THE RANGE OF RANDOM WALKS UP TO THE TIME OF EXIT FROM A DOMAIN

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Contents

1. Abstract

We find an expression for the expected range of a symmetric, nearest neighbor random walk; that is, the number of distinct locations visited on the two dimensional integer lattice up to the time of first exit from a square region of side length N, generalizing a known result for the analogous problem in one dimension. Further, we show that the expected range before exit from the region is of order $\frac{N^2}{ln(N)}$. The constant associated with this limit is computed as a function of the starting point.

2. Introduction

Random walks have been the subject of a great deal of study over the past century. They have extensive applications in Physics, and have even been used to study properties of long polymer chains. Random walks are a type of stochastic process describing the random movements of a particle through some space of possible locations. While this characterization is quite general, we will only be concerned with random walks on the one and two dimensional integer lattices. In particular, we consider random walks which are nearest-neighbor and symmetric (i.e. at each time step, the particle travels to an adjacent point on the lattice, chosen uniformly). Erdős and Dvoretzky [1] studied random walks of this type in 1951, proving some results for the limiting behavior of R_n; the range up to time n (i.e. the number of distinct points hit by the random walk in the first n time steps). For example, they showed that for simple random walks in dimension at least two, $R_n/ER_n \to 1$ a.s. This is known as a "strong law" for the range. They also showed that ER_n is of order $\sqrt{n}$ in one dimension, $n/ln(n)$ in two dimensions, and n for all higher dimensions. Since then, their results have been expanded upon by a number of others, including Spitzer [9], Jain and Pruitt [5], and Le Gall and Rosen [6].

In this paper we will consider a variation of this problem in which the range of a random walk is observed up to the time of first exit from a square region (instead of up to a fixed time). This is equivalent to finding the total number of points hit by a Random Walk which is absorbed on a square boundary. Conceptualizing the problem in this way, it becomes clear that it is also related to the so called "trapping problem", in which a random walk on the d dimensional integer lattice is absorbed at (possibly randomly located) "traps". For such random walks, the expected time to absorption and the probability of survival after n steps are often of interest. This problem has been studied extensively, for example by Hollander [4], Robledo and Woodhouse [7], and Ethier [2].

However, it seems that few have studied the range up to the time of absorption. Hollander and Weiss [3] compute the mean of R_n and estimate its variance when conditioning on the location of the random walk at time n. In the 1D case, Athreya, Sethuraman, and Tóth [8] showed that the proportion of points hit by a symmetric (or weakly asymmetric) random walk absorbed on the set $\{0, N\}$ converges in distribution, and computed the limiting probability density as a function of the starting point. This allowed them to compute the expected proportion of points hit directly. Let $R_n^{(N)}$ denote the number of distinct locations visited by a random walk on strip of length N before hitting an end-point, starting at the point n. Then,

$$\lim_{N\to\infty} \frac{E[R_{[\alpha N]}^{(N)}]}{N} = -(1-\alpha)\, ln(1-\alpha) - \alpha\, ln(\alpha).$$

Unfortunately, the method used in [9] to derive the limiting distribution in the 1D case is difficult to generalize to higher dimensions because it involves conditioning on the point of exit (either 0 or N in 1D), and utilizes the classic Gambler's Ruin Identity. In the 2D case, the number of potential exit points increases with the size of the rectangular boundary, and there is no well-known identity to exploit. This warrants pursuing a different approach.

We compute the expected range as a sum of the probabilities that each point in the square is hit before absorption. These probabilities are harmonic when expressed as functions of the starting point, and can be solved for explicitly by taking Discrete Fourier Transforms and solving for coefficients. The expressions for these functions (hereafter referred to as the "hitting probabilities") can be thought of as a 2D generalization of the Gambler's Ruin Identity, as they represent the probability of hitting a given point (winning the game) before hitting the boundary (going broke). As one might guess, we found that the expected proportion of points hit before absorption converges to 0 as the size of the absorbing square approaches infinity. Whereas in the 1D case a random walk starting halfway between the absorbing points is guaranteed to hit at least half of the points in the region, a random walk in 2D need not hit any fixed proportion of the region enclosed by the boundary, regardless of the starting point.

In order to find the appropriate scaling to quantify the limiting behavior of the expected range, we noted that a factor in the formulas for the hitting probabilities can be approximated (with acceptably small error terms) by a constant multiple of $1/ln(N)$, (at least for target points far enough away from the boundary). This motivated an attempt to compute the limit of the expected range multiplied by $ln(N)/N^2$, where N is the side length of the square boundary. One might also guess that this is the correct scaling for the expected range by noting that the time to hit the boundary in the 2D case is of order N^2. As was mentioned above, it is known [1] that for an unconstrained random walk, the range up to time N is of order $N/ln(N)$. Combining these results, we might expect the range up to absorption to be of order $N^2/ln(N^2) \approx N^2/ln(N)$.

Indeed, we found that this limit converges to a function of the starting point. Take any $(s,t) \in (0,1) \times (0,1)$. Letting $R^{(N)}_{([sN],[tN])}$ denote the range before hitting the boundary, starting at the nearest point on the integer lattice to the point (sN, tN), we obtained:

$$\lim_{N\to\infty} \frac{ln(N)}{N^2} E\left[R^{(N)}_{([sN],[tN])}\right] = \frac{128}{\pi^3} \sum_{m=1,\,\text{odd}}^{\infty} \sum_{n=1,\,\text{odd}}^{\infty} \frac{sin\,(n\pi s)\ sin\,(m\pi t)}{nm(n^2+m^2)} := u(s,t).$$

Although not discussed here, one can verify that $\Delta u = 8\pi$ for (s,t) away from the boundary. Also, the function $u(s,t)$ may be seen to equal the expected time of exit from the unit square for standard Brownian Motion starting at the point (s,t).

3. Overview and Results

3.1. **Set Up.** We begin by introducing some notation.

Let $\{X_k\}_{k\in\mathbb{N}}$ be a symmetric, nearest neighbor random walk on $\Omega = \{0,1,2,\dots,N\}^2$ (for some $N \in \mathbb{N}$), starting at some point $(a,b) \in \Omega$. Define the boundary of Ω by $D := \{(x,y) \in \Omega \,|\, x \in \{0,N\} \text{ or } y \in \{0,N\}\}$. Using this notation, we are interested in the behavior of the random walk up to τ, where $\tau = \inf\{t : X_t \in D\}$. With this in mind, we introduce some notation for the hitting time of each point:

For any $(x,y) \in \Omega$, define $T_{(x,y)} := \inf\{k \in \mathbb{N} \,|\, X_k = (x,y)\}$. Further, let $\tau = \min\{T_{(x,y)} \,|\, (x,y) \in D\}$. We now have enough notation to define the "hitting probabilities" concisely:

For each $(a,b) \in \Omega$, define a function $f^{(N)}_{(a,b)} : \Omega \backslash D \to [0,1]$ by

$$f^{(N)}_{(a,b)}(x,y) = P_{(a,b)}(T_{(x,y)} < \tau).$$

So $f^{(N)}_{(a,b)}(x,y)$ represents the probability, starting at (a,b), that the point (x,y) is hit before hitting the boundary. For clarity, this function's dependence on N is often omitted. The next step is to establish the relationship between these functions and the expected range before hitting the boundary.

Define a random variable $R^{(N)}_{(a,b)} : \Omega^\infty \to \mathbb{R}$ by

$$R^{(N)}_{(a,b)}\left(\{X_k\}_{k\in\mathbb{N}}\right) = \sum_{x=1}^{N-1}\sum_{y=1}^{N-1} 1_{\{T_{(x,y)}<\tau\}}\left(\{X_k\}_{k\in\mathbb{N}}\right).$$

So $R^{(N)}_{(a,b)}$ represents the number of distinct points hit by the random walk, starting at (a,b), before hitting the boundary. Taking the expected value and applying linearity, we see that

$$E\left[R^{(N)}_{(a,b)}\right] = \sum_{x=1}^{N-1}\sum_{y=1}^{N-1} P_{(a,b)}(T_{(x,y)} < \tau) = \sum_{x=1}^{N-1}\sum_{y=1}^{N-1} f_{(a,b)}(x,y).$$

Recall that our main goal is to compute $\lim_{N\to\infty} \frac{ln(N)}{N^2} E\left[R^{(N)}_{([sN],[tN])}\right]$ as a function of $(s,t) \in [0,1]\times[0,1]$. This computation can be divided roughly into three steps:

(1) Deriving expressions for hitting probabilities (Section 3.2)
(2) Simplifying expressions for hitting probabilities (Section 3.3)
(3) Using hitting probabilties to compute the desired limit (Section 3.4).

3.2. **Deriving Expressions for the Hitting Probabilities.** In this section, discrete Fourier transforms are used to find an explicit formula for the hitting probabilities $f^{(N)}_{(a,b)}$. Recall that we defined $f^{(N)}_{(a,b)} : \Omega\backslash D \to [0,1]$ by

$$f^{(N)}_{(a,b)}(x,y) = P_{(a,b)}(T_{(x,y)} < \tau).$$

We begin by noting that for any $(a,b) \in D$ and $(x,y) \in \Omega\backslash D$, we have:

$$f_{(a,b)}(x,y) = 0, \;\; f_{(x,y)}(x,y) = 1.$$

Indeed, if the random walk starts on the boundary, then the probability of hitting any other point *before* hitting the boundary is 0. Similarly, when starting from some point not on the boundary, the probability of hitting that same point is 1.

On the other hand, if $(a,b),(x,y) \in \Omega\backslash D$ and $(a,b) \neq (x,y)$, then first step analysis yields:

$$f_{(a,b)}(x,y) = \frac{1}{4}\left(f_{(a+1,b)}(x,y) + f_{(a-1,b)}(x,y) + f_{(a,b+1)}(x,y) + f_{(a,b-1)}(x,y)\right).$$

This motivates the definition of a function $g_N : \Omega\backslash D \to [0,1]$ by

$$g_N(x,y) = 1 - \frac{1}{4}\left(f_{(x+1,y)}(x,y) + f_{(x-1,y)}(x,y) + f_{(x,y+1)}(x,y) + f_{(x,y-1)}(x,y)\right),$$

so that for any $(a,b) \in \Omega$, $\vec{z} = (x,y) \in \Omega \backslash D$, we have

$$f_{(a,b)}(\vec{z}) - \frac{1}{4}\left(f_{(a+1,b)}(\vec{z}) + f_{(a-1,b)}(\vec{z}) + f_{(a,b+1)}(\vec{z}) + f_{(a,b-1)}(\vec{z})\right) = g_N(\vec{z})\, 1_{(a,b)}(\vec{z}). \quad (1)$$

Now consider discrete Fourier transforms of each term in the equation (1):

$$f_{(a,b)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right),$$

$$f_{(a+1,b)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} sin\left(\frac{n\pi(a+1)}{N}\right) sin\left(\frac{m\pi b}{N}\right),$$

$$f_{(a-1,b)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} sin\left(\frac{n\pi(a-1)}{N}\right) sin\left(\frac{m\pi b}{N}\right),$$

$$f_{(a,b+1)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi(b+1)}{N}\right),$$

$$f_{(a,b-1)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi(b-1)}{N}\right),$$

$$g_N(x,y)1_{(a,b)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} b_{mn} sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right).$$

Applying some trig identities, the left hand side of (1) may be expressed as

$$\sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)\left(1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]\right).$$

Orthogonality relations indicate that for any $n, n' \in \mathbb{N}$,

$$\sum_{a=1}^{N-1} sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{n'\pi a}{N}\right) = \frac{N\delta_{nn'}}{2}$$

Applying these relations to find b_k, we obtain

$$\sum_{b=1}^{N-1}\sum_{a=1}^{N-1} g_N(x,y)\, 1_{(a,b)}(x,y)\, sin\left(\frac{n'\pi a}{N}\right) sin\left(\frac{m'\pi b}{N}\right) = \frac{N^2 b_{m'n'}}{4},$$

so, relabeling ($n' \to n$, $m' \to m$):

$$b_{mn} = \frac{4 g_N(x,y) sin\left(\frac{n\pi x}{N}\right) sin\left(\frac{m\pi y}{N}\right)}{N^2}$$

Equating the coefficients on either side of (1),

$$a_{mn} = \frac{4g_N(x,y)sin\left(\frac{n\pi x}{N}\right)sin\left(\frac{m\pi y}{N}\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)}$$

Also, boundary conditions indicate that

$$f_{(x,y)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} sin\left(\frac{n\pi x}{N}\right) sin\left(\frac{m\pi y}{N}\right) = 1.$$

Thus,

$$g_N(x,y) = \left[\sum_{m=1}^{N-1}\sum_{n=1}^{N-1} \frac{4sin^2\left(\frac{n\pi x}{N}\right)sin^2\left(\frac{m\pi y}{N}\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)}\right]^{-1}$$

and

$$f_{(a,b)}^{(N)}(x,y) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} \frac{4g_N(x,y)sin\left(\frac{n\pi x}{N}\right)sin\left(\frac{m\pi y}{N}\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)}.$$

In order to simplify notation, we will usually be working with a closely related function $\alpha_N : (0,1)\times(0,1)\to\mathbb{R}$ defined by

$$\alpha_N(\beta,\gamma) = \left[\sum_{m=1}^{N-1}\sum_{n=1}^{N-1} \frac{4sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)}\right]^{-1}.$$

so that $g_N(x,y) = \alpha_N(\frac{x}{N},\frac{y}{N})$.

