 3.2].

Let k be an algebraic closure of a finite field of characteristic p. A linear algebraic group $\mathcal{T}$ is called a *torus* if $\mathcal{T} \cong \mathbf{G}_m \times \mathbf{G}_m \times \cdots \times \mathbf{G}_m$ where $\mathbf{G}_m \simeq k^\times$ the multiplicative group of k. We define the Weyl group of $\mathcal{T}$ by $W(\mathcal{T}) = N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$. Let $X := X(\mathcal{T})$ be the set of morphisms of algebraic groups from $\mathcal{T}$ to $\mathbf{G}_m$. Also let $Y := Y(\mathcal{T})$ be the set of morphisms of algebraic groups from $\mathbf{G}_m$ into $\mathcal{T}$. We call X the *character group* of $\mathcal{T}$ and Y the *cocharacter group* of $\mathcal{T}$.

Let $\mathcal{G}$ be a connected reductive algebraic group. A maximal connected solvable group $\mathcal{B}$ of a linear algebraic group $\mathcal{G}$ is called a *Borel subgroup*. Let $\mathcal{B}$ be a Borel subgroup of $\mathcal{G}$. Then $\mathcal{B}$ contains a maximal torus $\mathcal{T}$ of $\mathcal{G}$, and $\mathcal{B} = \mathcal{U} \rtimes \mathcal{T}$ where $\mathcal{U}$ is the unipotent subgroup of $\mathcal{B}$. For each Borel subgroup $\mathcal{B}$ of $\mathcal{G}$, there is a unique Borel subgroup $\mathcal{B}^-$ such that $\mathcal{B} \cap \mathcal{B}^- = \mathcal{T}$. We denote the unipotent subgroup of $\mathcal{B}^-$ by $\mathcal{U}^-$. Consider the minimal proper subgroups of $\mathcal{U}$ and $\mathcal{U}^-$ which are normalized by $\mathcal{T}$. They are connected unipotent of dimension 1 and so isomorphic to additive group $\mathbf{G}_a$. On each of these subgroups, $\mathcal{T}$ acts by conjugation, and it gives a homomorphism $\mathcal{T} \to \mathrm{Aut}(\mathbf{G}_a) \simeq \mathbf{G}_m$. Therefore, there is a element $\alpha \in X$ corresponding to the action of $\mathcal{T}$ on each of these subgroups. The finite set $\Phi \subseteq X$ of α's obtained from these subgroups are called the *root system* of $\mathcal{G}$. The connected unipotent subgroup of dimension 1 corresponding to α is denoted by $\mathcal{U}_\alpha$, and these $\mathcal{U}_\alpha$, $\alpha \in \Phi$ are called the *root subgroups* of $\mathcal{G}$. Consider a subgroup $\langle \mathcal{U}_\alpha, \mathcal{U}_{-\alpha} \rangle$ of $\mathcal{G}$. For each α, there is a homomorphism $\rho_\alpha : \mathrm{SL}_2(k) \to \langle \mathcal{U}_\alpha, \mathcal{U}_{-\alpha} \rangle$. Let $D = \{\mathrm{diag}(\lambda, \lambda^{-1}) | \lambda \in k^\times\}$. Note that the image of D under ρ_α lies in $\mathcal{T}$. Since $D \simeq \mathbf{G}_m$, we get a homomorphism $\alpha^\vee : \mathbf{G}_m \to \mathcal{T}$ and $\alpha^\vee \in Y(\mathcal{T})$. The set of such cocharacters are denoted by $\Phi^\vee$, i.e., $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ and called the *coroots* of $\mathcal{G}$. The quadruple $(X, \Phi, Y, \Phi^\vee)$ forms a *root datum* of G. Now we have the following classification of semisimple algebraic groups.

Theorem (Chevalley Classification Theorem)**.** *[MT11, Theorem 9.13] Two semisimple algebraic groups are isomorphic if and only if their root data are isomorphic.*

2.1.2 Simple Algebraic Groups

If $\mathcal{G}$ is a simple algebraic group, then Φ can be identified with one of indecomposable root systems, and there are 8 different types of connected root systems:

$$A_n (n \geq 1), \quad B_n (n \geq 2), \quad C_n (n \geq 3), \quad D_n (n \geq 4), \quad E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2$$

If $\mathcal{G}$ is of type either A_n, B_n, C_n, or D_n, then $\mathcal{G}$ is said to be a classical group. Otherwise, we say $\mathcal{G}$ is an *exceptional* group.

There could be multiple simple algebraic groups with the same root system type. Let $\Omega := \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$ and $\Lambda(\mathcal{G}) := \Omega/X$. We call $\Lambda(\mathcal{G})$ the *fundamental group* of $\mathcal{G}$. Note that $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$. If $X = \Omega$, i.e., $\Lambda(\mathcal{G}) = 1$, then we say $\mathcal{G}$ is of *simply connected* type, and if $X = \mathbb{Z}\Phi$, then we say $\mathcal{G}$ is of *adjoint* type. Between simple algebraic groups with isomorphic root systems, there is an *isogeny*, a surjective homomorphism with finite kernel. In particular, if $\widetilde{\mathcal{G}}$ is a semisimple algebric group of simply connected type with root system Φ, then there exists an isogeny from $\widetilde{\mathcal{G}}$ to $\mathcal{G}$ where $\mathcal{G}$ is any semisimple algebraic group with root system isomorphic to Φ.

2.2 Finite Groups of Lie-Type

Let $\mathcal{G}$ be a connected reductive group over k an algebraic closure of a finite field of characteristic p. Note that any linear algebraic group can be embedded as a closed subgroup of $\mathrm{GL}_n(k)$. Let $F_q : x \mapsto x^q$ be a field automorphism on k. Then there is a induced group homomorphism $F_q : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$, $(x_{ij}) \mapsto (x_{ij}^q)$. We call $F : \mathcal{G} \to \mathcal{G}$ a Frobenius endomorphism if there exists $\rho : \mathcal{G} \hookrightarrow \mathrm{GL}_n(k)$ a closed embedding such that the following diagram

$$\begin{array}{ccc} \mathcal{G} & \overset{\rho}{\hookrightarrow} & \mathrm{GL}_n(k) \\ \downarrow F & & \downarrow F_q \\ \mathcal{G} & \overset{\rho}{\hookrightarrow} & \mathrm{GL}_n(k) \end{array}$$

commutes. (See [Gec13, Definition 4.1.9, Theorem 4.1.11].) Also, if some power of F is a Frobenius endomorphism on $\mathcal{G}$, then we call F a Steinberg endomorphism. Consider a fixed point subgroup of $\mathcal{G}$ under its Steinberg endomorphism F,

$$\mathcal{G}^F = \{x \in \mathcal{G} | F(x) = x\}.$$

Then $\mathcal{G}^F$ is a finite subgroup of $\mathcal{G}$, and the finite groups realized as $\mathcal{G}^F$ for some connected reductive gorup G and Steinberg endomorphism F are called the *finite groups of Lie-type*.

In general, $\mathcal{G}$ being a simple algebraic group does not guarantee that $\mathcal{G}^F$ is a simple (abstract) group. However, under certain conditions, we get $\mathcal{G}^F$ is a quasisimple group, i.e., a perfect central extension of a simple group.

Theorem (Tits)**.** *[MT11, Theorem 24.17] Let $\mathcal{G}$ be a simple algebraic group of simply connected type with its Frobenius map F. If $\mathcal{G}^F$ is non-solvable and $\mathcal{G}^F \neq \mathrm{Sp}_4(2)$, $G_2(2)$, ${}^2G_2(3)$, ${}^2F_4(2)$, then $\mathcal{G}^F$ is perfect and $\mathcal{G}^F/Z(\mathcal{G}^F)$ is simple.*

2.2.1 Centralizer of an Element in Groups of Lie Type

Any linear algebraic group $\mathcal{G}$ can be embedded in $\mathrm{GL}_n(k)$ for some n as its closed subgroup. We can define semisimple elements and unipotent elements in $\mathcal{G}$ using it. Let ρ be an embedding of $\mathcal{G}$ into $\mathrm{GL}_n(k)$. An element $x \in \mathcal{G}$ is semisimple (unipotent) if $\rho(x)$ is semisimple (unipotent) in GL_n, and it does not depend on the choice of the embedding ([MT11, Theorem 2.5]).

Let $\mathcal{G}$ be a connected reductive algebraic group. We say an element $x \in \mathcal{G}$ is regular if $\dim(C_{\mathcal{G}}(x)) = \mathrm{rank}(\mathcal{G})$ where $\mathrm{rank}(\mathcal{G})$ is the dimension of a maximal torus in $\mathcal{G}$. In particular, if x is semisimple, then there exists at least one maximal torus $\mathcal{T}$ containing x, and so $\mathcal{T} \leq C_{\mathcal{G}}(x)$. Therefore, if x is regular semisimple, there exists a unique maximal torus $\mathcal{T}$ containing x and $C_G(x)^\circ = \mathcal{T}$.

For a fixed point subgroup $\mathcal{G}^F$, an element $x \in \mathcal{G}^F$ is semisimple (unipotent or regular respectively) if x is semisimple (unipotent or regular respectively) in its corresponding algebraic group $\mathcal{G}$.

2.2.2 Character Theory for Finite Groups of Lie Type

Let $G = \mathcal{G}^F$ where $\mathcal{G}$ is a connected reductive algebraic group with connected center. For such a group G and its irreducible character χ, the average character value of χ over the regular unipotent elements of G is either ± 1 or 0. In particular, $\chi(u) = \pm 1$ or 0 for any regular unipotent element u in G if the defining characteristic p is good (see [Car93, p.28] for good primes for $\mathcal{G}$). An irreducible character $\chi \in \mathrm{Irr}(G)$ is *semisimple* if the average character value of χ over the regular unipotent elements of G is nonzero. (See [Car93, Section 8.3] for more details.)

Let $\mathcal{T}$ be an F-stable maximal torus. We also let $\mathcal{G}^*$ be a dual group of $\mathcal{G}$ with corresponding Frobenius morphism F^*. From the Deligne-Lusztig theory, there exists a one-to-one correspondence between the geometric conjugacy classes (see [Car93, p.109]) of the pairs $(\mathcal{T}, \theta)$ where $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ and the conjucacy classes of semisimple elment in the dual group $\mathcal{G}^{*F^*}$. Also, there exists a unique semisimple character corresponding to each geometric conjugacy class of pairs $(\mathcal{T}, \theta)$ ([Car93, Proposition 8.4.6]).

We will list some basic facts on semisimple characters we could get from the Deligne-Lusztig theory.

The degrees of semisimple characters can be obtained from the index of the centralizer of a semisimple element in the dual group. If χ is a semisimple character of $\mathcal{G}^F$, there exists a unique conjugacy class of semisimple element $x \in \mathcal{G}^{*F^*}$ such that

$$\chi(1) = |\mathcal{G}^{*F^*} : C_{\mathcal{G}^{*F^*}}(x)|_{p'}$$

where p is the defining characteristic (see [Car93, Theorem 8.4.8]).

To see what exactly the semisimple characters are, we need the Deligne-Lusztig characters $R_{\mathcal{T},\theta}$ (see [Car93, p.206,207] for the definition). For each geometric conjugacy class of pairs $(\mathcal{T}_0, \theta_0)$, there is a unique semisimple character corresponding to it, and it can be expressed as follows:

$$\pm \sum_{(T,\theta)\in K} \frac{R_{\mathcal{T},\theta}}{(R_{\mathcal{T},\theta}, R_{\mathcal{T},\theta})}$$

where K is a set of representatives for the $\mathcal{G}^F$-conjugacy classes in the gometric conjugacy class of $(\mathcal{T}_0, \theta_0)$ ([Car93, Theorem 8.4.6]).

We say $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ is in *general position*, if ${}^w\theta \neq \theta$ for any $1 \neq w \in W(\mathcal{T})^F$ where $W(\mathcal{T})$ is the Wyle group of $\mathcal{T}$, i.e. $W(\mathcal{T}) = N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$. If θ is in general position, then $\pm R_{\mathcal{T},\theta}$ is an irreducible character of $\mathcal{G}^F$. Evaluation of $R_{\mathcal{T},\theta}$ on semisimple elements can be done using the following formula ([Car93, Theorem 7.5.1, 7.5.3]):

$$R_{\mathcal{T},\theta}(1) = \pm|\mathcal{G}^F : \mathcal{T}^F|_{p'}$$
$$R_{\mathcal{T},\theta}(x) = \frac{\epsilon_{\mathcal{T}}\epsilon_{C^0(x)}}{|\mathcal{T}^F||C^0(x)^F|_p} \sum_{g\in\mathcal{G}^F, g^{-1}xg\in\mathcal{T}^F} \theta(g^{-1}xg)$$

2.3 Character Theory

Let G be a finite group. A group homomorphism $\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C})$ is called a (complex) representation of G. Also, $\chi = \mathrm{tr}(\mathfrak{X})$ is called the character of G afforded by $\mathfrak{X}$. (See [Isa94].)

2.3.1 Character Formula

Two representations $\mathfrak{X}_1$ and $\mathfrak{X}_2$ of G are said to be equivalent if there exists a $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P^{-1}\mathfrak{X}_1 P = \mathfrak{X}_2$. Two representations are equivalent if and only if they afford the same character.

Let $g_1, \ldots, g_r$ be the representatives of G-conjugacy classes in G. We define a class sum $K_i = \sum_{x \in g_i^G} x$ for each i. Then we get the following formula:

$$K_i K_j = \sum_{i=1}^{r} a_{i,j,k} K_k$$

where $a_{i,j,k} = |\{x \in K_k | x \in g_i^G \cdot g_j^G\}|$ ([Isa94, Theorem 2.4]). In particular, $a_{i,j,k}$ is called a structure constant and is a nonnegative integer. The following lemma shows that there is an explicit way to compute $a_{i,j,k}$ using the character table.

Lemma 2.1. *Let G be a finite group. For class sums $K_i = \sum_{x \in g_i^G} x$ for $i = 1, 2, 3$, we can get the structure contstant $a_{1,2,3}$ as follow:*

$$a_{1,2,3} = \frac{|g_1^G|\,|g_2^G|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_1)\chi(g_2)\overline{\chi(g_3)}}{\chi(1)}$$

Proof. Consider $Z(\mathbb{C}G)$ the center of the group algebra. The class sums of G form a basis of $Z(\mathbb{C}G)$, and we define $\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}$ by $\omega_\chi(K) = \chi(g)|g^G|/\chi(1)$ where $g \in G$, $K = \sum_{x \in g^G} x$. Then ω_χ is an algebra homomorphism. Now we get the following:

$$\begin{aligned}
\omega_\chi(K_1)\omega_\chi(K_2) &= \sum_{i=1}^{r} a_{1,2,i}\omega_\chi(K_i) \\
\frac{|g_1^G|\chi(g_1)|g_2^G|\chi(g_2)}{\chi(1)^2} &= \sum_{i=1}^{r} a_{1,2,i}\frac{|g_i^G|\chi(g_i)}{\chi(1)} \\
\frac{|g_1^G|\chi(g_1)|g_2^G|\chi(g_2)\overline{\chi(g_3)}}{\chi(1)} &= \sum_{i=1}^{r} a_{1,2,i}|g_i^G|\chi(g_i)\overline{\chi(g_3)} \\
|g_1^G|\,|g_2^G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_1)\chi(g_2)\overline{\chi(g_3)}}{\chi(1)} &= \sum_{i=1}^{r} a_{1,2,i}|g_i^G| \sum_{\chi \in \mathrm{Irr}(G)} \chi(g_i)\overline{\chi(g_3)} \\
|g_1^G|\,|g_2^G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_1)\chi(g_2)\overline{\chi(g_3)}}{\chi(1)} &= a_{1,2,3}|g_3^G||C_G(g_3)|
\end{aligned}$$

$$a_{1,2,3} = \frac{|g_1^G|\,|g_2^G|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_1)\chi(g_2)\overline{\chi(g_3)}}{\chi(1)}$$

□

2.3.2 p-defect of a Character

Let p be a prime. A character $\chi \in \mathrm{Irr}(G)$ is said to be of p-defect 0 if p does not divide $|G|/\chi(1)$. We also say χ has positive p-defect if p divides $|G|/\chi(1)$.

We have the following theorem:

Lemma 2.2. *[Isa94, Theorem 8.17] Let $\chi \in \mathrm{Irr}(G)$ be a p-defect 0 character. Then χ vanishes on any element $g \in G$ of order divisible by p.*

Chapter 3

Characterization of Finite Groups Containing a Composition Factor of Order Divisible by pq

Let us restate Theorem A from Chapter 1.

Theorem A (Main Theorem)**.** *Let p, q be distinct odd primes and let G be a finite group. The following statements are equivalent:*
(i) G contains a composition factor whose order is divisible by pq;
(ii) G contains a $(2, p, q)$-triple.

The proof of Theorem A relies on the classification of minimal finite simple groups of order divisible by two distinct odd prime divisors p and q, a finite simple group of order divisible by pq which does not admit a proper simple section of order divisible by pq. Throughout this thesis, we use the notion of a *minimal simple* (p, q)*-section*. For a finite group G, we call X a *section* of G if there is a subgroup H of G and a normal subgroup N of H such that $X \cong H/N$. We denote it by $X \prec G$. A section X of a finite group G is called a *minimal simple* (p, q)*-section* of G if X is simple and $pq \big| |X|$ but X itself does not admit any proper simple section of order divisible by pq.

Now we state the base theorem. In Chapter 8, it will be proven that it is sufficient to show the following theorem to show Theorem A.

Theorem B (Base Theorem)**.** *Let G be a finite quasisimple group and let p, q be distinct odd primes such that $G/Z(G)$ is a minimal simple (p, q)-section of G. Then G admits a $(2, p, q)$-triple.*

From now on we use r and s to denote distinct odd prime divisors. We use p and q for finite Lie-type groups $G(q)$ over a finite field $\mathbb{F}_q$ where $q = p^f$ for a prime p as a convention instead. The proof of the reduction theorem relies on the following classification of minimal simple (r, s)-sections.

We use $\mathrm{L}_n^\epsilon(q)$ to denote $\mathrm{L}_n(q)$ if $\epsilon = +$ and $\mathrm{U}_n(q)$ if $\epsilon = -$. Also, $E_6^\epsilon(q)$ is $E_6(q)$ if $\epsilon = +$ and ${}^2E_6$ if $\epsilon = -$. For the further notation used for Theorem C, see the notation page.

A prime r is called a *primitive prime divisor* of $q^n - 1$ if r divides $q^n - 1$ but none of $q^i - 1$ for $1 \le i < n$ (cf. [Car13; Yab01]). We denote the set of primitive prime divisors of $q^n - 1$ by $\zeta(q, n)$.

Theorem C (Classification of Minimal Simple (r, s)-sections)**.** *Let r and s be distinct odd prime divisors of $|S|$. If S itself is a minimal simple (r, s)-section of S, i.e., S admits no proper simple section of order divisible by rs, then we may assume that it is one of the following cases:*

(i) $S = \mathsf{A}_r$ for $r > s$.

(ii) $S = \mathrm{L}_2(q)$.

(iii) $S = \mathrm{L}_n^\epsilon(q)$ for $n \ge 3$ and $r \in \zeta(\epsilon q, n)$. In particular, if $s = p$, then n is a prime.

(iv) $S = \mathrm{S}_{2n}(q)$ for $n \ge 2$, $(p, 2rs) = 1$, and $r \in \zeta(q^2, n)$.

(v) $S = \mathrm{O}_{2n+1}(q)$ for $n \ge 3$, $(p, 2rs) = 1$, and $r \in \zeta(q^2, n)$.

(vi) $S = \mathrm{O}_{2n}^\epsilon(q)$ for $n \ge 4$, $r \in \zeta(q^2, n)$, and $s \ne p$. Also, if $\epsilon = +$, then n is odd.

(vii) $S = {}^2B_2(q)$

(viii) $S = {}^2G_2(q)$ and $r \in \zeta(q, 6)$

(ix) $S = G_2(q)$, $r \in \zeta(q, 6)$, and $s \in \zeta(q, 3)$

(x) $S = {}^3D_4(q)$ and $r \in \zeta(q, 12)$.

(xi) $S = {}^2F_4(q)$ for $q > 2$ and either $r \in \zeta(q, 12)$, or $r \in \zeta(q, 6)$, $s \in \zeta(q, 4)$.

(xii) $S = F_4(q)$, $r \in \zeta(q, 12)$, and $s \in \zeta(q, 8) \cup \zeta(q, 4)$.

(xii) $S = E_6^\epsilon(q)$, $r \in \zeta(\epsilon q, k)$, and $s \in \zeta(\epsilon q, l)$ for $(k, l) = (12, 9)$, $(12, 5)$, $(9, 8)$, $(9, 5)$, or $(9, 4)$.

(xiv) $S = E_7(q)$, $r \in \zeta(q, k)$, and $s \in \zeta(q, l)$ for $(k, l) = (18, 14)$, $(18, 9)$, $(18, 7)$, $(18, 5)$, $(14, 12)$, $(14, 9)$, $(14, 7)$, $(14, 5)$, $(9, 10)$, $(9, 7)$, $(7, 12)$, or $(7, 10)$.

(xv) $S = E_8(q)$ and $r \in \zeta(q, k)$ for $k \in \{30, 24, 20, 15\}$. Furthermore, if $p = s$, then $k = 30$ or 15. If $p = 2$ and $k = 24$, then $s \in \zeta(q, l)$ for $l \in \{30, 20, 18, 15, 14, 10, 9, 8, 7\}$. If $p = 2$ and $k = 21$, then $s \in \zeta(q, l)$ for $l \in \{30, 24, 18, 15, 14, 10, 9, 7, 6, 3\}$.

(xvi) S is one of the sporadic groups with r, s listed in the following table:

S	(r,s)	G
M_{22}	$(11,7)$	$12M_{22}$
M_{23}	$(23,s),\ s\in\{3,5,7,11\}$	M_{23}
J_1	$(19,s),\ s\in\{3,5,7,11\}$ $(7,3),(7,5),(11,7)$	J_1
J_2	$(7,5)$	$2J_2$
J_3	$(19,17)$	$3J_3$
J_4	$(29,s),\ s\in\{3,5,7,11,23\}$, $(31,29),(31,23)$, $(37,s),\ s\in\{7,23,29,31\}$, $(43,s),\ s\in\{3,5,7,11,23,29,31,37\}$	J_4
Co_1	$(23,13)$	Co_1
Fi_{22}	$(13,11)$	$6Fi_{22}$
Fi_{23}	$(17,11),(17,13),(23,13),(23,17)$	Fi_{23}
Fi'_{24}	$(29,s),\ s\in\{3,5,7,11,13,17,23\}$	$3Fi'_{24}$
He	$(17,7)$	He
Ru	$(29,13)$	$2Ru$
Suz	$(11,7),(13,11)$	$6Suz$
$O'N$	$(31,s),\ s\in\{19,11,7\}$	$3O'N$
HN	$(19,11),(19,5)$	HN
Ly	$(67,s),\ s\in\{3,5,7,11,31,37\}$ $(37,s),\ s\in\{3,5,7,11,31\}$ $(31,11)$	Ly
Th	$(31,19),(31,13),(19,13),(13,5)$	Th
B	$(47,s)\ s\in\{3,5,7,11,13,17,19,23,31\}$ $(31,23),(31,17),(31,11),(19,13)$	$2B$
M	$(71,s),\ s\in\{11,13,17,19,23,29,31,41,47,59\}$ $(59,s),\ s\in\{7,11,13,17,19,23,31,41,47\}$ $(47,41),(47,29)$ $(41,s),\ s\in\{11,17,19,23,29,31\}$ $(31,29),(29,19)$	M

Theorem B and Theorem C will be proved in Chapters 5 to 7, and Theorem A will be proved in Chapter 8.

Chapter 4

Essential Lemmas

In this chapter, we show the lemmas that will be used throughout this thesis for the proofs of Theorem C and Theorem B.

Lemma 4.1. *Let G be a finite group with a unique non-abelian composition factor S. If X is a non-abelian simple section of G, then X is a section of S.*

Proof. If G is simple, then there is nothing to show.

We assume that G is not a simple group with a unique non-abelian composition factor S. Let M be a maximal normal proper subgroup of G. Either G/M is a cyclic group of prime order or $G/M \simeq S$. Let X be a non-abelian simple section of G such that $X = H/N$ for some $N \trianglelefteq H \leq G$.

