APPLICATION OF FEEDBACK ANALYSIS IN HYDROLOGICAL MODELING

by

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A Thesis Submitted to the Faculty of the

DEPARTMENT OF

HYDROLOGY AND ATMOSPHERIC SCIENCES

In Partial Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

In the Graduate College

THE UNIVERSITY OF ARIZONA

2018

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ACKNOWLEDGMENTS

This has been a truly fun and exciting project to be a part of, but without the work Jesse Dickinson had already done on it, I would probably still be years from completion. Watching Ty Ferré and Jesse debate different points of the project over the course of the last year encouraged me to start thinking out loud and asking my own questions – without fear of being "wrong". This is the single greatest thing I could have hoped for from this degree program, and I am so grateful to have been included in those discussions and to have had the opportunity to work with both of them.

This document would not be what it is without Ty's amazingly efficient and thorough editing, and insight into the broad-scale meaning of each small piece put together here. Jesse has been a solid resource for all points of this project, and has always been available for meetings, discussions, and editing this document – even though he has a fulltime job that requires him to do none of those things!

I would also like to acknowledge the time and effort my husband has put into supporting me, in all ways great and small, throughout this entire process. He has handled my stress and distraction with kindness and has been stepping up to whatever plate I may have abandoned with grace for years now – I owe you so many parachutes!

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Abstract

Feedback analysis has been a standard element of the design of electrical systems for almost one hundred years and has been an important tool in climate change studies for decades. This study is motivated by the lack of feedback analysis in hydrologic science. I propose that feedback analysis can be applied to hydrologic models to identify the most and least stable parts of a hydrologic system. In this study, I focus on developing a feedback framework for a simple hydrological model and quantify a feedback strength, which is a relative index of the importance of a feedback mechanism in regulating the system response to an external stress. Resource managers may use the feedback strength as a tool to identify and protect the least stable areas, or identify the feedback mechanisms in stable areas that could be developed to enhance stability in more vulnerable areas. This project builds on feedback analysis of simple electronic circuits, borrows from advances in the atmospheric sciences, and develops a new feedback analysis for groundwater systems and models.

Introduction

Hydrological systems are, in many ways, very similar to other natural systems. They can be described using boundary value analyses: the conditions on the boundary of a domain are defined, the connections within the domain are described, and the state of the system is calculated. This can be straightforward if only one process is considered and if that process is linear. Or, it can be more complicated if multiple, non-linear, interacting processes are at play. In this study, I consider a specific type of process interaction: feedback. Specifically, I hope to establish how feedback concepts can be applied to improve our ability to analyze complex hydrological systems and to communicate these analyses to non-experts.

The concept of feedback was formalized by Harold S. Black in 1927, but his idea proved so difficult to grasp that his patent took nine years to be approved – by which time he had many of the devices his patent described already in operation. He deserves credit for formalizing and determining how to harness feedback in system design, but mechanical devices such as float valves have been in use for centuries. These devices rely on a feedback process to achieve a desired equilibrium state, despite consistent perturbations. Quantifying a feedback can provide an understanding about how a system will respond to and “absorb” perturbations to an input and can reduce the uncertainty of responses to additional perturbations. In nature, feedback processes can cause a system to oscillate and adjust towards or away from equilibrium. On the right timescale, the response of a system with feedbacks can appear to be seamless and steady through time. Feedbacks that nudge toward equilibrium lead to steady state conditions, and it can be informative to understand the processes that maintain or justify a steady state analysis.

Feedback analyses can also identify an extreme change in perturbation that nudges a system away from an initial equilibrium. This can cause the system to cross a threshold where the system processes change, or even disappear. This ultimately leads to a new-normal equilibrium. The same feedback processes could be present at the new equilibrium but have different strengths. This underlies the popular concept of threshold behaviors in hydrology: crossing a threshold means the system moves from the initial equilibrium, and toward the establishment of a new-normal condition. The outcome depends on the nature and strengths of the feedback processes that persist in the final condition.

One example of a feedback that pushes away from equilibrium is the ice-albedo feedback in the global climate system. The process can be modeled by an increase in atmospheric CO_2 concentration, which leads to an increase in global temperature. Ice melting due to heightened average surface temperature shrinks ice cover in polar regions. This decrease in ice cover decreases the albedo of the Earth’s surface, which allows more energy to be absorbed, nudging the temperature up even more. This process began with a temperature increase and the result of the feedback process was a further increase in temperature. This action to amplify the initial change to the system is a hallmark of what is referred to as a positive feedback process. A decrease in temperature that leads to a further temperature decrease would also be an example of positive feedback process. In the opposite case, a decrease (increase) in temperature with feedbacks that lead to an increase (decrease) in temperature is negative feedback; this pushes the system back towards its initial equilibrium. Negative feedbacks act to dampen perturbations to the system. Positive and negative feedbacks can act simultaneously, embodied in different system processes. The ice-albedo effect is one of multiple atmospheric feedback processes that interact to create such a combined response. To forecast the cumulative effects of these processes, climate scientists have developed methods to quantify the feedback strength of each process. This allows them to project how temperature will change in response to different projected atmospheric CO_2 concentrations.

Identifying feedback processes and incorporating them into climate models is an ongoing challenge. Many of the recent changes to climate predictions are based on improved recognition of feedbacks, such as the ice-albedo effect or the release of methane from melting permafrost. To date, and somewhat surprisingly, these approaches have not been introduced to subsurface hydrological analysis. Rather, hydrological models tend to be described based on uncertainties in model conceptualization and/or model parameter values. Unfortunately, many of the methods developed to quantify these uncertainties are directly or indirectly based on assumptions of model linearity or they ignore parameter interactions. It is possible that analyses based on feedbacks can improve our ability to quantify forecast uncertainties. As important, it is possible that the concept of feedback could be useful for explaining forecast uncertainties. That is, if presented thoughtfully, we may be able to draw on widespread public understanding of simple examples of positive and negative feedbacks in everyday life to communicate the interactions of complicated, unfamiliar hydrological processes.

Feedback Concepts

Feedback is simply the idea that the output of a system informs some internal process that then feeds back to and affects the input. To quantify how a feedback process affects a system, it is necessary to compare it to a similar system where feedbacks are not present or are deactivated. An input propagates through a system without feedback – a reference system – as a straight line of events (Figure 1).

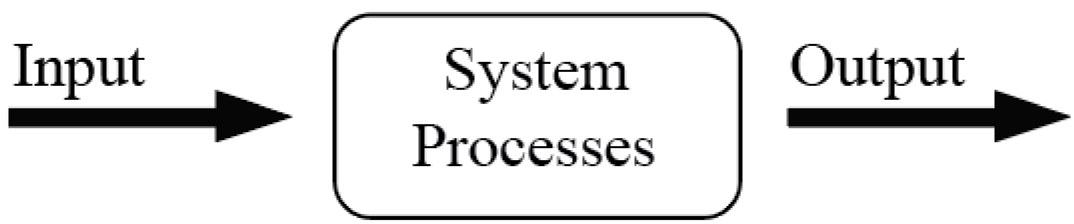


Figure 1. A reference system can be thought of as a linear chain of events, where once a signal leaves some part of the system, it does not return to it.

The simplest of systems have one unique response to each given level of input. If the output is independent of the state of the system, the system is said to be linear. Linear systems are additive, allowing for solution by superposition. A hydrological example of a simple linear system is flow through a confined aquifer system. As long as the equipotential surface is above the base of the confining unit, the saturated thickness will not change as a function of the water pressure. As a result, for example, a doubling of the head gradient would lead to a doubling of the flow through the system. Furthermore, if two processes that lead to changes in the gradient occur simultaneously, such as a change on a constant head boundary and a change in the pumping rate of a well, their combined response can be determined by summing the responses to the two individual changes.

Consider a simple example related to a confined aquifer. A simple system undergoing steady state flow is subjected to a change in the inflow, while the head at the outflow end is held constant. From Darcy's Law, the head gradient ∇H is linearly related to the steady state flow Q:

$$\nabla H = \frac{dH}{dL} = -\frac{Q}{KA} \quad \text{Eqn. 1}$$

where dH is a difference in head H at two points separated by distance dL, K is hydraulic conductivity, and A is cross-sectional area perpendicular to flow. The negative sign on the right-hand side of Equation 1 arises due to the convention that the change in H is calculated by subtracting the upstream value from the downstream value.

To draw analogies with electrical systems, we reorganize Equation 1 as:

$$\frac{dH}{Q} = -\frac{dL}{KA} \quad \text{Eqn. 2}$$

and restate Equation 2 in terms of voltage V and current I, with the subscript added to represent a reference system:

$$\frac{V}{I} = \beta_0 \quad \text{Eqn. 3}$$

Voltage is a measure of the electric potential difference between two points. This potential difference is analogous to a difference in the total head dH across distance dL in a groundwater system. However, the convention in electrical flow is to subtract the downstream value from the upstream, so there is an additional negative sign in the hydrological analysis. Then, the relation between a change in the voltage (ΔV) and a change in the input flow (ΔI) is:

$$\frac{\Delta V}{\Delta I} = \beta_0 \qquad \text{Eqn. 4}$$

A change in voltage in response to a change in applied electrical current is analogous to a change in head difference due to a change in applied water flow. This change in head can be written as $dH_1 - dH_2$, where the subscripts refer to initial (1) and perturbed (2) states.

Contrast the result above, for a confined aquifer system, to the responses of a system that is identical in all ways except that it is unconfined. For this system, both the saturated thickness (b) and the head gradient can change in response to a perturbation to the flow. (Note that the saturated thickness is equal to H if the datum elevation is placed at the base of the aquifer.) This can be described as a dependence of the transmissivity (Kb) of the system on the head at each location. Or, it can be described as a system with feedback (Figure 2). In this context, an increase in head at the inflow boundary would lead to an increase in the saturated thickness at that boundary. This would reduce the gradient needed to transmit a prescribed flow, thereby acting to reduce a head change $dH_1 - dH_2$ of the unconfined system compared to a confined reference system. As described above, this constitutes a negative feedback.

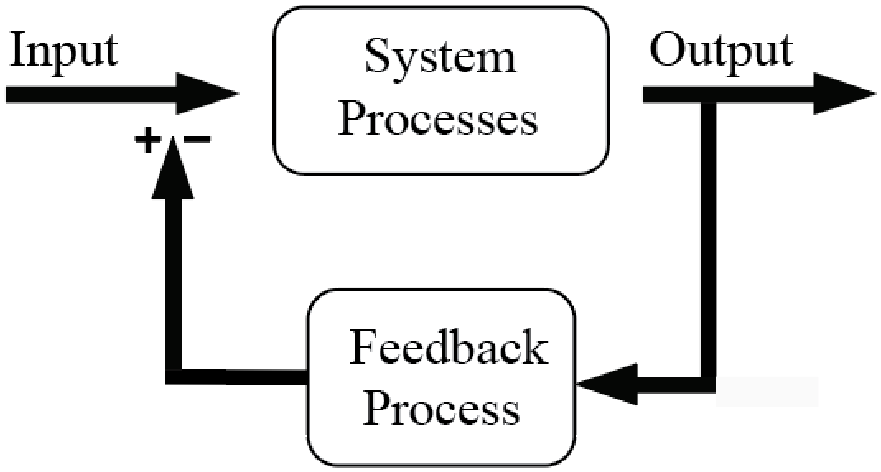


Figure 2. A more complicated system with an information loop that takes a signal from the output and then influences the input.

Because of the feedback introduced by changes in saturated thickness in the unconfined system, the ratio of the change in response to the change in the input will no longer be constant. For a given change in the system, this ratio can be defined as:

$\frac{\Delta V}{\Delta I} = \beta_f$, where f indicates the feedback system. Eqn. 5

The ratio of the β values for the systems with and without feedback can be used to define the strength of the feedback process. This ratio, called the gain, reflects what is "gained" by the inclusion of a feedback:

$G = \frac{\beta_f}{\beta_0}$ Eqn. 6

The strength of the feedback process is a non-dimensional quantity and is based on the gain:

$f = 1 - \frac{1}{G}$ Eqn. 7

Similarly, if the feedback strength is known, the gain can be defined as:

$$G = \frac{1}{1-f} \qquad \text{Eqn. 8}$$

Quantifying Feedback

Feedbacks are quantified by comparing the responses of the reference and feedback systems to a specific perturbation applied to both. Figure 3 shows the nonlinear relationship between the gain and the feedback.

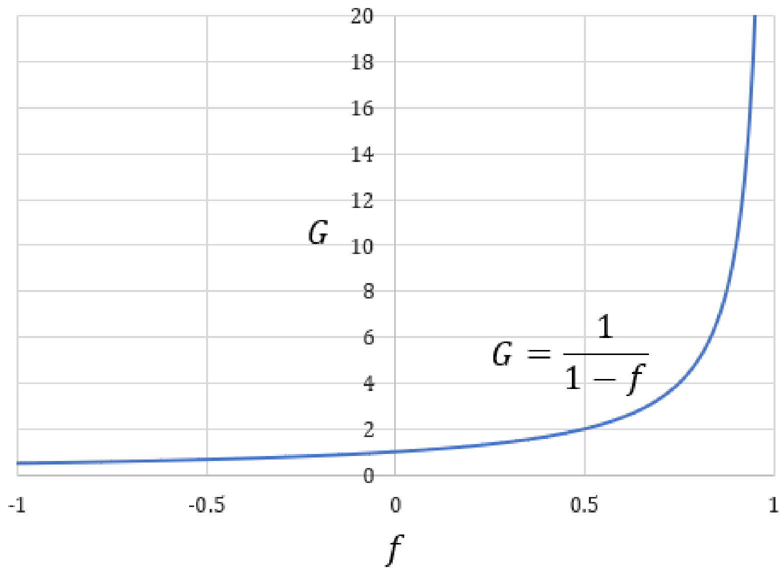


Figure 3. The gain, given by the ratio of the system responses ($G = \frac{\beta_f}{\beta_0}$), vs. the feedback value found from Equation 7.