For comparison, the analogous results from applying this method in the 1D case are:

$$f_n^{(N)}(y) = \sum_{k=1}^{N-1} \frac{2g_N(y)sin\left(\frac{k\pi y}{N}\right)sin\left(\frac{k\pi n}{N}\right)}{N\left[1-cos\left(\frac{k\pi}{N}\right)\right]}$$

and

$$g_N(y) = \left[\sum_{k=1}^{N-1} \frac{2sin^2\left(\frac{k\pi y}{N}\right)}{N\left[1-cos\left(\frac{k\pi}{N}\right)\right]}\right]^{-1}.$$

For a derivation of these expressions, see section (4.4).

3.3. **Simplification of Hitting Probabilities.** We begin this section with some motivation for its contents. The formulas for the hitting probabilities derived in the previous section allow us to write an explicit expression for the expected range before hitting the boundary. Indeed, we have that

$$E\left[{R_{(a,b)}}^{(N)}\right] = \sum_{x=1}^{N-1}\sum_{y=1}^{N-1} P_{(a,b)}(T_{(x,y)}<\tau) = \sum_{x=1}^{N-1}\sum_{y=1}^{N-1} f_{(a,b)}(x,y)$$

$$= \sum_{x=1}^{N-1}\sum_{y=1}^{N-1}\sum_{m=1}^{N-1}\sum_{n=1}^{N-1} \frac{4\alpha_N(\frac{x}{N},\frac{y}{N}) sin\left(\frac{n\pi x}{N}\right) sin\left(\frac{m\pi y}{N}\right) sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)}.$$

However, some additional simplification is necessary before the limit of this expression as $N \to \infty$ (after multiplying by $\frac{N^2}{ln(N)}$ and scaling the starting point $(a,b) = (sN, tN)$) can be computed. In particular, we would like to obtain an approximation for $\alpha_N(\frac{x}{N},\frac{y}{N})$ with error terms small enough to disappear in the limit. We were able to obtain such an approximation, but only for (x, y) sufficiently far away from the boundary. As we will see in section (3.4), the region near the boundary for which the approximation may not hold makes no contribution to the limit of the expected range, and can thus be safely ignored.

With this context in mind, we proceed by presenting the desired approximation for α_N as a Lemma:

Lemma 3.1.

$$\lim_{N\to\infty} \sup\left\{\left|\alpha_N(\beta,\gamma)ln(N) - \frac{\pi}{2}\right| \; : \; \frac{\delta}{ln(N)} \le \beta,\gamma \le 1 - \frac{\delta}{ln(N)}\right\} = 0.$$

Proof. The proof of this result can be divided into three steps:

(1) First, $(\alpha_N)^{-1}$ is decomposed into a dominant term of order $ln(N)$ and various error terms depending on an intermediate parameter $\epsilon \in (0, 1)$. Letting E_N denote the sum of the error terms as a function of ϵ, β, γ, we have:

(a) $$\frac{1}{\alpha_N(\beta,\gamma)ln(N)} = \frac{2}{\pi} + \frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}$$

(2) Next, we find a bound for $\frac{E_N}{ln(N)}$ which is uniform in β, γ within the desired region. In particular, we will construct a function $h_N(\epsilon, \delta)$ such that, for any $\frac{\delta}{ln(N)} \le \beta, \gamma \le N - \frac{\delta}{ln(N)}$,

(b) $$\left|\frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}\right| \le h_N(\epsilon,\delta), \qquad \text{and} \qquad \lim_{\epsilon\to 0}\lim_{N\to\infty} h_N(\epsilon,\delta) = 0.$$

(3) Lastly, (a) and (b) are used to derive the desired result.

Steps (1) and (2) involve a rather long sequence of approximations and error bounds (see section (4.1) for details). Step (3) is more straightforward, and will be presented here in full:

We begin by fixing $\frac{\delta}{ln(N)} \le \beta, \gamma \le N - \frac{\delta}{ln(N)}$. From (a), we obtain

$$\left|\frac{1}{\alpha_N(\beta,\gamma)ln(N)} - \frac{2}{\pi}\right| = \left|\frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}\right| \le h_N(\epsilon,\delta).$$

Thus, since α_N is nonnegative,

(c) $$\left|\alpha_N(\beta,\gamma)ln(N) - \frac{\pi}{2}\right| \le \frac{\pi}{2}\alpha_N(\beta,\gamma)ln(N)h_N(\epsilon,\delta).$$

We now require an upper bound for $\alpha_N(\beta,\gamma)ln(N)$. Using (a) again, we have that

$$\frac{1}{\alpha_N(\beta,\gamma)ln(N)} = \frac{2}{\pi} + \frac{E_N(\epsilon,\beta,\gamma)}{ln(N)} \geq \frac{2}{\pi} - \left|\frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}\right|$$

$$\geq \frac{2}{\pi} - h_N(\epsilon,\delta).$$

But since $h_N(\epsilon,\delta) \to 0$ as $N \to \infty$, $\epsilon \to 0$, we may choose N large enough and ϵ small enough (say $\epsilon < \epsilon_0, N > N_0$) that $h_N(\epsilon,\delta) \leq \frac{1}{\pi}$. Then,

$$\frac{1}{\alpha_N(\beta,\gamma)ln(N)} \geq \frac{2}{\pi} - \frac{1}{\pi} = \frac{1}{\pi}$$

so that

$$\alpha_N(\beta,\gamma)ln(N) \leq \pi.$$

Applying this approximation to (c), we obtain:

If $\epsilon < \epsilon_0$, $N > N_0$, and $\frac{\delta}{ln(N)} \leq \beta,\gamma \leq N - \frac{\delta}{ln(N)}$, then

$$\left|\alpha_N(\beta,\gamma)ln(N) - \frac{\pi}{2}\right| \leq \frac{\pi^2}{2}h_N(\epsilon,\delta).$$

Taking the supremum of both sides over β,γ in the desired region,

$$\sup\left\{\left|\alpha_N(\beta,\gamma)ln(N) - \frac{\pi}{2}\right| \ : \ \frac{\delta}{ln(N)} \leq \beta,\gamma \leq 1 - \frac{\delta}{ln(N)}\right\} \leq \frac{\pi^2}{2}h_N(\epsilon,\delta).$$

Taking limits as $N \to \infty$ and $\epsilon \to 0$ (in that order),

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\sup\left\{\left|\alpha_N(\beta,\gamma)ln(N) - \frac{\pi}{2}\right| \ : \ \frac{\delta}{ln(N)} \leq \beta,\gamma \leq 1 - \frac{\delta}{ln(N)}\right\}$$

$$\leq \frac{\pi^2}{2}\lim_{\epsilon\to 0}\lim_{N\to\infty} h_N(\epsilon,\delta) = 0.$$

But the left hand side does not depend on ϵ, so we obtain the desired result:

$$\lim_{N\to\infty}\sup\left\{\left|\alpha_N(\beta,\gamma)ln(N) - \frac{\pi}{2}\right| \ : \ \frac{\delta}{ln(N)} \leq \beta,\gamma \leq 1 - \frac{\delta}{ln(N)}\right\} = 0.$$

□

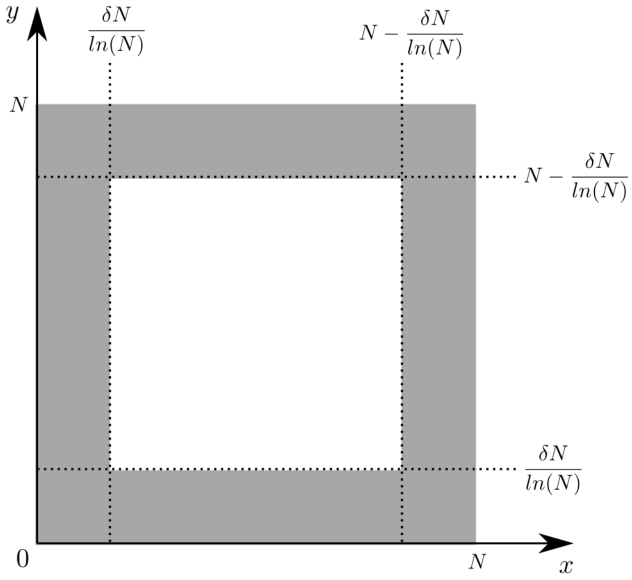

FIGURE 1. A graphical representation of the regions described below.

As we shall see in the next section, this estimate for α_N will greatly simplify the computation of the scaling of the expected range.

3.4. **Computation of the Expected Range.** We begin with the expression for the expected range before hitting the boundary from section (3.1) with starting point $(x, y) = (sN, tN)$:

$$E\left[{R_{(sN,tN)}}^{(N)}\right] = \sum_{x=1}^{N-1}\sum_{y=1}^{N-1} P_{(sN,tN)}(T_{(x,y)} < \tau) = \sum_{x=1}^{N-1}\sum_{y=1}^{N-1} f_{(sN,tN)}(x, y)$$

This sum over x and y may be split up into a region close to the boundary with area small enough to ignore, and a region far enough away from the boundary to apply our above result to simplify α_N (see Figure 1). Let $0 < \delta < \frac{1}{2}$. Then the region close to the boundary is covered (with some overlap on the corners) by:

- $$0 \leq x \leq \frac{\delta N}{ln(N)},\ \ 0 \leq y \leq N$$
- $$0 \leq y \leq \frac{\delta N}{ln(N)},\ \ 0 \leq x \leq N$$
- $$N \geq x \geq N - \frac{\delta N}{ln(N)},\ \ 0 \leq y \leq N$$
- $$N \geq y \geq N - \frac{\delta N}{ln(N)},\ \ 0 \leq x \leq N$$

Note that the area of each of these rectangles is $\delta\frac{N^2}{ln(N)}$. Thus, overbounding $f_{(sN,tN)}$ by 1 (since it represents a probability), the contribution of this region to the scaling of the range $\lim_{N\to\infty}\frac{ln(N)}{N^2}E[R_{(a,b)}]$ is no more than 4δ. δ was chosen arbitrarily, so the error from ignoring this region in the rest of our computations will not affect the result.

For the remaining region (i.e. the interior), we have that

$$\frac{\delta N}{ln(N)} \leq x \leq N - \frac{\delta N}{ln(N)},$$
$$\frac{\delta N}{ln(N)} \leq y \leq N - \frac{\delta N}{ln(N)},$$

and our result for the scaling of α_N (Lemma 3.1) may be applied.

We will now introduce some additional notation. Label the interior region $A_N(\delta)$ and the boundary (shaded in Figure 1) region $B_N(\delta)$.