We first show that show that if G/G_1 is cyclic of order p, then G_1 also has a section isomorphic to X. Since G/G_1 is isomorphic to a cyclic group of order p, either $HG_1 = G_1$ or $HG_1 = G$. If $HG_1 = G_1$ then $H \leq G_1$, so we are done. Assume that $HG_1 = G$. We let $H_1 := H \cap G_1$, and $N_1 := N \cap G_1 = N \cap H_1$. Then N_1 is a normal subgroup of H_1. Note that H_1 is a normal subgroup of H with index p as $G/G_1 = HG_1/G_1 \simeq H/H_1$. Since H/N is non-abelian simple, N is not contained in H_1. It follows that $H = H_1N$. Now we have the following

$$H_1/N_1 = H_1/(N \cap H_1) \simeq H_1N/N = H/N,$$

and so H_1/N_1 gives a section of G_1, which is isomorphic to X.

Applying the above argument successively to normal subgroups K in G with G/K solvable, we may assume that $G/M \simeq S$. Note that H is not solvable, and so it is not contained in M. Therefore, $H \cap M$ is a proper normal subgroup of H. Since X is a simple quotient of H, at least one of $H/(H \cap M)$ or $H \cap M$ has a composition factor isomorphic to X. However, M is solvable, and so $H \cap M$ is also solvable. It follows that $H/(H \cap M)$ has a composition factor isomorphic to X. Since HM/M is a subgroup of $S = G/M$ and $HM/M \simeq H/(H \cap M)$, S has a section isomorphic to X. $\square$

We use the notion of $\pi(m)$ for $m \in \mathbb{N}$ for the set of prime divisors of m in the following lemma.

Corollary 4.2. *Let G be a finite quasisimple group and $S = G/Z(G)$. Suppose that p and q are distinct odd prime divisors of $|G|$. Then S is a minimal simple (p, q)-section of G if and only if S is a minimal simple (p, q)-section of S.*

Lemma 4.3. *For finite groups S and G, let G be a central extension of S such that $|G| = m|S|$ where $\pi(m) \subseteq \{p, q, r\}$. If there is a (p, q, r)-triple in S, then there is a (p, q, r)-triple in G as well.*

Proof. The idea of the proof is based on a part of the proof of [GT15, Theorem 3.4]. If $m = 1$, there is nothing to show.

Suppose m is greater than 1. Let G be a central extension of S with $\sigma : G \twoheadrightarrow S$, and N the kernel of σ.

We first assume that $|\pi(m)| = 1$. Without loss of generality we may assume $\pi(m) = \{p\}$. Then N is a p-group. If there is a (p, q, r)-triple in $S \simeq G/N$, there exist x a p-element, y a q-element, and z an r-element in $G \setminus N$ such that $xyz = n$ for some $n \in N$. Since N is central in G, $n^{-1}x$ is a nontrivial p-element, and so $(n^{-1}x, y, z)$ forms a (p, q, r)-triple in G.

Now we consider that $|\pi(m)| > 1$. Note that N is abelian, and so it is a direct product of its Sylow subgroups. Let P be the Sylow p-subgroup of N, Q the Sylow q-subgroup of N, and R the Sylow r-subgroup of N. Consider the following maps induced from σ

$$G \to G/P \to G/(P \times Q) \to G/(P \times Q \times R) \simeq S.$$

Each of them is a central extension, and so we can lift a (p, q, r)-triple in S to G by repeating the above argument for the case $|\pi(m)| = 1$. □

The existence of triples in a finite group can be determined by a well-known character formula. For a finite group G, if there are $x, y, z \in G$ such that

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)} \neq 0$$

then $z \in x^G \cdot y^G$ (Lemma 2.1).

Now the following lemma immediately follows.

Lemma 4.4 (Character Formula). *Let G be a finite group. If there exists a nontrivial 2-element x, a p-element y, and a q-element z in G such that*

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)} \neq 0,$$

then there exists $x' \in x^G$, $y' \in y^G$, and $z' \in z^G$ such that $x'y'z' = 1$, i.e. G admits a $(2, p, q)$-triple (x', y', z').

We can get a corollary of Lemma 4.4 based on that for $N \lhd G$ and $x \in N$, we have $x^G \subset N$.

Corollary 4.5. *Let G be a finite group and N a normal subgroup of G. If there exists a nontrivial 2-element x, a p-element y, and a q-element z in N such that*

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)} \neq 0,$$

then N admits a $(2, p, q)$-triple.

We first prove the following lifting lemma.

For the groups of Lie type, there are mainly two different cases depending on the relation between the defining characteristic p and primes 2, r, and s.

The following lemma is proven in [GT15, Lemma 5.1]. It plays a crucial role in the case of $(p, 2rs) = 1$.

Lemma 4.6. *Let G be a quasisimple Lie-type group of simply connected type. If $x, y \in G$ are regular semisimple elements, then any noncentral semisimple element $z \in G$ is in $x^G \cdot y^G$.*

Suppose that G is a quasisimple Lie-type group of simply connected type with the defining characteristic p. If $(p, 2rs) = 1$, then any 2-element, r-element, or s-element is semisimple. Since $2rs \big| |G/Z(G)|$, there exists noncentral 2-elements, r-elements, and s-elements in G. Hence, if we can show that any two of a 2-element, an r-element, or an s-element in G are regular, we can conclude by Lemma 4.6 that G has a $(2, r, s)$-triple.

Lemma 4.7. *Let G be one of $\mathrm{SL}_n^\epsilon(q)$ for $n \geq 3$, $\mathrm{Sp}_{2n}(q)$ for $n \geq 2$, $\mathrm{Spin}_{2n+1}(q)$ for $n \geq 3$, $\mathrm{Spin}_{2n}^\epsilon(q)$ for $n \geq 4$, ${}^2G_2(3^{2n+1})$, $G_2(q)$ for $q > 3$, ${}^3D_4(q)$, and $E_8(q)$, where $q = p^f$ for p an odd prime. Then there exists a regular semisimple 2-element in G*

Proof. The existence of a regular 2-element in each group listed above is shown in [Gur+15, Section 7.2] for classical groups except for $\mathrm{Spin}_n^\epsilon(q)$ and [Gur+15, Lemma 7.16] for exceptional groups. However, they show that there is a regular 2-element g in $\Omega_n^\epsilon(q) \leq \mathrm{SO}_n^\epsilon(q)$. Let $\mathcal{G} := \mathrm{SO}_n^\epsilon(q)$ with F a Frobenius endomorphism such that $\mathcal{G}^F = \mathrm{SO}_n^\epsilon(q)$. Then there is an isogeny $\pi : \mathcal{G}_{sc} \to \mathcal{G}$ where $\mathcal{G}_{sc} := \mathrm{Spin}_n$ and an induced Frobenius endomorphism on $\mathcal{G}_{sc}$, and we denote it by F as well. Note that $\mathcal{G}_{sc}^F = \mathrm{Spin}_n^\epsilon(q)$ projects onto the subgroup $\Omega_n^\epsilon(q)$ of $\mathcal{G}^F$, with kernel a 2-group. Hence, we can find a 2-element $g_{sc} \in \mathrm{Spin}_n^\epsilon(q)$ such that $\pi(g_{sc}) = g$ such that $|g_{sc}|$ is a 2-power. To show that g_{sc} is regular, we compute the dimension of $C_{\mathcal{G}_{sc}}(g_{sc})$. Since the kernel of π is a finite set, $\dim C_{\mathcal{G}_{sc}}(g_{sc}) = \dim \pi(C_{\mathcal{G}_{sc}}(g_{sc}))$. Also, note that $\pi(C_{\mathcal{G}_{sc}}(g_{sc})) \leq C_{\mathcal{G}}(\pi(g_{sc})) = C_{\mathcal{G}}(g)$. Thus, we get

$$\dim C_{\mathcal{G}_{sc}}(g_{sc}) \leq \dim C_{\mathcal{G}}(g) = \mathrm{rank}\ \mathcal{G} = n = \mathrm{rank}\ \mathcal{G}_{sc},$$

and therefore, g_{sc} is a regular 2-element in $\mathcal{G}_{sc}^F$, i.e., $\mathrm{Spin}_n^\epsilon(q)$.

□

Proposition 4.8 describes another character formula which can be applied if p divides $2rs$. We use the notion of *semisimple characters* in Section 2.2.2.

Proposition 4.8. *Let $\mathcal{G}$ be a connected reductive algebraic group with connected center defined over an algebraic closure of a finite field of characteristic p. Let F be its Frobenius map. We assume $p \nmid |\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|$. Suppose that there exists $x, y \in [\mathcal{G}^F, \mathcal{G}^F]$ such that*

$$\sum_{\substack{\chi \in \mathrm{Irr}_s(\mathcal{G}^F) \\ \chi(1) \neq 1}} \frac{|\chi(x)\chi(y)|}{\chi(1)} < |\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|$$

where $\mathrm{Irr}_s(\mathcal{G}^F)$ is the set of semisimple characters of $\mathcal{G}^F$. Then there exists a unipotent element $u \in [\mathcal{G}^F, \mathcal{G}^F]$ such that

$$\sum_{\chi \in \mathrm{Irr}(\mathcal{G}^F)} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \neq 0.$$

Proof. We assume $\mathcal{G}$, F as they are in the statement. Suppose that

$$\sum_{\substack{\chi \in \mathrm{Irr}_s(\mathcal{G}^F) \\ \chi(1) \neq 1}} \frac{|\chi(x)\chi(y)|}{\chi(1)} < |\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|$$

for some $x, y \in [\mathcal{G}^F, \mathcal{G}^F]$. Note that $\lambda(x) = \lambda(y) = 1$ for any linear character $\lambda \in \mathrm{Irr}(\mathcal{G}^F)$ because $x, y \in [\mathcal{G}^F, \mathcal{G}^F]$. Let U be the set of regular unipotent elements in $\mathcal{G}^F$. Then U is not empty and is a union of $\mathcal{G}^F$-conjugacy classes by [Car93, Proposition 5.1.7]. Also, $U \subseteq [\mathcal{G}^F, \mathcal{G}^F]$ since $p \nmid |\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|$, and so $\lambda(u) = 1$ for any linear character $\lambda \in \mathrm{Irr}(\mathcal{G}^F)$ and $u \in U$. On the other hand, the average character value of $\chi \in \mathrm{Irr}(\mathcal{G}^F)$ over U is either 1, 0, or -1 by [Car93, Proposition 8.3.3]. In particular, if χ is semisimple, then $\sum_{u \in U} \chi(u) = \pm |U|$, and if χ is not semisimple, then $\sum_{u \in U} \chi(u) = 0$. Now

$$\begin{aligned}
\left| \sum_{u \in U} \sum_{\chi \in \mathrm{Irr}(\mathcal{G}^F)} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \right| &\geq \left| \sum_{u \in U} \sum_{\substack{\lambda \in \mathrm{Irr}(\mathcal{G}^F) \\ \lambda(1)=1}} \frac{\lambda(u)\lambda(x)\lambda(y)}{\lambda(1)} \right| - \left| \sum_{u \in U} \sum_{\substack{\chi \in \mathrm{Irr}(\mathcal{G}^F) \\ \chi(1)\neq 1}} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \right| \\
&= |U|\,|\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]| - \left| \sum_{\substack{\chi \in \mathrm{Irr}(\mathcal{G}^F) \\ \chi(1)\neq 1}} \frac{\chi(x)\chi(y)}{\chi(1)} \sum_{u \in U} \chi(u) \right| \\
&\geq |U|\,|\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]| - |U| \sum_{\substack{\chi \in \mathrm{Irr}_s(\mathcal{G}^F) \\ \chi(1)\neq 1}} \frac{|\chi(x)\chi(y)|}{\chi(1)} > 0.
\end{aligned}$$

Thus there exists a regular unipotent element u in $[\mathcal{G}^F, \mathcal{G}^F]$ such that

$$\sum_{\chi \in \mathrm{Irr}(\mathcal{G}^F)} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \neq 0.$$

$\square$

Corollary 4.9. *Let $\mathcal{G}$ be a connected reductive algebraic group with connected center defined over an algebraic closure of a finite field of characteristic p. Let F be its Frobenius map. We assume $p \nmid |\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|$. Suppose that there exists $x, y \in [\mathcal{G}^F, \mathcal{G}^F]$ such that*

$$\sum_{\substack{\chi \in \mathrm{Irr}(\mathcal{G}^F) \\ \chi(1)\neq 1}} \frac{|\chi(x)\chi(y)|}{\chi(1)} < |\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|.$$

Then there exists a regular unipotent element $u \in [\mathcal{G}^F, \mathcal{G}^F]$ such that

$$\sum_{\chi \in \mathrm{Irr}(\mathcal{G}^F)} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \neq 0.$$

Lemma 4.10. *Let $\mathcal{G}$ be a simple algebraic group in characteristic p. Let F be its Frobenius morphism. If $\mathcal{G}^F$ is non-solvable and $\mathcal{G}^F \neq \mathrm{Sp}_4(2)$, ${}^2F_4(2)$, ${}^2G_2(3)$, $G_2(2)$, then $p \nmid |\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|$.*

Proof. Let $\mathcal{G}$ and F be an algebraic group and its Frobenius morphism described in the statement. We let $\mathcal{G}_{sc}$ be a simple algebraic group of simply connected type such that there exists a natural isogeny $\pi : \mathcal{G}_{sc} \to \mathcal{G}$. Let $Z = \ker(\pi)$. Note that $Z \leq Z(\mathcal{G}_{sc})$ since $\mathcal{G}$ is connected [MT11, p.71]. Denote an induced Frobenius morphism on $\mathcal{G}_{sc}$ by F as well. Then $\mathcal{G}_{sc}^F$ is non-solvable and $\mathcal{G}^F \neq \mathrm{Sp}_4(2)$, ${}^2F_4(2)$, ${}^2G_2(3)$, $G_2(2)$. By [MT11, Theorem 24.17], $\mathcal{G}_{sc}^F$ is perfect. Then $\mathcal{G}^F/[\mathcal{G}^F, \mathcal{G}^F] \simeq Z/L(Z)$ where $L : Z \to Z$ defined by $L(z) = F(z)z^{-1}$ by [MT11, Theorem 24.21]. Note that Z is a p'-group because Z is contained in a torus. Thus p is coprime to $|\mathcal{G}^F : [\mathcal{G}^F, \mathcal{G}^F]|$. □

Now we show a few lemmas that will be used in the computation of the above character formulas.

Let G be a finite group. Note that $\chi \in \mathrm{Irr}(G)$ is called a *p-defect* 0 character if $\chi(1)_p = |G|_p$. We use the notion of a p-defect positive character χ if $\chi(1)_p < |G|_p$. If χ is p-defect 0, then $\chi(g) = 0$ for any $g \in G$ such that $p \big| |g|$ (Lemma 2.2).

For a finite group G and a prime l, we say that an element $g \in G$ is *l-singular*, if $l \big| |g|$.

Lemma 4.11. *Let G be a finite group and let l be a prime divisor of $|G|$. If B is the principal l-block with a cyclic defect group, then for any $\chi \in \mathrm{Irr}(B)$ nonexceptional and any $g \in G$ an l-singular element, $\chi(g) = \pm 1$.*

Proof. Let $\chi_1, \ldots, \chi_m$ be the nonexceptional characters in $\mathrm{Irr}(B)$, and χ_λ, $\lambda \in \Lambda$ the exceptional characters in $\mathrm{Irr}(B)$. Also we denote the sum of exceptional characters $\chi_\Lambda := \sum_{\lambda \in \Lambda} \chi_\lambda$. Then each of $\chi_1, \ldots, \chi_m, \chi_\Lambda$ corresponds to distinct nodes in the Brauer tree of B. Without loss of generality, we may assume $\chi_1 = 1_G$. Let g be an l-singular element in G. Clearly $\chi_1(g) = 1$. Suppose that there exists $i \in I := \{1, 2, \ldots, m, \Lambda\}$ such that $\chi_i(g) \neq \pm 1$. Let S be the set of such χ_i's. We define $\delta(\chi_i, \chi_j)$ for $i, j \in I$ by the length of a shortest path between the nodes corresponding to χ_i and χ_j on the Brauer tree of B. Then there exists $\chi_i \in S$ such that $\delta(1_G, \chi_i) = \min_{\chi \in S} \delta(1_G, \chi) =: \delta$. Note that 1_G is not in S therefore $\delta > 0$.

Let χ_j be a node adjacent to χ_i on a path between 1_G and χ_i of length δ. It follows that $\delta(1_G, \chi_j) = \delta - 1$. However, there is a unique $\varphi \in \mathrm{IBr}(B)$ such that the projective indecomposable character Φ associated with φ can be written as $\Phi = \chi_i + \chi_j$ (see [Dad66, Theorem 1. Part 2]). Since Φ vanishes on the l-singular elements, we get $\chi_i(g) + \chi_j(g) = 0$, and so χ_j is also in S, a contradiction. Thus for any nonexceptional $\chi \in \mathrm{Irr}(B)$, $\chi(g) = \pm 1$. □

Even when we do not have much condition on the irreducible characters, we still can get an upper bound of character values at an element based on the order of the centralizer of the element.

Lemma 4.12. *Let G be a finite group and x an element of G. If $\chi \in \mathrm{Irr}(G)$, we get*

$$|\chi(x)| \leq \sqrt{|C_G(x)|}.$$

In our case, we are more interested in upper bounds for character values at semisimple elements of prime order. The following gives a criterion for the possible centralizer orders of such an element.

Lemma 4.13. *Let $\mathcal{G}$ be a simple algebraic group of simply connected type defined over an algebraic closure of a finite field of characteristic p with Frobenius map F. If $x \in \mathcal{G}^F$ is of order a prime $s \neq p$, then s divides either $|(Z(C_{\mathcal{G}}(x))^\circ)^F|$ or $|Z([C_{\mathcal{G}}(x), C_{\mathcal{G}}(x)])^F|$.*

Proof. Note that the centralizer of any semisimple element of a simply connected algebraic group is connected reductive ([Car93, Theorem 3.5.4, 3.5.6]). Denote $\mathcal{C} := C_{\mathcal{G}}(x)$ and $\mathcal{S} := [\mathcal{C}, \mathcal{C}]$. Then $\mathcal{C} = Z(\mathcal{C})^\circ \mathcal{S}$ ([MT11, Theorem 8.22]) and in particular, $|\mathcal{C}^F| = |\mathcal{Z}^F||\mathcal{S}^F|$ (see [FJ93]). Suppose that s does not divide $|\mathcal{Z}^F|$. Consider the central product $\mathcal{Z}^F * \mathcal{S}^F = \mathcal{Z}^F \mathcal{S}^F$. It is a normal subgroup of $\mathcal{C}^F$ of index coprime to s because $(s, |\mathcal{Z}^F|) = 1$ and

$$[\mathcal{C}^F : \mathcal{Z}^F * \mathcal{S}^F] = |\mathcal{Z}^F||\mathcal{S}^F|/|\mathcal{Z}^F * \mathcal{S}^F| = |\mathcal{Z}^F \cap \mathcal{S}^F| \, \big| \, |\mathcal{Z}^F|.$$

It follows that $x \in \mathcal{Z}^F * \mathcal{S}^F$. Furthermore, since s does not divide $|\mathcal{Z}^F|$, x is in $\mathcal{S}^F$. Since x is an F-fixed element in the center of $\mathcal{C}$, we get

$$x \in Z(\mathcal{C})^F \cap \mathcal{S}^F \leq Z(\mathcal{S})^F$$

and the result follows. □

Chapter 5

Proof of Theorem B and theorem C for Alternating Groups and Sporadic Groups

5.1 Alternating Groups

Lemma 5.1. *Let $S = \mathsf{A}_n$ and let $r > s$ be distinct odd prime divisors of $|S|$. If S is minimal (r, s)-section of S then $n = r$.*

Proof. From $\mathsf{A}_r \leq \mathsf{A}_n$ for any $n \geq r$, (i) holds immediately if S is an alternating group. □

Lemma 5.2. *Let G be the Schur multiplier of $S = \mathsf{A}_n$ for some $n \geq 5$. If S is a minimal (r, s)-simple section for distinct odd primes $r, s \big| |S|$, $r > s$, then G admits a $(2, r, s)$-triple.*

Proof. Let $S = \mathsf{A}_n$ be a minimal (r, s)-simple section for distinct odd primes $r, s \big| |S|$, $r > s$. Then $S = \mathsf{A}_r$ by Lemma 5.1. Note that there exist $(2, r, s)$-triples in A_r for any distinct odd primes $r > s$ ([GT15, Lemma 5.5]). Let G be the Schur cover of S. Then $G = 6S$ if $S = \mathsf{A}_6, \mathsf{A}_7$ and $G = 2S$ otherwise. If $r \neq 7$, then we are done by Lemma 4.3. Suppose $r = 7$. Then $\pi(|G|) = \{2, 3, 5, 7\}$, and so s is either 3 or 5. For $s = 3$, S admits a $(2, 3, 7)$-triple by Lemma 4.3. If $s = 5$, one can directly check that $3S$ admits $(2, 5, 7)$-triple, x of order 8, y of order 5, and z of order 7 using the character table in GAP or [Con+85]. Now we can apply Lemma 4.3 to lift it to $G = 6S$. □

5.2 Sporadic Groups

Lemma 5.3. *Let S be a sporadic group or ${}^2F_4(2)'$ and r, s distinct odd prime divisors of $|S|$. If S itself is a minimal simple (r, s)-section of S, then it is one of the following cases in Table 5.1:*

Table 5.1: Minimal simple (r, s)-section $S = G/Z(G)$ where S is a Sporadic Group

S	(r, s)	G

M_{22}	$(11,7)$	$12M_{22}$
M_{23}	$(23,s),\ s\in\{3,5,7,11\}$	M_{23}
J_1	$(19,s),\ s\in\{3,5,7,11\}$ $(7,3),(7,5),(11,7)$	J_1
J_2	$(7,5)$	$2J_2$
J_3	$(19,17)$	$3J_3$
J_4	$(29,s),\ s\in\{3,5,7,11,23\}$, $(31,29),(31,23)$, $(37,s),\ s\in\{7,23,29,31\}$, $(43,s),\ s\in\{3,5,7,11,23,29,31,37\}$	J_4
Co_1	$(23,13)$	Co_1
Fi_{22}	$(13,11)$	$6Fi_{22}$
Fi_{23}	$(17,11),(17,13),(23,13),(23,17)$	Fi_{23}
Fi'_{24}	$(29,s),\ s\in\{3,5,7,11,13,17,23\}$	$3Fi'_{24}$
He	$(17,7)$	He
Ru	$(29,13)$	$2Ru$
Suz	$(11,7),(13,11)$	$6Suz$
$O'N$	$(31,s),\ s\in\{19,11,7\}$	$3O'N$
HN	$(19,11),(19,5)$	HN
Ly	$(67,s),\ s\in\{3,5,7,11,31,37\}$ $(37,s),\ s\in\{3,5,7,11,31\}$ $(31,11)$	Ly
Th	$(31,19),(31,13),(19,13),(13,5)$	Th
B	$(47,s)\ s\in\{3,5,7,11,13,17,19,23,31\}$ $(31,23),(31,17),(31,11),(19,13)$	$2B$
M	$(71,s),\ s\in\{11,13,17,19,23,29,31,41,47,59\}$ $(59,s),\ s\in\{7,11,13,17,19,23,31,41,47\}$ $(47,41),(47,29)$ $(41,s),\ s\in\{11,17,19,23,29,31\}$ $(31,29),(29,19)$	M

Proof. If S is one of 26 sporadic groups, then one can get a (partial in some cases) list of maximal subgroup of S given in [Con+85]. Table 5.2 compares $\pi(|S|)$ the set of prime divisors of $|S|$ with $\pi(|H|)$ the set of prime divisors of $|H|$ where H is a simple section of S. Now the result in Table 5.1 follows.