The mathematical relations and Figure 3 can be related to common understanding of feedback by considering systems with positive or negative feedback. Equations 7 and 8, and Figure 3, show that when the response of the system with feedback is larger than the response of the reference system ($\beta_f > \beta_0$), the feedback will be positive, and the gain will be greater than one. When the response of the perturbed system is smaller than that of the reference system ($\beta_f < \beta_0$), the feedback will be negative, and the gain will be less than one. In common terms, if the feedback is positive, the system gains more than would be expected without the feedback, leading to amplification. Alternatively, if the feedback is negative, the system gains less than would be expected for the same system without feedback, leading to dampening relative to the reference case.

Negative feedbacks act to nudge systems back toward equilibrium states when perturbed – they respond to a perturbation by nudging in the opposite direction. A centrifugal governor (Figure 4) is an example of a negative feedback mechanism that acts to regulate engine speed. The arms, weighted with flyballs, spin proportional to engine speed. Increased speed leads to increased angular momentum, which causes the arms to rise. As the arms rise, a lever attached to them incrementally closes the fuel valve, which causes the engine to decrease its speed. As the speed decreases, the governor spins more slowly and the flyball arms lower, opening the fuel valve. For every change in the speed, the governor feeds back a signal that causes a change in the opposite direction. This is a fundamental property of a negative feedback – it acts to decrease the amount of change applied to a system. This system will oscillate around the target speed by means of this constant response in the opposite direction of the input (Figure 5). Physically, this system can be designed so that those oscillations are not apparent on some timescale and the speed appears to be constant. This is generally true of all systems that we would consider to be steady state – they are constant over some averaging time due to the dampening effects of negative feedback processes.

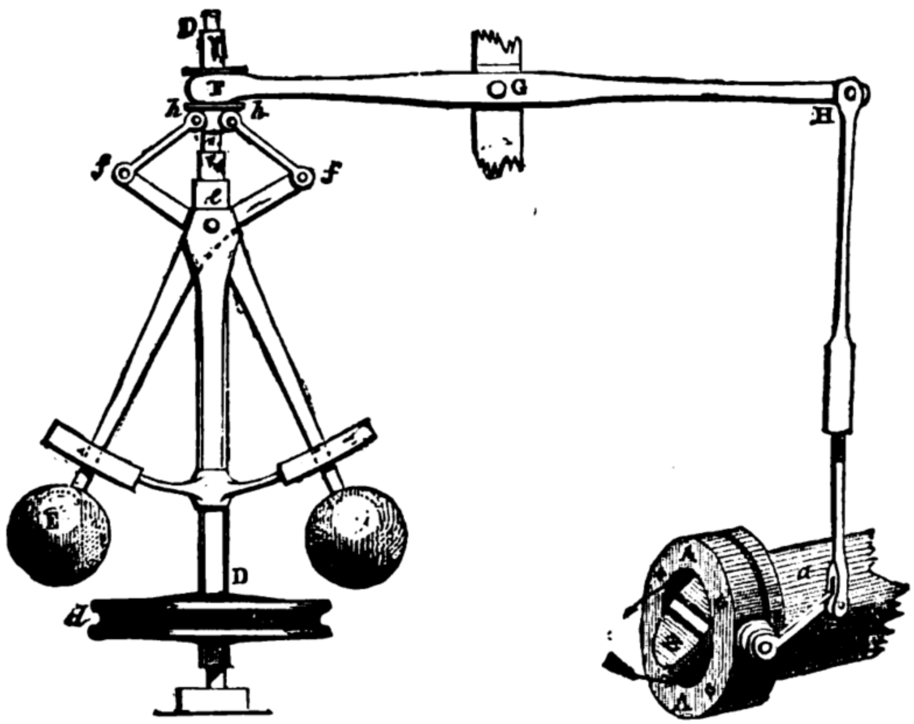


FIG. 4.—*Governor and Throttle-Valve.*

Figure 4. Centrifugal governor, first designed to moderate the speed of a steam engine by means of a negative feedback mechanism. Image from "Discoveries & Inventions of the Nineteenth Century" by R. Routledge, 13th edition, 1900.

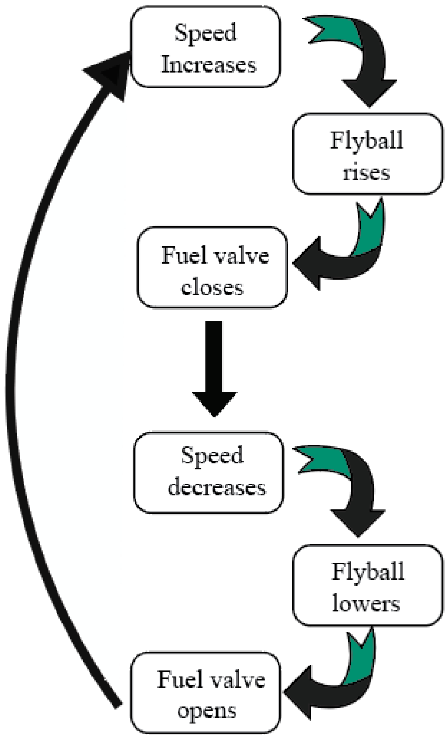


Figure 5. This conceptualization of the centrifugal governor's process of maintaining a constant target speed shows two possible loops that are connected in an endless back and forth to establish a stable, constant speed through a negative feedback.

The ice-albedo feedback, mentioned earlier, is a positive feedback. Positive feedback processes work to nudge a system away from equilibrium when perturbed, by generating a response in the same direction as the perturbation. A decrease in surface temperature causes more ice to grow and the albedo to increase, resulting in less energy absorbed at the surface, and a further nudge of the temperature downward. An increase in surface temperature will lead to an even greater increase in temperature. Figure 6 shows how this positive feedback amplifies a change in surface temperature regardless of which direction it is first perturbed.

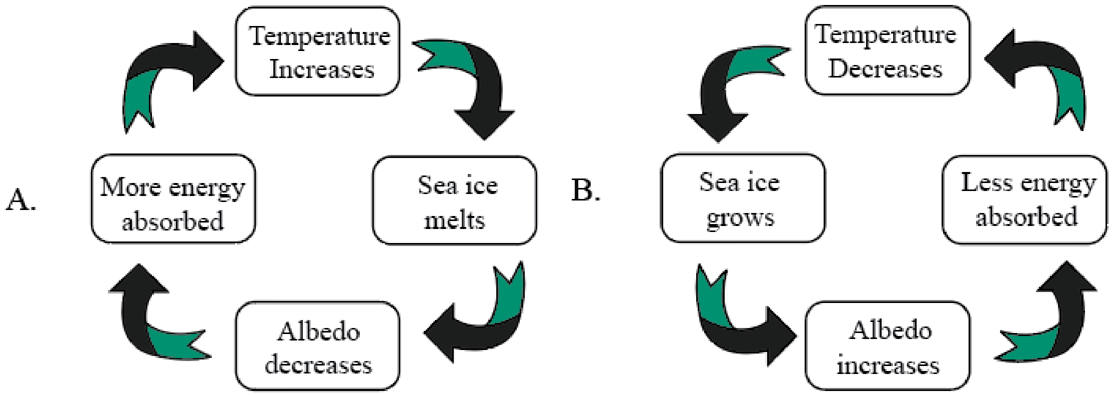


Figure 6. Positive feedback loops of sea ice and albedo for a change in the surface temperature of Earth.

The Reference System

A reference system is necessary to quantify a feedback. For the simple hydrological example presented above, the confined groundwater system is the reference system for quantifying the feedback strength of changes in the saturated thickness of the unconfined system. A system with no ice-albedo interaction is the reference for quantifying the strength of the ice-albedo feedback. In general, the reference system accounts for the influence of all system properties that are not subject to feedback (e.g. size and shape of the domain, intrinsic permeability, porosity, anisotropy, etc.). By accounting for all other influences, a reference system allows for a more direct description of the effects of the feedback process. In this context, the reference system can be any system, including one with feedbacks that cannot be excluded, or are not the focus of the study. The only requirement is that the reference and feedback systems can only differ by the exclusion or inclusion of the feedback process or processes being quantified.

To illustrate a feedback quantification that may be of interest to a water manager, consider the decision of whether to take steps to avoid the introduction of invasive plants that may replace native vegetation. The water manager may pose the question: how do both plant types affect the resilience of the water level to imposed changes in the water table elevation? Or, the water manager may ask the more direct question: what would be the change to the resilience of the system if native plants were replaced by invasive species? The invasive plants being considered have deep taproots and very high transpiration rates: that is, ET is only a very weak function of the water table depth. In contrast, the native plants have shallower roots, such that their ET rate decreases more strongly when water table depth increases. Both systems have negative feedback: the ET response of the plants acts to decrease the amount the water table is lowered when perturbed downward, compared to an identical system without plants. (Similarly, if the water table rises, the native plants will increase their ET, reducing the rate of water table rise.)

There are two strategies for a feedback analysis in this case. The first would be to set up a reference system that has vegetation with fixed ET, independent of the water level. (Note, the constant ET rate could be zero, representing no plants.) There would then be two cases to consider and each would be compared to the reference case separately. The first case would include ET as a function of the water table that approximates the use of groundwater by the native plants. The second would represent an ET response that reflects the water level dependence of the invasive plants. In this case, both plants would return negative feedback values, relative a perturbation to the water table (a gain less than 1.0). But, the native plant rate would be more negative. That is, the two feedback values can be compared to one another to assess the resilience of the water level to imposed changes in water level for both systems.

However, both values would be responses measured from the reference case, so the values obtained would have to be communicated from the context of that system.

The second way to conduct a feedback analysis is to answer the water manager's more direct question. For this analysis, we could set the existing, native plant community as the reference case. The feedback system would replace the water level dependence of the native species with that of the invasive species. In this case, the feedback would be positive because the response with invasive plants (test case) is larger than the response with the native plants (reference case), for perturbations to the water table. But, the result would only be meaningful with respect to the reference system. That is, I have just described the same feedback process (invasive plant ET rates) as negative and then as positive, merely by changing the reference case. Any such analysis then must include a clear description of the reference case. Either reference case given above can be applied, and each may be more appropriate for different analyses. But, it will require careful analysis and communication of results to avoid misunderstanding.

There are also different viewpoints from which the results of a feedback analysis can be viewed. Consider an extension of the problem described above. Now, the water manager may be interested in predicting the likelihood that invasive species will displace native species if the water table in the area is perturbed downward. If we take the rate of ET of a species to be positively correlated with species health, then a lowering of the water table will promote advancement of the invasive species. Consider this in the context of feedback. A lowering of the water table has a greater impact on the native species than on the invasive species. This leads to an increase in the health of the invasive species relative to the native species, leading to an increase in the relative abundance of the invasive species. This transition diminishes the negative ET feedback of the native plants, responsible for reducing the rate of water table decline. This then causes greater declines than would have occurred if the native plants maintained their numbers. The additional lowering of the water table would lead to a greater population advantage for the invasive plants, relative to the native plants. These linked feedback processes are common in natural systems, including hydrological systems. Therefore, it is important to develop methods that can assess the contributions of multiple, simultaneous, interacting processes.

Two major challenges for adopting feedback analyses for hydrological systems are: 1) to define the question as exactly as possible for the effective application of feedback analyses; and 2) to determine the most appropriate reference systems for the question posed.

Methods

The methods used in this work to quantify feedback are based on feedback analyses developed for both electrical systems and climate systems. The concept of feedback is universal, and there are feedbacks at work in every physical system. Understanding what they mean and how to use them has been a powerful tool for many decades. Unfortunately, the methods of calculating feedback, and even the definitions of standard terminology, vary considerably between and even within disciplines. In fact, making sense of the feedback methods presented elsewhere, and applying them to hydrology, has been the major emphasis of this study.

The work completed here was devoted to unifying three different methods for calculating feedback, defining the underlying terms and concepts in hydrologically-relevant ways, and showing (for the first time) how they could be applied to hydrological systems. The Methods section first reviews previous approaches to feedback analysis, then reviews the strengths and weaknesses of different approaches. Then, in Results, the challenges of translating these to hydrological analysis are shown through an application to a simple one-dimensional groundwater flow system. Further consideration of the use of feedback analyses for more complicated hydrological systems, including transient conditions, is presented in the Discussions section.

Feedback in an electrical system is an element of the system design that is manipulated to achieve a desired outcome. In atmospheric climate studies, feedback is used to quantify the relative strengths of known feedback processes to enable better modeling of future climate scenarios. Due to the disparate nature of the two types of systems, the terminology, equations, and methods used for each type do not seem immediately relatable or transferrable (e.g. Hansen et al., 1984; Schlesinger, 1989; Roe, 2009; Schwartz, 2011). Lack of consistency in terminology across climate science alone can make it difficult to build on the work done in that field, and not all uses of feedback in electrical systems are apparently relevant in natural systems.

When an electrical system that includes feedback experiences a perturbation to an input, such as current, the response signal in one location prompts a return signal to be "fed back" to an "upstream" location. As a result, the output changes the input to that location. This feeding back of the signal can either reduce the overall perturbation to the input (negative feedback) or increase it (positive feedback). The feedback in an electrical system can generally be described through a set of linked equations (e.g. current flow as a function of resistance and voltage and resistance as a function of voltage).

Electrical systems are often used as analogs to understand and describe many complicated natural systems. In general, the electrical systems provide simple, transparent models that have a clear system design, well-developed governing equations, and associated mathematical models. The challenge lies in drawing the correct analogy for natural systems, for which the properties are uncertain, the governing equations are simplified representations of multiple interacting processes, and the effect of feedbacks is often unquantified.

Given the intense international interest in forecasting the impacts of human activities on future climate, the scientific disciplines related to climate forecasting have led the way in adopting feedback analyses. This interest has grown from the recognition that the proper inclusion of system feedbacks can dominate the accuracy and uncertainty of model forecasts. Unless climate models are built within a carefully and clearly developed framework for including feedback, efforts to plan adaptation and mitigation strategies will be ineffective, at best, and potentially counter-productive, at worst. Given the recognized impact of human activities on atmospheric CO_2 concentrations, it is useful to consider a climate-related analysis that includes feedbacks that affect surface warming due to changes in CO_2 levels. The feedback methods for climate models are a useful template for feedback methods for hydrological systems because both involve similar physical processes – and challenges.

