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Jim Schwiegerling, "Linear decomposition of the optical transfer function for annular pupils," Proc. SPIE 10375, Current Developments in Lens Design and Optical Engineering XVIII, 103750F (23 August 2017); doi: 10.1117/12.2274788

Event: SPIE Optical Engineering + Applications, 2017, San Diego, California, United States

Linear decomposition of the optical transfer function for annular pupils

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ABSTRACT

A technique for decomposing the Optical Transfer Function (OTF) into a novel set of basis functions has been developed. The decomposition provides insight into the performance of optical systems containing both wavefront error and apodization, as well as the interactions between the various components of the pupil function. Previously, this technique has been applied to systems with circular pupils with both uniform illumination and Gaussian apodization. Here, systems with annular pupils are explored. In cases of annular pupil with simple defocus, analytic expressions for the OTF decomposition coefficients can be calculated. The annular case is not only applicable to optical systems with central obscurations, but the technique can be extended to systems with multiple ring structures. The ring structures can have constant area as is often found in zone plates and diffractive lenses or the rings can have arbitrary areas. Analytic expressions for the OTF decomposition coefficients again can be determined for ring structures with constant and quadratic phase variations. The OTF decomposition provides a general tool to analyze and compare a diverse set of optical systems.

Keywords: Optical Transfer Function, Modulation Transfer Function, Annular Pupils, Mathematics, Decomposition

1. INTRODUCTION

The Optical Transfer Function(OTF) is a versatile metric of the performance of optical systems. In earlier research, a technique for decomposing the complex pupil function of the system into Zernike polynomials was developed. The expansion coefficients (and their complex conjugates) for this pupil function decomposition were then used to form a linear expansion of the OTF into a novel set of basis functions.[1-3] The advantage of this technique is that the complex pupil function is used so that both apodization and wavefront error can be assessed and that the form of the pupil function is directly connected to the components of the OTF. This technique has applications in aberration theory, lens design and optimization. The expansion coefficients for the decomposition into the basis functions typically need to calculated numerically. However, in a recent paper, it was shown that when the pupil function is restricted to having a quadratic form, meaning that only Gaussian apodization in the presence of defocus is considered, the expansion coefficients can be obtained analytically.[4] Here, this analytic technique is exploited to analyzed optical systems with annular pupils in the presence of defocus. The resultant expressions are general and this technique can easily be expanded to include multiple annular structures such as those found in Fresnel zone plates and other wavefront coding applications.

2. METHODS

The pupil function $P(\rho)$ of a rotationally symmetric optical system is in general a complex function. Here, ρ is the normalized radial coordinate. The magnitude of the pupil function describes the pupil apodization or transmission as a function of position. The argument of the pupil function describes the wavefront error of the system. The OTF for rotationally symmetric systems was first described in [1]. The techniques were then expanded to the non-rotationally symmetric case in [2-3]. Here, a subset of rotationally symmetric pupil functions over an annular region with inner radius ϵ_1 and outer radius ϵ_2 are considered. These pupil functions have the form

$$P(\rho) = \left[cyl\left(\frac{\rho}{2\epsilon_2}\right) - cyl\left(\frac{\rho}{2\epsilon_1}\right)\right] exp(ia\rho^2), \qquad (1)$$

where $cyl(\,)$ is the cylinder function defining the unit pupil and $a = -2\pi W_{20}$, is a real constant. The constant W_{20} is the amount of defocus in units of waves in the system. Following [1], the pupil function is decomposed into a set of rotationally symmetric Zernike radial polynomials such that

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Current Developments in Lens Design and Optical Engineering XVIII, edited by R. Barry Johnson,
Virendra N. Mahajan, Simon Thibault, Proc. of SPIE Vol. 10375, 103750F
 · doi: 10.1117/12.2274788

$$P(\rho) = \sum_{n=0}^{N} b_{n0} R_n^0(\rho) \quad \textbf{(2)}$$

where b_{n0} are the expansion coefficients, $R_n^0(\rho)$ are the rotationally symmetric Zernike radial polynomials and N is the maximum order of the polynomials.[5] The expansion coefficients are then given by

