Order Parameters and Color-Flavor Center Symmetry in QCD

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Common lore suggests that N-color QCD with massive quarks has no useful order parameters that can be nontrivial at zero baryon density. However, such order parameters do exist when there are n_f quark flavors with a common mass and $d \equiv gcd(n_f, N) > 1$. These theories have a $\mathbb{Z}_d$ color-flavor center symmetry arising from intertwined color center transformations and cyclic flavor permutations. The symmetry realization depends on the temperature, baryon chemical potential, and value of n_f/N, with implications for conformal window studies and dense quark matter.

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Introduction.—Gauge theories are ubiquitous in many areas of physics. But defining order parameters in gauge theories is notoriously subtle. For example, in pure $SU(N)$ Yang-Mills (YM) theory, the simplest nontrivial order parameter is the expectation value of a line operator,

$$\langle \mathrm{tr}\Omega \rangle = \langle \mathrm{tr}\mathcal{P}e^{i\int_0^L dx_1 A_1} \rangle, \tag{1}$$

when the x_1 dimension is compactified with circumference L. If x_1 is regarded as Euclidean time, then the gauge theory functional integral with periodic boundary conditions calculates the thermal partition function with temperature $T = 1/L$. The thermal expectation value (1) is the Polyakov loop confinement order parameter for the $\mathbb{Z}_N$ center symmetry of pure YM, whose realization changes with temperature.

Adding fundamental representation quarks $\{q_a\}$, $a = 1, 2, \ldots, n_f$ turns YM theory into QCD, the theory of strong interactions. But the quarks explicitly break the $\mathbb{Z}_N$ center symmetry. The Polyakov loop ceases to be an order parameter. With massless quarks, the flavor symmetry is $G = SU(n_f)_V \times SU(n_f)_A \times U(1)_Q$ and the chiral condensate $\langle \sum_a \bar{q}_a q_a \rangle$ is an order parameter for the $SU(n_f)_A$ chiral symmetry, whose realization depends on temperature. But if the quarks are massive, as in real-world QCD, then chiral symmetry is explicitly broken and $\langle \bar{q}_a q_a \rangle$ ceases to be an order parameter. The remaining vectorlike $U(n_f)$ symmetry cannot break spontaneously at vanishing baryon number density [1], leading to the common understanding that QCD with dynamical massive quarks lacks nontrivial order parameters at zero baryon density.

However, QCD with massive quarks can possess symmetries that intertwine center and flavor transformations. Our discussion generalizes earlier work [2–9], showing that special boundary conditions (BCs) for quarks can lead to an unbroken $\mathbb{Z}_3$ symmetry. These works interpreted this choice as defining a "QCD-like" theory that they termed "Z_3-QCD." (See also Refs. [10–13].) Here, we generalize and reinterpret these BCs and use them to define order parameters for both quantum and thermal phase transitions in QCD.

Color-flavor center symmetry.—Center symmetry [14] acts only on topologically nontrivial observables. It can be viewed as a topologically nontrivial gauge transformation, with the action

$$\langle \mathrm{tr}\Omega \rangle \to \omega \langle \mathrm{tr}\Omega \rangle, \qquad \omega \equiv e^{2\pi i/N}. \tag{2}$$

By itself, this is not a symmetry when fundamental representation fields are present in the theory.

We assume n_f quark flavors have a common mass m_q, so the theory (on $\mathbb{R}^4$) has a $U(n_f)_V$ flavor symmetry. We consider flavor-twisted quark boundary conditions,

$$q_a(x_1 + L) = \mathcal{U}^{ab} q_b(x_1), \tag{3}$$

where $\mathcal{U}$ is a $U(n_f)$ matrix, and regard x_1 as a spatial direction. The flavor twist $\mathcal{U}$ may be assumed diagonal without loss of generality. The $SU(n_f)_V$ flavor subgroup has center $\mathbb{Z}_{n_f}$, which motivates a $\mathbb{Z}_{n_f}$ symmetric choice of BCs [2–9] for which the set of eigenvalues of $\mathcal{U}$ is invariant under multiplication by elements of $\mathbb{Z}_{n_f}$. If the theory is to retain charge conjugation and x_1-reflection symmetries (suitably redefined), then the set of eigenvalues must also be invariant under complex conjugation. Two possibilities result, namely n_fth roots of $+1$ or -1,