One challenge in any system analysis is to define which aspects are internal to the system and which can be modeled as external drivers or stresses. There is typically a single response variable of interest, but it may be related to and affected by changes to multiple other variables in the governing equations of that system. Typically, an external stressor is used to manipulate an input variable that has a clear mathematical relationship to the response variable. In many climate analyses, the external stress is a change in atmospheric CO_2. Because CO_2 absorbs outgoing radiation (heat), this change perturbs the radiative balance at the top of the atmosphere. The resultant change in the radiative flux translates to a change in surface temperature. This system has no feedback if it is modeled as a blackbody radiator such that a change in the radiative balance directly yields a change in surface temperature, and no other processes that could affect the radiative balance are considered. The change in the input is the change in the radiative flux at the top of the atmosphere. The change in the response is the change in the surface temperature. For the case without feedback, both the radiative flux and the temperature response can be calculated directly. But, for the feedback case, the amount of change in radiative flux due to a change in CO_2 is not directly known. This is because the feedback processes in the system will nudge the radiative flux, relative to the reference case, and this nudge is unknown until a feedback analysis is completed.

Another challenge in analyzing or modeling a system is choosing the system boundary conditions and parameter values. It is recognized that many model parameters are not true representations of physical properties. Rather, the parameters provide the best fit of an imperfect model to observations made in the real system. In particular, systems that include feedback often subject the attribution of the effects of feedback to physical parameter values. Proper inclusion of feedback leads to a more accurate representation of many systems, leading to improved forecasts and a deeper understanding of system responses.

Modeling system boundaries requires a pathway for relating a change that is external to a system to an outcome of interest that is internal to the system. For example, climate scientists calculate the impact of the ice-albedo effect, initiated by changes in CO_2, on surface temperature. Here, a CO_2 change is external because its change is independent of the system state. The external change results in responses of both the radiative flux and the surface temperature, which are then used to quantify the feedback. (The changes in radiative flux and temperature are similar to changes in current ΔI and voltage ΔV, repectively, as developed for the electrical case.) As discussed below, this translation of external stresses of interest to processes that are relevant to the modeled system has been one of the most difficult aspects of adopting feedback analyses for hydrologic modeling.

The first step for the climate example is to represent the system with no atmospheric or environmental feedbacks. The change in surface temperature is calculated directly from the Stefan-Boltzmann Law, which relates the radiative flux to temperature. After this initial calculation, the second step requires the addition of the feedback mechanism or mechanisms of interest. There are numerous feedback mechanisms at work in this system, including ice-albedo, water vapor, and cloud effects. Each of these can be described through functional relationships with radiative fluxes and surface temperature, and several can interact with one another. When any or all of these feedbacks are considered, the Stefan-Boltzmann Law no longer provides an adequate model of the system. Rather, the response of the real system, with feedbacks, is approximated using numerical models.

One way to use these models, which has been applied widely in hydrology, is to perform a sensitivity analysis. In these approaches, the net change in the response, including those affected by feedbacks, is calculated over a range of inputs to form a sensitivity function. While this is useful, it does not separate the response to a perturbation directly to the feedback processes. Similarly, it is common in climate modeling to simply apply the radiative perturbation calculated from a change in atmospheric CO_2 concentration, and then model the response of the surface temperature using parameterizations of the feedback processes. But, this does not account for the action of those processes in the system. Instead, the expected relation between each process and a change in the flux is parameterized based on observed or independently modeled changes. This has the result of disabling the feedbacks in the system and accounting for only the partial effects of feedbacks in the parameter values.

There is also a limitation in a sensitivity analysis in that it does not recognize how the system responses themselves may change, it only gives information about the sensitivity of the model to specific parameterized responses. In other words, the model is constrained by limitations placed on feedback processes to simplify the model. This does not then account for the way in which the response may further nudge the radiative flux, which has led to several revisions of climate forecasts using global climate models. Headlines such as "Why Soil Could Make Climate Change Worse than Scientists Thought" (Worland, 2017), reflect this, to some extent. The parameter values that have been changed from their initial intended meaning to account for the effects of feedbacks become increasingly unrepresentative as the system changes from the calibrated conditions. When change to some system appears to be accelerating (or decelerating) compared to modeled predictions, it may prove useful to do additional modeling with the inclusion of feedbacks. For this case the additional changes in the surface temperature when the system is modeled with feedbacks may provide better prediction of the effect of additional perturbations.

One of the most difficult concepts to grasp in applying feedback approaches is that a feedback response can be measured without measuring its effect on the input. Here, we measure the feedback response as a comparison of the responses of a reference "no feedback" system and a feedback system – as shown above. But, it is also possible to calculate the nudge to the input, which is the amount of change to the input as a linear proportion of the feedback system response given as:

$$\Delta V_f = \beta_0 \left(\Delta I + c \Delta V_f \right) \qquad \text{Eqn. 9}$$

For the climate example, the first part of the feedback analysis is to represent the system with no atmospheric or environmental feedbacks. The change in surface temperature is calculated directly from the Stefan-Boltzmann Law, which relates the radiative flux to temperature. After this initial calculation, the second step requires the addition of the feedback mechanism or mechanisms of interest. There are numerous feedback mechanisms at work in this system, including ice-albedo, water vapor, and cloud effects. Each of these can be described through functional relationships with radiative fluxes and surface temperature, and several can interact with one another. When any or all of these feedbacks are considered, the Stefan-Boltzmann Law no longer provides an adequate model of the system. Rather, the response of the real system, with feedbacks, is approximated using numerical models.

To calculate the nudge to the input, the response of the feedback system ΔV_f must be calculated. Given that this response is a function of the nudged input, Equation 9 can be rearranged to solve for ΔV_f, the total response of the feedback system, by:

$$\Delta V_f = \frac{\beta_0 \Delta I}{1 - c_1 \beta_0} = \frac{\beta_0 \Delta I}{1 - f} \qquad \text{Eqn. 10}$$

There is additional information in Equation 10, that states that the feedback value is equal to some linear proportion of β_0 indicated as c_1. This is merely a rephrasing of the formulation we have already seen, which relates the feedback value to the ratio $\frac{\beta_f}{\beta_0}$. Simply put, feedback is a function of the distance between the feedback and reference systems. Finally, if we combine Equations 10 and 5, we arrive at the definition for feedback given in the previous section defining the gain as $G = \frac{1}{1-f}$.

Once we quantify the system response with feedback ΔV_f, and the feedback value f found from it, the nudge to the input can be calculated by summing the terms in parenthesis in Equation 9. Additionally, Equation 10 can be used to generate predicted responses to additional input perturbations (ΔI), without requiring further modeling. Here we will follow this method, first by finding the feedback by a comparison of the system responses, and then show that the resulting feedback can be used to predict those – and additional – system responses.

The specific challenge of this study was to derive a flexible set of methods for feedback analysis of hydrological systems. To provide this flexibility, which may be necessary to address the wide range of hydrological questions and data limitation that we face, multiple methods were developed and a study was undertaken to demonstrate that they all result in the same feedback value for a an example hydrological problem.

Results

The first step toward the overall goal of the project was to apply the multiple feedback analysis approaches to a simple electronic system. This effort to untangle the methods of various studies and efforts in sciences unrelated to hydrology was much aided by the work of Roe (2009), who specifically undertook such an analysis to support future work in atmospheric sciences. Each method represented in this work has a particular use or gives more information about how the feedback and reference systems differ and why.

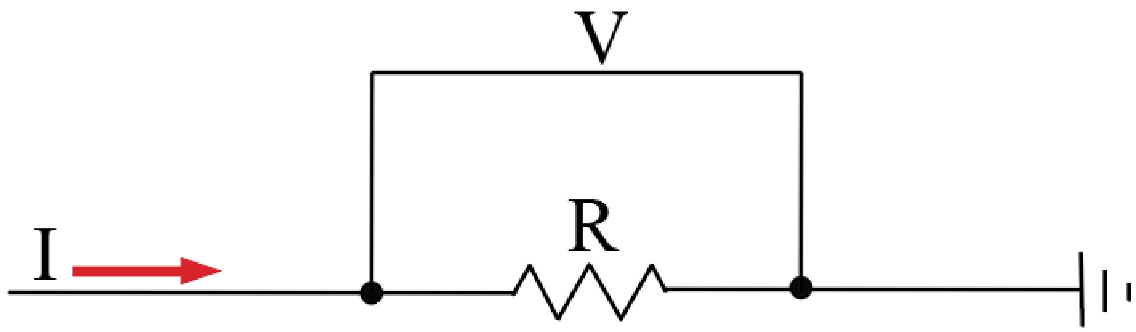


Figure 7. A simple electrical circuit with voltage shown across a distance with resistance R and current I.

To illustrate the three approaches to calculate feedback, consider a simple electrical circuit (Figure 7), which is frequently used to explain feedback in electrical systems. Consider this system with and without feedback. The no-feedback system, or reference case, has a single resistor in which the resistance is constant. The relation between current I, voltage V, and resistance R is $I = \frac{V}{R}$. In the feedback system, the resistance is a function of the voltage across it, here assigned as: $R(V) = 26 - 2(V - 12)$. Both systems start from the same state. That is, the resistance for the initial applied voltage of 12 is the same for both cases, and the resistance R decreases with increasing voltage V for the feedback case. The response to a perturbation applied to each system can then be compared directly.

Before continuing, let's draw an analogy between the mechanical governor and the simple electrical circuit with variable resistance. The governor, in a general sense, decreases the energy applied to the system (rate of fuel added) when the rate of revolution increases. But, it does not change the work required to achieve a given rate of rotation. If a directly analogous electrical system was designed with feedback to achieve a constant current, it would have a mechanism that could decrease the applied voltage as the current increased beyond the desired value. The simple electrical system is similar, but not identical. In the electrical circuit shown, the voltage is not changed, but the resistance (representing the work required to achieve a given current) decreases with increasing current. If the voltage increases beyond the base condition, the current also increases. This will decrease the resistance, which will lead to an additional increase in current, constituting a positive feedback.

Now consider the same system that is designed to maintain a constant voltage regardless of the current applied to the system. Starting from the equilibrium condition, if the current is increased, it will result in an increase in voltage. But, the feedback mechanism causes the resistance to decrease for the increased applied current. As a result, the voltage will increase less in the feedback case than in the case without feedback, constituting a negative feedback. The only difference between this system and the very similar system described in the previous paragraph is which quantity is considered the stressor and which the response. In the 'design' sense, these are related to the goal of the system: to maintain constant voltage or constant current. In a natural system, there is no intended outcome, but the response of interest and the applied stress can be different depending upon the purpose of the investigation. The critical point here is that the value, and even the sign, of feedback depends on the specific definition of feedback applied. This depends on which quantity is specified and which is free to vary. This has major implications for modeling decisions and is another startling example of the importance of clearly defining the reference case and the nature of the perturbation when presenting feedback analyses: the potential for misunderstanding is great!

For the non-feedback system, the ratio of the change in voltage across the resistor for a change in current β_0 is constant and equal to the resistance *R* (Table 1). For the feedback system, this ratio β_f is not constant for different changes in voltage. As defined above, the resistance in the feedback case decreases if the voltage increases.

no feedback: R constant:	V	I	R		Feedback: R decrease:	V	I	R
Base	**12**	**0.4615**	**26**		**Base**	**12**	**0.4615**	**26**
Perturbed	11.8	0.4538	26		Perturbed	11.8	0.447	26.4
	11.9	0.4577	26			11.9	0.4542	26.2
	12.1	0.4654	26			12.1	0.469	25.8
	12.2	0.4692	26			12.2	0.4766	25.6

Table 1. Results of perturbing the voltage for the case where *R* is constant, and *R* decreases with increasing voltage. Note that the resistance remains constant for the reference case but decreases with increasing voltage for the feedback case.

Here, we assess three approaches to quantifing the feedback strength. We will apply each to the second description of the system presented above: calculating the change in current in response to a perturbation of the voltage difference. The first method for calculating this feedback is based on the ratio $\frac{\beta_0}{\beta_f}$. First, we compute $\beta_0 = \frac{Vpert-Vbase}{Ipert-Ibase} = \frac{11.8-12}{0.4538-0.4615} = 26.$

Then, for the feedback system, $\beta_f = \frac{11.8-12}{0.447-0.4615} = 13.728.$ Finally, we can combine Equations 6 and 7 to calculate the feedback:

$$f = 1 - (\frac{\beta_0}{\beta_f}) = 1 - \left(\frac{26}{13.728}\right) = \mathbf{-0.89}. \qquad \text{Eqn. 11}$$

The term f is commonly called the feedback factor (Roe, 2009). We confirm the qualitative finding that the feedback is negative, but we further define the strength of the feedback using the feedback factor. This method, which we will refer to as the ratio or simple ratio method, only involves changes in input and response. However, this approach only returns 'lumped' feedbacks. That is, feedback factor f represents the total system response due to all feedbacks in the system. Here, we have only one feedback due to the relation between resistivity and voltage. In some cases, the ratio method may not provide sufficient insight to understand the importance of individual feedbacks in multi-feedback systems on the change of input and response.