$$b_{n0} = 2(n+1) \int_0^1 P(\rho) R_n^0(\rho) \rho d\rho. \quad \textbf{(3)}$$

The expansion coefficients b_{n0} are in general complex valued. The index n must be even due to the constraints on the Zernike Radial polynomials. The linear expansion of the OTF is then given by

$$OTF(\beta) = \frac{1}{c_N} \sum_{\substack{n=0 \\ n\ even}}^{N} \sum_{\substack{n'=0 \\ n'even}}^{N} (-1)^n b_{n0} b_{n'0}^{*} G_{n+1,n'+1,0}(\beta), \quad \textbf{(4)}$$

where $c_N = \sum_{\mathrm{even\ n}}^{N} |b_{n0}|^2/(n+1)$ and $G_{n+1,n'+1,0}(\beta)$ are a set of rotationally symmetric basis functions (G-Functions) derived in [1]. Note, in previous reports, these were described in the form $G_{n+1,n'+1,0}(-1,\beta)$. Here, the -1 in the argument will be dropped since it is only an artifact of the derivation of these functions. Table 1 summarizes the first few rotationally symmetric G-Functions. The preceding equations illustrate that the pupil function can be decomposed into a linear expansion with coefficients b_{n0}. The OTF is then described by a linear expansion in terms of G-Functions with expansion coefficients which are simply the products of the various b_{n0} and their complex conjugates.

Table 1. Rotationally symmetric G-Functions arising for quadratic pupils up to 5th order.

$G_{110}(\beta)$	$\frac{2}{\pi}\left[cos^{-1}\left(\frac{\beta}{2}\right) - \left(\frac{\beta}{2}\right)\sqrt{1-\frac{\beta^2}{4}}\right]$
$G_{130}(\beta)$	$\frac{1}{\pi}\left[-2\beta^2 cos^{-1}\left(\frac{\beta}{2}\right) + (\beta^2+4)\left(\frac{\beta}{2}\right)\sqrt{1-\frac{\beta^2}{4}}\right]$
$G_{330}(\beta)$	$\frac{1}{\pi}\left[\frac{2}{3} cos^{-1}\left(\frac{\beta}{2}\right) + \left(-\frac{2\beta^4}{9} + \frac{14\beta^2}{9} - \frac{10}{3}\right)\left(\frac{\beta}{2}\right)\sqrt{1-\frac{\beta^2}{4}}\right]$
$G_{150}(\beta)$	$\frac{1}{\pi}\left[(6\beta^2+6\beta^4) cos^{-1}\left(\frac{\beta}{2}\right) + \left(-\frac{4\beta^4}{3} + \frac{71\beta^2}{3} - 4\right)\left(\frac{\beta}{2}\right)\sqrt{1-\frac{\beta^2}{4}}\right]$
$G_{350}(\beta)$	$\frac{1}{\pi}\left[-4\beta^2 cos^{-1}\left(\frac{\beta}{2}\right) + \left(\frac{\beta^6}{6} - \frac{5\beta^4}{3} + 7\beta^2 + 4\right)\left(\frac{\beta}{2}\right)\sqrt{1-\frac{\beta^2}{4}}\right]$
$G_{550}(\beta)$	$\frac{1}{\pi}\left[\frac{2}{5} cos^{-1}\left(\frac{\beta}{2}\right) + \left(-\frac{2\beta^8}{25} + \frac{21\beta^6}{25} - \frac{236\beta^4}{75} + \frac{76\beta^2}{15} - \frac{18}{5}\right)\left(\frac{\beta}{2}\right)\sqrt{1-\frac{\beta^2}{4}}\right]$

The factor c_N in Eq. 4 is useful as well, as it converges to the relative transmission of the pupil. This factor enables the number of terms N in the expansions required for convergence. In the case of an annular pupil, c_N will converge to $\epsilon_2^2 - \epsilon_1^2$. The expansion coefficients b_{n0} for an annular pupil in the presence of defocus can be solved for analytically. Boivin [6] gives

$$exp(ia\rho^2) = \pm exp\left(\frac{ia}{2}\right)\sqrt{\frac{\pi}{a}}\sum_{s=0}^{\infty}(2s+1)i^s J_{s+1/2}\left(\frac{a}{2}\right)R_{2s}^0(\rho), \quad \textbf{(5)}$$