$$\mathcal{U} = \mathrm{diag}(1, \nu, \ldots, \nu^{n_f - 1}), \qquad \nu \equiv e^{2\pi i/n_f}, \tag{4a}$$

or

$$\mathcal{U} = \mathrm{diag}(\nu^{1/2}, \nu^{3/2}, \ldots, \nu^{n_f - 1/2}). \tag{4b}$$

With the BCs in (4), the finite L flavor symmetry is reduced to $G_L = U(1)_V^{n_f-1} \times U(1)_A^{n_f-1} \times U(1)_Q \subset G$.

The key observation is that if

$$d \equiv gcd(n_f, N) > 1, \tag{5}$$

then the circle-compactified theory, with either boundary condition (4), also remains invariant under an intertwined $\mathbb{Z}_d \subset \mathbb{Z}_N \times \mathbb{Z}_{n_f}^{\rm perm}$ color-flavor center (CFC) symmetry, generated by the combination of a center transformation with phase $\omega^{N/d} = e^{2\pi i/d}$ and a $\mathbb{Z}_d$ cyclic flavor permutation. To see this note that, for either choice (4), a $\mathbb{Z}_d$ center transformation effectively permutes the eigenvalues of $\mathcal{U}$. Combining the center transformation with the opposite cyclic flavor permutation [part of $U(n_f)_V$] leaves the boundary condition invariant.

CFC symmetry intertwines center and flavor transformations and so has both local and extended order parameters. CFC order parameters include Polyakov loops such as (1) with winding numbers that are nonzero mod d, as well as $\mathbb{Z}_{n_f}$ Fourier transforms of fermion bilinears, $\mathcal{O}_\Gamma^{(p)} \equiv \sum_{a=1}^{n_f} \nu^{-ap} \bar{q}_a \Gamma q_a$, where Γ is an arbitrary Dirac matrix and p mod $d \neq 0$. The action of the $\mathbb{Z}_d$ CFC symmetry is

$$\mathrm{tr}\Omega^p \to \omega^{Np/d} \mathrm{tr}\Omega^p, \qquad \mathcal{O}_\Gamma^{(p)} \to \nu^{n_f p/d} \mathcal{O}_\Gamma^{(p)}. \tag{6}$$

We note, however, that $\mathcal{O}^{(p)}$ also transforms nontrivially under pure flavor permutation transformations, while $\mathrm{tr}\Omega^p$ only transforms nontrivially under the combined CFC color-flavor transformations. In this sense, $\mathrm{tr}\Omega^p$ is the natural order parameter for CFC symmetry.

Center symmetry and confinement.—Consider the Polyakov loop connected correlator in QCD compactified on x_1 with circumference L,

$$\langle \mathrm{tr}\Omega(\vec{x}) \mathrm{tr}\Omega^\dagger(0) \rangle_{\rm conn} \equiv e^{-F(r)}, \qquad r = |\vec{x}|. \tag{7}$$

Suppose that $F(r) \sim Er$ as $r \to \infty$. When $n_f = 0$, the theory has a $\mathbb{Z}_N$ center symmetry. If the ground state is $\mathbb{Z}_N$ invariant, then no intermediate state created by a local operator acting on the vacuum can contribute to the correlator. All contributions to the correlator (7) must involve flux tubes that wrap the compactified dimension, so that $E = L\sigma$ with σ being the string tension.