Define lower and upper thresholds k_l and k_u by

$$k_l = \left\lceil \frac{\delta N}{ln(N)} \right\rceil, \quad k_u = \left\lfloor N - \frac{\delta N}{ln(N)} \right\rfloor.$$

For each $N \in \mathbb{N}$, define a function F_N by

$$F_N(s,t,x,y) = \frac{f_{(sN,tN)}(x,y)}{\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)},$$

so that

$$F_N(s,t,x,y) = \frac{4}{N^2}\sum_{m=1}^{N-1}\sum_{n=1}^{N-1}\frac{sin\left(\frac{n\pi x}{N}\right)sin\left(\frac{m\pi y}{N}\right)sin\left(n\pi s\right)sin\left(m\pi t\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

Then we have that

$$E[R_{(a,b)}] = \sum_{y=1}^{N-1}\sum_{x=1}^{N-1}\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)F_N(s,t,x,y).$$

(Separating the terms with (x,y) in the shaded region):

$$= \sum_{(x,y)\in\mathbb{Z}^2\cap A_N(\delta)}\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)F_N(s,t,x,y) + \sum_{(x,y)\in\mathbb{Z}^2\cap B_N(\delta)}\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)F_N(s,t,x,y)$$

By the area approximation discussed above, the second sum is of order $\frac{\delta N^2}{ln(N)}$. As such, we will focus on the first sum (over (x,y) in the interior), which may be rewritten as:

$$\sum_{(x,y)\in\mathbb{Z}^2\cap A_N(\delta)} \alpha_N\left(\frac{x}{N},\frac{y}{N}\right)F_N(s,t,x,y) = \sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)F_N(s,t,x,y)$$

$$= \frac{\pi}{2ln(N)}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}F_N(s,t,x,y) + \frac{1}{ln(N)}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\left(\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)ln(N)-\frac{\pi}{2}\right)F_N(s,t,x,y).$$

Thus,

$$(\star) \qquad \lim_{N\to\infty}\frac{ln(N)}{N^2}E[R_{(a,b)}] = \lim_{N\to\infty}\frac{\pi}{2}\left[\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}F_N(s,t,x,y)\right]$$

$$+\ \lim_{N\to\infty}\left[\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\left(\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)ln(N)-\frac{\pi}{2}\right)F_N(s,t,x,y)\right]+O(\delta).$$

All that remains is to compute these limits. We proceed by establishing the following Lemma:

Lemma 3.2.

$$\lim_{N\to\infty}\left[\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}F_N(s,t,x,y)\right] = \frac{256}{\pi^4}\sum_{m=1,\,odd}^{\infty}\sum_{n=1,\,odd}^{\infty}\frac{sin\,(n\pi s)\ sin\,(m\pi t)}{nm(n^2+m^2)}.$$

The main idea of the proof of this Lemma will be presented here, while the details can be found in section (4.3). Recall that

$$F_N(s,t,x,y) = \frac{4}{N^2}\sum_{m=1}^{N-1}\sum_{n=1}^{N-1}\frac{sin\left(\frac{n\pi x}{N}\right)sin\left(\frac{m\pi y}{N}\right)sin\,(n\pi s)\,sin\,(m\pi t)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

We begin by exchanging the order of summation, and estimating the denominator of the summand by n^2+m^2, just as in the proof of Lemma 3.1, to obtain:

$$\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}F_N(s,t,x,y) \approx \frac{64}{\pi^2N^2}\sum_{m=1}^{\lfloor\epsilon N\rfloor}\sum_{n=1}^{\lfloor\epsilon N\rfloor}\frac{\left(\sum_{x=k_l}^{k_u}sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u}sin\left(\frac{m\pi y}{N}\right)\right)sin\,(n\pi s)\,sin\,(m\pi t)}{n^2+m^2}.$$

The sums over x and y are then approximated by integrals:

$$\approx \frac{4}{N^2}\left[\frac{16}{\pi^2\left(1+O\left(\epsilon^2\right)\right)}\right]\sum_{m=1}^{\lfloor\epsilon N\rfloor}\sum_{n=1}^{\lfloor\epsilon N\rfloor}\frac{\left(\int_{k_l}^{k_u}sin\left(\frac{n\pi x}{N}\right)dx\right)\left(\int_{k_l}^{k_u}sin\left(\frac{n\pi y}{N}\right)dy\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{n^2+m^2}.$$

But these integrals may be computed exactly:

$$\int_{k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)dx = \frac{-N}{n\pi}cos\left(\frac{n\pi x}{N}\right)\Big|_{k_l}^{k_u} = \frac{N}{n\pi}\left[cos\left(\frac{n\pi k_l}{N}\right) - cos\left(\frac{n\pi k_u}{N}\right)\right]$$

$$= \frac{N}{n\pi}\left[cos\left(\frac{n\pi\delta}{ln(N)}\right) - cos\left(n\pi - \frac{n\pi\delta}{ln(N)}\right)\right]$$

$$= \frac{2N}{n\pi}sin\left(\frac{n\pi}{2}\right)\, sin\left(\frac{n\pi}{2} - \frac{n\pi\delta}{ln(N)}\right).$$

Plugging in these results and applying a Dominated Convergence argument, we obtain:

$$\lim_{N\to\infty}\left(\sum_{m=1}^{\lfloor\epsilon N\rfloor}\sum_{n=1}^{\lfloor\epsilon N\rfloor}\frac{\left[sin\left(\frac{n\pi}{2}\right)\, sin\left(\frac{n\pi}{2} - \frac{n\pi\delta}{ln(N)}\right)\right]\left[sin\left(\frac{m\pi}{2}\right)\, sin\left(\frac{m\pi}{2} - \frac{m\pi\delta}{ln(N)}\right]\right)sin\left(n\pi s\right)sin\left(m\pi t\right)}{nm(n^2+m^2)}\right).$$

$$= \sum_{m=1,\,\mathrm{odd}}^{\infty}\sum_{n=1,\,\mathrm{odd}}^{\infty}\frac{sin\left(n\pi s\right)\, sin\left(m\pi t\right)}{nm(n^2+m^2)}.$$

Reintroducing some constants which were omitted, we are done.

The remaining limit is an error term which can be shown to go to 0 in the limit as $N \to \infty$ as a straightforward consequence of the previous lemmas. As such, it is presented as a Corollary:

Corollary 3.3.

$$\lim_{N\to\infty}\left|\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\left(\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)ln(N) - \frac{\pi}{2}\right)F_N(s,t,x,y)\right| = 0$$

Proof. First, note that

$$\lim_{N\to\infty}\left|\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\left(\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)ln(N) - \frac{\pi}{2}\right)F_N(s,t,x,y)\right|$$

$$\leq \lim_{N\to\infty}\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\left|\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)ln(N) - \frac{\pi}{2}\right|\left|F_N(s,t,x,y)\right|$$

$$\leq \lim_{N\to\infty}\left[\sup_{(\beta,\gamma)\in A_N(\delta)}\left|\alpha_N\left(\frac{x}{N},\frac{y}{N}\right)ln(N) - \frac{\pi}{2}\right|\left(\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\left|F_N(s,t,x,y)\right|\right)\right]$$

$$= \left(\lim_{N\to\infty} \sup_{(\beta,\gamma)\in A_N(\delta)} \left| \alpha_N\left(\frac{x}{N}, \frac{y}{N}\right) ln(N) - \frac{\pi}{2} \right| \right) \left(\lim_{N\to\infty} \frac{1}{N^2} \sum_{y=k_l}^{k_u} \sum_{x=k_l}^{k_u} |F_N(s,t,x,y)| \right)$$

Remember that f_N is strictly nonnegative since it represents a probability. Also, it is clear that α_N is always nonnegative since each term in the sum which defines it cannot be negative.

Since

$$F_N(s,t,x,y) = \frac{f_{(sN,tN)}(x,y)}{\alpha_N(x,y)},$$

we have that F_N is nonnegative. Thus, we may remove the absolute value signs to obtain:

$$\lim_{N\to\infty} \left| \frac{1}{N^2} \sum_{y=k_l}^{k_u} \sum_{x=k_l}^{k_u} \left(\alpha_N\left(\frac{x}{N}, \frac{y}{N}\right) ln(N) - \frac{\pi}{2} \right) F_N(s,t,x,y) \right|$$

$$\leq \left(\lim_{N\to\infty} \sup_{(\beta,\gamma)\in A_N(\delta)} \left| \alpha_N\left(\frac{x}{N}, \frac{y}{N}\right) ln(N) - \frac{\pi}{2} \right| \right) \left(\lim_{N\to\infty} \frac{1}{N^2} \sum_{y=k_l}^{k_u} \sum_{x=k_l}^{k_u} F_N(s,t,x,y) \right)$$

By Lemma 3.1 (restated using the new notation $A_N(\delta)$ for the interior region), the term on the left equals zero. By Lemma 3.2, the sum on the right converges to a constant, and in particular is bounded. Thus, we have that

$$\lim_{N\to\infty} \left| \frac{1}{N^2} \sum_{y=k_l}^{k_u} \sum_{x=k_l}^{k_u} \left(\alpha_N\left(\frac{x}{N}, \frac{y}{N}\right) ln(N) - \frac{\pi}{2} \right) F_N(s,t,x,y) \right| = 0,$$

as desired.

□

Finally, applying Lemma 3.2 and Corollary 3.3 to $(\star)$, we obtain the desired result for the scaling of the expected range:

$$\lim_{N\to\infty} \frac{ln(N)}{N^2} E[R_{(a,b)}] = \frac{128}{\pi^3} \sum_{m=1,\,\text{odd}}^{\infty} \sum_{n=1,\,\text{odd}}^{\infty} \frac{sin\,(n\pi s)\; sin\,(m\pi t)}{nm(n^2+m^2)}.$$

4. Proofs

4.1. **Proof of Lemma 3.1.**

$$\lim_{N\to\infty} \sup \left\{ \left|\alpha_N(\beta,\gamma) ln(N) - \frac{\pi}{2}\right| \;:\; \frac{\delta}{ln(N)} \le \beta, \gamma \le 1 - \frac{\delta}{ln(N)} \right\} = 0.$$

Proof. Recall that we divided the proof of this Lemma into three steps:

1. First, $(\alpha_N)^{-1}$ is decomposed into a dominant term of order $ln(N)$ and various error terms depending on an intermediate parameter $\epsilon \in (0,1)$. Letting E_N denote the sum of the error terms as a function of ϵ, β, γ, we have:

$$\text{(a)} \qquad \frac{1}{\alpha_N(\beta,\gamma) ln(N)} = \frac{2}{\pi} + \frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}$$

2. Next, we find a bound for $\frac{E_N}{ln(N)}$ which is uniform in β, γ within the desired region. In particular, we will construct a function $h_N(\epsilon,\delta)$ such that, for any $\frac{\delta}{ln(N)} \le \beta, \gamma \le N - \frac{\delta}{ln(N)}$,

$$\text{(b)} \qquad \left|\frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}\right| \le h_N(\epsilon,\delta), \qquad \text{and} \qquad \lim_{\epsilon\to 0}\lim_{N\to\infty} h_N(\epsilon,\delta) = 0.$$

3. Lastly, (a) and (b) are used to derive the desired result.

Because step (3) was completed in section (3.3), all that remains are steps (1) and (2). It suffices to construct a function $h_N(\epsilon,\delta)$ satisfying

$$\text{(b)} \qquad \left|\frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}\right| \le h_N(\epsilon,\delta), \qquad \text{and} \qquad \lim_{\epsilon\to 0}\lim_{N\to\infty} h_N(\epsilon,\delta) = 0,$$

where

$$\text{(a)} \qquad \frac{1}{\alpha_N(\beta,\gamma) ln(N)} = \frac{2}{\pi} + \frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}.$$

4.1.1. *Identifying Error Terms.* We proceed by decomposing $(\alpha_N)^{-1}$ into simpler terms and labeling all error terms. At the end of the proof, we will find bounds for each error term, motivating the construction of h_N. Recall that α_N was defined by:

$$\frac{1}{\alpha_N(\beta,\gamma)} = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} \frac{4 sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{N^2\left(1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]\right)}.$$

To begin, take any $0 < \epsilon < 1$. We may split $[\alpha_N(\beta,\gamma)]^{-1}$ into four sums:

Dominant Term:

$$\sum_{m=1}^{\lfloor \epsilon N \rfloor} \sum_{n=1}^{\lfloor \epsilon N \rfloor} \frac{4 sin^2\left(n\pi\beta\right) sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)},$$

Error Term (1a):

$$\sum_{m=\lceil \epsilon N \rceil}^{N} \sum_{n=1}^{\lfloor \epsilon N \rfloor} \frac{4 sin^2\left(n\pi\beta\right) sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)},$$

Error Term (1b):

$$\sum_{m=1}^{\lfloor \epsilon N \rfloor} \sum_{n=\lceil \epsilon N \rceil}^{N} \frac{4 sin^2\left(n\pi\beta\right) sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)},$$

Error Term (1c):

$$\sum_{m=\lceil \epsilon N \rceil}^{N} \sum_{n=\lceil \epsilon N \rceil}^{N} \frac{4 sin^2\left(n\pi\beta\right) sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)}.$$

The dominant term may now be simplified as follows:

For sufficiently small ϵ, each term in the dominant term has $n << N$ and $m << N$. Thus, we may simplify the sum by using only the first several terms in the Taylor Series expansion for $cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)$ as a function of $\frac{n}{N}$ and $\frac{m}{N}$:

$$cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right) = 2-\frac{n^2\pi^2}{2N^2}-\frac{m^2\pi^2}{2N^2}+O\left(\left(\frac{n}{N}\right)^4+\left(\frac{m}{N}\right)^4\right).$$

Note that since $n<\epsilon N$, and $m<\epsilon N$ for any n, m in sum (1),

$$\frac{n^4}{N^4}+\frac{m^4}{N^4} \leq \epsilon^2\left(\frac{n^2}{N^2}+\frac{m^2}{N^2}\right)$$

Thus,

$$O\left(\left(\frac{n}{N}\right)^4+\left(\frac{m}{N}\right)^4\right) = O\left(\frac{\epsilon^2}{N^2}\left(n^2+m^2\right)\right)$$

Then,

$$\sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{4 sin^2\left(n\pi\beta\right) sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)} = \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{4 sin^2\left(n\pi\beta\right) sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[2-\frac{n^2\pi^2}{2N^2}-\frac{m^2\pi^2}{2N^2}+O\left(\frac{\epsilon^2}{N^2}\left(n^2+m^2\right)\right)\right]\right)}$$

$$= \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{4 sin^2\left(n\pi\beta\right) sin^2\left(m\pi\gamma\right)}{N^2\left(\frac{n^2\pi^2}{4N^2}+\frac{m^2\pi^2}{4N^2}+O\left(\frac{\epsilon^2}{N^2}\left(n^2+m^2\right)\right)\right)}$$

$$= \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{4 sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{\frac{\pi^2}{4}(n^2+m^2)(1+O(\epsilon^2))}.$$

$$= \frac{16}{\pi^2(1+O(\epsilon^2))} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{n^2+m^2}.$$

Since $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-x)^k$, we have that

$$\frac{1}{1+O(\epsilon^2)} = 1 - O(\epsilon^2).$$

Thus,

$$\frac{16}{\pi^2(1+O(\epsilon^2))} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{n^2+m^2}$$

$$= \frac{16}{\pi^2} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{(n^2+m^2)} - \left[\frac{16}{\pi^2} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{n^2+m^2}\right] O(\epsilon^2).$$

Label these terms:

Dominant Term:

$$\frac{16}{\pi^2} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{n^2+m^2}$$

Error Term 2:

$$-\left[\frac{16}{\pi^2} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{n^2+m^2}\right] O(\epsilon^2)$$

For clarity, we will not include the factor of $\frac{16}{\pi^2}$ on the dominant term in the following computations. It will be reintroduced later, when necessary.

This sum can be further decomposed using some trig identities:

$$\sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{n^2+m^2}$$

$$= \frac{1}{4} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{1}{n^2+m^2} - \frac{1}{4} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{cos(2\pi n\beta)}{n^2+m^2} - \frac{1}{2} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{sin^2(n\pi\beta)\, cos(2\pi m\gamma)}{n^2+m^2}.$$

Label these terms as follows:

Dominant Term:

$$\frac{1}{4}\sum_{m=1}^{\lceil\epsilon N\rceil}\sum_{n=1}^{\lceil\epsilon N\rceil}\frac{1}{n^2+m^2}$$

Error Term (3a):

$$-\frac{1}{4}\sum_{m=1}^{\lceil\epsilon N\rceil}\sum_{n=1}^{\lceil\epsilon N\rceil}\frac{cos\left(2\pi n\beta\right)}{n^2+m^2}$$

Error Term (3b):

$$-\frac{1}{2}\sum_{m=1}^{\lceil\epsilon N\rceil}\sum_{n=1}^{\lceil\epsilon N\rceil}\frac{sin^2\left(n\pi\beta\right)cos\left(2\pi m\gamma\right)}{n^2+m^2}.$$

We would like to change the sums in the dominant term to integrals. This can be accomplished by first splitting off the first several terms (to avoid complications when n, m are close to 0):

$$\sum_{m=1}^{\lceil\epsilon N\rceil}\sum_{n=1}^{\lceil\epsilon N\rceil}\frac{1}{n^2+m^2}=\sum_{m=2}^{\lceil\epsilon N\rceil}\sum_{n=2}^{\lceil\epsilon N\rceil}\frac{1}{n^2+m^2}+\sum_{n=1}^{\lceil\epsilon N\rceil}\frac{1}{n^2+1}+\sum_{m=2}^{\lceil\epsilon N\rceil}\frac{1}{1+m^2}.$$

Label these terms as follows:

Error Term (4a):

$$\sum_{n=2}^{\lceil\epsilon N\rceil}\frac{1}{n^2+1}+\sum_{m=2}^{\lceil\epsilon N\rceil}\frac{1}{1+m^2}.$$

Once again, we are leaving out a factor of $\frac{1}{4}$ from the previous simplifications. This leaves us (cumulatively) with a coefficient of $\frac{4}{\pi^2}$ on the dominant term.

Now, observe that

$$\sum_{m=2}^{\lceil\epsilon N\rceil}\sum_{n=2}^{\lceil\epsilon N\rceil}\frac{1}{n^2+m^2}=\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{dn\,dm}{\lceil n\rceil^2+\lceil m\rceil^2}$$

$$=\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{dn\,dm}{\lceil n\rceil^2+\lceil m\rceil^2}-\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{dn\,dm}{n^2+m^2}+\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{dn\,dm}{n^2+m^2}$$

$$=\int_1^{\epsilon N}\int_1^{\epsilon N}\left[\frac{1}{\lceil n\rceil^2+\lceil m\rceil^2}-\frac{1}{n^2+m^2}\right]dn\,dm+\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{dn\,dm}{n^2+m^2}.$$

Label these terms as follows:

Error Term (4b):

$$\int_1^{\epsilon N}\int_1^{\epsilon N}\left[\frac{1}{\lceil n\rceil^2+\lceil m\rceil^2}-\frac{1}{n^2+m^2}\right]dn\,dm.$$

We are now left with the dominant term

$$\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{dn\,dm}{n^2+m^2},$$

which can be simplified further by converting to polar coordinates.

First, Define

$$t_N = tan^{-1}\left(\frac{1}{\epsilon N}\right).$$

Then

$$\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{dn\,dm}{n^2+m^2} = 2\int_{t_N}^{\frac{\pi}{4}}\int_{\frac{1}{sin(\theta)}}^{\frac{\epsilon N}{cos(\theta)}}\frac{r\,dr\,d\theta}{r^2}$$

$$= 2\int_{t_N}^{\frac{\pi}{4}}\left[ln(\frac{\epsilon N}{cos(\theta)}) - ln(\frac{1}{sin(\theta)})\right]d\theta$$

$$= 2\int_{t_N}^{\frac{\pi}{4}} ln(\epsilon N)d\theta + 2\int_{t_n}^{\frac{\pi}{4}} ln(sin(\theta))d\theta - 2\int_{t_N}^{\frac{\pi}{4}} ln(cos(\theta))d\theta$$

Label these terms as follows:

Dominant Term:

$$2\int_{t_N}^{\frac{\pi}{4}} ln(\epsilon N)d\theta$$

Error Term (5a):

$$2\int_{t_n}^{\frac{\pi}{4}} ln(sin(\theta))d\theta$$

Error Term (5b):

$$-2\int_{t_N}^{\frac{\pi}{4}} ln(cos(\theta))d\theta.$$

Then observe that

$$2\int_{t_N}^{\frac{\pi}{4}} ln(\epsilon N)d\theta = \frac{\pi}{2}ln(N) - 2tan^{-1}\left(\frac{1}{\epsilon N}\right)ln(N) + \left[\frac{\pi}{2} - 2tan^{-1}\left(\frac{1}{\epsilon N}\right)\right]ln(\epsilon).$$

Label these terms as follows:

Dominant Term:

$$\frac{\pi}{2}ln(N)$$

Error Term (6):

$$-2tan^{-1}\left(\frac{1}{\epsilon N}\right) ln(N) + \left[\frac{\pi}{2} - 2tan^{-1}\left(\frac{1}{\epsilon N}\right)\right] ln(\epsilon).$$

Now is the time to reintroduce the factor of $\frac{4}{\pi^2}$ which we have been omitting. This results in a coefficient of $\frac{2}{\pi}$ for the dominant term.

Thus, we have that

$$\frac{1}{\alpha_N(\beta,\gamma) ln(N)} = \frac{2}{\pi} + \frac{\text{(Sum of all Error Terms listed above)}}{ln(N)}.$$

By comparison with

$$\text{(a)} \qquad \frac{1}{\alpha_N(\beta,\gamma) ln(N)} = \frac{2}{\pi} + \frac{E_N(\epsilon,\beta,\gamma)}{ln(N)},$$

we confirm that $E_N(\epsilon,\beta,\gamma)$ is equal to the sum of the labeled error terms. Recall that our goal is to construct a function $h_N(\epsilon,\delta)$ such that, for any $\frac{\delta}{ln(N)} \leq \beta, \gamma \leq N - \frac{\delta}{ln(N)}$,

$$\text{(b)} \qquad \left|\frac{E_N(\epsilon,\beta,\gamma)}{ln(N)}\right| \leq h_N(\epsilon,\delta), \qquad \text{and} \qquad \lim_{\epsilon\to 0}\lim_{N\to\infty} h_N(\epsilon,\delta) = 0.$$

By the triangle inequality, it suffices to construct a bound for each error term individually. Then, $h_N(\epsilon,\delta)$ may be defined as the sum of the individual bounds. We will see in the following section that every labeled error term except for Error Term (2) is in $o(ln(N)$, guaranteeing that the limit condition in (b) holds. On the other hand, Error Term (2) is of order $\epsilon^2 ln(N)$, and thus vanishes in the limit as $\epsilon \to 0$.

4.1.2. *Bounds for Labeled Error Terms.* We now derive bounds for each labeled error term. The error terms are presented in roughly the same order as they appear above with the exception of Error Term (2), which makes use of the analysis of the other error terms and is thus considered last.

Error Term (1c):

$$\sum_{m=\lceil \epsilon N\rceil}^{N} \sum_{n=\lceil \epsilon N\rceil}^{N} \frac{4sin^2(n\pi\beta)\, sin^2(m\pi\gamma)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)}.$$

First observe that for any $n > \epsilon N$,

$$\frac{n\pi}{N} > \epsilon\pi \implies cos(\epsilon\pi) > cos\left(\frac{n\pi}{N}\right)$$

since cosine is monotone decreasing on the interval $(0,\pi)$.