Schwartz (2011) describes the second method of calculating the feedback strength. To establish consistent terminology throughout this study, we have changed the terms used in that paper. Specifically, the author defines the feedback as $f = \frac{\beta_f}{\beta_0}$. He defines what we have termed feedback as $\Theta = 1 - \left(\frac{\beta_0}{\beta_f}\right)$. The following equations will use our notation, but feedback found using this method will be referred to as Θ to differentiate it from the other methods. This method explicitly includes information about how the resistance changes for a change in voltage, but no information about the perturbation. The feedback Θ for the example is:

$$\Theta = \frac{\partial lnR}{\partial lnV} = \frac{\ln(26.4) - \ln(26)}{\ln(11.8) - \ln(12)} = \mathbf{-0.91}. \qquad \text{Eqn. 12}$$

This method, which we will refer to as the functional dependence approach, has the great advantage that it does not require a reference case. This allows for more sophisticated incorporation of knowledge of the functional dependence of the feedback on the system response. For example, the change in feedback due to the change in the dependence of resistance on the voltage difference described above can be calculated directly, without having to run the system simulation. This can be particularly useful for systems that require significant computational effort to simulate. For example, testing the response of the feedback circuit where the resistance is twice as dependent on the voltage, so that $R(V) = 26 - 4 * (dV)$, would require setting up the equation only, and not require any modeling. When the voltage is 11.8, the resistance would be 26.8. For this case then:

$$\Theta = \frac{\ln(26.8) - \ln(26)}{\ln(11.8) - \ln(12)} = \mathbf{-1.80} \qquad \text{Eqn. 13}$$

Doubling the constant defining the dependence of the resistance on the voltage had the effect of doubling the feedback value. The β_0 value is constant, because the reference case is the same. However, the gain of this circuit does not exhibit a linear change – β_f for the initial case was 13.728, but the case with the stronger dependence had a β_f of 9.416. That is, β_f did not change by a factor of 2, although the feedback (Θ) did. This is a critical component of a feedback analysis, and it relates back to Figure 3, which shows that the gain does not change linearly with changes in feedback. This again highlights the potential value of using feedback to describe hydrological systems as well as demonstrating the potential advantages of the functional dependence approach for feedback analysis in some cases.

The final calculation method is the most complete and robust, incorporating elements of the other two approaches. As a result, it is the most difficult to compute, but it offers the most insight into all contributions to the system feedback. This method, typically referred to as the kernel method follows from Equations 9 and 10. The kernel method includes β_0 from the simple ratio method, a similar relationship from the functional dependence approach, and an additional term that describes how the response might change if only one feedback were acting, separate from other feedback processes. A major advantage of this approach is that it quantifies the strength of individual feedback processes in systems having multiple active feedbacks by defining the strength of each process separately.

Using the kernel approach and following Equation 9, the feedback factor for process i is

$$f_i = \beta_0 c_i \qquad \text{Eqn. 14}$$

Where $i = 1$ to N number of active feedback processes in the system. The constant c_i can be determined for each feedback process as (Roe, 2009),

$$c_i = \frac{\partial I}{\partial R_i}\bigg|_{R_{j,j\neq i}} \frac{dR}{dV} \qquad \text{Eqn. 15}$$

For the electrical case with a single feedback ($i = 1$), the first term, $\frac{\partial I}{\partial R}$, represents the change in the inflow that occurs in response to a slight change in the resistivity R. This first term is called the "radiative kernel" in applications to Earth's climate (Soden and Held, 2006). The second term, $\frac{dR}{dV}$, represents the change in resistance per change in voltage, which was defined for this system as $R(V) = 26 - 2(V - 12)$. For cases of multiple feedback processes, the first term is determined for each feedback process i, while holding all other feedback processes j to be inactive, with the exception of process i. Roe (2009) provides more details on the kernel method for multiple feedbacks.

The feedback f from the kernel approach for a single process is:

$$f = \beta_0 * \left[\left(\frac{\partial I}{\partial R}\right) * \frac{dR}{dV}\right] \qquad \text{Eqn. 16}$$

For multiple feedbacks, $f = \sum f_i$. The β_0 term has been defined above, and the functional dependence of R on V is defined in a manner similar to Equation 12 (although not based on the natural log of the values). The first term in Equation 15 remains to be determined. For the example problem considered here, the value is found using Ohm's law to calculate two values for current, keeping voltage constant and only changing the resistance value. Note that the change in resistance is equal to the change used in calculating the functional dependence term, allowing these to cancel such that the two bracketed terms provide a measure of $\frac{dI}{dV}$ for the specific resistance perturbation considered. Thus, the first term in Equation 15 is: $\frac{\partial I}{\partial R} = \frac{\left(\frac{Vbase}{Rbase}\right)-\left(\frac{Vbase}{Rpert}\right)}{Rpert-Rbase} = \frac{\left(\frac{12}{26.4}\right)-\left(\frac{12}{26}\right)}{26.4-26} = 0.02$. The kernel value for feedback is:

$$f_i = 26 * 0.02 * \frac{0.4}{-0.2} = \mathbf{-0.91}. \qquad \text{Eqn. 17}$$

The subscript *i* indicates that this is the feedback value due to a single feedback process. The total feedback, incorporating the effect of including all feedbacks, can then be found by summing the f_i terms. Multiple feedbacks at one location can be summed, and then used to find the gain in that location by:

$$G = \frac{1}{1-\sum f_i} \qquad \text{Eqn. 18}$$

This summing of feedbacks to calculate the gain at that location is necessary, due to the relationship between gain and feedback in Figure 3. Calculating a gain from the β_f of every f_i and then attempting to sum them would produce the incorrect gain. That is, it would yield an incorrect estimate of how much change the feedback process is causing within that system, relative to a reference system. The β values of different parts of a system – at different locations

– may be summed and their total will be equal to the β value found by considering the total change across the system. The gain itself, however, is not summed in that manner, and neither are the feedback values. Feedback values are only additive at a specific location. β values are additive across space. Nudges to the perturbation to the system are additive (Equation 9), where additional feedback processes would lead to additional $c_i\Delta V$ terms. This is an important distinction, with implications for more complex models.

The kernel method is strictly applicable for a given perturbation from a specific base condition. This is generally true for the simple ratio method as well, but it is only explicitly noted for the kernel method. It is explicit in this method because the first term in brackets in Equation 16 is measured directly from the base or equilibrium state that is perturbed. It is also proportional to β_0, which requires that the perturbation applied to obtain that value match that applied to the feedback system.

Each method shown above has specific strengths and requires slightly different input information. The simple ratio method is most useful when information is limited to system responses. Since both β values are obtained by dividing ΔV by the same ΔI, the only information required to calculate feedback by this method is ΔV_0 and ΔV_f. To reiterate the previous discussion, V_0 does not have to relate to a feedback-free system. Rather, it must be a system within which one feedback process (or one set of processes) is deactivated. The resulting feedback then represents the additional feedback of the excluded process(es). This strength is also the weakness of this system. The simple ratio system cannot separate the contributions of multiple feedback processes, nor can it separate the impact of the feedback from the change in stress that gives rise to the feedback.

The functional dependence method requires more information about the nature of the feedback. In many cases, this dependence will be known; but, for more complicated systems, it may be difficult to isolate the functional description of the feedback process. This method does not require that a reference system be established, however, it is still useful to conceptualize what that reference case would be, so that the resulting feedback value answers a defined question.

The kernel method requires the most information to calculate, but it also yields the most information. This is the only method described here that can isolate the portion of the overall feedback that can be attributed to each feedback process when more than one is present. This can be critical for understanding how a system as a whole responds to changes, and for identifying if there are both positive and negative feedbacks interacting. As will be discussed below, this could be a critical feature for communicating the results of a hydrological analysis to water managers.

Hydrological Case

The simplest possible hydrological case that is analogous to the electrical example is a comparison of confined and unconfined groundwater systems. A confined groundwater system (Figure 8) is one that exists between two relatively impermeable surfaces. If the water pressure is positive everywhere in the system, the saturated thickness at any given point is constant, regardless of how much water is flowing through or what the water pressure is.

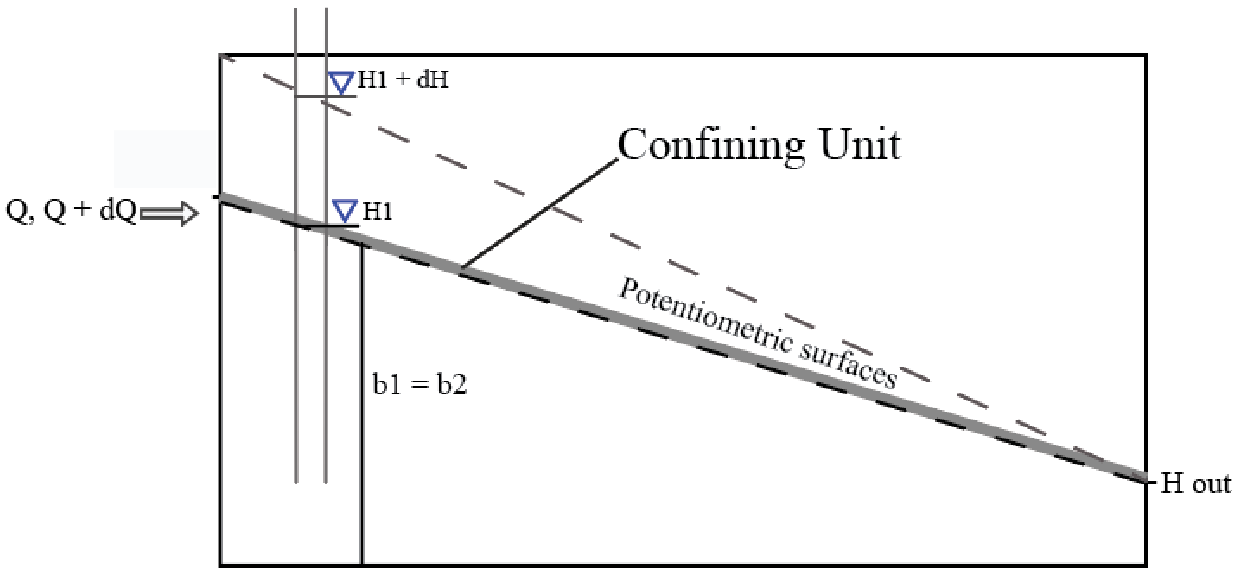


Figure 8. The confined aquifer system responds to changes in the amount of flow into the system by a change in the potentiometric head gradient, but the saturated thickness is constant.

The flow through a simple confined system is given by $Q = -Kba * \nabla H$, where b is the saturated thickness, a is the thickness of the system into the page, ∇H the hydraulic gradient, and K the hydraulic conductivity. To restate Darcy's Law in the form stated in Equation 1, $\frac{dH}{Q} = -\frac{dL}{T}$, we note that the transmissivity of the aquifer, T, is equal to $K * b$ and a = 100. As stated above, the negative sign results from the convention of calculating the hydraulic conductivity by subtracting the downstream head value from the upstream value. This is slightly different than the electrical case. For the electrical case, the voltage change is related to the current flow by the resistance, which is specified for a fixed distance and is expressed as $ohm * meters$. In contrast, for the hydraulic case, the hydraulic head gradient is defined as the head change over a specified distance and is related to the water flow through the conductivity, which is defined per unit length. We can then rephrase the relation from $\frac{\nabla H}{Q} = -\frac{1}{T}$ to $\frac{dH}{Q} = -\frac{dL}{T}$ and β is equal to $\frac{\Delta(\mathrm{d}H)}{\Delta Q}$.

The feedback system considered here is an unconfined aquifer system. In this case, Darcy's Law still applies, but is typically formatted as $Q = -KA\nabla H$. Here, the cross-sectional area is not fixed, or separated out as the saturated thickness multiplied by the thickness into the page. However, the cross-sectional area of flow will be a function of the saturated thickness, so the same equations given above for the confined case apply to the unconfined system (Figure 9) as well ($\frac{\Delta(dH)}{\Delta Q} = -\frac{dL}{T} = \beta_f$). But, in the unconfined case, the saturated thickness will change in response to a change in the flow and a change in the head gradient, so β_f of the unconfined system will vary spatially, with different perturbations, and at different initial heads.

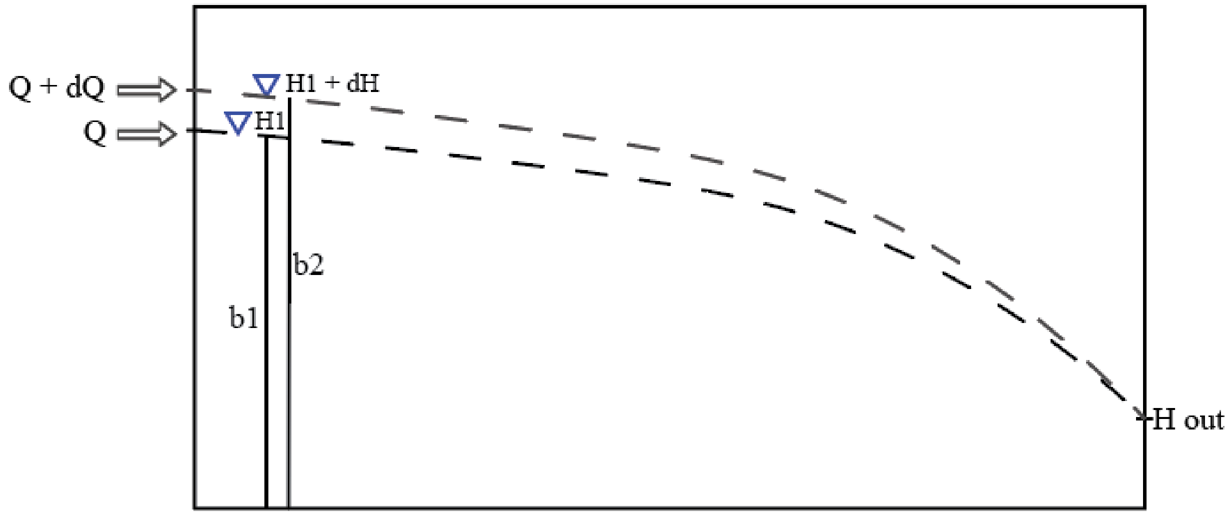


Figure 9. The unconfined aquifer system responds to a change in the flow in by changing both the head gradient and the saturated thickness.

The feedback of the unconfined system is the way in which both the saturated thickness and the head gradient change as a result of a perturbation to the flow into the left boundary. An increase in the flow would be expected to cause an increase in the gradient: if the right boundary is fixed, this would be achieved by an increase in the head at the left boundary. But, in this unconfined system, any increase in the head at the left boundary also increases the saturated thickness. This increases the cross-sectional area through which flow can pass, and increases the transmissivity. As a result of this increase, the gradient needed to admit the increased inflow is reduced. That is, the dependence of T on the head at the left boundary nudges the value of the head at the left boundary down. Contrast this with the confined case, where only the head gradient can change to adjust to a change in the flow into the system. This suggests that the head gradient for the confined case will change more than those in the unconfined case so that $\beta_0 > \beta_f$, and f will be negative.