On the other hand, if center symmetry is broken, explicitly or spontaneously, then intermediate states created by local operators can also contribute to the correlator (7). The minimal energy E need not grow with L. This is interpreted as a signal of string breaking. It is tempting to conclude that there is a tight link between unbroken center symmetry and confinement of static test quarks by unbreakable flux tubes.

Now suppose that $d = gcd(n_f, N) > 1$, all quarks have a common mass m_q, and we engineer the existence of $\mathbb{Z}_d$ CFC symmetry by using the BCs (4). As seen above, CFC symmetry acts on both Polyakov loops and appropriate local operators. Intermediate states created by local operators transforming the same as $\mathrm{tr}\Omega$ under all unbroken symmetries can contribute to the correlator (7). For example, states created by $\mathcal{A} \equiv \sum_{a=1}^{n_f} \nu^{-ap} \bar{q}_a \gamma_1 D_1 q_a$ and $\mathcal{B} \equiv \sum_{a=1}^{n_f} \nu^{-ap} \bar{q}_a \gamma_1 q_a$ can contribute to correlators of $\mathrm{Retr}\Omega^p$ and $\mathrm{Imtr}\Omega^p$, respectively, even when CFC symmetry is not spontaneously broken. Consequently, the string tension as defined by the asymptotic behavior of the correlator (7) vanishes regardless of the realization of the $\mathbb{Z}_d$ center symmetry.

Conformal window.—Let $x \equiv n_f/N$, and set to 0 the common quark mass and temperature, $m_q = T = 0$. If $x > \frac{11}{2}$, QCD becomes an infrared-free theory. For x below some $x_\chi < \frac{11}{2}$, chiral symmetry is believed to be spontaneously broken. In the intermediate range of values $x \in (x_\chi, \frac{11}{2})$, called the conformal window, QCD flows to a nontrivial infrared (IR) fixed point without chiral symmetry breaking. The value of x_χ has been the subject of intensive lattice investigations (see, e.g., Refs. [15–24]). The existence of an IR-conformal phase can be seen most easily in the Veneziano large N limit of QCD, where x is fixed along with the 't Hooft coupling $\lambda \equiv g^2 N$ as N increases. If $\epsilon \equiv \frac{11}{2} - x \to 0^+$, perturbation theory self-consistently implies the existence of an IR fixed point with a parametrically small coupling [25,26], $\lambda_{\rm IR} = \frac{64}{75}\pi^2 \epsilon \ll 1$.

One may show that $\mathbb{Z}_d$ CFC symmetry is spontaneously broken in the conformal window, at least at large N. The Veneziano limit is taken through a sequence of values for which $d = gcd(n_f, N)$ is fixed and greater than 1, while the ratio $x = n_f/N$ approaches a nonzero limit. Hence, the Polyakov loop (1) remains an order parameter for the intertwined $\mathbb{Z}_d$ center symmetry. The CFC realization may be determined by computing the quantum effective potential $V_{\rm eff}(\Omega)$. Here the loop expansion is controlled by the small value of λ at all scales when $\epsilon \ll 1$. The following small-ϵ analysis is thus valid for any circumference L.

Classically, $V_{\rm eff}(\Omega)$ is 0. Using standard methods [14,27], at one loop one finds $V_{\rm eff}(\Omega) = V_g(\Omega) + V_f(\Omega)$ with gluon and fermion contributions given by

$$V_g(\Omega) = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} (|\mathrm{tr}\Omega^n|^2 - 1), \tag{8}$$

and

$$V_f(\Omega) = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} (\mathrm{tr}\mathcal{U}^{-n} \mathrm{tr}\Omega^n + \mathrm{tr}\mathcal{U}^n \mathrm{tr}\Omega^{-n})$$
$$= \frac{2}{\pi^2 L^4 n_f{}^3} \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^4} (\mathrm{tr}\Omega^{n_f n} + \mathrm{H.c.}). \tag{9}$$

The upper or lower sign refers to BCs (4a)/(4b). As required, $V_{\rm eff}$ is invariant under CFC symmetry. To determine the minima of $V_{\rm eff}$ note that $V_g = O(N^2)$ while, due to our imposition of flavor-twisted BCs, $V_f = O(N^{-2})$.