Table 5.2: Prime divisors of the orders of sporadic groups and its simple sections

S	$\pi(\lvert S\rvert)$	H	$\pi(\lvert H\rvert)$
${}^2F_4(2)'$	$\{2,3,5,13\}$	$L_2(25)$	$\{2,3,5,13\}$
M_{11}	$\{2,3,5,11\}$	$L_2(11)$	$\{2,3,5,11\}$

M_{12}	$\{2,3,5,11\}$	M_{11}	$\{2,3,5,11\}$
M_{22}	$\{2,3,5,7,11\}$	$\mathrm{L}_3(4)$	$\{2,3,5,7\}$
		Ł$_2(11)$	$\{2,3,5,11\}$
M_{23}	$\{2,3,5,7,11,23\}$	M_{22}	$\{2,3,5,7,11\}$
M_{24}	$\{2,3,5,7,11,23\}$	M_{23}	$\{2,3,5,7,11,23\}$
J_1	$\{2,3,5,7,11,19\}$	$\mathrm{L}_2(11)$	$\{2,3,5,11\}$
J_2	$\{2,3,5,7\}$	$\mathrm{U}_3(3)$	$\{2,3,7\}$
J_3	$\{2,3,5,17,19\}$	$\mathrm{L}_2(16)$	$\{2,3,5,17\}$
		$\mathrm{L}_2(19)$	$\{2,3,5,19\}$
J_4	$\{2,3,5,7,11,23,29,31,37,43\}$	M_{24}	$\{2,3,5,7,11,23\}$
		$\mathrm{L}_5(2)$	$\{2,3,5,7,31\}$
		$\mathrm{L}_2(32)$	$\{2,3,11,31\}$
		$\mathrm{U}_3(11)$	$\{2,3,5,11,37\}$
Co_1	$\{2,3,5,7,11,13,23\}$	Co_2	$\{2,3,5,7,11,23\}$
		Suz	$\{2,3,5,7,11,13\}$
Co_2	$\{2,3,5,7,11,23\}$	M_{23}	$\{2,3,5,7,11,23\}$
Co_3	$\{2,3,5,7,11,23\}$	M_{23}	$\{2,3,5,7,11,23\}$
Fi_{22}	$\{2,3,5,7,11,13\}$	$\mathrm{U}_6(2)$	$\{2,3,5,7,11\}$
		$\mathrm{O}_7(3)$	$\{2,3,5,7,13\}$
Fi_{23}	$\{2,3,5,7,11,13,17,23\}$	Fi_{22}	$\{2,3,5,7,11,13\}$
		M_{23}	$\{2,3,5,7,11,23\}$
		$\mathrm{S}_8(2)$	$\{2,3,5,7,17\}$
Fi'_{24}	$\{2,3,5,7,11,13,17,23,29\}$	Fi_{23}	$\{2,3,5,7,11,13,17,23\}$
HS	$\{2,3,5,7,11\}$	M_{22}	$\{2,3,5,7,11\}$
McL	$\{2,3,5,7,11\}$	M_{22}	$\{2,3,5,7,11\}$
He	$\{2,3,5,7,17\}$	$\mathrm{S}_4(4)$	$\{2,3,5,17\}$
		$\mathrm{L}_3(4)$	$\{2,3,5,7\}$
Ru	$\{2,3,5,7,13,29\}$	${}^2F_4(2)'$	$\{2,3,5,13\}$
		${}^2B_2(8)$	$\{2,5,7,13\}$
		$\mathrm{L}_2(19)$	$\{2,3,5,7,29\}$
Suz	$\{2,3,5,7,11,13\}$	$G_2(4)$	$\{2,3,5,7,13\}$
		M_{11}	$\{2,3,5,11\}$
$O'N$	$\{2,3,5,7,11,19,31\}$	J_1	$\{2,3,5,7,11,19\}$
		$\mathrm{L}_2(31)$	$\{2,3,5,31\}$
HN	$\{2,3,5,7,11,19\}$	HS	$\{2,3,5,7,11\}$
		$\mathrm{U}_3(8)$	$\{2,3,7,19\}$
Ly	$\{2,3,5,7,11,31,37,67\}$	$G_2(5)$	$\{2,3,5,7,31\}$
		McL	$\{2,3,5,7,11\}$
Th	$\{2,3,5,7,13,19,31\}$	$\mathrm{L}_5(2)$	$\{2,3,5,7,31\}$
		$\mathrm{U}_3(8)$	$\{2,3,7,19\}$
		$\mathrm{L}_2(19)$	$\{2,3,5,19\}$
		${}^3D_4(2)$	$\{2,4,7,13\}$
B	$\{2,3,5,7,11,13,$	${}^2E_6(2)$	$\{2,3,5,7,11,13,17,19\}$

	$17, 19, 23, 31, 47\}$	Th	$\{2, 3, 5, 7, 13, 19, 31\}$
		Fi_{23}	$\{2, 3, 5, 7, 11, 13, 17, 23\}$
M	$\{2, 3, 5, 7, 11, 13, 17, 19$	B	$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 47\}$
	$23, 29, 31, 41, 47, 59, 71\}$	Fi'_{24}	$\{2, 3, 5, 7, 11, 13, 17, 23, 29\}$
		$\mathrm{O}_8^-(3)$	$\{2, 3, 5, 7, 13, 41\}$
		$\mathrm{L}_2(71)$	$\{2, 3, 5, 7, 71\}$
		$\mathrm{L}_2(59)$	$\{2, 3, 5, 29, 59\}$

□

Lemma 5.4. *Let G be the Schur multiplier of a sporadic group S or of $S = {}^2F_4(2)'$. If S is a minimal (r, s)-simple section for distinct odd primes $r, s \big| |S|$, $r > s$, then G admits a $(2, r, s)$-triple.*

Proof. One can directly compute the character formula in Lemma 4.4 for all the cases listed in Table 5.1 using GAP. See Appendix A for more details. □

Chapter 6

Proof of Theorem B and theorem C for Classical Groups

Recall the notation $X \prec G$ if X is (isomorphic to) a section of G.

Lemma 6.1. *Let S be a finite classical group and r, s be distinct odd prime divisors of $|S|$. If S itself is a minimal simple (r, s)-section of S, then we may assume that it is one of the following cases:*

(i) $S = \mathrm{L}_2(q)$

(ii) $S = \mathrm{L}_n^{\epsilon}(q)$ for $n \geq 3$ and $r \in \zeta(\epsilon q, n)$. In particular, if $s = p$, then n is a prime.

(iii) $S = \mathrm{S}_{2n}(q)$ for $n \geq 2$, $(p, 2rs) = 1$, and $r \in \zeta(q^2, n)$.

(iv) $S = \mathrm{O}_{2n+1}(q)$ for $n \geq 3$, $(p, 2rs) = 1$, and $r \in \zeta(q^2, n)$.

(v) $S = \mathrm{O}_{2n}^{\epsilon}(q)$ for $n \geq 4$, $r \in \zeta(q^2, n)$, and $s \neq p$. Also, if $\epsilon = +$, then n is odd.

Proof. The subgroup structure of finite classical groups is studied in [KL90]. We use the results in [KL90, Chapter 3] and Lemma 4.1 to obtain sections of S in the following argument.

- If $S = \mathrm{L}_n^{\epsilon}(q)$, then $\mathrm{L}_{n-1}^{\epsilon}(q) \prec S$, and so at least one of the primes r and s is in $\zeta(\epsilon q, n)$. We may assume $r \in \zeta(\epsilon q, n)$, in particular, $r \neq p$. Suppose that $p = s$. Note that $\mathrm{L}_{n/d}(q^d) \prec \mathrm{L}_n(q)$ for any $d|n$, $1 < d < n$. Also $\mathrm{U}_n(q)$ admits proper simple sections isomorphic to $\mathrm{U}_{n/d}(q^d)$ for odd $1 < d < n$ and $\mathrm{L}_{n/2}(q^2)$. Due to the minimality condition on S, n is a prime.

- If $S = \mathrm{S}_{2n}(q)$, then $\mathrm{S}_{2(n-1)}(q) \prec S$, and so at least one of the primes r and s is in $\zeta(q^2, n)$. We may assume $r \in \zeta(q^2, n)$, in particular, $r \neq p$. If $p = s$, then $\mathrm{L}_2(q^n) \prec S$ and $rs \big| |\mathrm{L}_2(q^n)|$. If $p = 2$, then $\mathrm{O}_{2n}^{\pm}(q) \prec S$ and $\mathrm{L}_2(q^n) \prec S$, and at least one of them is of order divisible by rs. Thus $(p, 2rs) = 1$.

- If $S = \mathrm{O}_{2n+1}(q)$, then $\mathrm{O}_{2n-1}(q) \prec S$, and so at least one of the primes r and s is in $\zeta(q^2, n)$. We may assume $r \in \zeta(q^2, n)$, in particular, $r \neq p$. If $p = s$, then $\mathrm{O}_{2n}^{\pm}(q) \prec S$, and at least one of them is of order divisible by rs. If $p = 2$, then S is isomorphic to $\mathrm{S}_{2n}(q)$.

- If $S = \mathrm{O}^{\epsilon}_{2n}(q)$, then $\mathrm{O}_{2n-1}(q) \prec S$, and so at least one of the primes r and s is in $\zeta(q^2, n)$. We may assume $r \in \zeta(q^2, n)$, in particular, $r \neq p$. Note that if $\epsilon = +$ and n is even, then there is no such prime. We show that if $p = s$, then S has a proper simple section of order divisible by rs. If $\epsilon = +$, then $\mathrm{O}^{\pm}_{2n-2}(q) \prec S$ and $\mathrm{L}_n(q) \prec S$. If $\epsilon = -$ and n is even, then $\mathrm{O}^{\pm}_{2n-2}(q) \prec S$ and $\mathrm{L}_2(q^n) \prec S$. If $\epsilon = -$ and n is odd, then $\mathrm{O}^{\pm}_{2n-2}(q) \prec S$ and $\mathrm{U}_n(q) \prec S$.

□

6.1 $\mathrm{SL}_2(q)$

The generic character table of $\mathrm{GL}_2(q)$ is known. We show Theorem B holds for $G = \mathrm{SL}_2(q)$ based on the tables given in Chevie [Gec+96].

Lemma 6.2. *Let $S = \mathrm{L}_2(q)$ for $q = p^f > 3$, p a prime and G the Schur cover of S. If r, s are distinct odd prime divisors of $|S|$, then G admits a $(2, r, s)$-triple.*

Proof. Since $\mathrm{L}_2(4) = \mathsf{A}_5$ or $\mathrm{L}_2(9) = \mathsf{A}_6$, we consider the cases where $q \neq 4, 9$. Note that $G = \mathrm{SL}_2(q)$.

Let r and s be distinct odd primes in the statement. Without loss of generality we may assume it is one of these two cases: (i) $(p, 2rs) = 1$, (ii) $(p, 2rs) \neq 1$.

If $(p, 2rs) = 1$, then r divides either $q - 1$ or $q + 1$, and in either case, there exists a regular semisimple r-element. Similarly, there exists a regular semisimple s-element. Thus G admits a $(2, r, s)$-triple Lemma 4.6.

Suppose that $(p, 2rs) \neq 1$. Let x be an element in G of order one of primes 2, r, or s which is not p. Let u be a regular unipotent element in G. Also we let y be an element in G of order one of primes 2, r, or s which is different from p and $|x|$. Since $\mathrm{SL}_2(q) \lhd \mathrm{GL}_2(q)$, it is enough to show that

$$\sum_{\chi \in \mathrm{Irr}(\mathrm{GL}_2(q))} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \neq 0$$

by Corollary 4.5. By the choice of x, the order of $C_{\mathrm{GL}_2(q)}(x)$ is either $(q-1)^2$ or q^2-1. In the Chevie character table, x^G is of type C_3 or C_4. Since x is in $\mathrm{SL}_2(q)$, we get $\chi(x)=1$ for any linear character $\chi \in \mathrm{GL}_2(q)$. It follows that $\chi(x)=1$ if x^G is of type C_3 and $\chi(x)=-1$ if x^G is of type C_4 for any $\chi \in \mathrm{GL}_2(q)$ of degree q. If x^G is of type C_3 then $\chi(x)=0$ for $\chi \in \mathrm{GL}_2(q)$ of degree $q-1$, and if x^G is of type C_4 then $\chi(x)=0$ for $\chi \in \mathrm{GL}_2(q)$ of degree $q+1$. The same results hold for y as well. Also note that $\chi(u)$ is one of $0, 1, -1$ and $\chi(u) \equiv \chi(1) \mod p$ for $\chi \in \mathrm{Irr}(\mathrm{GL}_2(q))$.

- If x^G and y^G are classes of different types, then $\chi(u)\chi(x)\chi(y) \neq 0$ if and only if χ is linear, in which case $\chi(u)=\chi(x)=\chi(y)=1$, and so the result follows.
- Suppose x^G and y^G are both classes of type C_3. Note that $(y^{-1})^G$ is also of type C_3. Since the order of x is different from the order of y, x^G and $(y^{-1})^G$ are different classes. Using the second orthogonality relation on the character table, we get

$$\sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q+1}} \chi(x)\chi(y) = -\sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=1}} \chi(x)\chi(y) - \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q}} \chi(x)\chi(y) = -2(q-1).$$

  Thus

$$\begin{aligned}
\sum_{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} &= \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=1}} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} + \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q+1}} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \\
&= (q-1) + \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q+1}} \frac{\chi(x)\chi(y)}{q+1} \\
&= q-1-\frac{2(q-1)}{q+1} > 0
\end{aligned}$$

- Suppose x^G and y^G both are classes of type C_4. We can make a similar argument in the previous case.

$$\sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q-1}} \chi(x)\chi(y) = -\sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=1}} \chi(x)\chi(y) - \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q}} \chi(x)\chi(y) = -2(q-1).$$

Thus

$$\begin{aligned}\sum_{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))}\frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} &= \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=1}}\frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} + \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q-1}}\frac{\chi(u)\chi(x)\chi(y)}{\chi(1)}\\ &= (q-1) - \sum_{\substack{\chi\in\mathrm{Irr}(\mathrm{GL}_2(q))\\ \chi(1)=q-1}}\frac{\chi(x)\chi(y)}{q-1}\\ &= q-1+2>0\end{aligned}$$

By Corollary 4.5, $\mathrm{SL}_2(q)$ admits a $(2,p,q)$-triple. □

6.2 Classical Groups with Rank ≥ 2

We prove that Theorem B holds for finite classical quasisimple groups of simply connected type: $\mathrm{SL}_n^\epsilon(q)$ for $n\geq 3$, $\mathrm{Sp}_{2n}(q)$ for $n\geq 2$, $\mathrm{Spin}_{2n+1}(q)$ for $n\geq 3$, and $\mathrm{Spin}_{2n}^\epsilon(q)$ for $n\geq 4$ where $q=p^f$ for p a prime.

We first consider degrees and character value bounds of irreducible characters χ in a finite group of Lie type where χ is r-defect positive for a prime r described in Lemma 6.1.

Lemma 6.3. *Let $r\in\zeta(\epsilon q,n)$ and $q=p^f$ for some prime p. If $\chi\in\mathrm{Irr}_s(\mathrm{GL}_n^\epsilon(q))$ is not r-defect 0, then*

$$\chi(1)=\begin{cases}|\mathrm{GL}_n(q)|_{p'}/|\mathrm{GL}_{n/d}(q^d)|_{p'}, & \epsilon=+1;\\ |\mathrm{GU}_n(q)|_{p'}/|\mathrm{GU}_{n/d}(q^d)|_{p'}, & \epsilon=-1, d \textit{ odd};\\ |\mathrm{GU}_n(q)|_{p'}/|\mathrm{GL}_{n/d}(q^d)|_{p'}, & \epsilon=-1, d \textit{ even}\end{cases}$$

for some $d\mid n$.

Proof. Let $\mathcal{G}=\mathrm{GL}_n^\epsilon(k)$ where k is an algebraic closure of a finite field of characteristic p. Let F be a Frobenius morphism such that $\mathcal{G}^F=\mathrm{GL}_n^\epsilon(q)$, and we denote $G:=\mathrm{GL}_n^\epsilon(q)$. Since G is self-dual, if χ is a semisimple irreducible character of G, there exists a semisimple element $x\in G$ such that

$$\chi(1)=|G:C_G(x)|_{p'}$$

([Car93, Theorem 8.4.8]). Suppose that $r\in\zeta(\epsilon q,n)$ and χ is not r-defect 0. We show that x is in a maximal non-split torus T of order $q^n-\epsilon^n$. Since χ is not r-defect 0, then r divides

$|G|/\chi(1)$. Thus r divides $|G|_{p'}/\chi(1) = |C_G(x)|_{p'}$, and so $r \Big| |C_G(x)|$. Thus there is an element of order r commuting with x.

We first show that for any element of order r is regular, its centralizer being a maximal torus of order $q^n - \epsilon^n$. Let g be an element of order r in G, and k be an algebraic closure of a finite field of characteristic p. Then there exists an eigenvalue $\lambda \in k$ of g such that $\lambda \neq 1$ and $\lambda^r = 1$. Note that $\lambda^{\epsilon q}$ is also an eigenvalue of g because g is an F-fixed point. Thus, we get distinct eigenvalues $\lambda, \lambda^{\epsilon q}, \lambda^{(\epsilon q)^2}, \dots, \lambda^{(\epsilon q)^{n-1}}$ of g because $r \in \zeta(\epsilon q, n)$. Since the characteristic polynomial of g is of degree n, $\lambda^{(\epsilon q)^k}$ for $k = 0, 1, \dots, n-1$ are the eigenvalues of g. Then the centralizer of t is of order $q^n - \epsilon^n$ by [TZ96, p.13] for $\epsilon = +1$ and [SF13, Section 2.4] for $\epsilon = -1$.

It follows that x is in a maximal torus of order $q^n - \epsilon^n$, because x is in the centralizer of an element of order r. We first consider the structure of the maximal non-split torus T of order $q^n - \epsilon^n$ (cf. [MT11, Section 25.1]). Let $\mathcal{G} = \mathrm{GL}_n^\epsilon$ with F Frobenius morphism such that $\mathcal{G}^F = G$. There exists $g \in \mathcal{G}$ such that $g^{-1}F(g) = w$ where w is the image of n-cycle σ under regular representation from S_n to $\mathcal{G}$. Then $T = (g\mathcal{T}_0 g^{-1})^F$ where $\mathcal{T}_0$ is the maximally split maximal torus of diagonal matrices in $\mathcal{G}$. Note that F acts on $g\mathcal{T}_0 g^{-1}$ as wF acts on $\mathcal{T}_0$, and

$$\mathcal{T}_0^{wF} = \{\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \mid \lambda_{(\sigma(i))} = \lambda_i^{\epsilon q} \text{ for } i = 1, \dots, n\}.$$

Let $t \in T$. Then the eigenvalues of t are $\lambda, \lambda^{\epsilon q}, \lambda^{(\epsilon q)^2}, \dots, \lambda^{(\epsilon q)^{n-1}}$ which are not necessarily all distinct and $\lambda^{(\epsilon q)^n} = \lambda$. Now let d be the smallest natural number such that $\lambda^{(\epsilon q)^d} = \lambda$. Clearly $d \mid n$. Then the centralizer of t is isomorphic to one the following groups depending on ϵ and d. (See [TZ96, p.13] for $\epsilon = +1$ and [SF13, Section 2.4] for $\epsilon = -1$.)

$$C_G(t) \simeq \begin{cases} \mathrm{GL}_{n/d}(q^d), & \epsilon = +1; \\ \mathrm{GU}_{n/d}(q^d), & \epsilon = -1, d \text{ odd}; \\ \mathrm{GL}_{n/2d}(q^{2d}), & \epsilon = -1, d \text{ even} \end{cases}$$

The result follows immediately. □

Lemma 6.4. *Let χ be a positive r-defect semisimple character for $r \in \zeta(\epsilon q, n)$ of $\mathrm{GL}_n^\epsilon(q)$ for $n \geq 3$ a prime, p a prime, and $q = p^f$. Denote the maximal non-split torus of order $q^n - \epsilon^n$ in $\mathrm{GL}_n^\epsilon(q)$ by T. Then the followings are true:*

(i) The degree of χ is either 1 or $|GL_n^\epsilon(q) : T|_{p'}$.

(ii) *There are* $q-1$ *linear characters in* $\mathrm{GL}_n^\epsilon(q)$*, and for such a linear character* χ*, we have* $\chi(x)=1$ *for any* x *in* $\mathrm{SL}_n^\epsilon(q)$*.*

(iii) *There are* $(q^n-q)/n$ *characters of degree* $|GL_n^\epsilon(q):T|_{p'}$ *in* $\mathrm{GL}_n^\epsilon(q)$*. Such a character* χ *vanishes on a regular semisimple element unless it is contained in* T*. In particular, if* x *is a regular semisimple element in* T*, then* $|\chi(x)|\le n$*.*

Proof. (i) follows from Lemma 6.3, and (ii) follows since $\mathrm{SL}_n^\epsilon(q)$ is the derived subgroup of $\mathrm{GL}_n^\epsilon(q)$ of index $q-1$.

We show that (iii) holds. Let $G=\mathrm{GL}_n^\epsilon(q)$ and T the maximal non-split torus of order $q^n-\epsilon^n$. From the Deligne-Lusztig theory [Car93, Theorem 8.4.6, 8.4.8], there are semisimple characters of degree $|G:T|_{p'}$ as many as the conjugacy classes of semisimple elements, centralizer of which is T. Thus we only need to count the number of such conjugacy classes. Any element in the maximally nonsplit torus T is similar to $\mathrm{diag}(\lambda,\lambda^{\epsilon q},\dots,\lambda^{(\epsilon q)^{n-1}})$ for some $\lambda\in\overline{\mathbb{F}_p}$, $\lambda^{(\epsilon q)^n}=\lambda$. Because n is a prime, λ, $\lambda^{\epsilon q}$, $\dots$, $\lambda^{(\epsilon q)^{n-1}}$ are all distinct unless $\lambda^{\epsilon q}=\lambda$. Therefore, there are q^n-q many regular semisimple elements in T, and each n of them, which have the same eigenvalues, are G-conjugate. Thus there are $(q^n-q)/n$ many semisimple characters of degree $|G:T|_{p'}$.

Let $\mathcal{G}$ be a connected reductive algebraic group with F its Frobenius morphism such that $G=\mathcal{G}^F$. Let $\mathcal{T}$ be the F-stable maximal torus obtained by twisting the maximally split torus $\mathcal{T}_0$ by $w\in W=W(\mathcal{T})$ such that $\mathcal{T}^F=T$. To get the bound for character values, we show that the semisimple characters of degree $|G:T|_{p'}$ are up to sign exactly the Deligne-Lusztig characters $R_{\mathcal{T},\theta}$ for $\theta\in\mathrm{Irr}(T)$ in general position (cf. [Car93, Theorem 8.4.6,8]). Since n is an odd prime, $\det(w)$ is 1. From the Deligne-Lusztig theory, $R_{\mathcal{T},\theta}$ up to sign is an irreducible character of G, and

$$\pm R_{\mathcal{T},\theta}(1)=|\mathcal{G}^F:\mathcal{T}^F|_{p'}=|G:T|_{p'}.$$

It is enough to show that there are exactly $(q^n-q)/n$ Deligne-Lusztig characters of this kind. There are q^n-q irreducible characters of T in general position. For such θ in general position, there is no nonidentity element fixing θ, and so $|\mathrm{Orb}_W(\theta)|=|C_W(w)|=n$. Since $R_{\mathcal{T},\theta}=R_{\mathcal{T},\theta'}$ if and only if $\theta'\in\mathrm{Orb}_W(\theta)$, the result follows.

Recall for a semisimple element $x \in G$ ([Car93, Proposition 7.5.3]),

$$R_{\mathcal{T},\theta}(x) = \frac{\epsilon_{\mathcal{T}}\epsilon_{C^0_{\mathcal{G}}(x)}}{|T||C^0_{\mathcal{G}}(x)^F|_p} \sum_{g \in G, x^g \in T} \theta(x^g)$$

It immediately follows that $R_{\mathcal{T},\theta}(x) = 0$ for $x \notin T$ regular. Suppose that $x \in T$ is regular. We claim that $x^g \in \mathcal{T}$ if and only if $g \in N_G(\mathcal{T})$. It is clear that $g \in N_G(\mathcal{T})$ implies $x^g \in \mathcal{T}$. Conversely, if $x^g \in \mathcal{T}$, then $x \in \mathcal{T}^{g^{-1}}$. Since x is regular, there is a unique maximal torus containing x. Thus, $\mathcal{T}^{g^{-1}} = \mathcal{T}$ and so $g \in N_G(\mathcal{T})$. Now,

$$|R_{\mathcal{T},\theta}(x)| = \left| \frac{\epsilon_{\mathcal{T}}\epsilon_{\mathcal{T}}}{|T||T|_p} \sum_{g \in G, x^g \in T} \theta(x^g) \right| \leq \frac{1}{|T|} \sum_{g \in G, x^g \in T} |\theta(x^g)| = \frac{1}{|T|} |N_G(\mathcal{T})| = |C_W(w)| = n.$$

□

Lemma 6.5. *Let $G = \Omega^{\epsilon}_{2n}(q)$ for $q = 2^f$. Let r be a prime such that $r \in \zeta(\epsilon q, n)$ and $r \big| |G|$. If $\chi \in \mathrm{Irr}_s(G)$ is not r-defect 0, then $\chi(1)$ is either 1, $|\mathrm{GO}^{\epsilon}_{2n}(q) : \mathrm{GU}_{n/d}(q^d)|_{2'}$, or $|\mathrm{GO}^{\epsilon}_{2n}(q) : \mathrm{GL}_{n/d}(q^d)|_{2'}$ for some $d > 1$, $d \mid n$.*

Proof. Let $G = \Omega^{\epsilon}_{2n}(q)$, $q = 2^f$ and $\chi \in \mathrm{Irr}_s(G)$. We make a similar argument as in the proof of Lemma 6.3. Since G is self-dual for q even, there exists a semisimple element $x \in G$ such that $\chi(1) = |G : C_G(x)|_{2'}$, and so χ is positive r-defect if and only if r divides the order of $C_G(x)$. Let $\widetilde{G} = \mathrm{GO}^{\epsilon}_{2n}(q)$. We consider $C_{\widetilde{G}}(x)$. As x being in the centralizer of an r-element, which is a maximal non-split torus, $C_{\widetilde{G}}(x)$ is either isomorphic to $\widetilde{G}$, $\mathrm{GU}_{n/d}(q^d)$, or $\mathrm{GL}_{n/d}(q^d)$ for some $d > 1$, $d \mid n$ by [Ngu10, Lemma 2.3]. Since $G = [\widetilde{G}, \widetilde{G}]$ and $\widetilde{G}/G \simeq \mathbb{Z}_2$, we get $|G : C_G(x)|_{2'} = |\widetilde{G} : C_{\widetilde{G}}(x)|_{2'}$. Now the result follows. □

Proposition 6.6. *Let S be one of $\mathrm{L}^{\epsilon}_n(q)$ for $n \geq 3$, $\mathrm{S}_{2n}(q)$ for $n \geq 2$, $\mathrm{O}_{2n+1}(q)$ for $n \geq 3$, and $\mathrm{O}^{\epsilon}_{2n}(q)$ for $n \geq 4$ where $q = p^f$ for p a prime, and let G be the Schur cover of S. If S is a minimal simple (r, s)-section of S itself, then G admits a $(2, r, s)$-triple.*

Proof. The following is the list of cases where S has an exceptional Schur cover.