Hydrological Results

The confined and unconfined cases are modeled as two-cell systems using MODFLOW (Harbaugh et al., 2017) in FloPy (Bakker et al., 2018), and the code for running the model and calculating results is in Appendix A. Flow occurs from left to right. The head in the right cell is fixed. The flow into the system is perturbed and the system response is defined as the head in the left cell (Table 2). The equilibrium state is represented as Q = 90, and every other value of Q represents a perturbation to the equilibrium state. The feedback values given in the last three columns are found by measuring from each perturbed response back to the equilibrium response – in other words, the perturbations to the flow are not incremental, each one represents a perturbation of a different magnitude from the common equilibrium state.

Q	H1, Unconf.	H1, Conf.			
90	**24.116**	**24.116**	f	θ	kernel
90.1	24.120	24.121	-0.08	-0.08	-0.08
90.2	24.124	24.125	-0.08	-0.08	-0.08
90.3	24.129	24.130	-0.08	-0.08	-0.08
90.4	24.133	24.134	-0.08	-0.08	-0.08
90.5	24.137	24.139	-0.08	-0.08	-0.08
90.6	24.141	24.143	-0.08	-0.08	-0.08
90.7	24.146	24.148	-0.08	-0.08	-0.08

Table 2. MODFLOW results for the head values at the start of the unconfined and confined systems, for a fixed head in the second cell ($H2 = 20$) and specified values of Q, and the feedback values calculated for each perturbation by each method.

The feedbacks calculated are slightly negative, as predicted by conceptually describing the way the head at the left boundary will be nudged in the opposite direction of its initial perturbation, forced by a change in flow. This is also consistent with the definitions given in Equations 6 and 7. The identical value of feedback calculated for all perturbations indicates that the feedback analysis could have been conducted once and the results would have been consistent over a larger range of perturbations. Such an analysis should be conducted to test the applicability of a feedback strength over the limits of the perturbations of interest. If confirmed, it allows the use of the feedback analysis to replace modeling to determine responses to any perturbation within that range. This may have direct utility for conducting parameter estimation on high dimensionality hydrological models.

This system can also be considered from a different perspective, acknowledging the interchangeability of response and input when setting up a model. Specifically, consider a confined system with a fixed thickness and an unconfined system for which the initial water table elevation coincides with the elevation of the top of the confined system. In this case, an increase in head at the left boundary will cause the unconfined system to have a larger saturated thickness than the confined system subjected to the same head boundary conditions. Therefore, setting the head values in the unconfined model equal to those of the confined model would yield greater flow volumes through the unconfined system. This would occur because both the head gradient and the saturated thickness of the unconfined system would be raised, causing a greater increase in Q.

The different perspective of this system presented above is most appropriate for addressing how the unconfined system, with feedback, is capable of transmitting flow for a specified head gradient relative to the confined case. Perturbing the head gradient of the unconfined system results in a greater increase in flow transmitted through the system relative to the confined case. Similarly, a decrease in the fixed head gradient would result in a more significant decrease in flow transmitted, relative the confined system. This response is different than the confined case because the saturated thickness in the unconfined system responds in the same direction as a change in the head gradient. The increase in gradient and the increase in saturated thickness complement one another, resulting in an amplification of any perturbation to the head gradient, constituting a positive feedback.

As suggested above, now that the feedback strength has been calculated and it has been demonstrated that it applies over the range of flow perturbations of interest, it can be used to predict the increment of head change for a given perturbation to the input by Equation 10 ($dH =$

$\frac{\beta_0}{1-f} * dQ)$. The results shown in Table 3 confirm that a simple calculation based on feedbacks can exactly replicate the results obtained by rerunning MODFLOW for the perturbations to flow listed in Table 2.

predicted dH1	modeled dH1
0.004	0.004
0.008	0.008
0.013	0.013
0.017	0.017
0.021	0.021
0.025	0.025
0.030	0.030

Table 3. The predicted changes in head compared to the modeled changes are identical within model precision using the feedback calculated in Table 2.

Adding Complexity

Part of the utility of a feedback analysis stems from being able to break apart a large and complex system into smaller parts and identifying how the feedback strength varies in space. As an example, the simple hydrological case presented above can be refined such that rather than including two model nodes spaced, for example, 100 meters apart, there are three nodes spaced 50 meters apart. (Nodes are located in the middle of cells in MODFLOW. Splitting the two-cell model into three maintains the total distance between the first and last nodes at 100 meters. Therefore, the system size is maintained.) Breaking apart the system also allows for another level of interaction, because flow in the middle cell relies on the solution in the first and last cells, and the solutions at those locations rely on that in the middle. The model not only resolves the interplay between saturated thickness and hydraulic gradient, but how the total system feedback is distributed over the domain.

As before, the head in the left cell is free to vary and is taken to represent the response of the system for the unconfined and confined cases (Table 4). The head gradient calculation requires the change in head found across some distance, so values used to calculate the feedback were obtained by calculating differences in the values at the nodes.

The values in Table 5 were found by determining the head gradient between nodes 1-2, 2-3, and between 1-3 (across the entire system). The values calculated between the first and third nodes in this case are essentially the same locations within the system as the two nodes from the two-cell model. However, the feedback found in this analysis is not exactly the same as was found in the two-cell case. This is because there is additional information in the three-cell model about the location of the water table between the first and last model nodes. To understand the effects of this difference, consider the representation of the unconfined water table shown in Figure 9. If this were represented with only two points, (i.e. for the two-cell model), it would have to represent the impact of this curvilinear surface with a straight line. When a third point is added for the three-cell analysis, the curve is represented as two straight segments, with the first having a shallower slope than the second. This improved resolution of the water table leads to a better approximation of the impacts of the feedback, which results in a better approximation of the head at the left boundary. Therefore, it is likely that the overall feedback defined with the three-cell model is a better representation of the true feedback strength. There will be a diminishing return if more cells are added until the resolution can be described as sufficient to capture the true feedback of the system. This is identical to the impact of resolution on solving the head distribution in any complex hydrological model.

The three-cell model offers more than just an improved estimate of the actual feedback strength. Importantly, it allows us to start identifying where within the system feedback is occurring. In this case, the feedback strength between the last two nodes is less than that calculated across the two-cell system, and the feedback strength is highest between the first two nodes. As stated above, the two linear segments of the three-cell representation of the water table have different slopes. The shallower slope (which occurs at the inflow end, where the head is higher) of the first segment leads to this segment having a higher average saturated thickness over its length. The second segment, bounded on the right by the fixed head boundary, has a steeper slope and begins with a lower head value where flow enters that segment, relative to the first segment's starting point. The second segment has a smaller average saturated thickness then the first segment or of the entire model. The change in saturated thickness is the mechanism that gives rise to feedback. From this, we can conclude qualitatively that the left segment contributes more to the overall feedback than the right segment. The calculated feedback strengths allow us to quantify these relative contributions.

The eventual goal is to produce a map of feedback values distributed across the system of interest. This will enable response prediction to perturbations on the local scale. However, this map can be used to garner additional information by considering the effect across the entire system as well. There are two pieces to consider here. The first is that the β values are additive when considered spatially. That is, the β value calculated within each model cell of a model can be summed across the model space. This should be equal to the β found by considering only the start and end points of the system setup. This is shown in Table 5, where the last row demonstrates that the sum of the β_f values in each cell is equal to the β_f found from between the first and last nodes of the model. The overall β values can then be used to calculate the gain, and subsequently the effective feedback of the entire system response – this is how the feedback indicated by f 1-3 was found. Note that the feedback values calculated for each model cell are not additive – the sum of them does not equal the effective feedback calculated across the system. However, the feedback values found by the kernel method, for processes that are all in the same cell, are additive. This is an important distinction, that across space, β values are additive, but feedbacks are not. In a single location, feedbacks are additive, but gains are not ($G = \frac{1}{1-\sum f_i}$).

Q	H1, Unconf.	H1, Conf.
90	**24.091**	**24.091**
90.1	24.096	24.096
90.2	24.100	24.100
90.3	24.104	24.105
90.4	24.108	24.110
90.5	24.112	24.114
90.6	24.116	24.119
90.7	24.121	24.123

Table 4. Heads in the first cell for the 3-cell model, for the same flow conditions that were applied to the 2-cell model.

f 1-2	f 2-3	f 1-3		Ɵ 1-2	Ɵ 2-3	Ɵ 1-3		kernel 1-2	kernel 2-3	kernel 1-3
-0.14	-0.05	-0.09		-0.14	-0.05	-0.09		-0.14	-0.05	-0.09

$\beta_{f,1-2}$	$\beta_{f,2-3}$	$\beta_{f,1-3}$	Sum $\beta_{f,1-2,2-3}$
0.0190	0.0228	0.0418	0.0418

Table 5. Feedback strengths for all three methods, calculated for between the first and second cells (1-2), second and third (2-3), and first to third (1-3). The feedback values were exactly the same for all perturbations to the flow, and so are not shown.

Additional equilibrium flow states were considered, to determine how the feedback response varies with equilibrium states, and the corresponding feedback strengths are shown in Table 6. Feedback values were identical for all three methods, so they are only listed once, as f. The feedback strengths calculated for the $Q = 21$ base case were then used to calculate the head at the left boundary for the higher Q cases. In this case, the feedback strengths were not identical for the different base Q values. But, when the $Q = 21$ feedback strengths were used to calculate heads for the other cases (Table 7), head changes showed good agreement with the MODFLOW results. It would depend on the required accuracy of the application to determine if the prediction accuracy was sufficient, thereby allowing for the use of feedback-based calculations rather than requiring complete model runs.

	f 1-2	2-3
Base Q = 21	-0.04	-0.01
Base Q = 60	-0.10	-0.03
Base Q = 90	-0.14	-0.05

Table 6. Each of the base Q values represent an equilibrium starting point from which perturbations of the same magnitude were applied, and feedback values calculated between model nodes, given here as f 1-2 and f 2-3. The same feedback value was found for perturbations greater than or less than the equilibrium Q.

Predicted dH	Q	modeled H1, Q = 21	predicted dH +H (21)	modeled H1, for Q's	difference
3.34	91.96	21.029	24.37	24.17	0.19
3.32	91.52	21.033	24.35	24.15	0.20
3.30	91.08	21.038	24.34	24.14	0.20
3.28	90.64	21.043	24.32	24.12	0.20
3.26	90.20	21.048	24.30	24.10	0.20
3.24	89.76	21.052	24.29	24.08	0.21
3.21	89.32	21.057	24.27	24.06	0.21

Table 7. Predicted head changes from specified perturbations to the equilibrium flow $Q = 21$.

Discussion

This study examined three different approaches for quantifying feedbacks in a simple, one-dimensional groundwater flow system. The results suggest that the feedback value is dependent on the reference state selected, proximity to model boundaries that experience transient conditions, equilibrium state from which it is perturbed, and spatial discretization of the model. These differences are due to the system parameters (e.g. resistivity and transmissivity) that vary as a function of a system state, the control of boundary conditions on state responses (e.g. head or voltage), and the design decisions that go into constructing the numerical model that represents the system. That is, the feedback analysis of the simple groundwater system provides insight into the controlling factors that should be considered for feedback analyses of more complex systems. This suggests that a certain level of sophistication will be necessary to apply feedback approaches to hydrological analyses. But, the ability of feedback approaches to forecast system responses to future changes in forcings holds promise for applications ranging from sensitivity analyses to support parameter estimation, to rapid calculations to support real-time interactive modeling tools for decision support.

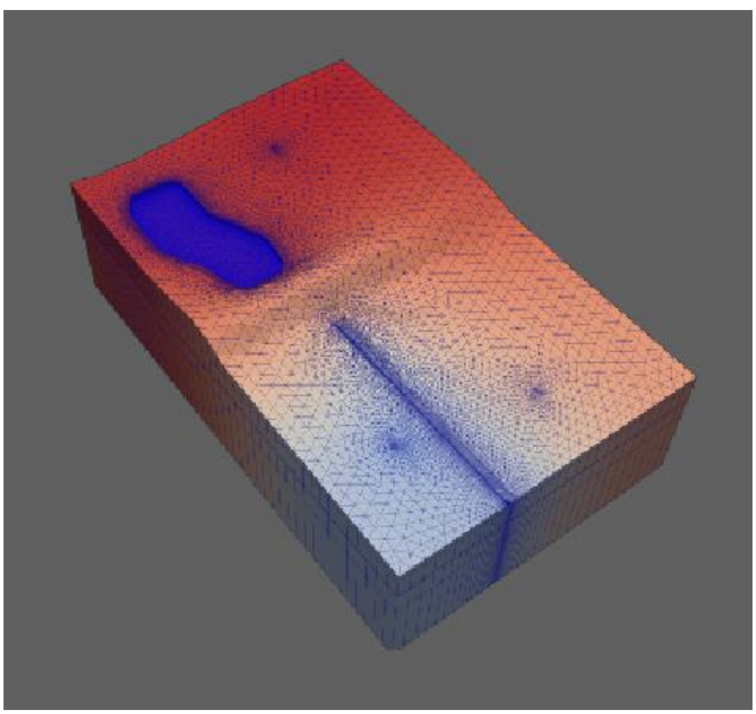


Figure 10. Illustrative model grid with ET flux shown as color flood.

Extending feedback approaches to more complex systems was beyond the scope of this study. But, based on what I have learned in developing these approaches, I will provide some discussion about further challenges that must be addressed to achieve this goal. For these discussions, consider a groundwater system shown as a model grid in Figure 10. There may be topographic variations, which define zones of recharge and evapotranspiration. Surface water features, such as lakes and streams, may interact with groundwater. They may also affect plant communities, which will influence patterns of ET. There may be anthropogenic stresses, such as pumping wells. All of these stresses may be overlain on a complex subsurface that includes both geologic structures and local hydrogeologic heterogeneity. If we consider the susceptibility of this system to climate change, then we must translate climate drivers to processes that are included in the model or that inform external stresses applied to the model. Climate change may be expressed as changes in the total and spatiotemporal distribution of precipitation, or in changes in ET, or in changes in human water use patterns, or any combination of the above. Feedbacks may be related to water levels in the unconfined system (through transmissivity as a function of water table depth), ET (through root water uptake as a function of water table depth and ET rates), and human use (through changes in pumping rates and seasonal timing of pumping as a function of water use preferences or reliance on groundwater).