At large N, the minima of $V_{\rm eff}$ are entirely determined by the gluonic contribution V_g, which favors coinciding eigenvalues, $\Omega \propto 1$. Consequently, when $\epsilon = \frac{11}{2} - x \ll 1$ the CFC symmetry is spontaneously broken at any L. On the other hand, at the pure Yang-Mill point, $x = 0$, center symmetry is certainly expected to be unbroken at large L, and standard large N counting arguments imply that the intertwined center symmetry should remain unbroken for sufficiently small x. Hence, there must be at least one transition at some $x = x_{\rm CFC}$ where the realization of the CFC symmetry changes. Logically possible phase diagrams are sketched in Fig. 1.

Turning on $m_q > 0$ or $T > 0$ gives a richer phase structure. With $\epsilon \ll 1$ and small m_q, the theory develops a new strong scale, $\Lambda_m \sim m_q e^{-75/(8\epsilon^2)}$. When L becomes comparable to Λ_m^{-1} we expect a $\mathbb{Z}_d$ center-restoring phase transition. We also expect a CFC-restoring phase transition at a nonzero temperature $T_* \sim 1/L$ when $m_q = 0$, similar to the large N deconfinement transition in $\mathcal{N} = 4$ super-Yang-Mills theory on $S^3 \times S^1$ [28,29].

Now consider $N = 3$ and massless quarks. If the $n_f = 15$ IR fixed point is weakly coupled, as widely believed, then our above calculation applies and $\mathbb{Z}_3$ center symmetry is spontaneously broken at $n_f = 15$. At $n_f = 3$, lattice calculations [6] with boundary conditions (4) are consistent with unbroken $\mathbb{Z}_3$ center symmetry when $L\Lambda \gg 1$. So for integer values of $x = n_f/3$, there must be a minimal value $2 \le x_{\rm CFC} \le 5$ where the $\mathbb{Z}_3$ CFC symmetry first becomes spontaneously broken.

Dense quark matter.—Consider the phase diagram of QCD with $N = n_f = 3$ and a common quark mass m_q, as a function of the $U(1)_Q$ chemical potential μ and temperature T. Previously known symmetry principles only suggest the existence of a curve $T(\mu)$ in the (T, μ) plane below which lies a superfluid phase with spontaneously broken $U(1)_Q$ symmetry, leading to a hypothesis of continuity of quark matter and hadronic nuclear matter [30]. Consideration of CFC symmetry implies the existence of additional phase structure when QCD is compactified with CFC-preserving BCs on a spatial circle large compared to other spatial scales.

First, consider the small T, small μ regime. Here lattice studies [6] imply that $\langle {\rm tr}\Omega\rangle = 0$ at large L. At high temperatures, $T \gg \max(\Lambda, \mu)$, $\langle {\rm tr}\Omega\rangle$ also vanishes, because the dynamics on spatial scales large compared to $(g^2T)^{-1}$ are described by pure three-dimensional YM theory [27], which confines. We expect this high-temperature region to be smoothly connected to the region near $T = \mu = 0$. However, as we next discuss the CFC symmetry realization behaves nontrivially when $T \to 0$ with $\mu \gg \max(\Lambda, m)$. A simple phase diagram consistent with our results is sketched in Fig. 2.

High density QCD, $\mu \gg \Lambda$, is believed to be in a "color-flavor-locked" (CFL) color-superconducting phase [31] when $T < T_{\rm CFL}$. The phase transition temperature $T_{\rm CFL}$ is comparable to the superconducting gap, $T_{\rm CFL} \sim \Delta \sim \mu g^{-5} e^{-(3\pi^2/\sqrt{2})/g}$. Electric and magnetic gluons develop Debye and Meissner static screening masses, respectively, both of order $g\mu$ in the CFL phase [32,33]. For $T_{\rm CFL} < T \lesssim g\mu$, low frequency magnetic fluctuations experience Landau damping. Consequently, for $T \lesssim g\mu$ the relevant gauge coupling is small, $g(\mu) \ll 1$, and cold dense quark matter is weakly coupled.