Thus, for any n, m in this sum,

$$0 < 1 - cos\left(\epsilon\pi\right) < 1 - \frac{1}{2}\left[cos\left(\frac{m\pi}{N}\right) + cos\left(\frac{n\pi}{N}\right)\right] < 2$$

and so

$$\frac{1}{2} \;\le\; \frac{1}{1 - \frac{1}{2}\left[cos\left(\frac{m\pi}{N}\right) + cos\left(\frac{n\pi}{N}\right)\right]} \;<\; \frac{1}{1 - cos\left(\epsilon\pi\right)}.$$

Thus, we have that

$$\sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=\lceil\epsilon N\rceil}^{N}\frac{4sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{N^2\left(1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]\right)} < \sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=\lceil\epsilon N\rceil}^{N}\frac{4sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{N^2\left[1 - cos\left(\epsilon\pi\right)\right]}.$$

Further, since $0 \le sin^2(x) \le 1$ for any $x \in \mathbb{R}$,

$$\sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=\lceil\epsilon N\rceil}^{N}\frac{4sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{N^2\left(1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]\right)} < \sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=\lceil\epsilon N\rceil}^{N}\frac{4}{N^2\left[1 - cos\left(\epsilon\pi\right)\right]}.$$

Simplifying,

$$\sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=\lceil\epsilon N\rceil}^{N}\frac{4}{N^2\left[1 - cos\left(\epsilon\pi\right)\right]} = \frac{4\left(N - \lceil\epsilon N\rceil\right)^2}{N^2\left[1 - cos\left(\epsilon\pi\right)\right]} \le \frac{4\left(1-\epsilon\right)^2}{1 - cos\left(\epsilon\pi\right)}.$$

Thus,

$$|\text{Error Term (1c)}| \;\le\; \frac{4\left(1-\epsilon\right)^2}{1 - cos\left(\epsilon\pi\right)}.$$

Error Term (1a):

$$\sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=1}^{\lfloor\epsilon N\rfloor}\frac{4sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{N^2\left(1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]\right)},$$

For any n, m in sum (2) (i.e. $1 \le n \le \epsilon N$ and $\epsilon N < m \le N$), observe that

$$cos\left(\frac{m\pi}{N}\right) + cos\left(\frac{n\pi}{N}\right) < cos(\epsilon\pi) + 1$$

$$\implies 0 \;<\; \frac{1 - cos(\epsilon\pi)}{2} \;<\; 1 - \frac{1}{2}\left[cos\left(\frac{m\pi}{N}\right) + cos\left(\frac{n\pi}{N}\right)\right]$$

$$\implies 0 \;<\; \frac{1}{1 - \frac{1}{2}\left[cos\left(\frac{m\pi}{N}\right) + cos\left(\frac{n\pi}{N}\right)\right]} \;<\; \frac{2}{1 - cos(\epsilon\pi)}.$$

Using again that $sin^2(x) \le 1$, we have that

$$\sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=1}^{\lfloor\epsilon N\rfloor}\frac{4sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{N^2\left(1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]\right)} \;<\; \sum_{m=\lceil\epsilon N\rceil}^{N}\sum_{n=1}^{\lfloor\epsilon N\rfloor}\frac{8}{N^2\left(1 - cos(\epsilon\pi)\right)}.$$

Simplifying,

$$\sum_{m=\lceil \epsilon N\rceil}^{N}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{8}{N^2\left(1-cos(\epsilon\pi)\right)}=\frac{8\epsilon N(N-\lceil \epsilon N\rceil)}{N^2\left(1-cos(\epsilon\pi)\right)}\leq\frac{8\epsilon(1-\epsilon)}{1-cos(\epsilon\pi)}.$$

Thus,

$$\text{Error Term (1a)}\ \leq\ \frac{8\epsilon(1-\epsilon)}{1-cos(\epsilon\pi)}.$$

Error Term (1b):

$$\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=\lceil \epsilon N\rceil}^{N}\frac{4sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{N^2\left(1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]\right)},$$

By symmetry, we may use the same bound as for Error Term (1a):

$$|\text{Error Term (1b)}|\ \leq\ \frac{8\epsilon(1-\epsilon)}{1-cos(\epsilon\pi)}.$$

Define

$$H_1(\epsilon)=\frac{4\left(1-\epsilon\right)^2}{1-cos\left(\epsilon\pi\right)}+\frac{16\epsilon(1-\epsilon)}{1-cos(\epsilon\pi)}.$$

Then H_1 is a bound for the absolute value of the sum of error terms (1a), (1b), and (1c). As desired, $H_1\in o(ln(N))$, and H_1 does not depend on β, γ.

Error Term (3a):

$$-\frac{1}{4}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{cos\left(2\pi n\beta\right)}{n^2+m^2}$$

Define

$$a_n=\sum_{m=1}^{\lceil \epsilon N\rceil}\frac{1}{n^2+m^2},\quad b_n=cos(2\pi n\beta),\quad B_k=\sum_{n=1}^{k}b_n.$$

Then

$$\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{cos\left(2\pi n\beta\right)}{n^2+m^2}=\sum_{n=1}^{\lceil \epsilon N\rceil}a_nb_n.$$

We begin by splitting the sum over n into two parts. (For now, some floor and ceiling functions will be omitted)

$$\sum_{n=1}^{\lceil \epsilon N\rceil}\sum_{m=1}^{\lceil \epsilon N\rceil}\frac{cos\left(2\pi n\beta\right)}{n^2+m^2}=\sum_{n=1}^{\epsilon ln(N)}\sum_{m=1}^{\lceil \epsilon N\rceil}\frac{cos\left(2\pi n\beta\right)}{n^2+m^2}+\sum_{n=\epsilon ln(N)}^{\epsilon N}\sum_{m=1}^{\lceil \epsilon N\rceil}\frac{cos\left(2\pi n\beta\right)}{n^2+m^2}.$$

Observe that

$$\left|\sum_{n=1}^{\epsilon ln(N)}\sum_{m=1}^{\lceil \epsilon N\rceil}\frac{cos\left(2\pi n\beta\right)}{n^2+m^2}\right| \leq \sum_{n=1}^{\epsilon ln(N)}\sum_{m=1}^{\lceil \epsilon N\rceil}\frac{1}{n^2+m^2}$$

$$=\sum_{n=1}^{\epsilon ln(N)}\int_0^{\epsilon N}\frac{dm}{n^2+\lceil m\rceil^2} \leq \sum_{n=1}^{\epsilon ln(N)}\int_0^{\epsilon N}\frac{dm}{n^2+m^2}$$

$$\leq \frac{\pi}{2}\sum_{n=1}^{\epsilon ln(N)}\frac{1}{n}$$

$$\leq \frac{\pi}{2}+\frac{\pi}{2}\int_1^{\epsilon \ln(N)}\frac{dn}{n}=\frac{\pi}{2}+\frac{\pi}{2}ln(\epsilon ln(N))$$

$$=\frac{\pi}{2}(1+ln(\epsilon))+\frac{\pi}{2}ln(ln(N))$$

We also have, by summation by parts, that

$$\sum_{n=1}^{\lceil \epsilon N\rceil}a_nb_n=a_{\epsilon N+1}\,B_{\epsilon N}+\sum_{n=1}^{\epsilon N}B_n\left(a_n-a_{n+1}\right),$$

$$\sum_{n=1}^{\lceil \epsilon ln(N)\rceil}a_nb_n=a_{\epsilon ln(N)+1}\,B_{\epsilon ln(N)}+\sum_{n=1}^{\epsilon ln(N)}B_n\left(a_n-a_{n+1}\right).$$

Thus,

$$\left|\sum_{n=\epsilon ln(N)}^{\lceil \epsilon N\rceil}a_nb_n\right|=\left|a_{\epsilon N+1}\,B_{\epsilon N}-a_{\epsilon ln(N)+1}\,B_{\epsilon ln(N)}+\sum_{n=\epsilon ln(N)}^{\epsilon N}B_n\left(a_n-a_{n+1}\right)\right|,$$

$$\leq \left|a_{\epsilon N+1}\,B_{\epsilon N}\right|+\left|a_{\epsilon ln(N)+1}\,B_{\epsilon ln(N)}\right|+\sum_{n=\epsilon ln(N)}^{\epsilon N}\left|B_n\right|\left(a_n-a_{n+1}\right),$$

Now, observe that for any $K\in\mathbb{N}$,

$$\left|B_K\right|=\left|\sum_{n=1}^{K}cos\left(2\pi n\beta\right)\right|=\left|\frac{1}{2}\sum_{n=1}^{K}\left[e^{i\theta n}+e^{-i\theta n}\right]\right|$$

$$=\frac{1}{2}\left|\sum_{n=1}^{K}e^{i\theta n}+\sum_{n=1}^{K}e^{-i\theta n}\right|.$$

These are partial sums of geometric series (with ratios on the complex unit circle).

$$=\frac{1}{2}\left|\frac{e^{i\theta}(1-e^{i\theta K})}{1-e^{i\theta}}+\frac{e^{-i\theta}(1-e^{-i\theta K})}{1-e^{-i\theta}}\right|$$

Upon further simplification, we obtain the following bound:

$$\leq \frac{1}{2} + \left| \frac{1}{2\,sin\,(\pi\beta)} \right|.$$

Since $\frac{\delta}{ln(N)} \leq \beta \leq 1 - \frac{\delta}{ln(N)}$, we have that

$$|sin(\pi\beta)| \geq sin\left(\pi \frac{\delta}{ln(N)}\right) = sin\left(\pi \left[1 - \frac{\delta}{ln(N)}\right]\right).$$

Applying the approximation $sin(\pi x) \geq 2x$ for $0 \leq x \leq \frac{\pi}{2}$, we have that

$$sin\left(\pi \frac{\delta}{ln(N)}\right) \geq \frac{2\delta}{ln(N)}.$$

Combining these results, we obtain:

$$|B_k| \leq \frac{1}{2} + \left| \frac{1}{2\,sin\,(\pi\beta)} \right| \leq \frac{1}{2} + \frac{1}{2sin\left(\pi \frac{\delta}{ln(N)}\right)} \leq \frac{1}{2} + \frac{ln(N)}{4\delta}.$$

Thus, we have that

$$\left| \sum_{n=\epsilon ln(N)}^{\lceil \epsilon N \rceil} a_n b_n \right| \leq |a_{\epsilon N+1}\,B_{\epsilon N}| + \left|a_{\epsilon ln(N)+1}\,B_{\epsilon ln(N)}\right| + \left[\frac{1}{2} + \frac{ln(N)}{4\delta}\right] \sum_{n=\epsilon ln(N)}^{\epsilon N} (a_n - a_{n+1}).$$

This sum is telescoping:

$$= |a_{\epsilon N+1}\,B_{\epsilon N}| + \left|a_{\epsilon ln(N)+1}\,B_{\epsilon ln(N)}\right| + \left[\frac{1}{2} + \frac{ln(N)}{4\delta}\right] (a_{\epsilon ln(N)} - a_{\epsilon N}).$$

$$\leq \left[\frac{1}{2} + \frac{ln(N)}{4\delta}\right] \left[a_{\epsilon N+1} + a_{\epsilon ln(N)+1} + a_{\epsilon N} + a_{\epsilon ln(N)}\right]$$

$\{a_n\}$ is decreasing:

$$\leq 4\left[\frac{1}{2} + \frac{ln(N)}{4\delta}\right] a_{\epsilon ln(N)}.$$

Now, observe that

$$a_{\epsilon ln(N)} = \sum_{m=1}^{\epsilon N} \frac{1}{(\epsilon ln(N))^2 + m^2} = \int_0^{\epsilon N} \frac{dm}{(\epsilon ln(N))^2 + \lceil m \rceil^2}$$

$$\leq \int_0^{\epsilon N} \frac{dm}{(\epsilon ln(N))^2 + m^2}$$

$$\leq \frac{\pi}{2\epsilon\, ln(N)}.$$

So

$$\left| \sum_{n=\epsilon ln(N)}^{\lceil \epsilon N \rceil} a_n b_n \right| \leq 4\left[\frac{1}{2} + \frac{ln(N)}{4\delta}\right] \left(\frac{\pi}{2\epsilon\, ln(N)}\right)$$

$$= \frac{\pi}{\epsilon ln(N)} + \frac{\pi}{2\epsilon\delta}.$$

Combining above results, we have shown that

$$|\text{Error Term (3a)}| \;\le\; \frac{\pi}{2}(1+ln(\epsilon)) + \frac{\pi}{2\epsilon\delta} + \frac{\pi}{2}ln(ln(N)) + \frac{\pi}{\epsilon ln(N)}.$$

Error Term (3b):

$$-\frac{1}{2}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{sin^2(n\pi\beta)\,cos\,(2\pi m\gamma)}{n^2+m^2}.$$

Note that if in the computation of Error Term (3a) we exchange the roles of m, n and β, γ (allowed by symmetry) and modify our definition of a_n in the following way:

$$a_m^* = \sum_{n=1}^{\epsilon N}\frac{sin^2(n\pi\beta)}{n^2+m^2},$$

Then this new sequence $\{a_m^*\}$ is still decreasing in m, and is clearly bounded above by the original sequence $\{a_m\}$ (without the sine squared in the numerator). Thus, the above argument also holds for this sum, and

$$\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{sin^2(n\pi\beta)\,cos\,(2\pi m\gamma)}{n^2+m^2}$$
$$\le \frac{\pi}{2}(1+ln(\epsilon)) + \frac{\pi}{2\epsilon\delta} + \frac{\pi}{2}ln(ln(N)) + \frac{\pi}{\epsilon ln(N)}.$$

So

$$|\text{Error Term (3b)}| \;\le\; \frac{\pi}{2}(1+ln(\epsilon)) + \frac{\pi}{2\epsilon\delta} + \frac{\pi}{2}ln(ln(N)) + \frac{\pi}{\epsilon ln(N)}.$$

Define

$$H_3(N,\epsilon,\delta) = \pi(1+ln(\epsilon)) + \frac{\pi}{\epsilon\delta} + \pi ln(ln(N)) + \frac{2\pi}{\epsilon ln(N)}.$$

Then H_3 is a bound for the absolute value of the sum of error terms (3a) and (3b). As desired, $H_3 \in o(ln(N))$, and H_3 does not depend on β, γ.