where the sign in front is the same as the sign of a. In developing the analytic form of the expansion coefficients, defining the following integral will be useful:

$$I_\epsilon(a) = 2\int_0^\epsilon exp(ia\rho^2)R_n^0(\rho)\rho d\rho, \quad \textbf{(6)}$$

where the subscript ϵ defines the upper limit of the integral. In the special case where $\epsilon \to 1$, this integral becomes

$$I_1(a) = 2\int_0^1 exp(ia\rho^2)R_n^0(\rho)\rho d\rho = \pm i^{n/2} exp\left(\frac{ia}{2}\right)\sqrt{\frac{\pi}{a}}J_{(n+1)/2}\left(\frac{a}{2}\right), \quad \textbf{(7)}$$

where Eq. 5 has been inserted into Eq. 6 and the derivation in Reference [4] has been followed to achieve this result, and $J_\nu(\,)$ is the Bessel function of the first kind. It is evident from Eq. 3 that the expansion coefficients are given by

$$b_{n0} = 2(n+1)\int_{\epsilon_1}^{\epsilon_2} exp(ia\rho^2)R_n^0(\rho)\rho d\rho = (n+1)\left[I_{\epsilon_2}(a) - I_{\epsilon_1}(a)\right]. \quad \textbf{(8)}$$

With a suitable change of variables, the integral in Eq. 6 can be rewritten as

$$I_\epsilon(a) = 2\epsilon^2\int_0^1 exp(ia\epsilon^2\rho^2)R_n^0(\epsilon\rho)\rho d\rho, \quad \textbf{(9)}$$

Using the relation in Eq. 6, this integral becomes

$$I_\epsilon(a) = \pm 2\epsilon exp\left(\frac{ia\epsilon^2}{2}\right)\sqrt{\frac{\pi}{a}}\sum_{s=0}^{\infty}(2s+1)i^s J_{s+1/2}\left(\frac{a\epsilon^2}{2}\right)\int_0^1 R_{2s}^0(\rho)R_n^0(\epsilon\rho)\rho d\rho, \quad \textbf{(10)}$$

The integral in this expression has been solved by Janssen and Dirksen[7]

$$\int_0^1 R_n^m(\rho)R_{n'}^m(\epsilon\rho)\rho d\rho = \frac{1}{2(n+1)}\left[R_{n'}^n(\epsilon) - R_{n'}^{n+2}(\epsilon)\right], \quad \textbf{(11)}$$

with the understanding that the second term is zero if $n' < n+2$. Combining Eqs. 10 and 11 gives

$$I_\epsilon(a) = \pm\epsilon exp\left(\frac{ia\epsilon^2}{2}\right)\sqrt{\frac{\pi}{a}}\sum_{s=0}^{n/2} i^s J_{s+1/2}\left(\frac{a\epsilon^2}{2}\right)\left[R_n^{2s}(\epsilon) - R_n^{2s+2}(\epsilon)\right], \quad \textbf{(12)}$$

where again, the sign is given by the sign of a. Note that the sum has now become finite since the upper index of the Zernike radial polynomial must be less than or equal to the lower index for the polynomial to be non-zero. Eqs. 8 and 12 now fully define the expansion coefficients for and annular pupil.

One final reduction is useful for the case of an annular pupil with no defocus. In this case, $a \to 0$. Both the radical and the Bessel function in Eq. 12 become problematic in this case. Using L'Hopital's rule, Eq. 12 reduces to the remarkably simple expression

$$I_\epsilon(a) = \epsilon^2[R_n^0(\epsilon) - R_n^2(\epsilon)]. \quad \textbf{(13)}$$

So, in the case of no defocus, Eqs. 8 and 13 fully define the OTF expansion coefficients.

3. EXAMPLES

The first example examined is the case of an annular pupil with no defocus. The OTF for this case was first examined by O'Neill.[8] Mahajan provides a derivation of the results along with tables of the calculated values.[9] For this example, Eqs. 8 and 13 were used to calculate the first 20 expansion coefficients. Values of $\epsilon_1 = 0.25$, 0.5 and 0.75 were used along with a fixed value of $\epsilon_2 = 1$. Figure 1 shows the calculated OTFs for the three different inner diameters. In comparing to the numeric values provided by Mahajan, the curves in Figure 1 match to about three decimals places. The main discrepancy occurs at the origin. To further reduce the difference between expansion and the numeric values, additional expansion coefficients are easily calculated.