In typical gauge-dependent language, CFL superconductivity is driven by an expectation value for diquark operators, $\langle q_i^a C\gamma_5 q_j^b\rangle \propto \epsilon^{abK}\epsilon_{ijK}$ [31]. The uncontracted flavor indices on the "condensate" might lead one to think that flavor permutation symmetry is broken, automatically implying accompanying spontaneous breaking of the CFC symmetry [7] when x_1 is compactified with BCs (4). But this gauge-dependent language is misleading. The true gauge-invariant order parameters for spontaneous breaking of chiral and $U(1)_Q$ symmetries, schematically $\langle \bar{q}C\gamma_5\bar{q}qC\gamma_5 q\rangle$ and $\langle (qC\gamma_5 q)^3\rangle$, are $SU(n_f)_V$ singlets [31]. So the development of CFL superconductivity does not, *ipso facto*, imply spontaneous breaking of CFC symmetry in the circle-compactified theory.

To study the CFC symmetry realization when $\mu \gg \Lambda$ and $T \ll g\mu$, we examine the one-loop effective potential for Ω, which is the sum of gluonic and fermionic contributions, $V_{\rm eff}(\Omega) = V_g(\Omega) + V_f(\Omega)$. The loop expansion

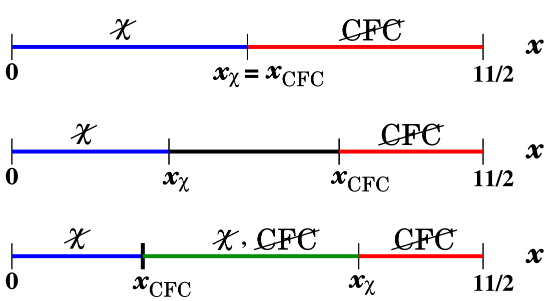

FIG. 1. Possible phase structures of massless QCD as a function of $x = n_f/N$. The chiral and CFC symmetry realizations change at some $x = x_\chi$ and $x = x_{\rm CFC}$, respectively.

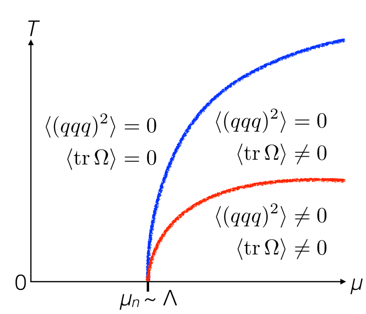

FIG. 2. Sketch of a possible phase diagram of circle-compactified $SU(3)_V$ symmetric QCD at $m_q > 0$, as a function of T and μ, in the large L limit.

is controlled by $g(\mu) \ll 1$ and is applicable for all L. For $T < T_{\rm CFL}$, gluons have effective masses $m_g \sim g\mu$ due to a combination of Debye screening and the Meissner effect from color superconductivity [32,33]. As with any one-loop holonomy effective potential contribution from massive adjoint bosons, we thus expect

$$V_g(\Omega) = -\frac{1}{L^4}\sum_{n=1}^{\infty} f_n(|{\rm tr}\Omega^n|^2 - 1), \tag{10}$$

with coefficients $f_n > 0$ which are exponentially small, $f_n \sim e^{-nm_g L}$, when $m_g L \gg 1$. At large μ, $V_g(\Omega)$ is highly suppressed compared to the $\mu = 0$ result (8).