Any feedback analysis would have to begin by defining the reference model in the context of the key feedback mechanism(s) to be investigated. The reference model for such an analysis could be one that represents the system at its current equilibrium. However, it may be more useful to establish a hydrological reference case that can be used for all feedback analyses – much like the blackbody radiator represents a universal reference case used in climate science. Like that system, this could be one modeled with no ET, or one that limits other groundwater fluxes. It may be useful to consider the aquifer portion of such a system to be homogeneous and isotropic, for best comparison across different feedback systems. In the end, the more accurately the system is described, the more accurate the calculated feedback strengths will be – as seen for the two-cell versus the three-cell analysis above. But, as for all numerical models, increased detail is dependent on available data and requires additional computational effort. A compromise will have to be made to provide sufficient accuracy of feedback strengths within the usual time and resource constraints of a hydrogeologic analysis.

Establishing a simple, standard reference system that applies to a wide range of model conditions may prove most useful for analyzing systems that are subject to multiple, simultaneous feedback processes. Drawing from examples in climate science, this would enable efficient examination of the role of feedback on water level stability to changes in plant community composition, or susceptibility of baseflow to climate change, or a number of other questions that are highly relevant to water managers. Application of the kernel method is key to such a study. Not only does it allow for definition of the individual feedback strengths, different combinations of the feedbacks could be studied more efficiently than would be possible using traditional modelling approaches. In fact, the computational efficiency of the feedback method may even make it possible to do these calculations in real time. This could allow for great flexibility for water managers to consider different approaches to climate adaptation without being limited by long model run times.

In the context of using feedback to support real-time scenario testing, it will be critical to first conduct a survey of interested parties to determine which system responses will be of interest and which system stresses will be considered. Unlike generic modeling efforts, feedback analyses rely critically on defining the context of the problem before undertaking the analysis.

As stated above, the reference system represents a choice that can be made to answer the best question in the most efficient manner. The utility of answering more efficiently-posed questions may outweigh the convenience of being able to apply a single model, such as a specific MODFLOW model, to many different questions.

One unique aspect of a feedback analysis is the ability to separate feedback effects from other drivers of hydrological change. It is possible that being able to identify feedback processes that lead to stabilizing or destabilizing conditions in specific areas within a model domain will be a very valuable tool. These maps could inform water managers about areas that are resilient or at risk, while identifying the key processes that underlie these conditions. However, as is true for all modeling efforts, it is even more true for feedback analyses that the value of the analysis will rely on the quality and transparency of the modeling decisions made. These decisions include which external forcing to apply, how the system response to that forcing is modeled, which parts of the system are fixed or free, and the specific definition of feedback.

The over-arching goal of this work was to make an initial effort to adapt feedback approaches to hydrologic analyses. Further, the aim was to do this to improve both our ability to forecast hydrological change and to communicate our findings. Before detailing some of the challenges that will be faced moving forward, it is worth considering how feedback results might be used.

Complex hydrological conceptualization

Consider that feedback analyses have been performed for a specific outcome of interest: climate impacts on the health of native vegetation from a hydrological perspective. Areas within a domain that show strong feedbacks may be particularly important for conservation or restoration efforts. The analyses are based on a complex hydrological model, with a solution grid like that shown in Figure 10. The feedback strengths for different processes could be indicated by different colors, distributed throughout the domain (Figure 11). For example, imagine that ET from native vegetation along a stream shows very high negative feedback as a function of winter groundwater extraction rates (blue area along the stream). That is, ET decreases with increased pumping, which in turn causes an increase in the groundwater levels. This would suggest that any impact of pumping that we would predict without considering relevant feedbacks is likely to be an overestimate. From a management perspective, however, this process by which ET is reduced represents a potential problem for the resilience of this part of the system. The negative feedback generated by the vegetation adjusting to groundwater levels may help maintain the overall system stability, but within that ecosystem, there could be negative impacts and associated costs that should be considered (e.g. erosion, if loss of vegetation occurs). This suggests that even though we may refer to negative feedbacks as indicators of stability within a system, these negative feedbacks may also highlight areas where system function or processes should be protected. It further suggests that we may want to pay particular attention to this outcome when considering future impacts of human water use, especially those that rely on winter pumping.

On the other hand, if invasive species have neutral or positive feedback of ET in that area on winter pumping (red areas along the stream), it may suggest that they would be relatively more advantaged by increased pumping than a simple analysis would indicate. For example, if erosion had the potential to be an issue, areas with invasive vegetation that are less susceptible to increasing depths to the water table may actually represent a resilient area, by that metric. In this way, feedback strength maps may be particularly useful to highlight areas, especially those with strong feedbacks, for special consideration for water managers, or they may identify areas that may be more resilient than expected. It may also prove useful to model areas that show little or no feedback after an initial analysis by using much simpler, less computationally expensive models.

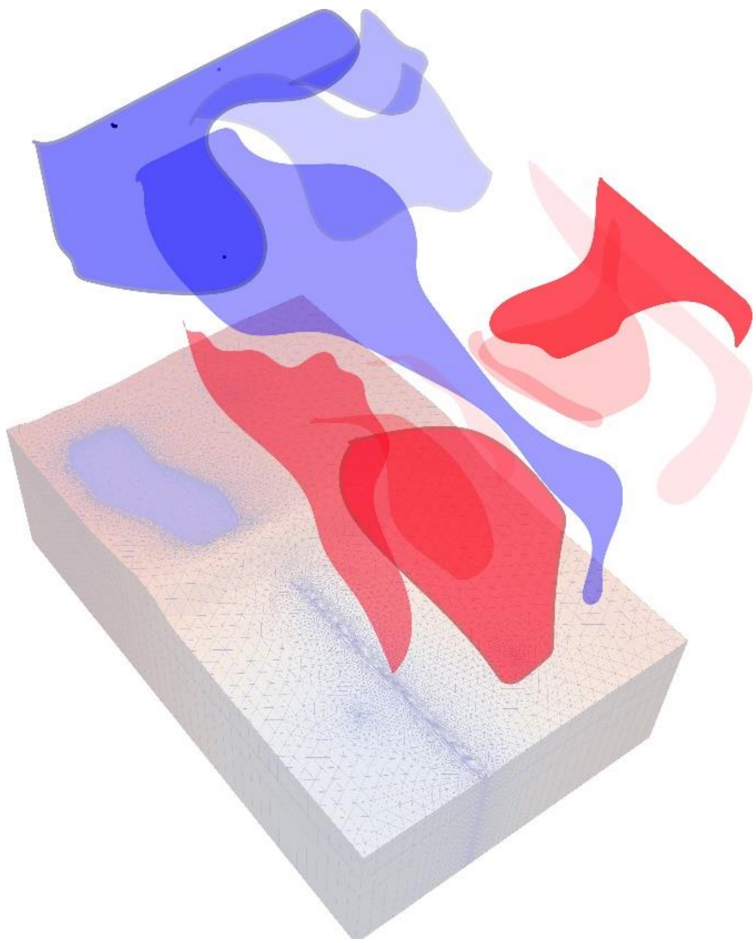


Figure 11. Feedback strengths for native (blue) and invasive (red) species to winter pumping from two wells in the domain.

Feedback and boundary conditions

Every groundwater model requires boundary conditions to isolate the region under study from its surroundings. For flow models, boundary conditions constrain the head or flow across a boundary element. If feedbacks are defined with respect to the condition that is fixed at a boundary, feedback at that boundary will approach zero. Conversely, feedback will be greatest where the change in the input is greatest. Therefore, while the choice of boundary conditions is important for all model-based analyses, it is especially critical for defining feedbacks. This is both a limit on the proper use of feedback analyses and, potentially, a way to provide insight into the impacts of modeling choices on model responses.

Feedback and medium properties

In addition to the location of boundary conditions, the medium properties may have a strong impact on system feedback strength. Heterogeneity of the subsurface can contribute to non-uniformly distributed recharge, focusing recharge in areas of higher conductivity and making those areas more susceptible to changes in water levels due to changes in recharge. The way that the recharge is routed through the system will ultimately define whether this heterogeneity leads to strong feedbacks. Similarly, the vadose zone itself can act to dampen temporal variations in infiltration to produce steady state recharge (Dickinson et al., 2014). One interesting outcome of the interaction of flow, water levels, and hydraulic properties is that changes in hydraulic conductivity and storage capacity may change a feedback strength map. In fact, being able to show the importance of knowing hydraulic properties to accurately characterize whether an area will experience positive, neutral, or negative feedback may be critical to water management.

Feedback and model resolution

The results of the analysis comparing two- and three-cell models suggest that the system response and feedback calculation are more accurate when model spatial resolution is increased. For example, the feedback values changed slightly when the system was modeled as three cells rather than two. The model calculates values at node locations – increasing the number of nodes in the system results in a better approximation of the curve of the water table. For models that can operate with a variable grid resolution, the model could be designed to have a finer grid resolution in specific areas of interest, where the strongest feedbacks occur, or where there are thought to be multiple feedback processes at work. This would increase the ability to create robust predicted responses and enable better understanding of the system response.

Feedback and equilibrium state variance

The feedback methods discussed here are predicated on the assumption that system responses are generally linear. The blackbody radiator used as a reference case in feedback analysis of climate systems is based on the Stefan-Boltzmann law, which is a non-linear function of temperature. The effects of this nonlinear response are dealt with in climate science studies by applying a series of perturbations to several different equilibrium conditions and considering whether the feedback changes linearly with changes to the equilibrium state. This is a good strategy if the linearity or degree of deviation from it is unknown, and it may show that a linear estimation of feedback can be made between equilibrium states. Estimated responses made using the feedback value may then be less uncertain and more accurate than those modeled based on non-linear dependencies of system responses.

The results in Table 7 show that a feedback value taken from one equilibrium state may show sufficient accuracy to replace additional model runs. If the kernel method can be shown to identify the feedback strengths of individual processes when multiple feedbacks are present, it may offer real benefits for scenario testing for water management and uncertainty estimation. We have shown that, for a simple system, the β values are additive when calculating the system-wide feedback. But, it remains to be seen if this holds if the strengths of different feedback processes change at different equilibrium states. This would complicate the choice of a reference model, at a minimum. More work is required to fully understand systems that may transition to new equilibrium states if thresholds are exceeded.

Feedback vs. sensitivity analysis in complex models

Anticipated climate changes – specifically average temperatures rising and precipitation patterns and quantities changing – represent additional sources of uncertainty in modeling natural system response into the future. A sensitivity analysis is one typical method for interpreting the effect of parameter uncertainty and model factors on model results. Such an analysis shows the relative importance of each internal system process to the modeled response, but for an uncertain climate future, would require a great amount of time and effort to fully analyze. Additionally, communicating the results of such an analysis in a way that is accessible to decision-makers and resource managers presents a challenge that, arguably, is as important as the analysis itself.

The first step of a model attempting to predict future responses to climate change would require predicting combinations of temperature and precipitation changes that may constitute a new normal – that alone represents a significant time investment. It may not even be possible to identify which patterns may represent a reasonable forecast for the future. Precipitation patterns alone could dominate new streamflow regimes that could alter plant communities, recharge, and baseflow. In areas like the Southwest, where surface flow is largely dependent on precipitation runoff, flashier events could cause a reduction in the number of days in a year a channel contains water, reducing infiltration through the streambed. Higher velocities near channel inputs may also affect recharge patterns and recharge through the channel bed. These variations represent changes in both space and time. Factoring in changes to temperature and amounts of precipitation complicate the possible scenarios that should be modelled even further.

Applying a variety of possible perturbations when doing a sensitivity analysis will only account for variability to the inputs, which will result in a limited representation of the system response. It will also not identify the way in which each individual system process may be interacting in response to perturbations, or to each other. Relations between parts of a system are often modeled by parameterizing them – for instance, relating different ET rates to groundwater levels, temperature, and precipitation. These parameterizations are reasonably effective when considering one equilibrium state and calibrating the model response to observations. However, modeling significant changes to both the magnitude and pattern of inputs may cause increased uncertainty in the resulting estimations of system responses.

The equilibrium state of the system is also a decision that will affect the model response. For example, if the equilibrium state is one where groundwater levels are relatively high, plants are transpiring at their maximum rate, and recharge is at a maximum, it will take longer for stresses applied to that state to manifest in system changes. This implies that the most representative models will not be linear, so that the response will be a function of the state. Non-linear system models are more difficult to parameterize, interpret, and have more uncertainty in model results.

A modeler could spend a large amount of time trying to estimate the ways the system may respond, in space and time, as changes to the inputs begin to manifest. In fact, a sensitivity analysis may not even recognize the layered interactions of different parts of the system in time. Changes in infiltration to groundwater through streambeds may be a slow change, manifesting significantly only after multiple timesteps. Changes in vegetation along riparian areas may occur more quickly in response to altered precipitation or temperature. These two things may even interact, as transpiration decreases, recharge may increase. This decreased transpiration could constitute a negative feedback that buffers changes to the groundwater levels.

In contrast to the difficulties with a standard model analysis listed above, a feedback analysis generates robust response estimations given only a few modeled conditions. If feedback values can be shown to be nearly linear across multiple equilibrium states and used to predict responses, uncertainty generated from parameterizing non-linear model responses may be reduced. Feedback analysis honors the complexity of natural system responses and interactions without reducing them to fixed or discrete parameterizations. This then produces an analysis that can be communicated more clearly and efficiently. Explaining model uncertainty involves explaining parameterization and sensitivity, and a general groundwater manager may not care, or comprehend, the implications of a model parameterization or sensitivity analysis. A map of feedback values with the simple explanation that negative values indicate stable parts of their system whose functioning should be protected, and weaker or positive values areas that should be monitored and are most vulnerable to change, may provide more accessible information that can result in more informed decision-making.