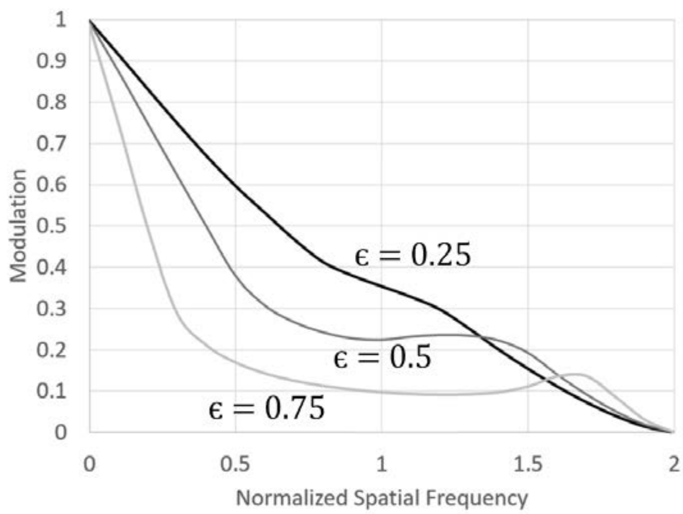

Figure 1. Calculated OTF for three different inner diameters of an annular aperture.

As a second example, the case of an annular pupil with $\lambda/2$ waves of defocus is examined. Again, values of $\epsilon_1 = 0.25$, 0.5 and 0.75 were used along with a fixed value of $\epsilon_2 = 1$. Figure 2 shows the calculated OTFs for the three different inner diameters. The expansion coefficients calculated with the right-hand side of Eq. 8 and Eq. 12 were compared to the expansion coefficients calculated with numerical integration of the left-hand side of Eq. 8 to validate the results. Again, 20 expansion terms capture the bulk of the OTF shape with the main discrepancy occurring at the origin. To further reduce the differences, additional expansion coefficients are easily calculated.

4. SUMMARY

A technique has been developed to decompose the OTF of an optical system into a set of basis function. While for the vast majority of cases, the expansion coefficients of this decomposition need to be calculated numerically, there are a handful of cases where the coefficients can be solved for analytically. The analytic case involves situations where the complex pupil function is quadratic. This includes cases where both defocus and Gaussian apodization are present. The case where Gaussian apodization in the presence of defocus has been previously explored.[4] Here, the case where only defocus is present, but the aperture is an annular ring is analyzed. The results demonstrate answers that are consistent with numerical calculations of the annular pupil. These results can also be extended to include more complex systems. For example, a Fresnel zone plate can be considered as a series of annular apertures. Expansion coefficients for each of the annular apertures can be combined to determine the net expansion coefficients for the OTF of the zone plate. Similarly, wavefront coding systems of concentric rings, each with its own phase shift, have been proposed to extend the depth of focus. Such systems are again easily analyzed with the results presented here.

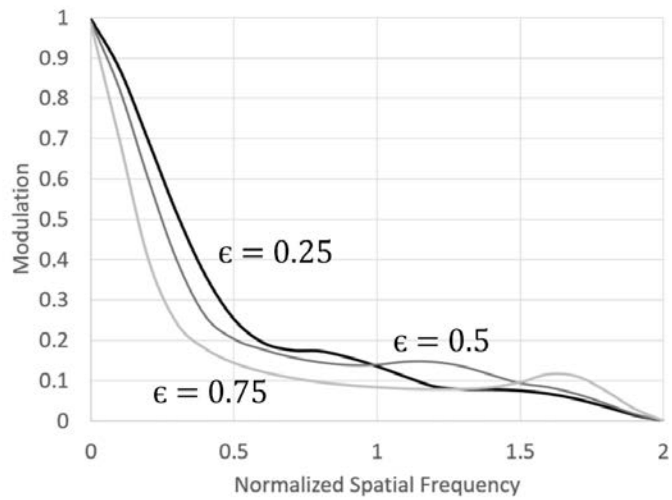

Figure 2. Calculated OTF for three different inner diameters of an annular aperture and a half-wave of defocus.

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