(i) $\mathrm{L}^{\epsilon}_n(q)$ with $(\epsilon, n, q) = (+, 3, 2)$, $(+, 3, 4)$, $(+, 4, 2)$, $(-, 4, 2)$, $(-, 4, 3)$, or $(-, 6, 2)$: Note that $\mathrm{L}_3(2) \simeq \mathrm{L}_2(7)$, $\mathrm{L}_4(2) \simeq \mathsf{A}_8$. In the remaining cases, there exist simple subgroups, which is of order with the exactly same set of prime divisors $\mathsf{A}_6 \leq \mathrm{U}_4(2)$, $\mathsf{A}_7 \leq \mathrm{U}_4(3)$, $M_{22} \leq \mathrm{U}_6(2)$

(see [Con+85]). The character table for $L_3(4)$ is available in GAP, so one can check the statement holds for $L_3(4)$ (see Appendix A.2.1).

(ii) $S_{2n}(q)$ with $(n, q) = (2, 2)$, $(2, 3)$, or $(3, 2)$: Note that $S_4(2) \simeq \mathsf{S}_6$, $S_4(3) \simeq U_4(2)$. Also, $\mathsf{A}_8 \leq S_6(2)$ and $\pi(|\mathsf{A}_8|) = \pi(|S_6(2)|)$.

(iii) $O_{2n+1}(q)$ with $(n, q) = (3, 3)$: Note that $G_2(3)$, $S_6(2)$, $L_4(3) \prec O_7(3)$. There is no pair (r, s) of distinct prime divisors of $O_7(3)$ such that $O_7(3)$ is a minimal simple (r, s)-section.

(iv) $O^{\epsilon}_{2n}(q)$ with $(\epsilon, n, q) = (+, 8, 2)$: Note that $\mathsf{A}_9 \leq O^+_8(2)$ and $\pi(|\mathsf{A}_9|) = \pi(|O^+_8(2)|)$.

Now we suppose S is not one of the groups listed above. The Schur cover G of S is the one of $SL^{\epsilon}_n(q)$, $Sp_{2n}(q)$, $Spin_{2n+1}(q)$, and $Spin^{\epsilon}_{2n}(q)$.

Suppose that we have one of the cases (ii-vi) in Lemma 6.1. Then there exists a regular semisimple element of order r ([MT08, Lemma 2.4]). Thus if $(p, 2rs) = 1$, we are done by Lemma 4.6 and Lemma 4.7. Now it is left to show that $SL^{\epsilon}_n(q)$ admits a $(2, r, s)$-triple for $p = s$ or 2 and that $Spin^{\epsilon}_{2n}(q)$ admits a $(2, r, s)$-triple for $p = 2$.

First we consider the case $G = SL^{\epsilon}_n(q)$ and $p = s$. From Lemma 6.1, $n > 2$ is a prime. Let u be a regular unipotent element, x a regular r-element, and y a regular 2-element in G. (Such regular 2-element y exists. See Lemma 4.7.) It is sufficient to show that there is a $(2, r, s)$-triple in $GL^{\epsilon}_n(q)$ formed by some $GL^{\epsilon}_n(q)$-conjugates of u, x, and y because all $GL^{\epsilon}_n(q)$-conjugates of u, x, and y are in G as well. Denote $GL^{\epsilon}_n(q)$ by $\widetilde{G}$. For $\chi \in \mathrm{Irr}(\widetilde{G})$, $\chi(u) = 0$ if and only if χ is not semisimple by the definition (cf. [Car93, p.280]). Also, for $\chi \in \mathrm{Irr}_s(\widetilde{G})$, we have $|\chi(u)| = 1$. For $\chi \in \mathrm{Irr}(\widetilde{G})$, $\chi(x) = 0$ if χ is r-defect 0. Now we see $\chi(u)\chi(x)\chi(y) \neq 0$ only if $\chi \in \mathrm{Irr}(\widetilde{G})$ is semisimple and positive r-defect, and since n is a prime, such characters are listed in Lemma 6.4. We compute character the formula as follow:

$$\begin{aligned}
\left| \sum_{\chi \in \mathrm{Irr}(\widetilde{G})} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \right| &\geq |\widetilde{G} : G| - \sum_{\substack{\chi \in \mathrm{Irr}(\widetilde{G}) \\ \chi(1) \neq 1}} \frac{|\chi(u)\chi(x)\chi(y)|}{\chi(1)} \\
&= (q-1) - \sum_{\substack{\chi \in \mathrm{Irr}_s(\widetilde{G}) \\ \chi(1) \neq 1}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \\
&\geq q - 1 - \frac{q^n - q}{n} \frac{n^2}{|GL^{\epsilon}_n(q) : T|_{p'}} \\
&= q - 1 - \frac{nq}{(q^{n-2} - \epsilon^{n-2})(q^{n-3} - \epsilon^{n-3}) \cdots (q - \epsilon)}.
\end{aligned}$$

If $n \geq 5$, we get

$$q - 1 - \frac{nq}{(q^{n-2} - \epsilon^{n-2})(q^{n-3} - \epsilon^{n-3}) \cdots (q - \epsilon)} \geq q - 1 - \frac{nq}{(q^{n-2} - 1)(q^{n-3} - 1) \cdots (q - 1)}$$
$$\geq q - 1 - \frac{2n}{(q^{n-2} - 1)(q^{n-3} - 1)} > 0.$$

If $n = 3$, from the initial assumption on (ϵ, n, q), we have $q \geq 5$. Therefore, we get

$$q - 1 - \frac{3q}{q - \epsilon} \geq q - 1 - \frac{3q}{q - 1} > 0.$$

If $p = 2$, we choose u a regular unipotent element, x a regular semisimple r-element, and y a nontrivial element of order s in G. In particular, we choose y to be noncentral. Suppose that n is not a prime. Then due to the minimality of S, we see that s does not divide the order of $|\mathrm{L}^{\epsilon^d}_{n/d}(q^d)|$ for any $d|n$, $1 < d < n$, because $\mathrm{L}^{\epsilon^d}_{n/d}(q^d) \prec S$. Let $\widetilde{G} := \mathrm{GL}^{\epsilon}_n(q)$. We again show that there is a $(2, r, s)$-triple in $\widetilde{G}$ formed by $\widetilde{G}$-conjugates of u, x, and y. If χ does not vanish on u nor x, then χ is a semisimple character with positive r-defect. Thus $\chi(1)$ is one of the degrees listed in Lemma 6.3, and among them only the linear characters are not s-defect 0. Since $G = \widetilde{G}'$ is in the kernel of each linear characters of $\widetilde{G}$, $\chi(u) = \chi(x) = \chi(y) = 1$ for all of $\chi \in \mathrm{Irr}(\widetilde{G})$ linear. Therefore, we get

$$\sum_{\chi \in \mathrm{Irr}(\widetilde{G})} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} = |\widetilde{G} : G| \neq 0$$

Now we assume $n > 2$ is a prime. If s does not divide $q^n - \epsilon^n$, then by Lemma 6.3, the only characters that are of positive r-defect and s-defect are the linear characters, and so by the similar argument as in the previous case, we are done. Suppose that s divides $q^n - \epsilon^n$. Then y is a nontrivial s-element in the maximal non-split torus of order $q^n - \epsilon^n$. From the proof of Lemma 6.3, the centralizer of y is either $\mathrm{GL}^{\epsilon}_n(q)$ or T since n is a prime. Since y is noncentral, the centralizer of y is T, and so y is a regular semisimple element in T. Therefore, we compute the character formula using Lemma 6.4 and Lemma 6.3 as above:

$$\left| \sum_{\chi \in \mathrm{Irr}(\widetilde{G})} \frac{\chi(u)\chi(x)\chi(y)}{\chi(1)} \right| \geq q - 1 - \frac{nq}{(q^{n-2} - \epsilon^{n-2})(q^{n-3} - \epsilon^{n-3}) \cdots (q - \epsilon)} > 0$$

and conclude that G admits a $(2, r, s)$-triple.

Lastly we consider the case $G = \mathrm{Spin}^{\epsilon}_{2n}(q)$ with q even. Note that $\mathrm{Spin}^{\epsilon}_{2n}(q) = \Omega^{\epsilon}_{2n}(q)$ for q even. Since $S = \mathrm{O}^{\epsilon}_{2n}(q)$ is a minimal simple (r, s)-section, we get the following conditions on s. (The subgroup structure used in below argument can be found in [KL90].)

(i) $s \nmid (q^n - \epsilon^n)$.

Otherwise, $\mathrm{L}_n(q) \prec S$ if $\epsilon = +$ and $\mathrm{U}_n(q) \prec S$ or $\mathrm{L}_2(q^n) \prec S$ if $\epsilon = -$, all of which are of order divisible by rs.

(ii) $s \nmid q^{2n/m} - 1$ for any prime $m|2n$.

Otherwise S admits a proper simple section of order divisible by rs. First we consider the case of $\epsilon = +$. Note that n is odd (see Lemma 4.1). If $m = 2$, we have already shown that $s \nmid (q^n - 1)$. If $s|(q^{2n/m} - 1)$ for an odd prime $m|n$ and $2 < m < n$, then $\mathrm{O}^{+}_{2n/m}(q^m) \prec S$ which is of order divisible by rs. If $m = n$ and so $s|(q^2 - 1)$, then $\mathrm{L}_n(q) \prec S$ of order divisible by rs. Now we consider the case of $\epsilon = -$. If $s|(q^{2n/m} - 1)$ for a prime $m|n$ and $1 < m < n$, then $\mathrm{O}^{-}_{2n/m}(q^m) \prec S$ which is of order divisible by rs. If $m = n$ and so $s|(q^2 - 1)$, then $\mathrm{U}_n(q) \prec S$ of order divisible by rs.

In particular, (ii) implies $s \nmid |\mathrm{GL}_{n/d}(q^d)|$ and $s \nmid |\mathrm{GU}_{n/d}(q^d)|$ for $d|n$, $d > 1$. Let u be a regular unipotent element, x a regular semisimple r-element, and z a nontrivial s-element in G. Note that $\chi(u)\chi(x) \neq 0$ for $\chi \in \mathrm{Irr}(G)$ only if χ is semisimple and r-defect positive. However, such χ is s-defect 0 unless $\chi(1) = 1$ by Lemma 6.5. Thus we get

$$\sum_{\substack{\chi \in \mathrm{Irr}_s(G) \\ \chi(1) \neq 1}} \left| \frac{\chi(x)\chi(y)}{\chi(1)} \right| = 0$$

and conclude G admits a $(2, r, s)$-triple using Proposition 4.8. □

Chapter 7

Proof of Theorem B and theorem C for Exceptional Groups of Lie type

Recall that the case $S = {}^2F_4(2)'$ has been treated in Lemma 5.4.

Lemma 7.1. *Let S be a finite exceptional group of Lie-type and r, s be distinct odd prime divisors of $|S|$. If S itself is a minimal simple (r, s)-section of S, then we may assume that it is one of the following cases:*

(i) $S = {}^2B_2(q)$

(ii) $S = {}^2G_2(q)$ and $r \in \zeta(q, 6)$

(iii) $S = G_2(q)$, $r \in \zeta(q, 6)$, and $s \in \zeta(q, 3)$

(iv) $S = {}^3D_4(q)$ and $r \in \zeta(q, 12)$.

(v) $S = {}^2F_4(q)$ and either $r \in \zeta(q, 12)$, or $r \in \zeta(q, 6)$, $s \in \zeta(q, 4)$.

(vi) $S = F_4(q)$, $r \in \zeta(q, 12)$, and $s \in \zeta(q, 8) \cup \zeta(q, 4)$.

(vii) $S = E_6^{\epsilon}(q)$, $r \in \zeta(\epsilon q, k)$, and $s \in \zeta(\epsilon q, l)$ for $(k, l) = (12, 9)$, $(12, 5)$, $(9, 8)$, $(9, 5)$, or $(9, 4)$.

(viii) $S = E_7(q)$, $r \in \zeta(q, k)$, and $s \in \zeta(q, l)$ for $(k, l) = (18, 14)$, $(18, 9)$, $(18, 7)$, $(18, 5)$, $(14, 12)$, $(14, 9)$, $(14, 7)$, $(14, 5)$, $(9, 10)$, $(9, 7)$, $(7, 12)$, or $(7, 10)$.

(ix) $S = E_8(q)$ and $r \in \zeta(q, k)$ for $k \in \{30, 24, 20, 15\}$. Furthermore, if $p = s$, then $k = 30$ or 15. If $p = 2$ and $k = 24$, then $s \in \zeta(q, l)$ for $l \in \{30, 20, 18, 15, 14, 10, 9, 8, 7\}$. If $p = 2$ and $k = 20$, then $s \in \zeta(q, l)$ for $l \in \{30, 24, 18, 15, 14, 10, 9, 7, 6, 3\}$.

Proof. The subgroup structure of exceptional groups of Lie-type is studied in [LSS92]. In the following argument, we use simple sections of S obtained from [LSS92, Table 5.1] and Lemma 4.1.

- If $S = {}^2G_2(q)$, then $\mathrm{L}_2(q) \prec S$. Thus, at least one of the primes r and s, say r, is in $\zeta(q, 6)$.
- If $S = G_2(q)$, then $\mathrm{L}_3(q) \prec S$ and $\mathrm{U}_3(q) \prec S$. Thus at least one of the primes r and s, say r, is in $\zeta(q, 6)$. Then $s \in \zeta(q, 3)$.

- If $S = {}^3D_4(q)$, then $\mathrm{L}_2(q^3) \prec S$. Hence, at least one of the primes r and s, say r, is in $\zeta(q, 12)$.
- If $S = {}^2F_4(q)$, then $\mathrm{U}_3(q) \prec S$, ${}^2B_2(q) \prec S$, and $\mathrm{S}_4(q) \prec S$. Since rs does not divide the order of any proper simple section of S, either at least one of r and s is in $\zeta(q, 12)$, or one of r and s is in $\zeta(q, 6)$ and the other is in $\zeta(q, 4)$.
- If $S = F_4(q)$, then $\mathrm{O}_9(q) \prec S$ and ${}^3D_4(q) \prec S$. Thus at least one of the primes r and s is in $\zeta(q, 12)$. Let r be such a prime. Since $r \big| |{}^3D_4(q)|$, it follows that $s \nmid |{}^3D_4(q)|$, and so $s \in \zeta(q, 8) \cup \zeta(q, 4)$.
- If $S = E_6^\epsilon(q)$, then $\mathrm{O}_{10}^\epsilon(q) \prec S$, $F_4(q) \prec S$, and $\mathrm{L}_3^\epsilon(q^3) \prec S$. If $S = E_7(q)$, then $E_6^\epsilon(q) \prec S$, $\mathrm{U}_8(q) \prec S$, $\mathrm{L}_8(q) \prec S$, and $\mathrm{O}_{12}^+(q) \prec S$. If $S = E_8(q)$, then $E_7(q) \prec S$, $\mathrm{O}_{16}^+(q) \prec S$, $\mathrm{U}_5(q^2) \prec S$, and ${}^3D_4(q^2) \prec S$. Since rs does not divide the order of any proper simple section of S, the result follows.

$\square$

7.1 Exceptional Groups with Generic Character Tables

The generic character tables for the simple exceptional groups ${}^2B_2(q)$, ${}^2G_2(q)$, $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$ are known and available in Chevie [Gec+96].

We first show that Theorem B holds for ${}^2B_2(q)$ by directly computing the character formula in Lemma 4.4.

Lemma 7.2. *Let $S = {}^2B_2(q)$ for $q = 2^{2n+1}$, and G the Schur cover of S. If r, s are distinct odd prime divisors of $|S|$, then G admits a $(2, r, s)$-triple.*

Proof. One can check the case of ${}^2B_2(8)$ by computing the character formula using GAP. Let $S = {}^2B_2(q)$ for $q > 8$. We use the character table in Chevie and its notation for the types of conjugacy classes and the type of characters. The Schur multiplier of S is trivial, and so $G = S$.

- Odd prime divisors r and s divide $(q-1)(q^2+1)$. Thus the conjucacy class of any nontrivial element of order r or s is of type 5, 6, or 7.

- The nontrivial semisimple characters of G are the irreducible characters of type 5, 6, or 7. In particular, there are total $q-1$ nontrivial semisimple characters, and the degrees of such characters are at least $(q-1)(q-\sqrt{2q}+1)$.
- For any semisimple character χ, we have $|\chi(g)| \leq 4$ if g^G is a conjugacy class of type 5, 6, or 7.

Thus we have

$$\sum_{\substack{\chi\in\mathrm{Irr}_s(G)\\ \chi(1)\neq 1}} \left|\frac{\chi(x)\chi(y)}{\chi(1)}\right| < (q-1)\cdot\frac{16}{(q-1)(q-\sqrt{2q}+1)} < 1$$

and so G admits a $(2, r, s)$-triple by Proposition 4.8. □

For the remaining exceptional groups of Lie-type with known generic character tables, we check the possible degrees of positive r-defect semisimple characters of G and their character bounds where (S, r) is one of the cases in Lemma 7.1 and G is the Schur cover of S.

Lemma 7.3. *Let G be one of the exceptional groups: ${}^2G_2(q)$, $G_2(q)$ for $q > 4$ even, ${}^3D_4(q)$, or ${}^2F_4(q)$ where $q = p^f$. Let r and s be two distinct odd primes such that S is a minimal simple (r, s)-section. In particular, we assume that we have one of the cases listed in Theorem C. Then the Schur cover G of S is S itself, and we have the following for $\chi \in \mathrm{Irr}_s(G)$, $\chi \neq 1_G$.*

(i) For $G = {}^2G_2(q)$, χ is r-defect positive only if it is either of type 12 (of degree $(q-1)(q+1)(q+\sqrt{3q}+1)$) or of type 14 (of degree $(q-1)(q+1)(q-\sqrt{3q}+1)$). There are $(q-\sqrt{3q})/6$ characters of type 12 and $(q+\sqrt{3q})/6$ characters of type 14.

(ii) For $G = G_2(q)$, χ is r-defect 0 or s-defect 0.

(iii) For $G = {}^3D_4(q)$ with q odd, χ is r-defect 0 or 2-defect 0.

(iv) For $G = {}^3D_4(q)$ with q even, χ is r-defect positive, only if it is of type 28 (of degree $|G|_{2'}/(q^4-q^2+1)$). There are $q^2(q-1)(q+1)/4$ characters of type 28.

(v) For $G = {}^2F_4(q)$ and $r \in \zeta(q, 12)$, χ is r-defect positive only if it is either of type 40 (of degree $|G|_{2'}/(q^2+q+1-\sqrt{2q}(q+1))$) or of type 41 (of degree $|G|_{2'}/(q^2+q+1+\sqrt{2q}(q+$

1))). There are $(q-\sqrt{2q})(q+1)/12$ characters of type 40 and $(q+\sqrt{2q})(q+1)/12$ characters of type 41.

(vi) For $G = {}^2F_4(q)$, $r \in \zeta(q,6)$, and $s \in \zeta(q,4)$, χ is either r-defect 0 or s-defect 0.

Proof. We use the Chevie character table for each G. Note that semisimple characters are the irreducible characters such that the average character value over regular unipotent elements is 0.

The conjugacy classes of regular unipotent elements in ${}^2G_2(q)$ are the classes with centralizer of order $3q$, and so they are the classes of types 5, 6, and 7. It follows that the nontrivial semisimple characters are the irreducible characters of types 9, 11, 12, 13, or 14. From Theorem C, $r \in \zeta(q,6)$, and so r does not divide $\Phi_n(q)$ for $n < 6$ but divides $\Phi_6(q) = (q+\sqrt{3q}+1)(q-\sqrt{3q}+1)$. The nontrivial semisimple characters of order not divisible by $q-\sqrt{3q}+1$ [resp. $q+\sqrt{3q}+1$] are of type 12 [resp. 14]. Thus we have (i).

Similarly, one can check the rest. The regular unipotent classes of $G_2(q)$ for $q = 2^f$ are the classes with centralizer of order $2q^2$. The regular unipotent classes $G = {}^3D_4(q)$ for q even are the classes with centralizer of order $2q^4$. The regular unipotent classes of ${}^2F_4(q)$ are the classes with centralizer of order $4q^2$. □

Lemma 7.4. *Let G be one of the exceptional groups: ${}^2G_2(q)$, ${}^3D_4(q)$ for q even, or ${}^2F_4(q)$. For $\chi \in \mathrm{Irr}_s(G)$ described in Lemma 7.3, the following is true:*

(i) For $G = {}^2G_2(q)$, if χ is of type either 12 or 14, then $\chi(u) = -1$ for any $u \in G$ regular unipotent and $|\chi(g)| \leq 6$ for any $g \in G$ regular semisimple.

(ii) For $G = {}^3D_4(q)$ with q even, if χ is of type 28, then $\chi(u) = 1$ for any $u \in G$ regular unipotent and $|\chi(g)| \leq 4$ for any $g \in G$ element of order l where $l \in \zeta(q,12)$.

(iii) For $G = {}^2F_4(q)$, if χ is of type either 40 or 41, $\chi(u) = 1$ for any $u \in G$ regular unipotent and $|\chi(g)| \leq 12$ for any $g \in G$ element of order $l \in \zeta(q,12)$.

Lemma 7.5. *Let S be one of ${}^2G_2(q)$ for $q = 3^{2n+1}$, $G_2(q)$ for $q > 2$, ${}^3D_4(q)$, or ${}^2F_4(q)$ for $q = 2^{2n+1} > 2$. If S is a minimal simple (r,s)-section of S itself, then the Schur cover G of S admits a $(2,r,s)$-triple.*

Proof. We first consider the cases where S has an exceptional Schur multiplier: $S = G_2(3)$, or $G_2(4)$. First, $G_2(3)$ has a simple subgroup isomorphic to $\mathrm{L}_2(13)$ and $\pi(|G_2(3)|) = \pi(|\mathrm{L}_2(13)|)$. For $G_2(4)$, there exists simple subgroups isomorphic to J_2, $\mathrm{U}_3(4)$, and $\mathrm{L}_2(13)$. Thus, there is no pair (r, s) of distinct prime divisors of $G_2(4)$ such that $G_2(4)$ is a minimal simple (r, s)-section.

Now we assume S is one of the following finite simple groups: ${}^2G_2(q)$ for $q = 3^{2n+1}$, $n \geq 1$, $G_2(q)$ for $q = p^f > 4$, ${}^3D_4(q)$ for $q = p^f$, or ${}^2F_4(q)$ for $q = 2^{2n+1}$ for $n \geq 1$. Note that the Schur multiplier in these cases is trivial, i.e., $G = S$.

We assume that we have one of the cases (viii-xi) in Theorem C. In all cases, there exists a regular semisimple r-element in G ([MT08, Lemma 2.3]).