Fermion excitations near the Fermi surface are nearly gapless as $T \to 0$, up to nonperturbative corrections from quark pairing. In the cold dense limit, $V_{\rm eff}(\Omega)$ is dominated by the fermion contribution,

$$V_f(\Omega) = \frac{1}{\pi\beta L^3 {n_f}^2}\sum_{n=1}^{\infty}\frac{(\pm 1)^n}{n^3}\bigg[({\rm tr}\Omega^{n_f n} + {\rm H.c.}) \times \sum_{k\in\mathbb{Z}+\frac{1}{2}} (1 + n_f n \mathfrak{m}_k L) e^{-n_f n \mathfrak{m}_k L}\bigg], \tag{11}$$

where $\mathfrak{m}_k^2 \equiv (2\pi k T + i\mu)^2 + m_q^2$, the upper or lower sign refers to BCs (4a)/(4b), and (11) is manifestly invariant under the CFC symmetry, as required. (Derivation is detailed in Supplemental Material [34].) To examine the realization of CFC symmetry, we work in the simplifying limit $m_q \ll \mu$ and focus on the regime $\mu L \gg 1$. If $TL \gg 1$, then the sum (11) is dominated by the $k = \pm\frac{1}{2}$, $n = 1$ terms, giving

$$V_f(\Omega) = \frac{\pm 2T e^{-n_f \pi L T}}{n_f \pi L^2}[\mu \sin(n_f \mu L) + \pi T \cos(n_f \mu L)] \times ({\rm tr}\Omega^{n_f} + {\rm H.c.}) + (\text{holonomy independent}), \tag{12}$$

up to exponentially small corrections. The $e^{-\pi T L n_f}$ factor arises from the lowest fermionic Matsubara frequency and our twisted boundary conditions. Alternatively, if $TL \to 0$ then the prefactor in (12) becomes $\pm({n_f}^2 \pi^2 L^3)^{-1}$. In either regime of TL, neglecting subdominant contributions, $V_{\rm eff}(\Omega) \propto {\rm Re}{\rm tr}\Omega^{n_f}$ with an amplitude that oscillates as a function of $n_f \mu L$.

For $n_f = N = 3$, extrema of $V_f(\Omega)$ fall into four categories: (a) one center-symmetric extremum at $\Omega = {\rm diag}(1, e^{2\pi i/3}, e^{4\pi i/3})$, where the $SU(3)$ gauge symmetry is "broken" down to $U(1)^2$ with the holonomy playing the role of an adjoint Higgs field; (b) three center-broken extrema with "residual" gauge group $U(1)^2$ where $\Omega = {\rm diag}(e^{(2k-1)i\pi/3}, e^{2ki\pi/3}, e^{(2k+1)i\pi/3})$, $k = 0,\ 1,\ 2$; (c) nine center-broken "$SU(2) \times U(1)$" extrema at $\Omega = {\rm diag}(e^{ki\pi/9}, e^{ki\pi/9}, e^{-2ki\pi/9})$ with k mod $6 = 2$, 3 or 4; (d) three center-broken "$SU(3)$" extrema, $\Omega = {\rm diag}(e^{2ki\pi/3}, e^{2ki\pi/3}, e^{2ki\pi/3})$, $k = 0,\ 1,\ 2$. These "$SU(3)$" extrema are also minima of V_g.

The form (12) implies that the locations of the minima of $V_{\rm eff}(\Omega)$ change as a function of μL, as illustrated in the contour plots in Fig. 3 for two nearby values of μL. There are quantum oscillations in the phase structure of cold dense QCD on a circle, with the minima of $V_{\rm eff}$ cycling through two inequivalent sets of local minima as μL varies. These come in two groups within which $V_f(\Omega)$ is degenerate. One group consists of the center-symmetric and three $SU(3)$ extrema. The other consists of the six $SU(2) \times U(1)$ extrema with $\Omega = {\rm diag}(e^{ki\pi/9}, e^{ki\pi/9}, e^{-2ki\pi/9})$ and k mod $6 = 2$ or 4. (The remaining six extrema are always saddle points for $TL \gg 1$.)