Error Term (4a):

$$\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{1}{n^2+1} + \sum_{m=2}^{\lceil \epsilon N\rceil}\frac{1}{1+m^2}.$$
$$\le 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3}.$$

Thus,

$$|\text{Error Term (4a)}| \;\le\; \frac{\pi^2}{3}.$$

Error Term (4b):

$$\int_1^{\epsilon N}\int_1^{\epsilon N}\left[\frac{1}{\lceil n\rceil^2+\lceil m\rceil^2}-\frac{1}{n^2+m^2}\right]dn\,dm.$$

Observe that for any n, m positive real numbers greater than 1,

$$n \le \lceil n\rceil \le n+1$$

$$\implies n^2 \le \lceil n\rceil^2 \le (n+1)^2$$

(and similarly for m)

$$\implies n^2+m^2 \le \lceil n\rceil^2+\lceil m\rceil^2 \le (n+1)^2+(m+1)^2$$

$$\implies \frac{1}{(n+1)^2+(m+1)^2} \le \frac{1}{\lceil n\rceil^2+\lceil m\rceil^2} \le \frac{1}{n^2+m^2}$$

$$\implies \frac{1}{(n+1)^2+(m+1)^2}-\frac{1}{n^2+m^2} \le \frac{1}{\lceil n\rceil^2+\lceil m\rceil^2}-\frac{1}{n^2+m^2} \le 0$$

$$\implies \left|\frac{1}{\lceil n\rceil^2+\lceil m\rceil^2}-\frac{1}{n^2+m^2}\right| \le \frac{1}{n^2+m^2}-\frac{1}{(n+1)^2+(m+1)^2}$$

$$= \frac{(n+1)^2+(m+1)^2-n^2-m^2}{(n^2+m^2)\left[(n+1)^2+(m+1)^2\right]}$$

$$\le \frac{2n+2m+2}{(n^2+m^2)^2} = \frac{2n}{(n^2+m^2)^2}+\frac{2m}{(n^2+m^2)^2}+\frac{2}{(n^2+m^2)^2}$$

So

$$\left|\int_1^{\epsilon N}\int_1^{\epsilon N}\left[\frac{1}{\lceil n\rceil^2+\lceil m\rceil^2}-\frac{1}{n^2+m^2}\right]dn\,dm\right|$$

$$\le \int_1^{\epsilon N}\int_1^{\epsilon N}\frac{2n\,dn\,dm}{(n^2+m^2)^2}+\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{2m\,dm\,dn}{(n^2+m^2)^2}+\int_1^{\epsilon N}\int_1^{\epsilon N}\frac{2\,dn\,dm}{(n^2+m^2)^2}$$

Converting integrals to polar form and covering square with a quarter annulus:

$$\le \int_0^{\frac{\pi}{2}}\int_{\sqrt{2}}^{\sqrt{2}\epsilon N}\frac{2r^2\,cos(\theta)\,dr\,d\theta}{r^4}+\int_0^{\frac{\pi}{2}}\int_{\sqrt{2}}^{\sqrt{2}\epsilon N}\frac{2r^2\,sin(\theta)\,dr\,d\theta}{r^4}+\int_0^{\frac{\pi}{2}}\int_{\sqrt{2}}^{\sqrt{2}\epsilon N}\frac{2r\ dr\,d\theta}{r^4}$$

$$\le \int_{\sqrt{2}}^{\sqrt{2}\epsilon N}\frac{\pi\,dr}{r^2}+\int_{\sqrt{2}}^{\sqrt{2}\epsilon N}\frac{\pi\,dr}{r^2}+\int_{\sqrt{2}}^{\sqrt{2}\epsilon N}\frac{\pi\,dr\,d\theta}{r^3}$$

$$= 2\pi\left[\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}\epsilon N}\right]+\pi\left[\frac{1}{2}-\frac{1}{2\epsilon^2N^2}\right]$$

$$= \sqrt{2}\pi+\frac{\pi}{2}-\frac{\sqrt{2}\pi}{\epsilon N}-\frac{\pi}{2\epsilon^2N^2}.$$

Thus,

$$|\text{Error Term (4b)}| \ \le\ \sqrt{2}\pi+\frac{\pi}{2}-\frac{\sqrt{2}\pi}{\epsilon N}-\frac{\pi}{2\epsilon^2N^2}.$$

Define

$$H_4(\epsilon, N) = \frac{\pi}{3} + \sqrt{2}\pi + \frac{\pi}{2} - \frac{\sqrt{2}\pi}{\epsilon N} - \frac{\pi}{2\epsilon^2 N^2}.$$

Then H_4 is a bound for the absolute value of the sum of error terms (4a) and (4b). As desired, $H_4 \in o(ln(N))$, and H_4 does not depend on β, γ.

Error Term (5a):

$$2\int_{t_n}^{\frac{\pi}{4}} ln(sin(\theta))d\theta$$

Recall that

$$t_N = tan^{-1}\left(\frac{1}{\epsilon N}\right).$$

Note that t_N is a decreasing sequence, and in particular

$$\lim_{N\to\infty} t_N = 0.$$

So

$$\left|\int_{t_n}^{\frac{\pi}{4}} ln(sin(\theta))d\theta\right| \leq \left|\int_{0}^{\frac{\pi}{4}} ln(sin(\theta))d\theta\right|$$

and

$$\left|\int_{t_n}^{\frac{\pi}{4}} ln(cos(\theta))d\theta\right| \leq \left|\int_{0}^{\frac{\pi}{4}} ln(cos(\theta))d\theta\right|$$

since $ln(sin(\theta))$ and $ln(cos(\theta))$ are both non-positive for any $\theta \in [0, \frac{\pi}{4}]$.

Also, note that for any $\theta \in [0, \frac{\pi}{4}]$,

$$sin(\theta) \geq \frac{2\sqrt{2}}{\pi}\theta$$

since $sin(0) = 0$, $sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, and sin is concave down for $\theta \in [0, \pi]$.

$$\implies 0 \geq ln(sin(\theta)) \geq ln(\theta) + ln\left(\frac{2\sqrt{2}}{\pi}\right)$$

since ln is increasing on $[0, \infty]$.

$$\implies \left|\int_{0}^{\frac{\pi}{4}} ln(sin(\theta))d\theta\right| \leq \left|\int_{0}^{\frac{\pi}{4}} ln(\theta)d\theta\right| + \frac{\pi}{4}ln\left(\frac{2\sqrt{2}}{\pi}\right)$$

$$= \lim_{t\to 0}\left|\theta ln(\theta) - \theta\right|_{t}^{\frac{\pi}{4}} + \frac{\pi}{4}ln\left(\frac{2\sqrt{2}}{\pi}\right)$$

$$= \left|\frac{\pi}{4}ln(\frac{\pi}{4}) - \frac{\pi}{4}\right| + \frac{\pi}{4}ln\left(\frac{2\sqrt{2}}{\pi}\right).$$

Thus,

$$|\text{Error Term (5a)}| \leq \left|\frac{\pi}{4}ln(\frac{\pi}{4}) - \frac{\pi}{4}\right| + \frac{\pi}{4}ln\left(\frac{2\sqrt{2}}{\pi}\right).$$

Error Term (5b):

$$-2\int_{t_N}^{\frac{\pi}{4}} ln(cos(\theta))d\theta.$$

Similarly, for any $\theta \in [0, \frac{\pi}{4}]$, we have that

$$1 \geq cos(\theta) \geq \frac{\sqrt{2}}{2}$$

$$\implies 0 \geq ln(cos(\theta)) \geq ln\left(\frac{\sqrt{2}}{2}\right)$$

$$\implies \left|\int_0^{\frac{\pi}{4}} ln(cos(\theta))d\theta\right| \leq \left|\frac{\pi}{4}ln\left(\frac{\sqrt{2}}{2}\right)\right|$$

Thus,

$$|\text{Error Term (5b)}| \leq \left|\frac{\pi}{4}ln\left(\frac{\sqrt{2}}{2}\right)\right|.$$

Define

$$H_5 = \left|\frac{\pi}{4}ln(\frac{\pi}{4}) - \frac{\pi}{4}\right| + \frac{\pi}{4}ln\left(\frac{2\sqrt{2}}{\pi}\right) + \left|\frac{\pi}{4}ln\left(\frac{\sqrt{2}}{2}\right)\right|.$$

Then H_5 is a bound for the absolute value of the sum of error terms (5a) and (5b). As desired, $H_5 \in o(ln(N))$, and H_5 does not depend on β, γ.

Error Term (6):

$$-2tan^{-1}\left(\frac{1}{\epsilon N}\right)ln(N) + \left[\frac{\pi}{2} - 2tan^{-1}\left(\frac{1}{\epsilon N}\right)\right]ln(\epsilon).$$

Note that

$$\lim_{N\to\infty} tan^{-1}\left(\frac{1}{\epsilon N}\right) = 0,$$

so Error Term (6) is in o(ln(N)).

Define

$$H_6(\epsilon, N) = \left|-2tan^{-1}\left(\frac{1}{\epsilon N}\right)ln(N) + \left[\frac{\pi}{2} - 2tan^{-1}\left(\frac{1}{\epsilon N}\right)\right]ln(\epsilon)\right|.$$

Then H_6 is a bound for (and is, in fact, equal to) the absolute value of error term (6). As desired, $H_6 \in o(ln(N))$, and H_6 does not depend on β, γ.