For $G = {}^2G_2(q)$, if $s \neq 3$, then G admits a $(2, r, s)$-triple by Lemma 4.6 and Lemma 4.7. Now we suppose $s = 3$. Let x be a regular semisimple r-element and y a regular 2-element. Then using Lemma 7.3 and Lemma 7.4, we get the following

$$\sum_{\substack{\chi \in \mathrm{Irr}_s(G) \\ \chi(1) \neq 1}} \left| \frac{\chi(x)\chi(y)}{\chi(1)} \right| \leq \frac{q + \sqrt{3q}}{6} \cdot \frac{36}{(q-1)(q+1)(q - \sqrt{3q} + 1)} < 1$$

and so G admits a $(2, r, s)$-triple by Proposition 4.8.

For $G = G_2(q)$, if q is odd, then G admits a $(2, r, s)$-triple by Lemma 4.6 and Lemma 4.7. We assume that q is even. Let x be a regular semisimple r-element and y a nontrivial s-element. Note that $\chi(x)\chi(y) \neq 0$ for $\chi \in \mathrm{Irr}_s(G)$ only if χ is both r-defect positive and s-defect positive. By Lemma 7.3, the principal character is the only semisimple character which is not r-defect 0 nor s-defect 0. Thus G admits a $(2, r, s)$-triple by Proposition 4.8.

For $G = {}^3D_4(q)$, if $(p, 2s) \neq 1$, then G admits a $(2, r, s)$-triple by Lemma 4.6 and Lemma 4.7. We assume $p = s$. Let x be a regular semisimple r-element and y a regular semisimple 2-element. Note that $\chi(x)\chi(y) \neq 0$ for $\chi \in \mathrm{Irr}_s(G)$ only if χ is both r-defect positive and 2-defect positive. By Lemma 7.3, the principal character is the only such character. Thus G admits a $(2, r, s)$-triple by Proposition 4.8. Now we suppose $p = 2$. If s does not divide $q^4 - q^2 + 1$, then the principal character is the only semisimple character that is both r-defect positive and s-defect positive by Lemma 7.3. Again we are done by Proposition 4.8. If s divides $q^4 - q^2 + 1$, then $s \in \zeta(q, 12)$, and so there exists a regular semisimple s-element [MT08, Lemma 2.3] Let x be a regular semisimple r-element and y a regular semisimple s-element. By Lemma 7.3 and Lemma 7.4, we get the following

$$\sum_{\substack{\chi\in\mathrm{Irr}_s(G)\\ \chi(1)\neq 1}} \left|\frac{\chi(x)\chi(y)}{\chi(1)}\right| \leq \frac{q^2(q-1)(q+1)}{4}\cdot\frac{16}{|G|_{2'}/(q^4-q^2+1)} < 1$$

and so G admits a $(2,r,s)$-triple by Proposition 4.8.

For $G = {}^2F_4(q)$, we first consider the case where $r \in \zeta(q,6)$ and $s \in \zeta(q,4)$. Then the principal character is the only character of positive r-defect and positive s-defect by Lemma 7.3. Thus G admits a $(2,r,s)$-triple by Proposition 4.8. Now we ssume that $r \in \zeta(q,12)$. If s does not divide q^4-q^2+1, then the principal character is the only semisimple character that is both r-defect positive and s-defect positive, so we are done by Proposition 4.8. If s divides q^4-q^2+1, then $s \in \zeta(q,12)$, and so there exists a regular semisimple s-element. Let x be a regular semisimple r-element and y a regular semisimple s-element. By Lemma 7.3 and by Lemma 7.4, we get the following

$$\sum_{\substack{\chi\in\mathrm{Irr}_s(G)\\ \chi(1)\neq 1}} \left|\frac{\chi(x)\chi(y)}{\chi(1)}\right| \leq \frac{(q+\sqrt{2q})(q+1)}{12}\cdot\frac{144}{|G|_{2'}/(q^2+q+1-\sqrt{2q}(q+1))} < 1$$

and so G admits a $(2,r,s)$-triple by Proposition 4.8. □

7.2 Exceptional Groups of Type F_4

In this section we work on the exceptional groups of type F_4.

First we consider the possible degrees of positive r,s-defect characters of G where (S,r,s) is one of the cases given in Lemma 7.1 and G is the Schur cover of $S = F_4(q)$.

The following lemma is a direct consequence from the list of character degrees from Frank Lübeck's webpage ([Lüb07]). The label for the unipotent characters of $F_4(q)$ follows [Car93, p.479].

Lemma 7.6. *Let G be $F_4(q)$ with $q > 2$ and $r \in \zeta(q,k)$ and $s \in \zeta(q,l)$. Then $\chi \in \mathrm{Irr}(G)$ is neither r-defect 0 nor s-defect 0, only if χ is the principal character ($\phi_{1,0}$), the Steinberg* *character ($\phi_{1,24}$), or one of the following:*

k	l	χ	$\chi(1)$
12	8	$F_4[i]$	$\frac{1}{4}q^4\Phi_1(q)^4\Phi_2(q)^4\Phi_3(q)^2\Phi_6(q)^2$
		$F_4[-i]$	$\frac{1}{4}q^4\Phi_1(q)^4\Phi_2(q)^4\Phi_3(q)^2\Phi_6(q)^2$
	4	$B_{2,1}$	$\frac{1}{2}q\Phi_1(q)^2\Phi_3(q)^2\Phi_8(q)$
		$\phi_{4,1}$	$\frac{1}{2}q\Phi_2(q)^2\Phi_6(q)^2\Phi_8(q)$
		$F_4[i]$	$\frac{1}{4}q^4\Phi_1(q)^4\Phi_2(q)^4\Phi_3(q)^2\Phi_6(q)^2$
		$F_4[-i]$	$\frac{1}{4}q^4\Phi_1(q)^4\Phi_2(q)^4\Phi_3(q)^2\Phi_6(q)^2$
		$B_{2,r}$	$\frac{1}{4}q^4\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)^2\Phi_6(q)^2\Phi_8(q)$
		$B_{2,\varepsilon}$	$\frac{1}{2}q^{13}\Phi_1(q)^2\Phi_3(q)^2\Phi_8(q)$
		$\phi_{4,13}$	$\frac{1}{2}q^{13}\Phi_2(q)^2\Phi_6(q)^2\Phi_8(q)$

Now we consider bounds for character values that we could use in the computation of character formula in Lemma 4.4.

Lemma 7.7. *Let $G = F_4(q)$. For χ listed in Lemma 7.6, $\chi(g) = \pm 1$ for any nontrivial r-element $g \in G$ where $r \in \zeta(q, 12)$. Furthermore, if $\chi = F_4[i]$ or $F_4[-i]$, then $\chi(g) = \pm 1$ for any nontrivial s-element $g \in G$ where $s \in \zeta(q, 8)$.*

Proof. It follows from [HL98, Theorem 2.1] and Lemma 4.11. □

We can also get an upper bound for character values at an element based on the order of its centralizer.

Lemma 7.8. *Let $G = F_4(q)$ and $s \in \zeta(q, 4)$. If q is odd, then there exists $g \in G$ of order s such that $|\chi(g)| \leq q^6$ for any $\chi \in \mathrm{Irr}(G)$. If q is even, then there exists $g \in G$ of order s such that $|\chi(g)| \leq q^2 + 1$ for any $\chi \in \mathrm{Irr}(G)$.*

Proof. Let G, s be as described in the statement. If q is odd, we get $q^4(q^4 - 1)^2$ as the maximum possible order of the centralizer of a nontrivial s-element from [SI74b]. If q is even, we choose an s-element $g \in G$ from a class whose representative is of type h_{63} in a maximal torus of type $H(22)$ in [SI74a]. Then $|C_G(g)| = (q^2 + 1)^2$. By Lemma 4.12, we get the results. □

Lemma 7.9. *Let S be $F_4(q)$ with $q = p^f$, p a prime. If S is a minimal simple (r, s)-section of S itself, then the Schur cover G of S admits a $(2, r, s)$-triple.*

Proof. For the $F_4(2)$ case, we have $G = 2S$, and so if S admits a $(2, r, s)$-triple, then G admits a $(2, r, s)$-triple as well by Lemma 4.3. Since $O_9(2) \prec S$ and ${}^3D_4(2) \prec S$, we may assume $r \in \zeta(q, 12)$ and $s \in \zeta(q, 8) \cup \zeta(q, 2)$. It follows that (r, s) is either $(13, 17)$, or $(13, 5)$. Using GAP character tables, one can check the elements g_1, g_2, g_3 from conjugacy classes `16a, 13a, 17a`, or `16a, 13a, 5a` respectively, satisfy the character formula in Lemma 4.4. We consider $S = F_4(q)$ with $q > 2$, and then we have $G = S$.

We assume that we have the case (vi) in Lemma 7.1. Then there exists a regular semisimple r-element ([MT08, Lemma 2.3]). Also if $s \in \zeta(q, 8)$ then there exists a regular s-element ([MT08, Lemma 2.3]). Thus if $p \neq 2$, then G admits a $(2, r, s)$-triple by Lemma 4.6.

Suppose that $s \in \zeta(q, 8)$ but $p = 2$. Let x be a regular semisimple r-element and y a regular semisimple s-element. Note that $\chi(x)\chi(y) \neq 0$ for $\chi \in \mathrm{Irr}(G)$ only if χ is neither r-defect 0 nor s-defect 0. For such χ (listed in Lemma 7.6), we get $|\chi(x)| = |\chi(y)| = 1$ by Lemma 7.7. Now we get

$$\sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} = \sum_{\chi \in S} \frac{|\chi(x)\chi(y)|}{\chi(1)} = \frac{8}{q^4(q^6-1)^2(q^2-1)^2} + \frac{1}{q^{24}} < 1$$

and so G admits a $(2, r, s)$-triple by Corollary 4.9.

Suppose that $s \in \zeta(q, 4)$ and $p \neq 2$. Let x be a regular r-element. Pick a 2-element y and a s-element z so that $|\chi(y)| \leq \sqrt{|C_G(y)|} = \sqrt{(q+1)^3 q^3}$ and $|\chi(z)| \leq q^6$ (cf. [Gur+15, Lemma 7.16] and Lemma 7.8). From Lemma 7.6, there are at most 8 nontrivial irreducible characters of G that are both r-defect positive and s-defect positive. Such characters are of degree at least $\frac{1}{2} q \Phi_1(q)^2 \Phi_3(q)^2 \Phi_8(q)$. Using Lemma 7.7, we get

$$\begin{aligned} \left| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)} \right| &\geq 1 - \left| \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) \neq 1}} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)} \right| \geq 1 - 8 \cdot \frac{\sqrt{(q+1)^3 q^3} \cdot q^6}{\frac{1}{2} q (q^4+1)(q^3-1)^2} \\ &\geq 1 - 16 \cdot \frac{1.40 q^8}{0.98 q^{10}} \geq 1 - \frac{16 \cdot 1.40}{0.98 \cdot 25} > 0 \end{aligned}$$

for $q \geq 5$. For $q = 3$, one can check that the character formula is nonzero by using the values of the character degrees listed in Lemma 7.6. Hence, G admits a $(2, r, s)$-triple by Lemma 4.4.

Suppose that $s \in \zeta(q, 4)$ and $p = 2$. Let x be a regular r-element and y a nontrivial s-element such that $|\chi(y)| \leq q^2 + 1$. Such a y exists by Lemma 7.8. From Lemma 7.6, there are at most 8 nontrivial irreducible characters of G that are both r-defect positive and s-defect positive. Such characters are of degree at least $\frac{1}{2}q\Phi_1(q)^2\Phi_3(q)^2\Phi_8(q)$. Thus, we get

$$\sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq 8 \cdot \frac{q^2+1}{\frac{1}{2}q(q^4+1)(q^3-1)^2} < 1$$

and so G admits a $(2, r, s)$-triple by Corollary 4.9. □

7.3 Exceptional Groups of Type E_6

In this section we work on the exceptional groups of type E_6.

First we consider the possible degrees of positive r, s-defect characters of G where (S, r, s) is one of the cases given in Lemma 7.1 and G is the Schur cover of $S = E_6^{\pm}(q)$.

The following lemma is based on the list of character degrees from Frank Lübeck's webpage ([Lüb07]). The label for the unipotent characters of $(E_6^{\epsilon})_{sc}(q)$ follows [Car93, p.479].

Lemma 7.10. *Let G be $(E_6^{\epsilon})_{sc}(q)$, $r \in \zeta(\epsilon q, k)$, and $s \in \zeta(\epsilon q, l)$. Then $\chi \in \mathrm{Irr}(G)$ is neither r-defect 0 nor s-defect 0, only if χ is the principal character ($\phi_{1,0}$), the Steinberg character ($\phi_{1,36}$ if $\epsilon > 0$, and $\phi_{1,24}$ if $\epsilon < 0$), or one of the following:*

k	l	χ	$\chi(1)$
12	9	$E_6^{\epsilon}[\theta]$	$\frac{1}{3}q^7\Phi_1(q)^4\Phi_1(\epsilon q)^2\Phi_2(q)^4\Phi_4(q)^2\Phi_5(\epsilon q)\Phi_8(q)$
		$E_6^{\epsilon}[-\theta]$	$\frac{1}{3}q^7\Phi_1(q)^4\Phi_1(\epsilon q)^2\Phi_2(q)^4\Phi_4(q)^2\Phi_5(\epsilon q)\Phi_8(q)$
	5	$\phi_{6,1}$ *if* $\epsilon > 0$, $\phi_{2,4'}$ *if* $\epsilon < 0$	$q\Phi_8(q)\Phi_9(\epsilon q)$
		$\phi_{6,25}$ *if* $\epsilon > 0$, $\phi_{2,16'}$ *if* $\epsilon < 0$	$q^{25}\Phi_8(q)\Phi_9(\epsilon q)$
9	8	*(none)*	*(none)*
	5	$\phi_{64,4}$ *if* $\epsilon > 0$, ${}^2A_{5,1}$ *if* $\epsilon < 0$	$\epsilon q^4\Phi_2(\epsilon q)^3\Phi_4(q)^2\Phi_6(\epsilon q)^2\Phi_8(q)\Phi_{12}(q)$
		$\phi_{64,13}$ *if* $\epsilon > 0$, ${}^2A_{5,\varepsilon}$ *if* $\epsilon < 0$	$\epsilon q^{13}\Phi_2(\epsilon q)^3\Phi_4(q)^2\Phi_6(\epsilon q)^2\Phi_8(q)\Phi_{12}(q)$
	4	$\phi_{20,2}$ *if* $\epsilon > 0$, $\phi_{4,1}$ *if* $\epsilon < 0$	$q^2\Phi_4(q)\Phi_5(\epsilon q)\Phi_8(q)\Phi_{12}(q)$
		$\phi_{90,8}$ *if* $\epsilon > 0$, $\phi_{6,6''}$ *if* $\epsilon < 0$	$\frac{1}{3}q^7\Phi_3(\epsilon q)^3\Phi_5(\epsilon q)\Phi_6(\epsilon q)^2\Phi_8(q)\Phi_{12}(q)$
		$\phi_{20,20}$ *if* $\epsilon > 0$, $\phi_{4,13}$ *if* $\epsilon < 0$	$q^{20}\Phi_4(q)\Phi_5(\epsilon q)\Phi_8(q)\Phi_{12}(q)$

Now we consider bounds for character values that we will use in the computation of the character formula (in Lemma 4.4).

Lemma 7.11. *Let $G = (E_6^\epsilon)_{sc}(q)$ and $k \in \{12, 9\}$. For each χ and k listed in Lemma 7.10, we have $\chi(g) = \pm 1$ for any nontrivial r-element $g \in G$ where $r \in \zeta(\epsilon q, k)$. Furthermore, if $\chi = E_6^\epsilon[\theta]$ or $E_6^\epsilon[-\theta]$, then $\chi(g) = \pm 1$ for any nontrivial s-element $g \in G$ where $s \in \zeta(\epsilon q, 9)$.*

Proof. It follows from [HLM95, Theorem 3.1] for $(E_6)_{sc}(q)$, [HL98, Theorem 2.2] for $({}^2E_6)_{sc}(q)$, and Lemma 4.11. □

Lemma 7.12. *If $G = (E_6^\epsilon)_{sc}(q)$ and s a prime in L, then there exists an element g of order s such that $|C_G(g)| \le C$ where $L = \zeta(\epsilon q, l)$ and C as follows.*

L	C
$\zeta(\epsilon q, 12)$	$(q^2 + \epsilon q + 1)(q^4 - q^2 + 1)$
$\zeta(\epsilon q, 9)$	$q^6 + \epsilon q^3 + 1$
$\zeta(\epsilon q, 8)$	$(q^4 + 1)(q^2 - 1)$
$\zeta(\epsilon q, 5)$	$(q^5 - \epsilon^5)\lvert \mathrm{SL}_2(q)\rvert$
$\zeta(\epsilon q, 4)$	$(q - \epsilon)(q^2 + 1)\lvert \mathrm{SL}_4^{-\epsilon}(q)\rvert$

In particular, such g is regular semisimple for $l = 12, 9, 8$.

Proof. For the cases $l = 12, 9, 8$, it is proven in [MT08, Lemma 2.3] or [Gur+15, Lemma 6.3].

Centralizers of semisimple elements in $(E_6^\epsilon)_{sc}(q)$ are studied in [FJ93], and we shall use their results in the remaining proof.

Let $G = (E_6^\epsilon)_{sc}(q)$ and let $x \in G$ be semisimple. We let $\mathcal{G}$ be a simple algebraic group with F a Frobenius morphism such that $\mathcal{G}^F = G$. Note that $C_{\mathcal{G}}(x)$ is connected reductive. We denote $\mathcal{C} := C_{\mathcal{G}}(x)$, $\mathcal{Z} := Z(\mathcal{C})^\circ$, and $\mathcal{S} := [\mathcal{C}, \mathcal{C}]$. Let s be a prime in $\zeta(\epsilon q, l)$. By Lemma 4.13, if $|x| = s$, then s divides either $|\mathcal{Z}^F|$ or $|Z(\mathcal{S})^F|$.

We show that s does not divide $|Z(\mathcal{S})^F|$. First we assume $s \ge 7$. Note that $\mathrm{rank}(\mathcal{S}) \le \mathrm{rank}(\mathcal{G}) = 6$. It follows that $s \nmid |Z(\mathcal{S})^F|$. If $s = 5$ so that $l = 4$, then there are cases where $\mathrm{rank}(\mathcal{S}) \ge 4$. However, in all cases, one can check the center of $\mathcal{S}^F$ is coprime to 5, because $5 \nmid (q \pm 1)$ due to $5 \in \zeta(\epsilon q, 4)$. Now the result follows because the values of C in the table are the maximum possible order of $C_G(x)$ such that $s \nmid |Z(\mathcal{S})^F|$ but $s \,\big|\, |\mathcal{Z}^F|$ in [FJ93]. □

Corollary 7.13. *If $G = (E_6^\pm)_{sc}(q)$ then there exists an element $g \in G$ of order s where*

$s \in \zeta(\epsilon q, l)$ such that $|\chi(g)| \le B$ for any $\chi \in \mathrm{Irr}(G)$:

l	5	4
B	q^4	$1.3q^9$

Proof. It follows from Lemma 7.12 and Lemma 4.12. □

Lemma 7.14. *Let S be $E_6^\epsilon(q)$ with $q = p^f$, p a prime. If S is a minimal simple (r, s)-section of S itself, then the Schur cover G of S admits a $(2, r, s)$-triple.*

Proof. We assume that we have the case (vii) in Lemma 7.1. Note that $(p, rs) \neq 1$, and there exists a regular r-element by Lemma 7.12. Let x be a regular r-element. Let y be a regular s-element if possible ($l = 9$ or 8) or a nontrivial s-element described in Lemma 7.12. Pick z a regular unipotent element if $p = 2$, a regular semisimple 2-element if $q \equiv \epsilon \mod 4$, or a 2-element with its centralizer of order at most $q^7(q-\epsilon)$ if $q \equiv -\epsilon \mod 4$ ([Gur+15, Lemma 7.16]). In all cases, we have $|\chi(z)| \le \sqrt{q^7(q+1)} \le 1.3q^4$ for all $\chi \in \mathrm{Irr}(G)$.

We compute the character formula using Lemma 7.10 and Lemma 7.11, for $(k, l) = (12, 9)$. There are at most 3 nontrivial irreducible characters of G that are neither r-defect 0 nor s-defect 0. Such characters are of degree at least $\frac{1}{3}q^7\Phi_1(q)^4\Phi_1(\epsilon q)^2\Phi_2(q)^4\Phi_4(q)^2\Phi_5(\epsilon q)\Phi_8(q)$ and their values on x and y are ± 1 by Lemma 7.11.

$$\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| \ge 1 - \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)\neq 1}} \frac{|\chi(x)|\,|\chi(y)|\,|\chi(z)|}{\chi(1)}$$
$$\ge 1 - 3\cdot \frac{1.3q^4}{\frac{1}{3}q^7\Phi_1(q)^4\Phi_1(\epsilon q)^2\Phi_2(q)^4\Phi_4(q)^2\Phi_5(\epsilon q)\Phi_8(q)} > 0$$

For the case of $(k, l) = (12, 5)$, we compute the character formula using Lemma 7.10, Lemma 7.11, and Corollary 7.13. There are at most 3 nontrivial irreducible characters of G that are neither r-defect 0 nor s-defect 0. Such characters are of degree at least $q\Phi_8(q)\Phi_9(\epsilon q)$. If $\chi \in \mathrm{Irr}(G)$ is of r-defect positive and s-defect positive, then $\chi(x) = \pm 1$. Also, by Corollary 7.13, we have $|\chi(y)| \le q^4$.

$$\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| \ge 1 - \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)\neq 1}} \frac{|\chi(x)|\,|\chi(y)|\,|\chi(z)|}{\chi(1)}$$
$$\ge 1 - 3\cdot \frac{q^4\cdot 1.3q^4}{q\Phi_8(q)\Phi_9(\epsilon q)} \ge 1 - \frac{4.2}{q^3} > 0$$

Suppose $(k,l)=(9,8)$. If $p\neq 2$, we are done by Lemma 4.6 because x and y are regular semisimple. Suppose that $p=2$. By Lemma 7.10, the principal character is the only irreducible character which is nonzero at both x and y. Note that the Steinberg character St vanishes on z. Therefore,

$$\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}=1\neq 0.$$

Now we show the case of $(k,l)=(9,5)$. By Corollary 7.13, $|\chi(y)|\leq q^4$. There are at most 3 nontrivial irreducible characters of G that are neither r-defect 0 nor s-defect 0. Such characters are of degree at least $\epsilon q^{13}\Phi_2(\epsilon q)^3\Phi_4(q)^2\Phi_6(\epsilon q)^2\Phi_8(q)\Phi_{12}(q)$. Any such a character is a nonexceptional character in the principal r-block ([HLM95; HL98]), and so $\chi(x)=\pm 1$. Using all of these facts, we estimate the value of the character formula. One can check that

$$\begin{aligned}\left|\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| &\geq 1-3\cdot\frac{q^4\cdot 1.3q^4}{q^4(q+\epsilon)^3(q^2+1)^2(q^2-\epsilon q+1)^2(q^4+1)(q^4-q^2+1)}\\ &\geq 1-\frac{9.8}{q^{15}}>0\end{aligned}$$

Therefore, G admits a $(2,r,s)$-triple.

Lastly, we suppose $(k,l)=(9,4)$. By Corollary 7.13, $|\chi(y)|\leq 1.3q^9$. There are at most 4 nontrivial irreducible characters of G that are neither r-defect 0 nor s-defect 0. Such characters are of degree at least $q^2\Phi_4(q)\Phi_5(\epsilon q)\Phi_8(q)\Phi_{12}(q)$. Any such a character is a nonexceptional character in the principal r-block ([HLM95; HL98]), and so $\chi(x)=\pm 1$. Using all of these facts, we estimate the value of the character formula for $q\geq 3$. One can check that

$$\begin{aligned}\left|\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| &\geq 1-4\cdot\frac{1.3q^9\cdot 1.3q^4}{q^2(q^2+1)(q^4+\epsilon q^3+q^2+\epsilon q+1)(q^4+1)(q^4-q^2+1)}\\ &\geq 1-\frac{9.2}{q^3}>0\end{aligned}$$

In case of $q=2$ we can get the same result, but we need a sharper estimate. For y, we use its centralizer order as in Lemma 7.12, and so $|\chi(y)|\leq\sqrt{|C_G(y)|}=\sqrt{q^6(q^4-1)^2(q^3+\epsilon)(q-\epsilon)}$.