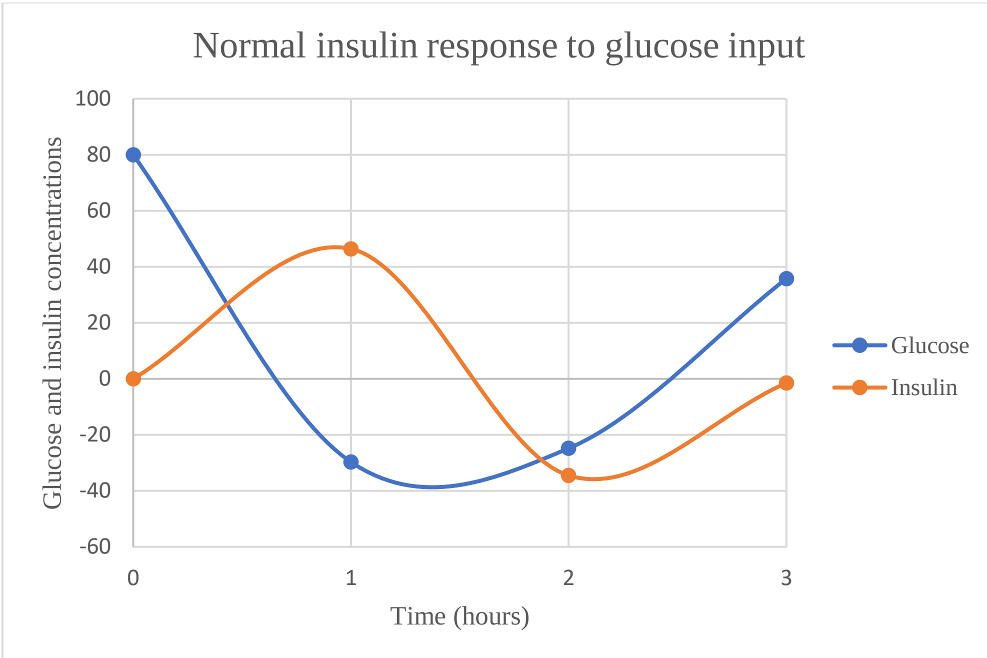
Timescale of response

This study only considers feedbacks in steady-state systems. Most hydrological analyses consider transient conditions. Extending feedbacks to transient analyses will require considerable work. In the simplest sense, a time series of responses must be compared for the reference and perturbed models, rather than just considering single values. In addition, both the magnitude and the timing of the system response may be affected by feedback. That is, a change applied at one model boundary will take time to propagate through the system, while other changes may be felt more immediately. For example, altered precipitation patterns may affect vegetation and ET patterns rapidly, but changes to infiltration beneath the root zone may take considerably longer to affect groundwater flow. Care must be taken to ensure that the observed changes are correctly attributed to the right feedback processes in these cases.

Ultimately, it is the timescale of multiple feedbacks that maintains system equilibrium, when multiple feedback processes are at work. Imbalances in the timescales of different feedbacks can lead to system oscillations. One well-described example is the insulin response (Figure 12). The insulin response is a negative feedback that reduces (increases) the glucose concentration in the bloodstream in response to an increase (decrease) in blood sugar. There is a range of blood sugar that is healthy, but too much or too little causes health issues, so maintaining that optimal equilibrium is of significant concern. Eating food – a cupcake for example – may cause a rapid rise in blood sugar. This triggers the release of insulin, which acts to remove the glucose from the bloodstream. Similarly, if the blood sugar falls too low, a reverse process is prompted, raising the blood glucose back to its healthy equilibrium state.

A healthy insulin response acts quickly, oscillates briefly, and arrives back at its equilibrium state without outside aid. The diabetic response – still a negative feedback – is a weaker response that oscillates about equilibrium more slowly, takes more time and may even require help to fully return to the equilibrium state. These responses have a similar pattern, but a diabetic person requires frequent monitoring and great attention to food intake to avoid very significant health consequences. The overall magnitude of changes of these systems are roughly the same, but the difference in response time has significant consequences.

This suggests another possible use of feedback analyses in hydrology – describing the time scale of response of a system in the context of recovering from short duration, relatively high magnitude perturbations. A classic example of this issue is the difficulty of communicating legacy drawdown from a well after it has stopped pumping – it is often surprising and disappointing to stakeholders when they learn that the drawdown cone will continue to expand long after a pump stops operating. It could be very useful to have a simple measure of the timescale of such impacts. Ideally, the feedback strength and feedback timing could be combined to determine whether proposed adaptation strategies will have the desired effects in an acceptable time frame. This use may help managers respond more appropriately to infrequent stresses by considering the resilience of the system through time. It is possible that this vulnerability or resiliency could be described simply and communicated concisely by a single pair of feedback values.



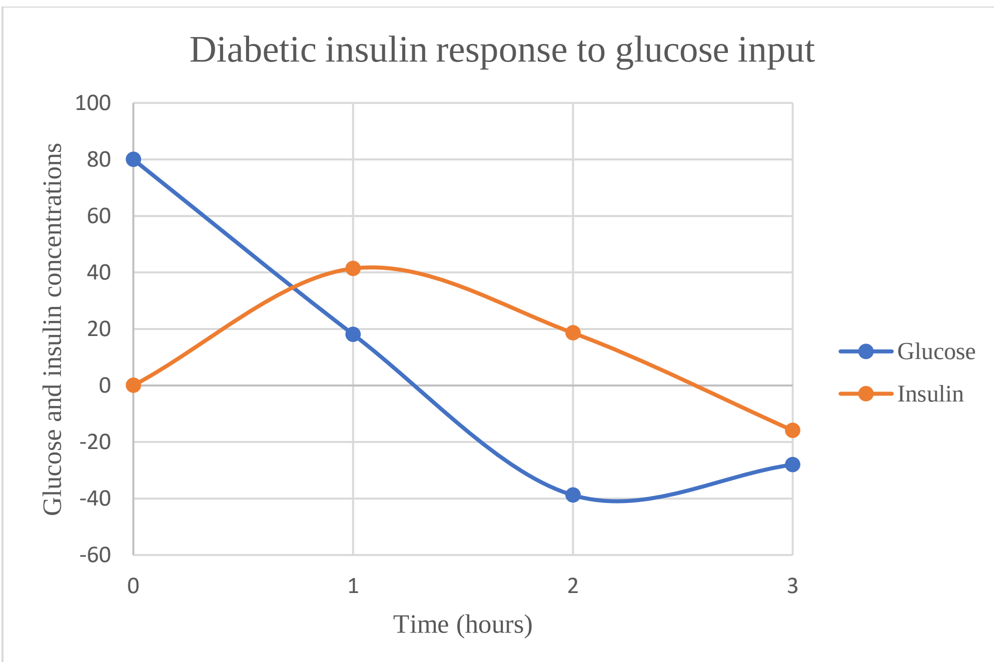


Figure 12. Healthy and diabetic feedback responses to glucose ingestion with time, data from Shiang & Kandeel (2010).

Conclusions

What parts of the hydrological system will respond to change first, which areas will respond most drastically, and how will each part interact and distribute the effects of the additional system changes? These are all questions that a thorough feedback analysis may be able to answer. It may also prove to be a more efficient alternative for exploring responses to wide ranges of perturbations, when compared to developing specific models for each possible perturbation. That is, a feedback analysis would require fewer model runs to understand how various combinations of perturbations will change the responses of the system.

Communicating modeling results and conclusions may also be simplified by using feedbacks. Identifying areas of positive feedbacks that are subject to instability may highlight locations that should be protected from resource development or show what processes may be used elsewhere to promote stability. Rehabilitation and monitoring efforts could be focused at those locations. Understanding where the weaker feedbacks are could aid in identifying areas likely to see the least change. These may represent parts of the system worth monitoring, since changes to such locations may indicate significant deviations from previous equilibrium conditions. Or, these may indicate areas that can sustain development that would be unacceptable in less hydrologically resilient areas. A feedback analysis of a model like the one shown in Figure 10 and presented as feedback zones (Figure 11) could provide a map of the location of feedback strengths. This then may prove easier to communicate to a water manager who may otherwise have to rely on outside interpretation of model results for information about the system being managed.

There may be several ways to create a feedback strength map, depending on the information available and the objective of the analysis. The simple ratio method of calculating feedback could be applied in a wide variety of cases, where only the responses of the reference and feedback cases are known. These could be modeled or observed responses, compared between two different system types, such as the example of the invasive plant feedback, relative to that of the native plant. The functional dependence case, taken directly from the electrical analysis, may be most useful when a reference case response is irrelevant, or when only the internal system relations are observable – the input is unknown. The kernel method shows the greatest promise for moving forward in hydrology, due to its ability to calculate the relative importance of each feedback process. It may also prove useful for connecting surface and groundwater models, or even linking such hydrological models to atmospheric models. The code created for this study enabled multiple model iterations to be run and calculations of feedback performed very efficiently. Perhaps models of groundwater, surface water, and atmospheric systems could be run within computer code that would allow each model to iteratively interact with and be affected by the others.

Ultimately, all parts of the Earth system influence all other parts. The closer we come to modeling those complex interactions, the better our forecasts can be. The more we understand them – as feedback shows promise to improve – the better we can communicate to decision-makers and managers of systems without relying on isolated models or work-intensive model interpretation. The methods outlined and discussed in this work provide the first step to using feedback analysis in hydrological systems.

Appendix A

```python
# coding: utf-8

# # Feedback calculation between confined and unconfined groundwater models

## 1. Import necessary libraries

# 2. Create model object settings (MODFLOW version) and other things that will remain constant

# 3. Name the confined and unconfined models and set then up

# 4. Use a function to run the model and store results of each iteration separately for confined and unconfined

# In[2]:

import os

import numpy as np

import flopy

import matplotlib.pyplot as plt

import flopy.utils.binaryfile as bf

import itertools as it

import pandas as pd

# # User Inputs

# 1. Specify equilibrium flow value, whether perturbations should be positive or negative from equilibrium (doesn't matter, gives same feedback value), how many cells to model, the delr between model nodes, and the name of the output file containing results of all feedback calculations and model output.

# In[58]:

Base_Flow = 90  #enter desired base flow here

numit = 7 #enter number of perturbations to test here

Flows = []

for i in range(numit):

    flow = Base_Flow

    Flows.append(flow)

    #set flag for pos or neg pert from equilibrium
```

Flag = 1 #1 for pos pert, -1 for neg pert

numcell = 3 #set number of model cells

delr = 50 #set distance between model nodes

name = 'Output' #specify what you want to name the output file - spits out all feedback values, predicted head changes and

#modeled heads for the unconfined case. Anything included in the parenthesis at the end will also be added to this.

Set up first unconfined case to get top elevations for reference case

In[4]:

modelname = 'unconf'

m = flopy.modflow.Modflow(version = 'mf2005',exe_name = 'mf2005.exe',modelname = modelname)

Lx = 100

Ly = 100

nlay = 1

nrow = 1

ncol = numcell

nper = 1

delc = Ly/nrow

laycbd = 0

laytyp = 1 #zero for confined

botm = 0

top = 100 #for confined, top equals lowest top from unconfined case at H1, for unconfined, top = 100

nstp = 1

tsmult = 1

steady = True

strt = 20. #constant head boundary

dis = flopy.modflow.ModflowDis(m,nlay=nlay,nrow=nrow,ncol=ncol,nper=nper,delr=delr,delc=delc,laycbd=laycbd,top=top

,botm=botm,nstp=nstp,tsmult=tsmult,steady=steady)

flow = []