Error Term (2):

$$-\left[\frac{16}{\pi^2}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{n^2+m^2}\right]O(\epsilon^2)$$

First, we fix a constant $M \in \mathbb{N}$ to quantify the scaling of the $O(\epsilon^2)$ factor, so that this error term may be bounded in absolute value by

$$\frac{16\,M\,\epsilon^2}{\pi^2}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{sin^2\left(n\pi\beta\right)sin^2\left(m\pi\gamma\right)}{n^2+m^2}$$

$$\leq \frac{16\,M\,\epsilon^2}{\pi^2}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{1}{n^2+m^2}.$$

Now, note that this sum differs from a dominant term in the previous section only by a factor of $M\epsilon^2$. Because all of the above error terms are in $o(ln(N)$, equation (a) implies that

$$\lim_{N\to\infty}\frac{16}{\pi^2\,ln(N)}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{1}{n^2+m^2}=\frac{2}{\pi}.$$

Thus, we have

$$\lim_{\epsilon\to 0}\left[\lim_{N\to\infty}\frac{16\,M\,\epsilon^2}{\pi^2\,ln(N)}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{1}{n^2+m^2}\right]=\frac{2M}{\pi}\lim_{\epsilon\to 0}\epsilon^2=0.$$

Define

$$H_2(N,\epsilon)=\frac{16\,M\,\epsilon^2}{\pi^2}\sum_{m=1}^{\lceil \epsilon N\rceil}\sum_{n=1}^{\lceil \epsilon N\rceil}\frac{1}{n^2+m^2}.$$

Then H_2 is a bound for the absolute value of Error Term (2). As desired, H_2 is independent of β, γ, and

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{H_2}{ln(N)}=0.$$

4.1.3. *Combining Error Bounds.* We begin by listing all of the error bounds derived above:

(1) $$H_1(\epsilon) = \frac{4\,(1-\epsilon)^2}{1 - cos\,(\epsilon\pi)} + \frac{16\epsilon(1-\epsilon)}{1 - cos(\epsilon\pi)}$$

(2) $$H_2(N,\epsilon) = \frac{16\,M\,\epsilon^2}{\pi^2} \sum_{m=1}^{\lceil \epsilon N \rceil} \sum_{n=1}^{\lceil \epsilon N \rceil} \frac{1}{n^2+m^2}$$

(3) $$H_3(N,\epsilon,\delta) = \pi(1 + ln(\epsilon)) + \frac{\pi}{\epsilon\delta} + \pi ln(ln(N)) + \frac{2\pi}{\epsilon ln(N)}$$

(4) $$H_4(N,\epsilon) = \frac{\pi}{3} + \sqrt{2}\pi + \frac{\pi}{2} - \frac{\sqrt{2}\pi}{\epsilon N} - \frac{\pi}{2\epsilon^2 N^2}$$

(5) $$H_5 = \left|\frac{\pi}{4} ln(\frac{\pi}{4}) - \frac{\pi}{4}\right| + \frac{\pi}{4} ln\left(\frac{2\sqrt{2}}{\pi}\right) + \left|\frac{\pi}{4} ln\left(\frac{\sqrt{2}}{2}\right)\right|$$

(6) $$H_6(N,\epsilon) = \left|-2tan^{-1}\left(\frac{1}{\epsilon N}\right) ln(N) + \left[\frac{\pi}{2} - 2tan^{-1}\left(\frac{1}{\epsilon N}\right)\right] ln(\epsilon)\right|$$

Now, define

$$h_N(\epsilon,\delta) = \frac{1}{ln(N)} \sum_{k=1}^{6} H_k.$$

Since h_N was constructed to satisfy both conditions in (b), we are done.

□

4.2. **Proof of Lemma 3.2.**

$$\lim_{N\to\infty} \left[\frac{1}{N^2} \sum_{y=k_l}^{k_u} \sum_{x=k_l}^{k_u} F_N(s,t,x,y)\right] = \frac{256}{\pi^4} \sum_{m=1,\,\text{odd}}^{\infty} \sum_{n=1,\,\text{odd}}^{\infty} \frac{sin\,(n\pi s)\ sin\,(m\pi t)}{nm(n^2+m^2)}.$$

Proof.

$$\frac{1}{N^2} \sum_{x=1}^{N} \sum_{y=1}^{N} F_N(a,b,x,y).$$

$$= \frac{4}{N^4} \sum_{y=k_l}^{k_u} \sum_{x=k_l}^{k_u} \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} \frac{sin\left(\frac{n\pi x}{N}\right) sin\left(\frac{m\pi y}{N}\right) sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)}{\left(1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]\right)}$$

To begin, pick an arbitrary $0 < \epsilon \leq 1$ and split the sum into four parts, just as in the simplification of α_N:

(1)
$$\frac{4}{N^4} \sum_{y=k_l}^{k_u} \sum_{x=k_l}^{k_u} \sum_{m=1}^{\lfloor \epsilon N \rfloor} \sum_{n=1}^{\lfloor \epsilon N \rfloor} \frac{sin\left(\frac{n\pi x}{N}\right) sin\left(\frac{m\pi y}{N}\right) sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)}{1 - \frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right) + cos\left(\frac{m\pi}{N}\right)\right]}.$$

(2)

$$\frac{4}{N^4}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\sum_{m=\lceil \epsilon N\rceil}^{N}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{sin\left(\frac{n\pi x}{N}\right)sin\left(\frac{m\pi y}{N}\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

(3)

$$\frac{4}{N^4}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=\lceil \epsilon N\rceil}^{N}\frac{sin\left(\frac{n\pi x}{N}\right)sin\left(\frac{m\pi y}{N}\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

(4)

$$\frac{4}{N^4}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}\sum_{m=\lceil \epsilon N\rceil}^{N}\sum_{n=\lceil \epsilon N\rceil}^{N}\frac{sin\left(\frac{n\pi x}{N}\right)sin\left(\frac{m\pi y}{N}\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

By exchanging the order of summation, we may rewrite the above sums in the following way:

(1)

$$\frac{4}{N^4}\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{\left(\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u} sin\left(\frac{m\pi y}{N}\right)\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

(2)

$$\frac{4}{N^4}\sum_{m=\lceil \epsilon N\rceil}^{N}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{\left(\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u} sin\left(\frac{m\pi y}{N}\right)\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

(3)

$$\frac{4}{N^4}\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=\lceil \epsilon N\rceil}^{N}\frac{\left(\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u} sin\left(\frac{m\pi y}{N}\right)\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

(4)

$$\frac{4}{N^4}\sum_{m=\lceil \epsilon N\rceil}^{N}\sum_{n=\lceil \epsilon N\rceil}^{N}\frac{\left(\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u} sin\left(\frac{m\pi y}{N}\right)\right)sin\left(\frac{n\pi a}{N}\right)sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

We will proceed by showing that the latter three of these sums go to zero in the limit as $N \to \infty$.

Observe that for any $1 \leq n \leq N$, $R \in \mathbb{N}$, letting $\theta = \frac{n\pi}{N}$ for clarity,

$$\left|\sum_{x=1}^{R} sin\left(\frac{n\pi x}{N}\right)\right| = \left|\sum_{x=1}^{R}\frac{e^{i\theta x}-e^{-i\theta x}}{2i}\right|$$

$$\leq \frac{1}{2}\left|\sum_{x=1}^{R}\left[e^{i\theta}\right]^x\right| + \frac{1}{2}\left|\sum_{x=1}^{R}\left[e^{-i\theta}\right]^x\right|$$

$$= \left|\frac{e^{i\theta}}{2}\left(\frac{1-e^{i\theta R}}{1-e^{i\theta}}\right)\right| + \left|\frac{e^{-i\theta}}{2}\left(\frac{1-e^{-i\theta R}}{1-e^{-i\theta}}\right)\right|$$

$$\leq \left|\frac{1-e^{i\theta R}}{1-cos(\theta)-isin(\theta)}\right| + \left|\frac{1-e^{-i\theta R}}{1-cos(\theta)+isin(\theta)}\right|$$

$$\leq 2\left|\frac{1-cos(\theta)+isin(\theta)}{(1-cos(\theta))^2+sin^2(\theta)}\right| + 2\left|\frac{1+cos(\theta)-isin(\theta)}{(1-cos(\theta))^2+sin^2(\theta)}\right|$$

$$\leq 12\left|\frac{1}{2-2cos(\theta)}\right| = \frac{6}{1-cos\left(\frac{n\pi}{N}\right)}.$$

If $\epsilon N \leq n \leq N$, then $\epsilon\pi \leq \frac{n\pi}{N} \leq \pi$, and so

$$cos\left(\frac{n\pi}{N}\right) \leq cos(\epsilon\pi)$$

$$\implies 1-cos\left(\frac{n\pi}{N}\right) \geq 1-cos(\epsilon\pi)$$

$$\implies \frac{1}{1-cos\left(\frac{n\pi}{N}\right)} \leq \frac{1}{1-cos(\epsilon\pi)}.$$

Thus, we have that

$$\left|\sum_{x=1}^{R} sin\left(\frac{n\pi x}{N}\right)\right| \leq \frac{6}{1-cos(\epsilon\pi)}.$$

In particular,

$$\left|\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right| \leq 2\left|\sum_{x=1}^{k_l} sin\left(\frac{n\pi x}{N}\right)\right| + \left|\sum_{x=1}^{N} sin\left(\frac{n\pi x}{N}\right)\right|$$

$$\leq \frac{18}{1-cos(\epsilon\pi)}.$$

Recall, from our computation of α_N, the following bound for the denominator of sums (2), (3), and (4):

$$\frac{1}{1-\frac{1}{2}\left[cos\left(\frac{m\pi}{N}\right)+cos\left(\frac{n\pi}{N}\right)\right]} < \frac{2}{1-cos(\epsilon\pi)}.$$

Combining these bounds, we have that for any n, m in sums (2), (3), (4),

$$\left| \frac{\left(\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u} sin\left(\frac{m\pi y}{N}\right)\right) sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]} \right|$$

$$\leq N\left(\frac{2}{1-cos(\epsilon\pi)}\right)\left(\frac{18}{1-cos(\epsilon\pi)}\right)$$

It follows that sums (2), (3), and (4) are each bounded in absolute value by

$$\frac{1}{N}\left(\frac{2}{1-cos(\epsilon\pi)}\right)\left(\frac{18}{1-cos(\epsilon\pi)}\right).$$

Thus, the sums go to zero in the limit as $N \to \infty$.

This leaves us with sum (1), which can be simplified just as in the computation for α:

$$\frac{4}{N^4}\sum_{m=1}^{\lfloor \epsilon N \rfloor}\sum_{n=1}^{\lfloor \epsilon N \rfloor} \frac{\left(\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u} sin\left(\frac{m\pi y}{N}\right)\right) sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)}{1-\frac{1}{2}\left[cos\left(\frac{n\pi}{N}\right)+cos\left(\frac{m\pi}{N}\right)\right]}.$$

$$= \frac{4}{N^2}\left[\frac{16}{\pi^2\left(1+O\left(\epsilon^2\right)\right)}\right]\sum_{m=1}^{\lfloor \epsilon N \rfloor}\sum_{n=1}^{\lfloor \epsilon N \rfloor} \frac{\left(\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)\right)\left(\sum_{y=k_l}^{k_u} sin\left(\frac{m\pi y}{N}\right)\right) sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)}{n^2+m^2}.$$

To simplify this sum, first note that by the definition of the ceiling function,

$$\sum_{x=k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right) = \int_{k_l}^{k_u} sin\left(\frac{n\pi\lceil x\rceil}{N}\right)dx = \int_{k_l}^{k_u}\left[sin\left(\frac{n\pi\lceil x\rceil}{N}\right) - sin\left(\frac{n\pi x}{N}\right)\right]dx + \int_{k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)dx.$$

Now observe that, since sine satisfies the Lipschitz condition,

$$\begin{aligned}\left|sin\left(\frac{n\pi\lceil x\rceil}{N}\right) - sin\left(\frac{n\pi x}{N}\right)\right| &\leq \frac{n\pi}{N}\left|\lceil x\rceil - x\right| \\ &\leq \frac{n\pi}{N}\end{aligned}$$

.

Since $1 \leq k_l \leq k_u \leq N$, this implies (by monotonicity of integral):

$$\int_{k_l}^{k_u}\left|sin\left(\frac{n\pi\lceil x\rceil}{N}\right) - sin\left(\frac{n\pi x}{N}\right)\right|dx \leq \int_0^N \left|sin\left(\frac{n\pi\lceil x\rceil}{N}\right) - sin\left(\frac{n\pi x}{N}\right)\right|dx \leq n\pi.$$

Thus, when we replace the sums over x and y by integrals, the error term is bounded in absolute value by

$$\frac{1}{N^2}\left[\frac{64}{(1+O(\epsilon^2))}\right]\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{nm}{n^2+m^2}$$

But $nm \leq n^2+m^2$, so $\frac{nm}{n^2+m^2} \leq 1$ and

$$\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{nm}{n^2+m^2} \leq \epsilon^2 N^2$$

$$\Longrightarrow \frac{1}{N^2}\left[\frac{64}{(1+O(\epsilon^2))}\right]\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{nm}{n^2+m^2} \leq \frac{64\epsilon^2}{(1+O(\epsilon^2))}.$$

In the end of our computation, after taking the limit as $N \to \infty$, we shall let $\epsilon \to 0$, and so this term will have no contribution to the result.