As z is a regular 2-element, we have $|\chi(z)| \leq \sqrt{2 \cdot 2^6} \leq 11.4$. Also $\mathrm{St}(z) = 0$. Therfore,

$$\begin{aligned}
&\left|\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| \\
&\geq 1 - 3 \cdot \left.\frac{11.4 \cdot 1 \cdot \sqrt{q^6(q^4-1)^2(q^3+\epsilon)(q-\epsilon)}}{q^2(q^2+1)(q^4+\epsilon q^3+q^2+\epsilon q+1)(q^4+1)(q^4-q^2+1)}\right|_{q=2} \\
&\geq 1 - 3 \cdot \left.\frac{11.4\sqrt{q^6 \cdot q^8(q^3+1)(q+1)}}{q^2 \cdot q^2(q^4-q^3+q^2-q+1) \cdot q^4 \cdot (q^4-q^2+1)}\right|_{q=2} \\
&= 1 - 3 \cdot \left.\frac{11.4q^7\sqrt{(q^3+1)(q+1)}}{q^8(q^4-q^3+q^2-q+1)(q^4-q^2+1)}\right|_{q=2} \\
&= 1 - 3 \cdot \frac{11.4\sqrt{27}}{2 \cdot 11 \cdot 13} \geq 1 - 0.63 > 0
\end{aligned}$$

Therefore, G admits a $(2, r, s)$-triple by Lemma 4.4. □

7.4 Exceptional Groups of Type E_7

In this section we work on the exceptional groups of type E_7.

First we consider the possible degrees of positive r, s-defect characters of G where (S, r, s) is one of the cases given in Lemma 7.1 and G is the Schur cover of $S = E_7(q)$.

We use the list of character degrees from Frank Lübeck's webpage ([Lüb07]) to get the following result.

Lemma 7.15. *Let G be $(E_7)_{sc}(q)$, $r \in \zeta(q,k)$, and $s \in \zeta(q,l)$. Then $\chi \in \mathrm{Irr}(G)$ is neither r-defect 0 nor s-defect 0, only if χ is the principal character ($\phi_{1,0}$), the Steinberg character ($\phi_{1,63}$), or one of the following:*

k	l	$\chi(1)$	$\lvert\{\varphi \mid \varphi(1) = \chi(1)\}\rvert$
18	14	$\frac{1}{2}q^{11}\Phi_1(q)^7\Phi_3(q)^3\Phi_4(q)^2\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)$	2
	9	$q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$	1
		$\frac{1}{3}q^7\Phi_1(q)^6\Phi_2(q)^6\Phi_4(q)^2\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_{10}(q)\Phi_{14}(q)$	2
		$\frac{1}{3}q^{16}\Phi_1(q)^6\Phi_2(q)^6\Phi_4(q)^2\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_10(q)\Phi_{14}(q)$	2
		$q^{46}\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$	1
	7	*(none)*	*(none)*
	5	$q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$	1
		$\frac{1}{2}q^3\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)$	1
		$\frac{1}{2}q^{30}\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)$	1
		$q^{46}\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$	1

14	12	$\frac{1}{2}q^4\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)\Phi_7(q)\Phi_9(q)\Phi_{10}(q)\Phi_{18}(q)$	1
		$\frac{1}{2}q^{25}\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)\Phi_7(q)\Phi_9(q)\Phi_{10}(q)\Phi_{18}(q)$	1
	9	*(none)*	*(none)*
	7	$q^2\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q)$	1
		$q^{37}\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q)$	1
	5	$q^2\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q)$	1
		$\frac{1}{2}q^8\Phi_3(q)^2\Phi_6(q)^3\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{18}(q)$	1
		$\frac{1}{2}q^{15}\Phi_3(q)^2\Phi_6(q)^3\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{18}(q)$	1
		$q^{37}\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q)$	1
12	7	$\frac{1}{2}q^4\Phi_2(q)^4\Phi_5(q)\Phi_6(q)^2\Phi_9(q)\Phi_{10}(q)\Phi_{14}(q)\Phi_{18}(q)$	1
		$\frac{1}{2}q^{25}\Phi_2(q)^4\Phi_5(q)\Phi_6(q)^2\Phi_9(q)\Phi_{10}(q)\Phi_{14}(q)\Phi_{18}(q)$	1
10	9	$q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$	1
		$\frac{1}{2}q^3\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	1
		$\frac{1}{2}q^{30}\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	1
		$q^{46}\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$	1
	7	$q^2\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q)$	1
		$\frac{1}{2}q^8\Phi_3(q)^3\Phi_5(q)\Phi_6(q)^2\Phi_8(q)\Phi_9(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	1
		$\frac{1}{2}q^{15}\Phi_3(q)^3\Phi_5(q)\Phi_6(q)^2\Phi_8(q)\Phi_9(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	1
		$q^{37}\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q)$	1
9	7	$\frac{1}{2}q^{11}\Phi_2(q)^7\Phi_4(q)^2\Phi_6(q)^4\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$	2

Lemma 7.16. *If $G = (E_7)_{sc}(q)$ and s a prime in L, then there exists an element g of order s such that $|C_G(g)| \leq C$ where $L = \zeta(q, l)$ and C as follows.*

L	C
$\zeta(q, 18)$	$(q+1)(q^6 - q^3 + 1)$
$\zeta(q, 14)$	$q^7 + 1$
$\zeta(q, 12)$	$(q^4 - q^2 + 1)\lvert\mathrm{SL}_2(q^3)\rvert$
$\zeta(q, 10)$	$(q^5 + 1)\lvert\mathrm{SU}_3(q)\rvert$
$\zeta(q, 9)$	$(q-1)(q^6 + q^3 + 1)$
$\zeta(q, 7)$	$q^7 - 1$
$\zeta(q, 5)$	$(q^5 - 1)\lvert\mathrm{SL}_3(q)\rvert$

In particular, such g is regular semisimple for $l = 18, 14, 9, 7$.

Proof. For the cases $l = 18, 14, 9, 7$, it is proven in [MT08, Lemma 2.3] or [Gur+15, Lemma 6.3].

Centralizers of semisimple elements in $(E_7)_{sc}(q)$ are studied in [FJ93], and we use their results in the remaining proof for $l = 12, 10, 5$.

Let $G = (E_7)_{sc}(q)$ and $x \in G$ semisimple. We let $\mathcal{G}$ be a simple algebraic group with F a

Frobenius morphism such that $\mathcal{G}^F = G$. Note that $C_{\mathcal{G}}(x)$ is connected reductive. We denote $\mathcal{C} := C_{\mathcal{G}}(x)$, $\mathcal{Z} := Z(\mathcal{C})^\circ$, and $\mathcal{S} := [\mathcal{C}, \mathcal{C}]$. Let s be a prime in $\zeta(q,l)$. By Lemma 4.13, if $|x| = s$, then s divides either $|\mathcal{Z}^F|$ or $|Z(\mathcal{S})^F|$.

We show that s does not divide $|Z(\mathcal{S})^F|$. Then we get the values of C in the table as the maximum of the centralizer orders for which $\Phi_l(q)$ is a factor of $|\mathcal{Z}^F|$ from [FJ93].

Note that $s \geq l+1$ if $s \in \zeta(q,l)$. Therefore, if $l = 12$ or 10, then we are done since $s \nmid |Z(\mathcal{S})^F|$ for any $\mathcal{S}$ of rank at most $\mathrm{rank}(\mathcal{G}) = 7$. Suppose $l = 5$. Since $s \in \zeta(q,5)$, $5|(s-1)$ by Fermat's little Theorem. Thus we get $s \geq 11$, and so were are done by the same argument for $l = 12, 10$. □

Corollary 7.17. *If $G = (E_7)_{sc}(q)$, then there exists an element $g \in G$ of order s where $s \in \zeta(q,l)$ such that $|\chi(g)| \leq B$ for any $\chi \in \mathrm{Irr}(G)$:*

l	18	14	12	10	9	7	5
B	$1.3q^{7/2}$	$1.004q^{7/2}$	$q^{13/2}$	$q^{13/2}$	$q^{7/2}$	$q^{7/2}$	$q^{13/2}$

Proof. It follows from Lemma 7.16 and Lemma 4.12. □

Lemma 7.18. *Let S be $E_7(q)$ with $q = p^f$, p a prime. If S is a minimal simple (r,s)-section of S itself, then the Schur cover G of S admits a $(2,r,s)$-triple.*

Proof. We assume that we have the case (viii) in Lemma 7.1. Note that $(p,rs) = 1$, and there exists a regular r-element by Lemma 7.16. Let x be a regular r-element. Let y be a regular s-element if possible ($l = 14$, 9 or 7) or a nontrivial s-element described in Lemma 7.16 and Corollary 7.17 for each l.

First we consider the cases of $(k,l) = (18,14), (18,9), (18,7), (14,9), (14,7),$ or $(9,7)$. Note that x and y are regular semisimple. If $p \neq 2$, then we are done by Lemma 4.6. Thus, we suppose $p = 2$, and choose z to be a regular unipotent element. Note that $G = S$, and so we may use Corollary 4.9.

(i) $(k,l) = (18,14)$:

From the choice of x and y, we get $|\chi(x)| \leq 1.3q^{7/2}$, $|\chi(y)| \leq 1.004q^{7/2}$ for any $\chi \in \mathrm{Irr}(G)$ by Lemma 7.15. There are three nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Now it follows:

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq 3\cdot \frac{1.3052q^7}{\frac{1}{2}q^{11}\Phi_1(q)^7\Phi_3(q)^3\Phi_4(q)^2\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)} < 1$$

(ii) $(k,l)=(18,9)$:

From the choice of x and y, we get $|\chi(x)|\leq 1.3q^{7/2}$, $|\chi(y)|\leq q^{7/2}$ for any $\chi\in\mathrm{Irr}(G)$ by Corollary 7.17. There are seven nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Now it follows:

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq 7\cdot \frac{1.3q^7}{q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)} = \frac{9.1q^7}{1.5q^{17}} < 1$$

(iii) $(k,l)=(18,7),(14,9)$:

The Steinberg character is the only nontrivial irreducible character that is neither r-defect 0 nor s-defect 0 from Corollary 7.17. However, $\mathrm{St}(z)=0$, and so G admits a $(2,r,s)$-triple by Lemma 4.4.

(iv) $(k,l)=(14,7)$:

From the choice of x and y, we get $|\chi(x)|\leq 1.004q^{7/2}$, $|\chi(y)|\leq q^{7/2}$ for any $\chi\in\mathrm{Irr}(G)$ by Corollary 7.17. There are three nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Now it follows that:

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq 3\cdot \frac{1.004q^7}{q^2\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q)} < 1$$

(v) $(k,l)=(9,7)$:

From the choice of x and y, we get $|\chi(x)|\leq q^{7/2}$, $|\chi(y)|\leq q^{7/2}$ for any $\chi\in\mathrm{Irr}(G)$ by Corollary 7.17. There are three nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Now it follows that:

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq 3\cdot \frac{q^7}{\frac{1}{2}q^{11}\Phi_2(q)^7\Phi_4(q)^2\Phi_6(q)^4\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)} < 1$$

Now we consider the case of $(k,l)=(18,5)$, $(14,12)$, $(14,5)$, $(12,7)$, $(10,9)$, $(10,7)$. Pick z a regular unipotent element if $p=2$ or a 2-element with its centralizer of order at most

$q^7(q+1)^2$ if $p \neq 2$ ([Gur+15, Lemma 7.16]). In all cases, $|\chi(z)| \leq \sqrt{q^7(q+1)^2} \leq \frac{4}{3}q^{9/2}$ for any $\chi \in \mathrm{Irr}(G)$.

(vi) $(k,l) = (18,5)$:

There are at most 5 nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Furthermore, such characters are of degree at least $q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$. From the choice of x, y, and z, we get $|\chi(x)| \leq 1.3q^{7/2}$, $|\chi(y)| \leq q^{13/2}$, and $|\chi(z)| \leq \frac{4}{3}q^{9/2}$ for any $\chi \in \mathrm{Irr}(G)$. For $q \geq 3$, we now show that G admits a $(2,r,s)$-triple by Lemma 4.4.

$$\begin{aligned}\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| &\geq 1 - \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)\neq 1}} \frac{|\chi(x)|\,|\chi(y)|\,|\chi(z)|}{\chi(1)} \geq 1 - 5\cdot\frac{1.3q^{7/2}\cdot q^{13/2}\cdot 1.4q^{9/2}}{q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)}\\ &\geq 1 - 5\cdot\frac{1.82q^{14.5}}{q(q^4-q^2+1)(q^{14}-1)/(q^2-1)} \geq 1 - 5\cdot\frac{1.82q^{14.5}}{q\cdot 0.75q^4\cdot q^{12}}\\ &\geq 1 - \frac{12.2}{q^{2.5}} > 0\end{aligned}$$

For $q = 2$, we use Corollary 4.9.

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq 5\cdot\frac{1.3q^{10}}{q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)} \leq \frac{6.5q^{10}}{1.5q^{17}} < 1$$

(vii) $(k,l) = (14,12)$:

There are 3 nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Furthermore, such characters are of degree at least

$$\frac{1}{2}q^4\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)\Phi_7(q)\Phi_9(q)\Phi_{10}(q)\Phi_{18}(q) = \frac{1}{2}(q^7-1)(q^9-1)(q^3-1)(q^5-1)\Phi_{10}(q)\Phi_{18}(q).$$

From the choice of x, y, and z, we get $|\chi(x)| \leq 1.004q^{7/2}$, $|\chi(y)| \leq q^{13/2}$, and $|\chi(z)| \leq \frac{4}{3}q^{9/2}$ for any $\chi \in \mathrm{Irr}(G)$. Now it follows that

$$\begin{aligned}\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| &\geq 1 - 3\cdot\frac{1.004q^{7/2}\cdot q^{13/2}\cdot \frac{4}{3}q^{9/2}}{\frac{1}{2}(q^7-1)(q^9-1)(q^3-1)(q^5-1)\Phi_{10}(q)\Phi_{18}(q)}\\ &= 1 - \frac{4.016q^{14.5}}{0.4q^{24}\Phi_{10}(q)\Phi_{18}(q)} \geq 1 - \frac{10.04}{q^{9.5}\Phi_{10}(q)\Phi_{18}(q)} > 0\end{aligned}$$

(viii) $(k,l) = (14,5)$:

There are at most 5 nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Furthermore, such characters are of degree at least

$$q^2\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q) = q^2(q^4+q^2+1)(q^8+q^4+1)(q^{12}+q^6+1).$$

From the choice of x, y, and z, we get $|\chi(x)| \leq 1.004q^{7/2}$, $|\chi(y)| \leq q^{13/2}$, and $|\chi(z)| \leq \frac{4}{3}q^{9/2}$ for any $\chi \in \mathrm{Irr}(G)$.

$$\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| \geq 1 - 5 \cdot \frac{1.004q^{7/2} \cdot q^{13/2} \cdot \frac{4}{3}q^{9/2}}{q^2(q^4+q^2+1)(q^8+q^4+1)(q^{12}+q^6+1)}$$
$$\geq 1 - \frac{6.7q^{14.5}}{q^2 \cdot q^4 \cdot q^8 \cdot q^{12}} = 1 - \frac{6.7}{q^{11.5}} > 0$$

(ix) $(k,l) = (12,7)$:

There are 3 nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Such characters are of degree at least

$$\frac{1}{2}q^4\Phi_2(q)^4\Phi_5(q)\Phi_6(q)^2\Phi_9(q)\Phi_{10}(q)\Phi_{14}(q)\Phi_{18}(q) = \frac{1}{2}q^4(q^7+1)(q^3+1)(q^5+1)(q^9+1)\Phi_9(q)\Phi_5(q).$$

From the choice of x, y, and z, we get $|\chi(x)| \leq q^{13/2}$, $|\chi(y)| \leq q^{7/2}$, and $|\chi(z)| \leq \frac{4}{3}q^{9/2}$ for any $\chi \in \mathrm{Irr}(G)$.

$$\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| \geq 1 - 3 \cdot \frac{q^{13/2} \cdot q^{7/2} \cdot \frac{4}{3}q^{9/2}}{\frac{1}{2}q^4(q^7+1)(q^3+1)(q^5+1)(q^9+1)\Phi_9(q)\Phi_5(q)}$$
$$= 1 - \frac{8q^{14.5}}{q^4 \cdot q^7 \cdot q^3 \cdot q^5 \cdot q^9\Phi_9(q)\Phi_5(q)} = 1 - \frac{8}{q^{13.5}\Phi_9(q)\Phi_5(q)} > 0$$

(x) $(k,l) = (10,9)$:

There are at most 5 nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Furthermore, such characters are of degree at least $q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)$. From the choice of x, y, and z, we get $|\chi(x)| \leq q^{13/2}$, $|\chi(y)| \leq q^{7/2}$, and $|\chi(z)| \leq \frac{4}{3}q^{9/2}$ for any $\chi \in \mathrm{Irr}(G)$.

$$\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| \geq 1 - 5 \cdot \frac{q^{13/2} \cdot q^{7/2} \cdot \frac{4}{3}q^{9/2}}{q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q)}$$
$$= 1 - \frac{6.67q^{14.5}}{1.5q^{17}} > 1 - \frac{4.5}{q^{2.5}} > 0$$

(xi) $(k,l) = (10,7)$:

There are at most 5 nontrivial irreducible characters that are neither r-defect 0 nor s-defect 0 from Lemma 7.15. Furthermore, such characters are of degree at least

$$q^2\Phi_3(q)^2\Phi_6(q)^2\Phi_9(q)\Phi_{12}(q)\Phi_{18}(q) = q^2(q^4+q^2+1)(q^8+q^4+1)(q^{12}+q^6+1).$$

From the choice of x, y, and z, we get $|\chi(x)| \leq q^{13/2}$, $|\chi(y)| \leq q^{7/2}$, and $|\chi(z)| \leq \frac{4}{3}q^{9/2}$ for any $\chi \in \mathrm{Irr}(G)$.

$$\left|\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)}\right| \geq 1 - 5 \cdot \frac{q^{13/2}\cdot q^{7/2}\cdot \frac{4}{3}q^{9/2}}{q^2(q^4+q^2+1)(q^8+q^4+1)(q^{12}+q^6+1)}$$
$$= 1 - \frac{6.67q^{14.5}}{q^2\cdot q^4\cdot q^8\cdot q^{12}} > 1 - \frac{6.67}{q^{11.5}} > 0$$

Hence, in all cases, G admits a $(2, r, s)$-triple by Lemma 4.4 and Corollary 4.9. □

7.5 Exceptional Groups of Type E_8

In this section we work on the exceptional groups of type E_8.

First we consider the possible degrees of positive r, s-defect characters of G where (S, r, s) is one of the cases given in Lemma 7.1 and G is the Schur cover of $S = E_8(q)$.

Lemma 7.19. *Let G be $E_8(q)$, $r \in \zeta(q,k)$, and $s \in \zeta(q,l)$ for some $k, l \in \{30, 24, 20, 18, 15, 14, 12, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$.*

(i) There exists a unique irreducible character χ_0 such that

$$\chi_0(1) = q\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q).$$

Moreover, $\chi_0(1) = \min\{\chi(1) | \chi \in \mathrm{Irr}(G), \chi \neq 1\}$.

(ii) There are at most $27.2q^8$ irreducible characters.

For q even, let J be the set of nontrivial irreducible characters $\chi \in \mathrm{Irr}(G)$ such that χ is both r-defect positive and s-defect positive.

(iii) If $k = 30$ or 15 and $l \neq k$, then $|J| \leq 29$.

(iv) If $k = 30$ and $l \in \{6, 4, 3\}$, then the degree of $\chi \in J$ is at least $\frac{1}{2}q^3\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{24}(q)$ unless $\chi = \chi_0$.

(v) If $k = 30$ and $l \in \{2, 1\}$, then $|J| \leq 28$, and the degree of $\chi \in J$ is at least $\frac{1}{2}q^3\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{24}(q)$.

(vi) *If* $k = 15$ *and* $l \in \{6, 4, 3\}$, *then the degree of* $\chi \in J$ *is at least* $\frac{1}{2}q^3\Phi_4(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q)$ *unless* $\chi = \chi_0$.

(vii) *If* $k = 15$ *and* $l \in \{2, 1\}$, *then* $|J| \leq 28$ *and the degree of* $\chi \in J$ *is at least* $\frac{1}{2}q^3\Phi_4(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q)$ *unless* $\chi = \chi_0$.

(vi) *If* $k = 24$ *and* $l \in \{30, 20, 18, 15, 14, 10, 9, 8, 7\}$, *then* $|J| \leq 23$. *The degree of* $\chi \in J$ *is at least* $q^2\Phi_5(q)\Phi_7(q)\Phi_{10}(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{20}(q)\Phi_{30}(q)$.

(vii) *If* $k = 20$ *and* $l \in \{30, 24, 18, 15, 14, 10, 9, 7, 6, 3\}$, *then* $|J| \leq 24$. *The degree of* $\chi \in J$ *is at least* $\frac{1}{2}q^3\Phi_1^4(q)\Phi_3^2(q)\Phi_5^2(q)\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{24}(q)$.

Proof. (ii) comes from [FG12, Theorem 1.1]. Most results can be obtained by looking at the list of irreducible characters with its degree and multiplicity that are r-defect positive from the list of character degrees from Frank Lübeck's webpage ([Lüb07]). If $l \neq k$ then the characters of degree $|G|_{p'}/\Phi_k(q)$ are s-defect 0. In addition, if l is either 1 or 2 then χ_0 is not s-defect 0 because $q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q) \not\equiv 0 \mod s$ ($q \equiv \pm 1 \mod s$). □

Lemma 7.20. *If* $G = E_8(q)$ *and* s *a prime in* L, *then there exists an element* g *of order* s *such that* $|C_G(g)| \leq C$ *where* $L = \zeta(q, l)$ *and* C *as follows.*

G	L	C
$E_8(q)$	$\zeta(q,30)$	$q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$
	$\zeta(q,24)$	$q^8 - q^4 + 1$
	$\zeta(q,20)$	$q^8 - q^6 + q^4 - q^2 + 1$
	$\zeta(q,15)$	$q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$
$E_8(q)$ $(q = 2^f)$	$\zeta(q,18)$	$(q^6 - q^3 + 1)\lvert\mathrm{SU}_3(q)\rvert$
	$\zeta(q,14)$	$(q+1)(q^6 - q^5 + q^4 - q^3 + q^2 - q + 1)\lvert\mathrm{SL}_2(q)\rvert$
	$\zeta(q,12)$	$(q^4 - q^2 + 1)\lvert{}^3D_4(q)\rvert$
	$\zeta(q,10)$	$(q^4 - q^3 + q^2 - q + 1)\lvert\mathrm{SU}_5(q)\rvert$
	$\zeta(q,9)$	$(q^6 + q^3 + 1)\lvert\mathrm{SL}_3(q)\rvert$
	$\zeta(q,8)$	$(q^4 + 1)\lvert\mathrm{O}_8^-(q)\rvert$
	$\zeta(q,7)$	$(q-1)(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)\lvert\mathrm{SL}_2(q)\rvert$
	$\zeta(q,6)$	$(q^2 - q + 1)\lvert{}^2E_6(q)\rvert$
	$\zeta(q,5)$	$(q^4 + q^3 + q^2 + q + 1)\lvert\mathrm{SL}_5(q)\rvert$
	$\zeta(q,4)$	$(q^2 + 1)\lvert\mathrm{O}_{12}^-(q)\rvert$
	$\zeta(q,3)$	$(q^2 + q + 1)\lvert E_6(q)\rvert$
	$\zeta(q,2)$	$(q+1)\lvert\mathrm{O}_{14}^-(q)\rvert$
	$\zeta(q,1)$	$(q-1)\lvert\mathrm{O}_{14}^+(q)\rvert$

In particular, such a g is regular semisimple for $l = 30, 24, 20, 15$.

Proof. In the cases $l = 30, 24, 20, 15$, the fact that g is regular semisimple was proved in [MT08, Lemma 2.3] or [Gur+15, Lemma 6.3].

The centralizers of semisimple elements in $E_8(q)$ are studied in [FJ94], and we use their results in the remaining proof.

Let $G = E_8(q)$ with q even and let $x \in G$ be a semisimple element. We let $\mathcal{G}$ be a simple algebraic group with F a Frobenius morphism such that $\mathcal{G}^F = G$. Note that $C_{\mathcal{G}}(x)$ is connected reductive. We denote $\mathcal{C} := C_{\mathcal{G}}(x)$, $\mathcal{Z} := Z(\mathcal{C})^{\circ}$, and $\mathcal{S} := [\mathcal{C}, \mathcal{C}]$. Let s be a prime in $\zeta(q, l)$. By Lemma 4.13, if $|x| = s$, then s divides either $|\mathcal{Z}^F|$ or $|Z(\mathcal{S})^F|$.