heads = []

```
# create ibound array with all cells active
ib = np.ones((nrow,ncol))
# set first cell to active
# set last cell in first row to contant head
ib[(nrow-1),(ncol-1)] = -1
bas = flopy.modflow.ModflowBas(m,ibound = ib,strt = strt)
lpf = flopy.modflow.ModflowLpf(m,laytyp=1,ipakcb=50)
# Define a function that will run MODFLOW every time it is called.
# In[5]:
def modelrun(name):
  for i in Flows:
    j = i#+0.01*i
    spd = {0:[[0,0,0,j]]}
    wel = flopy.modflow.ModflowWel(m,ipakcb = 50,stress_period_data=spd)
    gmg = flopy.modflow.ModflowGmg(m)
    oc = flopy.modflow.ModflowOc(m)
    oc.stress_period_data[0,0] = ['save head', 'save budget', 'print head', 'print budget']
    print(top)
    print(j)
    m.write_input()
    m.run_model()
    hds = bf.HeadFile(modelname+'.hds') #read binary head file output from modflow
    times = hds.get_times() #not sure
    head = hds.get_data(totim=times[-1]) #assign result of reading binary file to variable 'head'
    heads.append(head)
    levels = np.linspace(0, 10, 11) #not sure
    cbb = bf.CellBudgetFile(modelname+'.cbc') #this is how you read the budget file, which is binary
    kstpkper_list = cbb.get_kstpkper() #variable that contains the results....maybe?
    frf = cbb.get_data(text='FLOW RIGHT FACE', totim=times[-1])[0] #write to text for each result
    flow.append(frf)
```

```
    print(laytyp)


    return flow, heads;
# Create test model run to obtain head values for reference case.
# In[6]:
test = modelrun(modelname)
# Set "top" equal to the list of heads from simulation.
# In[7]:
list = []
for i in range(ncol):
    new = heads[0][0][0][i]
    list.append(new)
top = np.ones((nrow,ncol))
top *= list
# Setup the run for the first reference case model.
# In[8]:
modelname = 'conf'
m = flopy.modflow.Modflow(version = 'mf2005',exe_name = 'mf2005.exe',modelname = modelname)
laycbd = 0
laytyp = 0 #zero for confined
botm = 0
#top = 20 #28.54370689 #(Q = 90 for case where only testing perturbations from Q = 90)
#21.45 #for confined lowest (Q = 30), top equals lowest top from unconfined case at H1, for unconfined,
top = 100
nstp = 1
tsmult = 1
steady = True
strt = 20. #constant head boundary
dis =
flopy.modflow.ModflowDis(m,nlay=nlay,nrow=nrow,ncol=ncol,nper=nper,delr=delr,delc=delc,laycbd=la
ycbd,top=top
```

```
,botm=botm,nstp=nstp,tsmult=tsmult,steady=steady)

flow = []

heads = []

# create ibound array with all cells active

ib = np.ones((nrow,ncol))

# set first cell to active

# set last cell in first row to contant head

ib[(nrow-1),(ncol-1)] = -1

bas = flopy.modflow.ModflowBas(m,ibound = ib,strt = strt)

lpf = flopy.modflow.ModflowLpf(m,laytyp=0,ipakcb=50)

# # define flow in

# # Arrange outputs for feedback calculations

# ## this next cell takes the model output heads and flows and creates two matrices, of numit rows by ncol columns

# 1. create function, inputs are heads and flow from model

# 2. set up empty lists to fill within nested for loo

# 3. first layer of for loop iterates over the rows (number of model runs, numit)

# 4. second layer of for loop iterates over each element in the row (the heads or flows for a given run)

# 5. heads comes out as a list, turn it into an array and reshape for easy indexing

# In[9]:

def arrange(heads,flow):

    row = []

    newHd = []

    frow = []

    newFl = []

    for i in range(numit):

        for j in range(ncol):

            new = heads[i][0][0][j]

            row.append(new)

            new2 = flow[i][0][0][j]
```

```
            frow.append(new2)
        newHd = row
        newFl.append(frow[0:ncol])
    #now reshape the heads from a list to a array of numit by ncol:
    heads = np.asarray(newHd)
    heads = heads.reshape((numit,ncol))
    flows = newFl
     return [heads, flows];
# In[10]:
A = modelrun(modelname)
# In[11]:
CB = arrange(A[1],A[0])
CBhds = CB[0]
CBflws = CB[1]
# ## now I'm going to perturb the flows and run the model again
# In[12]:
#run confined, perturbed flows
pFlows = []
mult = [1,2,3,4,5,6,7]
if Flag == 1:
    for i in range(numit):
        new = mult[i]*0.1+Flows[i]
        pFlows.append(new)
elif Flag == -1:
    for i in range(numit):
        new = mult[i]*-0.1+Flows[i]
        pFlows.append(new)
Flows = pFlows
B = modelrun(modelname)
```

```
# redifine flows based on base flow indicated at start
# In[13]:
Flows = []
for i in range(numit):
    flow = Base_Flow
    Flows.append(flow)
# In[14]:
CP = arrange(B[1][numit:2*numit],B[0][numit:2*numit]) #don't ask, too painful
CPhds = CP[0]
CPflws = CP[1]
# In[15]:
plt.plot(CPhds[0],'ro')
plt.plot(CBhds[numit-1],'bo')
plt.title('confined heads')
plt.show()
# import matplotlib.pyplot as plt
# plt.plot([1,2,3,4], [1,4,9,16], 'ro')
# plt.axis([0, 6, 0, 20])
# plt.show()
# # Set up unconfined model:
# In[16]:
modelname = 'unconf'
m = flopy.modflow.Modflow(version = 'mf2005',exe_name = 'mf2005.exe',modelname = modelname)
laycbd = 0
laytyp = 1 #zero for confined
botm = 0
top = 100 #for confined, top equals lowest top from unconfined case at H1, for unconfined, top = 100
nstp = 1
tsmult = 1
```

```
steady = True

strt = 20. #constant head boundary

dis =
flopy.modflow.ModflowDis(m,nlay=nlay,nrow=nrow,ncol=ncol,nper=nper,delr=delr,delc=delc,laycbd=la
ycbd,top=top

,botm=botm,nstp=nstp,tsmult=tsmult,steady=steady)

flow = []

heads = []

# create ibound array with all cells active

ib = np.ones((nrow,ncol))

# set first cell to active

# set last cell in first row to contant head

ib[(nrow-1),(ncol-1)] = -1

bas = flopy.modflow.ModflowBas(m,ibound = ib,strt = strt)

lpf = flopy.modflow.ModflowLpf(m,laytyp=1,ipakcb=50)

# In[17]:

C = modelrun(modelname)

# In[18]:

UB = arrange(C[1],C[0])

UBhds = UB[0]

UBflws = UB[1]

# In[19]:

Flows = pFlows

D = modelrun(modelname)

# Redefine flows again, to reset from perturbed values.

# In[20]:

Flows = []

for i in range(numit):

   flow = Base_Flow

   Flows.append(flow)

# In[21]:
```

```
UP = arrange(D[1][numit:2*numit],D[0][numit:2*numit]) #don't ask, too painful

UPhds = UP[0]

UPflws = UP[1]

# In[22]:

modelname = 'conf'

m = flopy.modflow.Modflow(version = 'mf2005',exe_name = 'mf2005.exe',modelname = modelname)

laycbd = 0

laytyp = 0 #zero for confined

botm = 0

top = 20 #28.54370689 #(Q = 90 for case where only testing perturbations from Q = 90)

#21.45 #for confined lowest (Q = 30), top equals lowest top from unconfined case at H1, for unconfined,
top = 100

nstp = 1

tsmult = 1

steady = True

strt = 20. #constant head boundary

numit = len(Flows)

dis =
flopy.modflow.ModflowDis(m,nlay=nlay,nrow=nrow,ncol=ncol,nper=nper,delr=delr,delc=delc,laycbd=la
ycbd,top=top

,botm=botm,nstp=nstp,tsmult=tsmult,steady=steady)

flow = []

heads = []

# create ibound array with all cells active

ib = np.ones((nrow,ncol))

# set first cell to active

# set last cell in first row to contant head

ib[(nrow-1),(ncol-1)] = -1

bas = flopy.modflow.ModflowBas(m,ibound = ib,strt = strt)

lpf = flopy.modflow.ModflowLpf(m,laytyp=0,ipakcb=50)
```

```
# In[23]:
#plt.plot(UPhds[0],'ro')
plt.plot(UPhds[0],'bo')
plt.plot(CPhds[0], 'ko')
plt.title('confined (black) vs unconfined (blue) heads across system')
plt.show()
# In[24]:
def gradmake(base,pert):
    bgrad = []
    pgrad = []
    for i in range(numit):
        for j in range(ncol-1):
            b = (base[i][j+1] - base[i][j])/delr
            bgrad.append(b)
            p = (pert[i][j+1] - pert[i][j])/delr
            pgrad.append(p)
    return bgrad, pgrad;
 # In[25]:
def Xgradmake(base,pert):
    bgradX = []
    pgradX = []
    for i in range(numit):
        bX = (base[i][ncol-1] - base[i][0])/delr
        bgradX.append(bX)
        pX = (pert[i][ncol-1] - pert[i][0])/delr
        pgradX.append(pX)
    return bgradX, pgradX;
# In[26]:
Xgrad = Xgradmake(CBhds,CPhds)
# In[27]:
```

```
Xbase = np.asarray(Xgrad[0])
Xpert = np.asarray(Xgrad[1])
dXgrd = Xbase-Xpert
# In[28]:
XG0 = []
for i in range(numit):
    new = dXgrd[i]/(Flows[i]-pFlows[i])
    XG0.append(new)
# In[29]:
Ugrads = gradmake(UBhds,UPhds)
Cgrads = gradmake(CBhds,CPhds)
# In[30]:
UBgrad = np.asarray(Ugrads[0]).reshape((numit,ncol-1))
UPgrad = np.asarray(Ugrads[1]).reshape((numit,ncol-1))
CBgrad = np.asarray(Cgrads[0]).reshape((numit,ncol-1))
CPgrad = np.asarray(Cgrads[1]).reshape((numit,ncol-1))
# In[31]:
dUgrad = UPgrad - UBgrad
dUgrad = dUgrad.reshape((numit,ncol-1))
dCgrad = CPgrad - CBgrad
dCgrad = dCgrad.reshape((numit,ncol-1))
newFl = np.asarray(pFlows) - np.asarray(Flows)
# In[32]:
UPH = np.asmatrix(UPhds)
# Create csv outputs of some of the things calculated up to this point.
# In[33]:
import pandas as pd
df1 = pd.DataFrame(UBhds)
df1.to_csv("UBhds.csv")
```

```
df2 = pd.DataFrame(UPhds)
df2.to_csv("UPhds.csv")

df3 = pd.DataFrame(CBhds)
df3.to_csv("CBhds.csv")

df4 = pd.DataFrame(CPhds)
df4.to_csv("CPhds.csv")

df5 = pd.DataFrame(pFlows)
df5.to_csv("pFlows.csv")
# # Calculate feedback!
#
# 1. R = (H2 - H1)/Q
# 2. dR/dV = (Rpert - Rbase)/(Vpert - Vbase)
# 3. dQ/dR = ((Vbase/Rpert) - (Vbase/Rbase))/(Rpert - Rbase)
# 4. G0 = 1/T for confined case
# In[34]:
#1. calculate R for all four instances:
R_UB = []
R_UP = []
R_CB = []
R_CP = []
for i in range(numit):
    for col in range(ncol - 1):
        RUB = UBgrad[i][col]/Flows[i]
        R_UB.append(RUB)
        RUP = UPgrad[i][col]/pFlows[i]
        R_UP.append(RUP)
        RCB = CBgrad[i][col]/Flows[i]
```

```
        R_CB.append(RCB)
        RCP = CPgrad[i][col]/pFlows[i]
        R_CP.append(RCP)
R_UB = np.asarray(R_UB).reshape((numit,ncol-1))
R_UP = np.asarray(R_UP).reshape((numit,ncol-1))
R_CB = np.asarray(R_CB).reshape((numit,ncol-1))
R_CP = np.asarray(R_CP).reshape((numit,ncol-1))
# n[35]:
#2. Calculate dR/dV:
dRdV_U = (R_UP - R_UB)/(dUgrad)
dRdV_C = (R_CP - R_CB)/(dCgrad)
# In[36]:
#3. dQ/dR
dQdR_U = ((UBgrad/R_UB) - (UBgrad/R_UP))/(R_UP - R_UB)*-1
dQdR_C = ((CBgrad/R_CB) - (CBgrad/R_CP))/(R_CP - R_CB)*-1
# In[37]:
#4. G0
G0 = R_CB*-1
# In[38]:
#Calculate the kernel:
UKern = G0*dQdR_U*dRdV_U
CKern = G0*dQdR_C*dRdV_C
# In[39]:
Uk = []
Uk2 = []
for i in range(numit):
    new1 = XG0[i]*dQdR_U[i][0]*dRdV_U[i][0]
    new2 = G0[i][0]*dQdR_U[i][0]*dRdV_U[i][0]
    Uk.append(new1)
    Uk2.append(new2)
```

```
# # Calculate phi:
#First take absolute values of relevant variables, then difference of natural logs
# In[40]:
RUB = np.absolute(R_UB)
RUP = np.absolute(R_UP)
UBg = np.absolute(UBgrad)
UPg = np.absolute(UPgrad)
phi = (np.log(RUB) - np.log(RUP))/(np.log(UBg)- np.log(UPg))
# # Calculate feedback from gain formula:
# dgrad/dq
# In[41]:
dgrad = UBgrad - UPgrad
dflow = np.asarray(pFlows) - np.asarray(Flows)
Gf = []
for i in range(numit):
    for j in range(1):
        now = dgrad[i]/dflow[i]
        Gf.append(now)

eff = 1-(G0/Gf)
# # Calculate predicted head changes:
# 1. calculated the predicted head changes from Gf*dQ, where Gf = G0/(1-f)
# In[42]:
#for every value in the matrix of feedback values, calculated a predicted head change
Hdspred = []
for i in range(numit):
    for j in range(ncol-1):
        pred = (G0/(1-UKern[i][j]))*(pFlows[i] - Flows[i])*delr
        Hdspred.append(pred[i][j])
```

```
# In[43]:
#it output as a list, convert to an array and reshape it
Hdspred = np.asarray(Hdspred).reshape((numit,ncol-1))
# In[44]:
for i in range(numit):
   Hdspred[i][0] *= (ncol-1)
# In[45]:
Hdspred2 = []
for i in range(numit):
   pred2 = ((XG0[i])/(1-UKern[i][0])*(pFlows[i] - Flows[i])*delr)*-1
   Hdspred2.append(pred2)
# In[46]:
Bigflows = []
for i in range(numit):
   new5 = pFlows[i]*4.4
   Bigflows.append(new5)
# In[47]:
Bigf = np.asarray(Bigflows)
diff = Bigf - Flows
# In[48]:
Hdspred_new = []
for i in range(numit):
   pre = ((XG0[i])/(1-UKern[i][0])*(diff)*delr)*-1
Hdspred_new.append(pre)
# In[49]:
for i in range(numit):
   Hdspred[i][0] = Hdspred2[i]
# In[50]:
dU = UPhds - UBhds
print(dU)
```

```
# In[51]:

dC = CPhds - CBhds

# In[52]:

dgradUC = dC - dU

 # Write results to a file

# In[53]:

phi_f = pd.DataFrame(phi)

phi_f.to_csv("phi.csv")

Kern_f = pd.DataFrame(UKern)

Kern_f.to_csv("UKernel.csv")

Xf = pd.DataFrame(eff)

Xf.to_csv("Xf.csv")

Pred = pd.DataFrame(Hdspred_new)

Pred.to_csv("PredHds.csv")

newFlows = pd.DataFrame(Bigf)

newFlows.to_csv('bigflow.csv')

import csv

with open('all', 'w') as csvfile:

    writer=csv.writer(csvfile, delimiter=',')

    writer.writerows(zip(eff,phi,UKern,Hdspred))

# In[54]:

Xf = type(str(eff))

# Create an output with the name specified at the start.

# In[55]:

import csv

with open(name, 'w') as csvfile:

    writer=csv.writer(csvfile, delimiter=',')

    writer.writerows(zip(eff,phi,UKern,Hdspred,UBhds,UPhds))
```

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