This justifies our replacement of the sums over x and y with integrals.

$$\frac{4}{N^2}\left[\frac{16}{\pi^2(1+O(\epsilon^2))}\right]\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{\left(\int_{k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)dx\right)\left(\int_{k_l}^{k_u} sin\left(\frac{n\pi y}{N}\right)dy\right) sin\left(\frac{n\pi a}{N}\right) sin\left(\frac{m\pi b}{N}\right)}{n^2+m^2}.$$

But these integrals may be computed exactly:

$$\int_{k_l}^{k_u} sin\left(\frac{n\pi x}{N}\right)dx = \frac{-N}{n\pi}cos\left(\frac{n\pi x}{N}\right)\Bigg|_{k_l}^{k_u} = \frac{N}{n\pi}\left[cos\left(\frac{n\pi k_l}{N}\right) - cos\left(\frac{n\pi k_u}{N}\right)\right]$$

$$= \frac{N}{n\pi}\left[cos\left(\frac{n\pi\delta}{ln(N)}\right) - cos\left(n\pi - \frac{n\pi\delta}{ln(N)}\right)\right]$$

$$= \frac{2N}{n\pi}sin\left(\frac{n\pi}{2}\right)\, sin\left(\frac{n\pi}{2} - \frac{n\pi\delta}{ln(N)}\right).$$

Applying this result to the sum, we obtain:

$$\left[\frac{256}{\pi^4(1+O(\epsilon^2))}\right]\sum_{m=1}^{\lfloor \epsilon N\rfloor}\sum_{n=1}^{\lfloor \epsilon N\rfloor}\frac{\left[sin\left(\frac{n\pi}{2}\right)\, sin\left(\frac{n\pi}{2}-\frac{n\pi\delta}{ln(N)}\right)\right]\left[sin\left(\frac{m\pi}{2}\right)\, sin\left(\frac{m\pi}{2}-\frac{m\pi\delta}{ln(N)}\right]\right) sin\left(n\pi s\right) sin\left(m\pi t\right)}{nm(n^2+m^2)}.$$

Now, note that

$$\lim_{N\to\infty}\left[sin\left(\frac{n\pi}{2}\right)\ sin\left(\frac{n\pi}{2}-\frac{n\pi\delta}{ln(N)}\right)\right] = sin^2\left(\frac{n\pi}{2}\right)$$
$$=\begin{cases}1 & n\text{ odd,}\\ 0 & n\text{ even.}\end{cases}$$

Also, we have that

$$\left|sin\left(\frac{n\pi}{2}\right)\ sin\left(\frac{n\pi}{2}-\frac{n\pi\delta}{ln(N)}\right)\right| \leq 1$$

Define

$$\phi_N(m,n) = \frac{\left[sin\left(\frac{n\pi}{2}\right)\ sin\left(\frac{n\pi}{2}-\frac{n\pi\delta}{ln(N)}\right)\right]\left[sin\left(\frac{m\pi}{2}\right)\ sin\left(\frac{m\pi}{2}-\frac{m\pi\delta}{ln(N)}\right)\right] sin\left(n\pi s\right) sin\left(m\pi t\right)}{nm(n^2+m^2)},$$

It follows from the above observations that

$$\lim_{N\to\infty}\phi_N(m,n) = \begin{cases}\frac{sin(n\pi s)sin(m\pi t)}{nm(n^2+m^2)} & n, m\text{ odd,}\\ 0 & n\text{ or }m\text{ even.}\end{cases}$$

and for any $n, m, N \in \mathbb{N}$,

$$|\phi_N(m,n)| \leq \frac{1}{nm(n^2+m^2)} := G(m,n)$$

All that remains to apply the Dominated Convergence Theorem is to show convergence for sums of the bounding function G is summable:

For any n, m positive odd integers,

$$(n-m)^2 + nm > 0$$
$$\implies n^2+m^2-nm > 0$$
$$\implies n^2+m^2 > nm$$
$$\implies nm(n^2+m^2) > n^2m^2.$$

Thus,

$$\sum_{m=1}^{\infty}\sum_{n=1,}^{\infty}\frac{1}{nm\left(n^2+m^2\right)} < \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^2m^2} = \frac{\pi^4}{36}$$

So

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{nm\left(n^2+m^2\right)}<\infty.$$

Thus, by DCT, we have that (omitting some constants for clarity):

$$\lim_{N\to\infty}\left(\sum_{m=1}^{\lfloor\epsilon N\rfloor}\sum_{n=1}^{\lfloor\epsilon N\rfloor}\frac{\left[sin\left(\frac{n\pi}{2}\right)\,sin\left(\frac{n\pi}{2}-\frac{n\pi\delta}{ln(N)}\right)\right]\left[sin\left(\frac{m\pi}{2}\right)\,sin\left(\frac{m\pi}{2}-\frac{m\pi\delta}{ln(N)}\right]\right)sin\left(n\pi s\right)sin\left(m\pi t\right)}{nm(n^2+m^2)}\right).$$

$$=\sum_{m=1,\,\mathrm{odd}}^{\infty}\ \sum_{n=1,\,\mathrm{odd}}^{\infty}\frac{sin\left(n\pi s\right)\,sin\left(m\pi t\right)}{nm(n^2+m^2)}.$$

(Clearly, this series is convergent by comparison with the series of the dominating function)

Reincluding the omitted constants and letting $\epsilon\to 0$, we are done:

$$\lim_{N\to\infty}\left[\frac{1}{N^2}\sum_{y=k_l}^{k_u}\sum_{x=k_l}^{k_u}F_N(s,t,x,y)\right]=\frac{256}{\pi^4}\sum_{m=1,\,\mathrm{odd}}^{\infty}\ \sum_{n=1,\,\mathrm{odd}}^{\infty}\frac{sin\left(n\pi s\right)\,sin\left(m\pi t\right)}{nm(n^2+m^2)}.$$

$\square$

4.3. **Hitting Probabilities in the 1D Case.** The derivation for the hiting probabilities in one dimension may be computed using the same method as in two dimensions. We begin with some notation. Consider the typical gambler's ruin Markov chain with state space $\{0,1,2,3,...,N\}$ and equal probability $p=1/2$ of going in either direction (left or right) at each step. The boundaries, 0 and N, are absorbing states.

Define $f_n(y)=P_n\left(T_y<min\{T_0,T_N\}\right)$ for any $0<y<N$ and $0\le n\le N$.

Just as in two dimensions, these probabilties are harmonic as a function of the starting point, with boundary conditions $f_0(y)=f_N(y)=0$, $f_y(y)=1$. We obtain the following equation:

$$f_n(y)-\frac{1}{2}\left(f_{n+1}(y)+f_{n-1}(y)\right)=\alpha(y)\delta(n-y)\quad(1)$$

for some function $\alpha(y)$.

Now, take the discrete Fourier transform of each term in equation (1), using sine functions as a basis:

$$f_n(y)=\sum_{k=1}^{N-1}a_k\,sin\left(\frac{k\pi n}{N}\right),$$

$$\begin{aligned} f_{n+1}(y) &= \sum_{k=1}^{N-1} a_k\, sin\left(\frac{k\pi(n+1)}{N}\right), \\ f_{n-1}(y) &= \sum_{k=1}^{N-1} a_k\, sin\left(\frac{k\pi(n-1)}{N}\right), \\ \alpha(y)\delta(n-y) &= \sum_{k=1}^{N-1} b_k sin\left(\frac{k\pi n}{N}\right). \end{aligned}$$

Using orthogonality relations to solve for b_k:

$$\begin{aligned} \sum_{n=1}^{N-1} \alpha(y)\delta(n-y) sin\left(\frac{m\pi n}{N}\right) &= \sum_{n=1}^{N-1}\sum_{k=1}^{N-1} b_k sin\left(\frac{k\pi n}{N}\right) sin\left(\frac{m\pi n}{N}\right) \\ \implies \alpha(y) sin\left(\frac{k\pi y}{N}\right) &= \sum_{k=1}^{N-1} \frac{b_k\, N\, \delta(k-m)}{2} \\ \implies b_k &= \frac{2\alpha(y) sin\left(\frac{k\pi y}{N}\right)}{N}. \end{aligned}$$

Now, we may use trig identities to simplify the LHS of equation (1).

Plugging in the discrete Fourier transforms and combining sums:

$$\begin{aligned} & f_n(y) - \frac{1}{2}\left(f_{n+1}(y) + f_{n-1}(y)\right) \\ = & \sum_{k=1}^{N-1} \left[a_k sin\left(\frac{k\pi n}{N}\right) - \frac{a_k}{2}\left(sin\left(\frac{k\pi(n+1)}{N}\right) + sin\left(\frac{k\pi(n-1)}{N}\right)\right)\right] \end{aligned}$$

Observe (using sum of angles formula):

$$\begin{aligned} & sin\left(\frac{k\pi(n+1)}{N}\right) + sin\left(\frac{k\pi(n-1)}{N}\right) \\ &= sin\left(\frac{k\pi n}{N}\right) cos\left(\frac{k\pi}{N}\right) + sin\left(\frac{k\pi}{N}\right) cos\left(\frac{k\pi n}{N}\right) \\ &+ sin\left(\frac{k\pi n}{N}\right) cos\left(\frac{k\pi}{N}\right) - sin\left(\frac{k\pi}{N}\right) cos\left(\frac{k\pi n}{N}\right) \\ &= 2\, sin\left(\frac{k\pi n}{N}\right) cos\left(\frac{k\pi n}{N}\right). \end{aligned}$$

Thus, we have that

$$f_n(y) - \frac{1}{2}\left(f_{n+1}(y) + f_{n-1}(y)\right) = \sum_{k=1}^{N-1} a_k sin\left(\frac{k\pi n}{N}\right)\left[1 - cos\left(\frac{k\pi n}{N}\right)\right].$$

Equating coefficients in equation (1), we obtain

$$a_k\left[1 - cos\left(\frac{k\pi n}{N}\right)\right] = b_k$$

$$\Longrightarrow a_k = \frac{2\alpha(y) sin\left(\frac{k\pi y}{N}\right)}{N\left[1 - cos\left(\frac{k\pi n}{N}\right)\right]}.$$

Substituting a_k back into the discrete Fourier transform for $f_n(y)$:

$$f_n(y) = \sum_{k=1}^{N-1} \frac{2\alpha(y) sin\left(\frac{k\pi y}{N}\right) sin\left(\frac{k\pi n}{N}\right)}{N\left[1 - cos\left(\frac{k\pi}{N}\right)\right]}.$$

Since $f_y(y) = 1$ for any $0 < y < N$, we may plug in $y = n$ to obtain

$$1 = \sum_{k=1}^{N-1} \frac{2\,\alpha(y)\, sin^2\left(\frac{k\pi y}{N}\right)}{N\left[1 - cos\left(\frac{k\pi}{N}\right)\right]}.$$

$$\Longrightarrow \alpha(y) = \left[\sum_{k=1}^{N-1} \frac{2 sin^2\left(\frac{k\pi y}{N}\right)}{N\left[1 - cos\left(\frac{k\pi}{N}\right)\right]}\right]^{-1}.$$

As desired, we have derived an explicit formula for $f_n(y)$:

$$f_n(y) = \sum_{k=1}^{N-1} \frac{2\alpha(y) sin\left(\frac{k\pi y}{N}\right) sin\left(\frac{k\pi n}{N}\right)}{N\left[1 - cos\left(\frac{k\pi}{N}\right)\right]}, \alpha(y) = \left[\sum_{k=1}^{N-1} \frac{2 sin^2\left(\frac{k\pi y}{N}\right)}{N\left[1 - cos\left(\frac{k\pi}{N}\right)\right]}\right]^{-1}.$$

One can check that these expressions agree with the classic Gambler's Ruin Identity:

$$f_n^{(N)}(y) = \begin{cases} \frac{n}{y}, & \text{for } y \geq n \\ \\ \frac{N-n}{N-y}, & \text{for } y < n \end{cases}$$

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