For the cases $l \geq 5$, it is enough to show that $s \nmid |Z(\mathcal{S})^F|$. Then the result follows because $s \big| |\mathcal{Z}^F|$ and each value of C in the table is the maximum possible centralizer order such that $\Phi_l(q) \big| |\mathcal{Z}^F|$ ([FJ94]). Note that $s \nmid |Z(\mathcal{S})^F|$ if s is greater than 8, because $\mathrm{rank}(\mathcal{S}) \leq \mathrm{rank}(\mathcal{G}) = 8$. If $s \in \zeta(q, l)$, then $s \geq l + 1$. Thus if $l \geq 7$ then we are done. If $l = 6$ or 5, then $s \neq 7$, as $7 \notin \zeta(q, l)$ since $q^3 = 8^f \equiv 1 \mod 7$. Also, $l | (s - 1)$ by Fermat's little Theorem, so $s \geq 11$, and we are done as before.

For the cases $s \in \zeta(q, l)$ for $l = 3$ or 4, note that $s \geq 5$. If $s \geq 8$ then were are done as above. Since $l | (s - 1)$ by Fermat's little Theorem, so $(s, l) = (5, 4)$ or $(7, 3)$. If $s \big| |\mathcal{Z}^F|$, then each value of C in the table is the maximum possible centralizer order such that $\Phi_l(q) \big| |\mathcal{Z}^F|$ ([FJ94]). Now we assume $s \nmid |\mathcal{Z}^F|$. Then $s \big| |Z(\mathcal{S})^F|$ by Lemma 4.13. Since $\mathcal{S}$ is semisimple, it is a central product of simple algebraic groups $\mathcal{S}_i$'s, i.e., $\mathcal{S} = \mathcal{S}_1 * \mathcal{S}_2 * \cdots * \mathcal{S}_k$ such that $[\mathcal{S}_i, \mathcal{S}_j] = 1$ and $|\mathcal{S}_i \cap \Pi_{j \neq i} \mathcal{S}_j| < \infty$ for any $i \neq j$. Note that $\mathrm{rank}(\mathcal{S}) \leq \mathrm{rank}(G)$ and so $\sum \mathrm{rank}(\mathcal{S}_i) \leq 8$. Let $\mathcal{O}_1$, $\mathcal{O}_2$, $\ldots$, $\mathcal{O}_{k'}$ be the F-orbits on $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k\}$. We may assume $\mathcal{S}_i$ is a representative of $\mathcal{O}_i$ for each $i \leq k'$, and let $|\mathcal{O}_i| = n_i$. Then $Z(\mathcal{S}) = Z(\mathcal{S}_1) * Z(\mathcal{S}_2) * \cdots * Z(\mathcal{S}_k)$. Since $s \big| |Z(\mathcal{S})^F|$, s divides $|Z(\mathcal{S})|$, and therefore, there exists an $i \in \{1, \ldots, k\}$ such that $s \big| |Z(\mathcal{S}_i)|$ for some $i \in \{1, \ldots, k\}$. For such an $i \in \{1, \ldots, k\}$, $\mathcal{S}_i$ is a simple algebraic group with $\mathrm{rank}(\mathcal{S}_i) \leq 8$ and $s \big| |Z(\mathcal{S}_i)|$, and therefore, $\mathcal{S}_i = \mathrm{SL}_s$. Without loss of generality, we may assume that $\mathcal{S}_1 = \mathrm{SL}_s$. If there is no $i \neq 1$ such that $\mathcal{S}_i = \mathrm{SL}_s$, then $s \nmid |Z(\mathcal{S}_i)|$ for any $i \neq 1$. Then $\mathcal{S}_1$ is F-stable and the s-part of $Z(\mathcal{S})^F$ is contained in $Z(\mathcal{S}_1^F) = Z(\mathrm{SL}_s^{\pm}(q))$. Then $s | (q \pm 1)$ which contradicts $s \in \zeta(q, l)$ for $l = 3, 4$. If there is some $i \neq 1$ such that $\mathcal{S}_i = \mathrm{SL}_s$, then $k = i = 2$ and $(s, l) = (5, 4)$ because $\sum \mathrm{rank}(\mathcal{S}_i) \leq 8$. Then $\mathcal{S}$

is of type $2A_4$. In all cases that $\mathcal{S}$ is of type $2A_4$, one can check that $|\mathcal{C}^F| \leq (q^2+1)|\mathrm{O}_{12}^{-}(q)|$ from the list of centralizer orders in [FJ94]. Hence, the result holds.

Now we assume that $l = 1, 2$. There exists an $x \in G$ such that $|\mathcal{Z}^F| = q - \epsilon$ and $|\mathcal{S}^F| = |\mathrm{O}_{14}^{\epsilon}(q)|$ ([FJ94, Section 3]). We show that $\mathcal{C}^F = \mathcal{Z}^F \times \mathcal{S}^F$ and $Z(\mathcal{C}^F) = \mathcal{Z}^F$. Note that $\mathcal{Z}^F * \mathcal{S}^F \leq \mathcal{C}^F$. Since $Z(\mathcal{S}^F) = Z(\mathrm{O}_{14}^{\epsilon}(q)) = 1$ for q even, it follows that $\mathcal{Z}^F \cap \mathcal{S}^F \leq Z(\mathcal{S}^F) = 1$. Thus $\mathcal{Z}^F * \mathcal{S}^F = \mathcal{Z}^F \times \mathcal{S}^F$, and since $|\mathcal{C}^F| = |\mathcal{Z}^F|\,|\mathcal{S}^F|$, we get $\mathcal{C}^F = \mathcal{Z}^F \times \mathcal{S}^F$. It is sufficient to prove that $C_G(z) = C_G(x)$ for any $1 \neq z \in \mathcal{Z}^F$ since $s \big| |\mathcal{Z}^F|$, and so there exists a nontrivial element $x \in \mathcal{Z}^F$ of order s. Now we show that $C_G(z) = C_G(x)$ for any $1 \neq z \in \mathcal{Z}^F$. Suppose y is a semisimple element in G such that $x^G \neq y^G$ but $C_G(x) = C_G(y)^g$ for some $g \in G$. Then $C_G(x) = C_G(y^g)$. Thus, there exists a G-conjugate of y, the centralizer of which is exactly $C_G(x)$. In particular, there are at least two such conjugates of y since any semisimple element in G is real ([TZ05, Proposition 3.1]). Note that there are $(q-\epsilon-1)/2$ semisimple classes in the genus of x (see [FJ94, Section 4]). Since $|\mathcal{Z}^F| = q - \epsilon$ and $C_G(1) = G$, all $q - \epsilon - 1$ nonidentity elements in $\mathcal{Z}^F$ are semisimple elements whose centralizer is exactly $C_G(x)$. □

Corollary 7.21. *If $G = E_8(q)$, then there exists an element $g \in G$ of order s where $s \in \zeta(q,l)$ such that $|\chi(g)| \leq B$ for any $\chi \in \mathrm{Irr}(G)$:*

(i) $G = E_8(q)$

l	30	24	20	15
B	$1.3q^4$	q^4	q^4	q^4

(ii) $G = E_8(q)$, *q even*

l	18	14	12	10	9	8	7	6	5	4	3	2	1
B	q^7	q^5	q^{16}	q^{14}	q^7	q^{16}	q^5	q^{40}	$1.3q^{14}$	q^{34}	$1.3q^{40}$	$1.12q^{46}$	q^{46}

Proof. It follows from Lemma 7.20 and Lemma 4.12. □

For the $\chi_0 \in \mathrm{Irr}(G)$ of degree $q\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$, we need better character bounds. The next few lemmas show how we can apply Lemma 4.11 to χ_0.

Lemma 7.22 (Burnside's Fusion Control Lemma)**.** *Let G be a finite group with P a Sylow p-subgroup. If x and x' are in $C_G(P)$, then $x \sim_G x'$ if and only if $x \sim_{N_G(P)} x'$.*

Proof. One direction is clear. We show the other direction. Suppose that $x \sim_G x'$ for x and x' in $C_G(P)$. Then there exists a $g \in G$ such that $x' = x^g$. Note that P is contained in $C_G(x)$ and $C_G(x^g)$. It follows that both P and $P^{g^{-1}}$ are Sylow p-subgroups in $C_G(x)$. Then there exists $h \in C_G(x)$ such that $P^{g^{-1}} = P^h$. Since hg normalizes P and $x^{hg} = x'$, we are done. □

Lemma 7.23. *Let $q = p^f$ for some prime p. If an odd prime r divides $\Phi_k(q)$ with $k \in \{15, 30\}$, then $r \in \zeta(q,k)$.*

Proof. It is equivalent to show that $\Phi_k(q)_r = (q^k - 1)_r$ if $r|\Phi_k(q)$.

We show the case $k = 15$ by way of contradiction. Suppose that r is an odd prime dividing $\Phi_{15}(q)$ but also dividing $\Phi_1(q)\Phi_3(q)\Phi_5(q)$. If r divides $\Phi_1(q)$, then $q \equiv 1 \mod r$, and so $\Phi_{15}(q) \equiv 1 \mod r$, a contradiction. If $\Phi_3(q) \equiv 0$, then $q^2 + q + 1 \equiv 0$ and $q^3 \equiv 1 \mod r$, which together imply $\Phi_{15}(q) \equiv -5q \mod r$. Since r is neither p nor 5 (5 does not divide $\Phi_3(q)$), we reached another contradiction. If r divides $\Phi_5(q)$, then $q^4+q^3+q^2+q+1 \equiv 0$ and $q^5 \equiv 1 \mod r$. It follows that $\Phi_{15}(q) \equiv 3q^3 - 3 \mod r$. Note that r does not divide $q^3 - 1$ by the above argument and that r is not 3 (3 does not divide $\Phi_5(q)$). Again it yields a contradiction.

Now we let r be an odd prime divisor of $\Phi_{30}(q)$. Then r divides $q^{30}-1 = (q^{15}-1)(q^{15}+1)$. Since r divides $q^{15}+1$ and $(q^{15}-1, q^{15}+1) = 2$, r does not divide $q^{15}-1$. Thus $(q^{30}-1)_r = (q^{15}+1)_r$. Since $q^{15}+1 = \Phi_2(q)\Phi_6(q)\Phi_{10}(q)\Phi_{30}(q)$, it is sufficient to show that r does not divide $\Phi_2(q)\Phi_6(q)\Phi_{10}(q)$. If r divides $\Phi_2(q)$, then $q \equiv -1 \mod r$ which implies $\Phi_{30}(q) \equiv 1 \mod r$, a contradiction. If r divides $\Phi_6(q)$, then $q^2 - q + 1 \equiv 0 \mod r$ and $q^3 \equiv -1 \mod r$ which together imply that $\Phi_{30}(q) \equiv 5q \mod r$. Clearly r does not divide q. Also, r is not 5 because 5 does not divide $\Phi_3(q)$, and so r does not divide $\Phi_{30}(q)$, a contradiction. If r divides $\Phi_{10}(q)$, then $q^4 - q^3 + q^2 - q + 1 \equiv 0 \mod r$ and $q^5 \equiv -1 \mod r$. It follows that $\Phi_{30}(q) \equiv -3q^3 + 3 \mod r$. Note that r does not divide $q^3 - 1$ because $(q^3-1)|(q^{15}-1)$ and r is not 3 because 3 does not divide $\Phi_{10}(q)$, a contradiction. □

Lemma 7.24. *Let $G = E_8(q)$ with $q = p^f$, p a prime. Let $k \in \{15, 30\}$.*

(i) Any nonidentity element t in a maximal torus of order $\Phi_k(q)$ is regular.

(ii) If $1 \neq g \in G$ has order dividing $\Phi_k(q)$, then g is regular and contained in a maximal torus of order $\Phi_k(q)$.

Proof. (i) Let $G = E_8(q)$ and T a maximal torus of order $\Phi_k(q)$ for $k \in \{15, 30\}$. Such a T exists by [Car72, G-20]. Let t be a nonidentity element in T. Since $|T| = \Phi_k(q)$ is odd, $|t|$ is odd as well. In particular, there exist an odd prime $r \big| \, |t|$ and $n \in \mathbb{N}$ such that $|t^n| = r$. By Lemma 7.23, $r \in \zeta(q, k)$. Now we use the list of possible centralizer orders of semisimple element in G given in [FJ94]. Since $r > 16$ divides the toral part of the centralizer (Lemma 4.13), the only possible order of $C_G(t^n)$ is $\Phi_k(q)$. Note that $T \leq C_G(t) \leq C_G(t^n)$. Since $|T| = |C_G(t^n)|$, we get $T = C_G(t)$, and so t is regular.

(ii) If $1 \neq g \in G$ has order dividing $\Phi_k(q)$ for $k \in \{15, 30\}$, then g is a semisimple element. There exists a maximal torus containing g of order divisible by $|g|$. By [Car72, G-20] and Lemma 7.23, such a torus is of order $\Phi_k(q)$. The result follows from part (i). □

Lemma 7.25. *Let $G = E_8(q)$ with $q = p^f$, p a prime, and let $r \in \zeta(q, k)$ for $k \in \{15, 30\}$. Suppose that χ is an irreducible character with positive r-defect such that $\chi(1) < |G|_{p'}/\Phi_k(q)$. Then χ is in the principal r-block.*

Proof. Let $r \in \zeta(q, k)$ for $k \in \{15, 30\}$, and let χ be an irreducible character of G with positive r-defect such that $\chi(1) < |G|_{p'}/\Phi_k(q)$. Let $\mathcal{G}$ be a simple algebraic group with F Frobenius morphism such that $\mathcal{G}^F = G$. Note that $\mathcal{G}$ is self-dual. There exists an r'-element $x \in G$ such that $\chi \in \mathcal{E}_r(\mathcal{G}^F, (x))$ where

$$\mathcal{E}_r(\mathcal{G}^F, (x)) = \bigcup_{y \in C_{\mathcal{G}}(x)^F_r} \mathcal{E}(\mathcal{G}^F, (xy)).$$

(For the definition of a *Lusztig series* $\mathcal{E}(\mathcal{G}^F, (g))$ where g is a a semisimple element in the dual group $\mathcal{G}^{*F^*}$, see [DM91, Definition 13.16].) Any irreducible character in $\mathcal{E}_r(\mathcal{G}^F, (x))$ is of degree divisible by $[G : C_G(x)]_{p'}$ ([HM01, Section 2]). Since χ has positive r-defect, $r \big| \, |C_G(x)|$. Thus there is a semisimple r-element y such that $y \in C_G(x)$, in particular, $x \in C_G(y)$. By Lemma 7.24, since $r \in \zeta(q, k)$, y is regular and contained in a maximal torus T of order $\Phi_k(q)$. Now it follows that $x \in T$. Again by Lemma 7.24, either $x = 1$ or $C_G(x) = T$. Since $\chi(1) < |G : T|_{p'}$, we get $\chi \in \mathcal{E}_r(\mathcal{G}^F, (1))$. On the other hand, one can show that there are as many r-blocks of G with maximal defect in G as distinct G-conjugacy classes of r'-elements in T. Indeed, let R be the Sylow r-subgroup of T, and let A be a subgroup of T such that $T = R \times A$. Note that R is a Sylow r-subgroup of G, $y \in R$, and $T \leq C_G(R) \leq C_G(y) = T$. Thus, $C_G(R) = R \times A$. By [NT12, Lemma 2.1], there are as many r-blocks with maximal defect in G as distinct $N_G(R)$-conjugacy classes of elements in A. We are done by Burnside's fusion control lemma.

For any $x \in G$ a semisimple r'-element, $\mathcal{E}_r(\mathcal{G}^F, (x))$ is a disjoint union of r-blocks ([BM89]). Note that if x is an r'-element in T, then $\mathcal{E}_r(\mathcal{G}^F, (x))$ contains at least one r-block of maximal defect since it contains the semisimple character labeled by x which has r'-degree. Also $\mathcal{E}_r(G, (x))$ and $\mathcal{E}_r(G, (x'))$ are disjoint for distinct G-conjugacy classes (x) and (x'). Thus, each $\mathcal{E}_r(G, (x))$ contains exactly one r-blocks of maximal defect. Now it follows that $\mathcal{E}_r(G, (1))$ contains the principal r-block and it is the only r-block with maximal defect contained in $\mathcal{E}_r(G, (1))$. Therefore, χ is in the principal r-block. □

Corollary 7.26. *Let $G = E_8(q)$, and let $r \in \zeta(q, k)$ for $k \in \{15, 30\}$. Then $\chi(g) = \pm 1$ for any nontrivial r-element $g \in G$ where $\chi \in \mathrm{Irr}(G)$ is an irreducible character with maximal r-defect such that it is a unique character of its degree and $\chi(1) < |G|_{p'}/\Phi_k(q)$.*

Proof. Let χ be a character described in the statement. Then χ is in the principal block by Lemma 7.25. Also, since it is a unique character of its degree, χ is nonexceptional. The result follows from Lemma 4.11. □

Lemma 7.27. *Let S be $E_8(q)$ with $q = p^f$, p a prime. If S is a minimal simple (r, s)-section of S itself, then the Schur cover G of S admits a $(2, r, s)$-triple.*

Proof. For $E_8(q)$, its Schur cover is itself, i.e., $G = S$. We assume that we have the case (ix) in Lemma 7.1. Then there exists a regular semisimple r-element by Lemma 7.20. Note that there is a regular 2-element if p is odd (Lemma 4.7). Therefore, if $(p, 2rs) = 1$, then G admits a $(2, r, s)$-triple by Lemma 4.6, so we are done.

Now we assume $p = s$. Then $r \in \zeta(q, 30) \cup \zeta(q, 15)$. Let x be a regular semisimple r-element, and let y be a regular semisimple 2-element as in [Gur+15, Lemma 7.16]. Then $|\chi(x)| \leq 1.3q^4$ by Corollary 7.21 and $|\chi(y)| \leq \sqrt{|C_G(y)|} \leq \sqrt{q^8 - 1} \leq q^4$. Using Lemma 7.19, we get the following:

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)|\,|\chi(y)|}{q\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)} \leq 27.2q^8\cdot\frac{1.3q^4\cdot q^4}{q^{29}} = \frac{35.36}{q^{13}} < 1$$

and so G admits a $(2, r, s)$-triple by Corollary 4.9.

Lastly, we consider the case $p = 2$. Let x be a regular semisimple r-element, and y a nontrivial s-element described in Lemma 7.20 for each l.

(i) $k = l = 30$:

There are $29+(1/30)q(q^4-1)(q^3+q^2-1)$ many nontrivial irreducible characters that are not r-defect 0, and their degrees are at least $q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$ by Lemma 7.19. Note that $|\chi(x)| \leq 1.3q^4$ and $|\chi(y)| \leq 1.3q^4$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21.

$$\begin{aligned}\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} &\leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)|\,|\chi(y)|}{q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)}\\ &\leq \left(29+\frac{1}{30}q(q^4-1)(q^3+q^2-1)\right)\cdot\frac{1.3q^4\cdot 1.3q^4}{q^{29}}\\ &\leq \left(29+\frac{1}{30}q\cdot q^4\cdot 2q^3\right)\cdot\frac{1.7}{q^{21}}\\ &\leq \left(29+\frac{1}{15}q^8\right)\cdot\frac{1.7}{q^{21}} \leq \frac{16}{15}q^8\cdot\frac{1.7}{q^{21}} \leq \frac{1.8}{q^{13}} < 1\end{aligned}$$

(ii) $k = 30$, $l \in \{24, 20, 18, 15, 14, 12, 10, 9, 8, 7, 5\}$:

There are at most 29 nontrivial irreducible characters that are not r-defect 0 nor s-defect 0. Any nontrivial irreducible character of G is of degree at least

$$q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$$

by Lemma 7.19. Also note that $|\chi(x)| \leq 1.3q^4$ and $|\chi(y)| \leq q^{16}$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21.

$$\begin{aligned}\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} &\leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)|\,|\chi(y)|}{q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)}\\ &\leq 29\cdot\frac{1.3q^4\cdot q^{16}}{q^{29}} = \frac{37.7}{q^9} < 1\end{aligned}$$

(iii) $k = 30$, $l \in \{6, 4, 3\}$:

By Lemma 7.19, there are at most 29 nontrivial irreducible characters that are not r-defect 0 nor s-defect 0. These characters are of degree at least

$$\begin{aligned}
&\frac{1}{2}q^3\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{24}(q)\\
&=\frac{1}{2}q^3(q^{15}-1)(q^9-1)(q^5-1)(q^{12}+1)\frac{q^{14}-1}{q+1}\\
&\geq \frac{1}{2}q^3\left(1-\frac{1}{2^{15}}\right)q^{15}\cdot\left(1-\frac{1}{2^9}\right)q^9\cdot\left(1-\frac{1}{2^5}\right)q^5\cdot q^{12}\cdot\frac{\left(1-\frac{1}{2^{14}}\right)q^{14}}{\frac{3}{2}q}\\
&\geq 0.32q^{57}
\end{aligned}$$

except for one character of degree $q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$, denote this character by χ_0. By Corollary 7.26, $\chi_0(x)=\pm 1$. Since χ_0 is nonlinear, we get $|\chi_0(y)|/\chi_0(1)\leq 19/20$ by [Glu95, Theorem 1.11]. Note that $|\chi(x)|\leq 1.3q^4$ and $|\chi(y)|\leq 1.3q^{40}$ for any $\chi\in\mathrm{Irr}(G)$ by Corollary 7.21. Thus one can get,

$$\begin{aligned}
\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}}\frac{|\chi(x)\chi(y)|}{\chi(1)} &\leq \frac{|\chi_0(y)|}{\chi_0(1)}+\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G,\chi_0}}\frac{|\chi(x)|\,|\chi(y)|}{0.32q^{57}}\\
&\leq \frac{19}{20}+28\cdot\frac{1.3q^4\cdot 1.3q^{40}}{0.32q^{57}}\leq\frac{19}{20}+\frac{148}{q^{13}}<1
\end{aligned}$$

(iv) $k=30$, $l\in\{2,1\}$:

By Lemma 7.19, there are at most 28 nontrivial irreducible characters that are not r-defect 0 nor s-defect 0. These characters are of degree at least

$$\frac{1}{2}q^3\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{24}(q)\geq 0.32q^{57}.$$

Note that $|\chi(x)|\leq 1.3q^4$ and $|\chi(y)|\leq 1.12q^{46}$ for any $\chi\in\mathrm{Irr}(G)$ by Corollary 7.21. Thus one can get,

$$\begin{aligned}
\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}}\frac{|\chi(x)\chi(y)|}{\chi(1)} &\leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G,\chi_0}}\frac{|\chi(x)|\,|\chi(y)|}{0.32q^{57}}\\
&\leq 28\cdot\frac{1.3q^4\cdot 1.12q^{46}}{0.32q^{57}}\leq\frac{127.4}{q^7}<1
\end{aligned}$$

(v) $k=l=15$:

There are $29+(1/30)q(q^4-1)(q^3-q^2+1)$ many nontrivial irreducible characters that are not r-defect 0, and their degrees are at least $q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$. Note that

$|\chi(x)| \leq q^4$ and $|\chi(y)| \leq q^4$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21.

$$\begin{aligned}\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} &\leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)|\,|\chi(y)|}{q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)} \\ &\leq \left(29+\frac{1}{30}q(q^4-1)(q^3-q^2+1)\right)\cdot\frac{q^4\cdot q^4}{q^{29}} \\ &\leq \left(29+\frac{1}{30}q\cdot q^4\cdot q^3\right)\cdot\frac{1}{q^{21}} \\ &\leq \left(29+\frac{1}{30}q^8\right)\cdot\frac{1}{q^{21}} \leq \frac{31}{30}q^8\cdot\frac{1}{q^{21}} \leq \frac{1.1}{q^{13}} < 1\end{aligned}$$

(vi) $k = 15$, $l \in \{24, 20, 18, 14, 12, 10, 9, 8, 7, 5\}$

By Lemma 7.19, there are at most 29 nontrivial irreducible characters that are not r-defect 0 nor s-defect 0. Any nontrivial irreducible character of G is of degree at least $q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$. Also note that $|\chi(x)| \leq q^4$ and $|\chi(y)| \leq q^{16}$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21.

$$\begin{aligned}\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} &\leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)|\,|\chi(y)|}{q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)} \\ &\leq 29\cdot\frac{q^4\cdot q^{16}}{q^{29}} = \frac{29}{q^9} < 1\end{aligned}$$

(vii) $k = 15$, $l \in \{6, 4, 3\}$:

By Lemma 7.19, there are at most 29 nontrivial irreducible characters that are not r-defect 0 nor s-defect 0. These characters are of degree at least

$$\begin{aligned}&\frac{1}{2}q^3\Phi_4(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q) \\ &= \frac{1}{2}q^3\cdot\frac{q^{14}-1}{q^2-1}\cdot\frac{q^{24}-1}{q^6-1}\cdot\frac{q^{20}-1}{q^{10}-1}\cdot(q^6+q^3+1)(q^8+q^7-q^5-q^4-q^3+q+1) \\ &\geq \frac{1}{2}q^3\cdot q^{12}\cdot q^{18}\cdot q^{10}\cdot q^6\cdot q^8 \\ &= 0.5q^{57}\end{aligned}$$

except for one character of degree $q^4\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$, denote this character by χ_0. Since $\chi_0(1) < |G|_{p'}/\Phi_k(q)$, we have $\chi_0(x) = \pm 1$ by Corollary 7.26. Since χ_0 is nonlinear, we get $|\chi_0(y)|/\chi_0(1) \leq 19/20$ by [Glu95, Theorem 1.11]. Note that $|\chi(x)| \leq q^4$

and $|\chi(y)| \leq 1.3q^{40}$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21. Thus one can get,

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq \frac{|\chi_0(y)|}{\chi_0(1)} + \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G,\chi_0}} \frac{|\chi(x)|\,|\chi(y)|}{0.5q^{57}}$$

$$\leq \frac{19}{20} + 28\cdot\frac{q^4\cdot 1.3q^{40}}{0.5q^{57}} \leq \frac{19}{20} + \frac{73}{q^{13}} < 1$$

(viii) $k = 15$, $l \in \{2, 1\}$:

By Lemma 7.19, there are at most 28 nontrivial irreducible characters that are not r-defect 0 nor s-defect 0. These characters are of degree at least

$$\frac{1}{2}q^3\Phi_4(q)^2\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q) \geq 0.5q^{57}$$

Note that $|\chi(x)| \leq q^4$ and $|\chi(y)| \leq 1.12q^{46}$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21. Thus one can get,

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G,\chi_0}} \frac{|\chi(x)|\,|\chi(y)|}{0.5q^{57}}$$

$$\leq 28\cdot\frac{q^4\cdot 1.12q^{46}}{0.5q^{57}} = \frac{62.72}{q^7} < 1$$

(ix) $k = 24$, $l \in \{20, 18, 14, 10, 9, 8, 7\}$:

By Lemma 7.19, there are at most 23 nontrivial irreducible characters which are not r-defect 0 nor s-defect 0. These characters are of degree at least

$$\begin{aligned} & q^2\Phi_5(q)\Phi_7(q)\Phi_{10}(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{20}(q)\Phi_{30}(q) \\ &= q^2\cdot\frac{q^{30}-1}{q^6-1}\cdot\frac{q^{14}-1}{q^2-1}\cdot\Phi_{20}(q) \\ &\geq q^2\cdot q^{24}\cdot q^{12}\cdot\frac{3}{4}q^8 \\ &\geq 0.75q^{46} \end{aligned}$$

Note that $|\chi(x)| \leq q^4$ and $|\chi(y)| \leq q^{16}$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21.

$$\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)\chi(y)|}{\chi(1)} \leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}} \frac{|\chi(x)|\,|\chi(y)|}{0.75q^{46}}$$

$$\leq 23\cdot\frac{q^4\cdot q^{16}}{0.75q^{46}} \leq \frac{31}{q^{26}} < 1$$

(x) $k = 20$, $l \in \{18, 14, 10, 9, 7, 6, 3\}$:

By Lemma 7.19, there are at most 24 non principal irreducible characters which are not r-defect 0 nor s-defect 0. These characters are of degree at least

$$\begin{aligned}
&\frac{1}{2}q^3\Phi_1^4(q)\Phi_3^2(q)\Phi_5^2(q)\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{24}(q)\\
&= \frac{1}{2}q^3(q^9-1)(q^{15}-1)(q-1)^2(q^{12}+1)\cdot\frac{q^{14}-1}{q^2-1}\cdot(q^4+q^3+q^2+q+1)\\
&\geq \frac{1}{2}q^3\cdot\left(1-\frac{1}{2^9}\right)q^9\cdot\left(1-\frac{1}{2^{15}}\right)q^{15}\cdot\frac{1}{4}q^2\cdot q^{12}\cdot q^{12}\cdot q^4\\
&\geq 0.12q^{57}
\end{aligned}$$

Note that $|\chi(x)| \leq q^4$ and $|\chi(y)| \leq 1.3q^{40}$ for any $\chi \in \mathrm{Irr}(G)$ by Corollary 7.21.

$$\begin{aligned}
\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}}\frac{|\chi(x)\chi(y)|}{\chi(1)} &\leq \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi\neq 1_G}}\frac{|\chi(x)|\,|\chi(y)|}{0.12q^{57}}\\
&\leq 24\cdot\frac{q^4\cdot 1.3q^{40}}{0.12q^{57}} = \frac{260}{q^{13}} < 1
\end{aligned}$$

Again, by Corollary 4.9, G admits a $(2, r, s)$-triple. $\square$

Chapter 8

Proof of The Main Theorem

Recall Theorem A.

Theorem A (Main Theorem)**.** *Let p, q be distinct odd primes and let G be a finite group. The following statements are equivalent:*
(i) G contains a composition factor whose order is divisible by pq;
(ii) G contains a $(2, p, q)$-triple.

Proof. Let G be a finite group and p, q distinct odd prime divisors of $|G|$.

We first prove that if G admits a $(2, p, q)$-triple, then it has a composition factor of order divisible by pq. Suppose that there is a finite group admitting a $(2, p, q)$-triple but not having a composition factor of order divisible by pq. Let G be such a group of minimal order. If G is simple, then G itself is the desired composition factor. Hence, we can assume that G is not simple. Let N be a nontrivial proper normal subgroup of G. In particular, neither N nor G/N has a composition factor of order divisible by pq. Let (x, y, z) be a $(2, p, q)$-triple of G. We assume that x is in N. Then $yz = x^{-1} \in N$ and so $yN = z^{-1}N$. Since $|yZ|$ is coprime to $|z^{-1}N|$, $|yN| = |z^{-1}N| = 1$, and so $y, z \in N$ as well. It contradicts the minimality of G. Similarly, one can show if $x \notin N$ then $y \notin N$ and $z \notin N$. Thus (xN, yN, zN) gives a $(2, p, q)$-triple of G/N, a contradiction to the minimality of G.

Now we show the other direction by contradiction. Suppose that there is a group G such that it has a composition factor of order divisible by pq but does not admit a $(2, p, q)$-triple. Let G be such a group of minimal order. Then any proper subgroup of G does not admit a $(2, p, q)$-triple, and so it does not have a composition factor of order divisible by pq by the minimality of G. It follows that G does not have a proper simple section of order divisible by pq. It is enough to show that G is quasisimple because then we reach a contradiction to Theorem B.

Consider $[G, G]$. Since $G/[G, G]$ is abelian, it does not have a composition factor divisible by pq. Therefore, $[G, G]$ must contain one, and so $[G, G] = G$, i.e., G is perfect.

Now we show that the Frattini subgroup $\Phi(G)$ is the largest proper normal subgroup by contradiction. Suppose that there exists N a proper normal subgroup of G and M a maximal subgroup of G such that $NM = G$. Then we get $M/(M \cap N) \simeq MN/N = G/N$. Note that N does not contain a $(2, p, q)$-triple, and so it does not have a composition factor of order divisible by pq due to the minimality of G. Therefore, G/N has a composition factor of order divisible by pq. It follows that M does have a composition factor of order divisible by pq, again contradicting the minimality of G. Hence $\Phi(G)$ is the largest proper normal subgroup of G, and so $G/\Phi(G)$ is simple.

Lastly, we show that $\Phi(G)$ is central. Suppose not. Then there exists a proper normal subgroup N in G which is noncentral. Let N be such a group of minimal order. Since $N \leq \Phi(G)$ and $\Phi(G)$ is nilpotent, N is nilpotent. Therefore, the minimality of N implies that N is an l-group for some prime l. Note that $\mathrm{O}_2(G) = \mathrm{O}_p(G) = \mathrm{O}_q(G) = 1$. Otherwise, for example, if $\mathrm{O}_2(G) \neq 1$, then $G/\mathrm{O}_2(G)$ has a composition factor of order divisible by pq since $\mathrm{O}_2(G)$ does not. By the minimality of G, $G/\mathrm{O}_2(G)$ admits a $(2, p, q)$-triple. Then by the similar argument we made in the proof of Lemma 4.3, it can be lifted to G, a contradiction. For the same reason we get $\mathrm{O}_p(G) = 1$ and $\mathrm{O}_q(G) = 1$. This implies that $l \neq 2, p, q$. Also note N is abelian by [GT15, Proposition 2.1]. Consider G/N. Since G/N has a composition factor of order divisible by pq, it admits a $(2, p, q)$-triple, i.e., there exist $x, y, z \in G$ such that x is a 2-element, y is a p-element, z is a q-element and $xyz = n$ for some $n \in N$. Let $L/N = \langle xN, yN, zN \rangle$ for some $L \leq G$. Then by the first part, L/N has a composition factor of order divisible by pq, and so does L. Therefore, $L = G$ by the minimality of G. Now we get $G = \langle x, y, z \rangle N$, and so $G = \langle x, y, z \rangle$ since $N \leq \Phi(G)$. By [GT15, Lemma 2.2], we get N-conjugates of x, y, z, say x_1, y_1, z_1 such that $x_1 y_1 z_1 = 1$, a contradiction. $\square$

Appendix A

Computation of Character Formula Based on GAP

Many calculations of the character formula in Lemma 4.4 (stated below) can be done using the GAP character table library.

Lemma 4.4 (Character Formula)**.** *Let G be a finite group. If there exists a nontrivial 2-element x, a p-element y, and a q-element z in G such that*

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)} \neq 0,$$

then there exists $x' \in x^G$, $y' \in y^G$, and $z' \in z^G$ such that $x'y'z' = 1$, i.e. G admits a $(2, p, q)$-triple (x', y', z').

For example, let `t` be the character table of a Mathieu group M_{23}. Then we can store the matrix of character table in `irr`.

```
gap> t:=CharacterTable("M23");
CharacterTable( "M23" )
gap> Display(t);
M23

      2   7   7  2  5  .  2  1  1  3   .   .   1   1   .   .   .   .
      3   2   1  2  .  1  1  .  .  .   .   .   .   .   1   1   .   .
      5   1   .  1  .  1  .  .  .  .   .   .   .   .   1   1   .   .
      7   1   1  .  .  .  .  1  1  .   .   .   1   1   .   .   .   .
     11   1   .  .  .  .  .  .  .  .   1   1   .   .   .   .   .   .
     23   1   .  .  .  .  .  .  .  .   .   .   .   .   .   .   1   1

         1a  2a 3a 4a 5a 6a 7a 7b 8a 11a 11b 14a 14b 15a 15b 23a 23b
     2P  1a  1a 3a 2a 5a 3a 7a 7b 4a 11b 11a  7a  7b 15a 15b 23a 23b
     3P  1a  2a 1a 4a 5a 2a 7b 7a 8a 11a 11b 14b 14a  5a  5a 23a 23b
     5P  1a  2a 3a 4a 1a 6a 7b 7a 8a 11a 11b 14b 14a  3a  3a 23b 23a
     7P  1a  2a 3a 4a 5a 6a 1a 1a 8a 11b 11a  2a  2a 15b 15a 23b 23a
    11P  1a  2a 3a 4a 5a 6a 7a 7b 8a  1a  1a 14a 14b 15b 15a 23b 23a
    23P  1a  2a 3a 4a 5a 6a 7a 7b 8a 11a 11b 14a 14b 15a 15b  1a  1a

X.1       1   1  1  1  1  1  1  1  1   1   1   1   1   1   1   1   1
X.2      22   6  4  2  2  .  1  1  .   .   .  -1  -1  -1  -1  -1  -1
```

```
X.3       45  -3  .  1  .  .  A /A -1   1   1  -A -/A   .   .  -1  -1
X.4       45  -3  .  1  .  . /A  A -1   1   1 -/A  -A   .   .  -1  -1
X.5      230  22  5  2  .  1 -1 -1  .  -1  -1   1   1   .   .   .   .
X.6      231   7  6 -1  1 -2  .  . -1   .   .   .   .   1   1   1   1
X.7      231   7 -3 -1  1  1  .  . -1   .   .   .   .   C  /C   1   1
X.8      231   7 -3 -1  1  1  .  . -1   .   .   .   .  /C   C   1   1
X.9      253  13  1  1 -2  1  1  1 -1   .   .  -1  -1   1   1   .   .
X.10     770 -14  5 -2  .  1  .  .  .   .   .   .   .   .   .   D  /D
X.11     770 -14  5 -2  .  1  .  .  .   .   .   .   .   .   .  /D   D
X.12     896   . -4  .  1  .  .  .  .   B  /B   .   .   1   1  -1  -1
X.13     896   . -4  .  1  .  .  .  .  /B   B   .   .   1   1  -1  -1
X.14     990 -18  .  2  .  .  A /A  .   .   .   A  /A   .   .   1   1
X.15     990 -18  .  2  .  . /A  A  .   .   .  /A   A   .   .   1   1
X.16    1035  27  . -1  .  . -1 -1  1   1   1  -1  -1   .   .   .   .
X.17    2024   8 -1  . -1 -1  1  1  .   .   .   1   1  -1  -1   .   .

A = E(7)+E(7)^2+E(7)^4
  = (-1+Sqrt(-7))/2 = b7
B = E(11)+E(11)^3+E(11)^4+E(11)^5+E(11)^9
  = (-1+Sqrt(-11))/2 = b11
C = -E(15)^7-E(15)^11-E(15)^13-E(15)^14
  = (-1+Sqrt(-15))/2 = b15
D = E(23)+E(23)^2+E(23)^3+E(23)^4+E(23)^6+E(23)^8+E(23)^9+E(23)^12+E(23)^13\
+E(23)^16+E(23)^18
  = (-1+Sqrt(-23))/2 = b23
gap> irr:=Irr(t);;
```

One of the case for M_{23} to be checked is the existence of a $(2,3,23)$-triple in M_{23} (see Theorem C). We can show that there exist a 2-element in the conjugacy class 8a, a 3-element in the conjugacy class 3a, and a 23-element in the conjugacy class 23a forming a $(2,3,23)$-triple in M_{23}. The 9th, 3rd, and 16th columns in the character table matrix `irr` are representing the conjugacy classes 8a, 3a, and 23a. Now we compute the character formula

$$\sum_{\chi\in\mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)} \neq 0,$$

for $x \in$8a, $y \in$3a, and $z \in$23a as follow.

```
gap> Sum([1..17], i -> irr[i][9] * irr[i][3] * irr[i][16] / irr[i][1]);
1
```

A.1 Sporadic Groups

In this sections, we will list the conjugacy classes that were used to compute character formula for the Sporadic groups listed in Theorem C as well as which column of character table corresponds to each conjugacy class.

A.1.1 Mathieu Group M_{22}

It is enough to show that $3.M_{22}$ contains a $(2, 7, 11)$-triple by Lemma 4.3.

```
# Character formula computation for "3.M22"

2-element 8a [26]
7-element 7a [20]
11-element 11a [29]
```

A.1.2 Mathieu Group M_{23}

We need to check the existence of $(2, s, 23)$-triple for $s \in \{3, 5, 7, 11\}$ in M_{23}.

```
# Character formula computation for "M23"

2-element 8a [9]
3-element 3a [3]
5-element 5a [5]
7-element 7a [7]
11-element 11a [10]
23-element 23a [16]
```

A.1.3 Janko Group J_1

We need to check $(2, r, s)$-triple in J_1 for the following cases:

S	(r, s)
J_1	$(19, s),\ s \in \{3, 5, 7, 11\}$ $(7, 3), (7, 5), (11, 7)$

```
# Character formula computation for "J1"

2-element  2a [2]
3-element  3a [3]
5-element  5a [4]
7-element  7a [7]
11-element 11a [10]
19-element 19a [13]
```

A.1.4 Janko Group J_2

By Lemma 4.3, it is enough to check the existence of a $(2, 5, 7)$-triple in J_2 instead of its Schur multiplier.

```
# Character formula computation for "J2"

2-element  8a [14]
5-element  5a [7]
7-element  7a [13]
```

A.1.5 Janko Group J_3

We check the existence of a $(2, 17, 19)$-triple in $3.J_3$.

```
# Character formula computation for "3.J3"

2-element  8a [23]
17-element 17a [44]
19-element 19a [50]
```

A.1.6 Janko Group J_4

We check the existence of $(2, r, s)$-triple for r and s listed below in J_4.

S	(r, s)
J_4	$(29, s),\ s \in \{3, 5, 7, 11, 23\}$, $(31, 29), (31, 23)$, $(37, s),\ s \in \{7, 23, 29, 31\}$, $(43, s),\ s \in \{3, 5, 7, 11, 23, 29, 31, 37\}$

```
# Character formula computation for "J4"

2-element 16a [29]
3-element 3a [4]
5-element 5a [8]
7-element 7a [12]
11-element 11a [19]
23-element 23a [36]
29-element 29a [41]
31-element 31a [43]
37-element 37a [50]
43-element 43a [57]
```

A.1.7 Conway Group Co_1

We check the existence of $(2, 13, 23)$-triple in Co_1.

```
# Character formula computation for "Co1"

2-element 16a [66]
13-element 13a [58]
23-element 23a [78]
```

A.1.8 Fischer Group Fi_{22}

It is enough to check the existence of $(2, 11, 13)$-triple in $3.Fi_{22}$ by Lemma 4.3.

```
# Character formula computation for "3.Fi22"

2-element 16a [129]
11-element 11a [86]
13-element 13b [120]
```

A.1.9 Fischer Group Fi_{23}

We check the existence of $(2, 11, 17)$, $(2, 13, 17)$, $(2, 13, 23)$, $(2, 17, 23)$-triples in Fi_{23}.

```
# Character formula computation for "Fi23"

2-element 16a [63]
11-element 11a [41]
13-element 13a [57]
17-element 17a [65]
23-element 23a [80]
```

A.1.10 Fischer Group Fi'_{24}

We need to check the existence of $(2, s, 29)$-triples for $s \in \{3, 5, 7, 11, 13, 17, 23\}$ in $3.Fi'_{24}$.

```
# Character formula computation for "3.Fi24'"

2-element 16a [139]
3-element 27a [202]
5-element 5a [28]
7-element 7a [58]
11-element 11a [89]
13-element 13a [121]
17-element 17a [142]
23-element 23a [180]
29-element 29a [208]
```

A.1.11 Held Group He

We check the existence of a $(2, 7, 17)$-triple in He.

```
# Character formula computation for "He"

2-element 8a [17]
7-element 7a [12]
17-element 17a [26]
```

A.1.12 Rudvalis Group Ru

It is enough to check the existence of a $(2, 13, 29)$-triple in Ru by Lemma 4.3.

```
# Character formula computation for "Ru"

2-element 16a [25]
13-element 13a [20]
29-element 29a [35]
```

A.1.13 Suzuki Sporadic Group Suz

It is enough to check the existence of $(2, 7, 11)$, $(2, 13, 11)$-triples in $3.Suz$ by Lemma 4.3.

```
# Character formula computation for "Suz"

2-element 8a [51]
7-element 7a [48]
11-element 11a [72]
13-element 13a [88]
```

A.1.14 O'Nan Group $O'N$

We check the existence of $(2, 7, 31)$, $(2, 11, 31)$, $(2, 19, 31)$-triples in $3.O'N$.

```
# Character formula computation for "O'N"

2-element 16a [42]
7-element 7a [18]
11-element 11a [33]
19-element 19a [54]
31-element 31a [75]
```

A.1.15 Harada-Norton Group HN

We check the existence of $(2, 5, 19)$, $(2, 11, 19)$-triples in HN.

```
# Character formula computation for "HN"

2-element 8a [18]
5-element 5a [9]
```

```
11-element 11a [29]
19-element 19a [37]
```

A.1.16 Lyons Group Ly

We check the existence of $(2, r, s)$-triples listed below in Ly.

S	(r, s)
	$(67, s),\ s \in \{3, 5, 7, 11, 31, 37\}$
Ly	$(37, s),\ s \in \{3, 5, 7, 11, 31\}$
	$(31, 11)$

```
# Character formula computation for "Ly"

2-element 8a [12]
3-element 9a [14]
5-element 25a [34]
7-element 7a [11]
11-element 11a [17]
31-element 31a [38]
37-element 37a [45]
```

A.1.17 Thompson Group Th

We check the existence of $(2, 5, 13), (2, 13, 19), (2, 13, 31), (2, 19, 31)$-triples in Th.

```
# Character formula computation for "Th"

2-element 8a [13]
5-element 5a [8]
13-element 13a [23]
19-element 19a [29]
31-element 31a [42]
```

A.1.18 Baby Monster Group

It is enough to check the existence of $(2, r, s)$-triples ((r, s) listed below) in B by Lemma 4.3.

S	(r, s)

B	$(47, s)$ $s \in \{3, 5, 7, 11, 13, 17, 19, 23, 31\}$ $(31, 23), (31, 17), (31, 11), (19, 13)$

```
# Character formula computation for "B"

2-element 32a [147]
3-element 27a [131]
5-element 25a [128]
7-element 7a [31]
11-element 11a [54]
13-element 13a [75]
17-element 17a [91]
19-element 19a [98]
23-element 23a [112]
31-element 31a [145]
47-element 47a [172]
```

A.1.19 Monster Group M

We check the existence of $(2, r, s)$-triple for r, s listed below in M.

S	(r, s)
M	$(71, s)$, $s \in \{11, 13, 17, 19, 23, 29, 31, 41, 47, 59\}$ $(59, s)$, $s \in \{7, 11, 13, 17, 19, 23, 31, 41, 47\}$ $(47, 41), (47, 29)$ $(41, s)$, $s \in \{11, 17, 19, 23, 29, 31\}$ $(31, 29), (29, 19)$

```
# Character formula computation for "M"
2-element 32a [107]
3-element 27a [91]
5-element 25a [88]
7-element 7a [19]
11-element 11a [34]
13-element 13a [45]
17-element 17a [57]
19-element 19a [63]
23-element 23a [76]
29-element 29a [97]
31-element 31a [105]
```

```
41-element 41a [127]
47-element 47a [139]
59-element 59a [152]
71-element 71a [169]
```

A.2 Finite Groups of Lie-type with Exceptional Schur Cover

A.2.1 $\mathrm{L}_3(4)$

The Schur cover of $\mathrm{L}_3(4)$ is $(3 \times 4 \times 4).\mathrm{L}_3(4)$. Also, $\pi(\mathrm{L}_3(4)) = \{2, 3, 5, 7\}$ and $\mathrm{L}_2(7), \mathsf{A}_6 \subseteq \mathrm{L}_3(4)$. Thus it is enough to show that there exists a $(2, 5, 7)$-triple in $3.\mathrm{L}_3(4)$ by Lemma 4.3. We use a 2-element in the conjugacy class `4a`, a 5-element in the conjugacy class `5a`, and a 7-element in the conjugacy class `7a`. The conjugacy classes `4a`, `5a`, and `7a` are shown in the 8th, 17th, and 23rd columns of the character table.

```
gap> t:= CharacterTable("3.L3(4)");
CharacterTable( "3.L3(4)" )
gap> irr:=Irr(t);;
gap> Sum([1..28], i -> irr[i][8] * irr[i][17] * irr[i][23] / irr[i][1]);
1
```

A.2.2 ${}^2B_2(8)$

The Schur multiplier of ${}^2B_2(8)$ is elementary abelian of order 4. Thus it is enough to check the existence of $(2, 5, 7), (2, 5, 13), (2, 7, 13)$-triples in ${}^2B_2(8)$. We use a 2-element in the conjugacy classes `4a`, a 5-element in the conjugacy classes `5a`, a 7-element in the conjugacy classes `7a`, and a 13-element in the conjugacy classes `13a`. The conjugacy classes `4a`, `5a`, `7a` and `13a` are shown in the 3rd, 5th, 6th, and 9th columns of the character table.

```
gap> t:=CharacterTable("Sz(8)");
CharacterTable( "Sz(8)" )
gap> irr:=Irr(t);;
gap> c:= EmptyPlist(13);
[  ]
gap> c[2]:=3; c[5]:=5; c[7]:=6; c[13]:=9;
3
5
6
```

```
9
gap> Sum([1..11], i -> irr[i][c[2]] * irr[i][c[5]] * irr[i][c[7]]
/ irr[i][1]);
1
gap> Sum([1..11], i -> irr[i][c[2]] * irr[i][c[5]] * irr[i][c[13]]
/ irr[i][1]);
1
gap> Sum([1..11], i -> irr[i][c[2]] * irr[i][c[7]] * irr[i][c[13]]
/ irr[i][1]);
1
```

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