At $T = 0$ there are quantum phase transitions when the minimum energy state switches from one set of extrema to another, with associated jumps in the ground state degeneracy. (Similar behavior in other circle-compactified theories has been seen in Refs. [37,38].) There are an infinite number of phase transitions in the cold dense limit as L increases and successive energy bands pass through the value of the chemical potential, with an accumulation point at $L = \infty$. Borrowing a term from the condensed-matter literature [39,40], each point in the (T, μ) phase diagram for QCD where this phenomenon occurs can be called a multiphase point [41]. As we discuss below, this behavior is expected in a finite area domain of the (T, μ) phase diagram, so in fact we find a multiphase region.

The small residual gluon contribution to $V_{\rm eff}$ favors configurations with clumped holonomy eigenvalues, lowering the energy of $SU(3)$ extrema relative to the center-symmetric point. Hence, we expect that all genuine minima of $V_{\rm eff}$ in this multiphase region are associated with broken CFC symmetry, with $\langle {\rm tr}\Omega \rangle \neq 0$ [42].

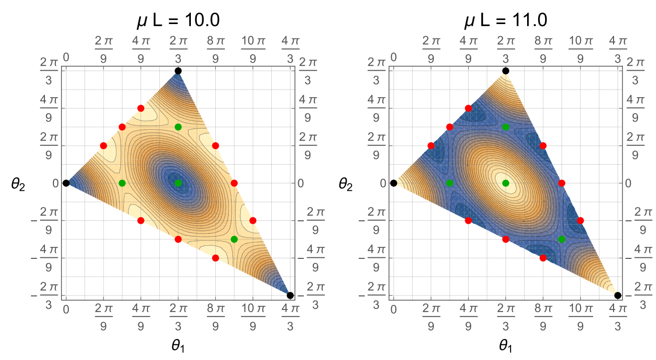

FIG. 3. Contour plots of V_f for $N = n_f = 3$, with BCs (4), as a function of θ_1, θ_2 for two nearby values of μL with $T/\mu = 10^{-3}$, illustrating the quantum oscillations described in the text. Darker colors indicate lower values of $L^4 V_{\rm eff}$. Regions outside the triangle shown are gauge equivalent to points within the triangle. The center-symmetric point $(\theta_1, \theta_2, \theta_3) = (0, 2\pi/3, 4\pi/3)$ lies at the center of the triangle while the corners are the coinciding eigenvalue points (0,0,0) and $\pm(2\pi/3, 2\pi/3, 2\pi/3)$. Dots denote critical points of $\hat{V}_f$. Results with BCs (4b) are similar.

Putting everything together, we conclude that there must be some curve $T = T_{\mathrm{CFC}}(\mu)$ below which the CFC symmetry is spontaneously broken with oscillatory multiphase behavior. We lack a definitive calculation of $V_g(\Omega)$ valid for $T > T_{\mathrm{CFL}}$. But when $T_{\mathrm{CFL}} \lesssim T \ll g\mu$ we expect that V_g will grow in size and continue to favor CFC breaking. The nonperturbative physics that favors CFC restoration is only expected to set in once $T \gtrsim g\mu$. Hence we expect that $T_{\mathrm{CFC}}(\mu)$ is $\mathcal{O}(g\mu)$, greatly exceeding T_{CFL} at large μ. The T_{CFC} curve must end at some point μ_c on the $T = 0$ axis. The simplest hypothesis is that μ_c coincides with $\mu_n \sim \Lambda$, the critical chemical potential needed to produce pressureless nuclear matter at $T = 0$, as illustrated in Fig. 2.

Conclusions.—We have shown that there are well-defined and nontrivial order parameters for quantum and thermal phase transitions in QCD, compactified on a circle, provided $gcd(n_f, N) > 1$ with quarks having a common mass m_q. This is a consequence of the existence of color-flavor center symmetry, and has interesting implications for the phase structure of QCD as a function n_f/N, μ, T, and m_q.

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Note added.—Recently, Ref. [43] appeared, and has some overlap with parts of our construction.

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