

Nuclear physics from an expansion around the unitarity limit

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Abstract. Many features of the structure of nuclei can be understood in the unitarity limit, where the two-nucleon S waves have bound states at zero energy. In this limit, the only dimensionful parameter, which is needed for proper renormalization of the relevant effective field theory, is set by the triton binding energy. While the complexity of some many-body systems may stem from a profusion of distinct scales, this one three-body scale is sufficient to generate rich structures already in few-body systems due to the anomalous breaking of continuous to discrete scale invariance. I discuss how the spectra of light nuclei arise from a controlled, perturbative expansion around the unitarity limit. I also present some implications of discrete scale invariance for nuclear matter.

1. Introduction

What is essential in the physics of nuclei? Somehow a perfect plate of *rigatoni alla genovese* at La Colombaia on the beach in Ischia seemed to me the ideal occasion to ponder this question.

The traditional answer is: a nuclear potential obtained by fitting nucleon-nucleon (NN) scattering (up to some arbitrarily chosen energy, such as that of pion production) and bound-state data nearly exactly and then appending some three-body forces (3BFs) as needed to fix the most glaring discrepancies in the description of three- and/or four-nucleon systems. In the last quarter of a century, the field-theoretical roots of this program have been revived by the use of effective field theory (EFT), particularly in the form of Chiral EFT, the generalization of Chiral Perturbation Theory to systems with $A \geq 2$ nucleons. The approach has taken hold in the community of focusing on “chiral potentials” and building them up according to increasing A . The essential features of the interactions get lost in the details of the high “orders” (NLO, N²LO, N³LO, N⁴LO, and counting...) deemed necessary to follow this program. In most reports we do not even find leading-order (LO) results, because they are supposed to be so poor.

This was not the original goal of the EFT program. EFT concerns the S matrix—the potential is only an intermediate step—and is organized by energy, not particle number. LO is supposed to provide an overall description of most low-energy observables—to represent the essential physics. One would like to have a rough but basically correct picture of few- and perhaps many-nucleon observables already at LO, with details obtained in perturbation theory. Apart from the difference in the magnitude of the expansion parameter, this is what we do in, say, non-relativistic QED, where the Coulomb interaction forms the basis on top of which finer interactions are accounted for with distorted-wave perturbation theory.

Here I discuss a recent proposal [1] for how to accomplish this through Pionless EFT [2]. I will argue that a good starting point is the limit where the $A = 2$ system is at unitarity and a single momentum scale Λ_* appears through the 3BF that ensures renormalizability for $A \geq 3$ systems. This scale arises from the anomalous breaking of scale symmetry down to a residual discrete scale invariance (DSI), which then determines the essential features of spectra and reactions. The emergence of complex spectra from such simple interactions is striking, and in the context of cold-atom physics has provided impetus to a vigorous experimental program. We now propose that it applies to nuclei as well. Our emphasis shifts from an accurate description of the NN system to an approximate description of three-, four- and hopefully more-nucleon bound states.

2. Why two-body unitarity?

For non-relativistic systems of identical particles of mass m , the $A = 2$ T matrix $T_2(k)$ at momentum k much smaller than the inverse of the potential range R is given by the effective-range expansion (ERE) [3]. When there is a shallow S -wave bound state with binding energy $B_2 \ll (mR^2)^{-1}$, the scattering length is large, $|a_2| \approx (mB_2)^{-1/2} \gg R$, while the effective range, shape parameter and higher ERE parameters have natural size, $|r_2| \sim |P_2| \sim \dots \sim R$. The unitarity limit, in which $a_2^{-1} = r_2 = P_2 = \dots = 0$, is an idealization of what I call the “unitarity window”,

$$T_2(a_2^{-1} \ll k \ll R^{-1}) = \frac{4\pi}{m} (ik)^{-1} \left[1 + \mathcal{O}\left(kR, (ka_2)^{-1}, \alpha mk^{-1}\right) \right], \quad (1)$$

where I allowed for electromagnetic effects $\alpha m \lesssim a_2^{-1}$, with α the fine-structure constant. In EFT, the expansion of T_2 in kR is rephrased as a more general expansion of *all* amplitudes in powers of $M_{\text{lo}}/M_{\text{hi}} < 1$, the ratio between the typical momentum of interest and the theory’s breakdown scale. The $(ka_2)^{-1}$ expansion is an additional expansion around the unitarity limit in powers of $\aleph/M_{\text{lo}} < 1$, where \aleph denotes a fine-tuned scale associated with the small two-body binding momentum.

$A \geq 3$ ground states, whose average binding momentum is identified with M_{lo} , fall within the unitarity window, $\aleph \ll Q_{A \geq 3} \ll M_{\text{hi}}$. An estimate of the typical binding momentum is

$$Q_A \sim \sqrt{2mB_A/A}, \quad (2)$$

since in this case each particle contributes an equal amount $-Q_A^2/(2m)$ to the total energy, while Q_2 reproduces the pole location in the complex relative-momentum plane. For nuclear systems, where the range of the interaction is given by the pion mass, $R \sim m_\pi^{-1} \simeq 1.4$ fm, we find $Q_A/m_\pi = 0.3, 0.5, 0.8, 0.7, 0.7, \dots, 0.9, \dots$ for $A = 2, 3, 4, 5, 6, \dots, 56, \dots$. An expansion in $M_{\text{lo}}/M_{\text{hi}} \sim 0.7$ should not be quickly convergent but is useful if we can calculate to many orders. (In fact, we have some indirect evidence that the expansion parameter is smaller than this.) The unitarity expansion [1] is likely no worse, since $\aleph_1/M_{\text{lo}} \sim 0.45$ if we take Q_2 , or alternatively the isospin-singlet inverse scattering length, as representative of the 3S_1 scale \aleph_1 . A more limited, 1S_0 unitarity expansion [4, 5] converges much faster because $\aleph_0/M_{\text{lo}} \sim 0.05$, using the middle component ($I_3 = 0$) of the isospin-triplet inverse scattering length as representative of the 1S_0 scale \aleph_0 . Isospin splitting does not spoil this expansion because the difference between $I_3 = +1$ and $I_3 = 0$ inverse scattering lengths is $\sim \alpha m \sim \aleph_0$, while that between $I_3 = -1$ and $I_3 = 0$ is $\sim m_d - m_u < \aleph_0$, where m_d (m_u) is the down (up) quark mass.

We do not have a precise explanation for why the NN system is close to unitarity. A possibility [6] is that the physical quark masses happen to be near the critical values where the 1S_0 virtual state (deuteron) goes (un)bound, similarly to cold atoms around a Feshbach resonance produced by an external magnetic field. Whatever the reason, it looks like nuclear systems are in the sweet spot of the unitarity window. The estimates above are admittedly very crude; a more convincing case is made by explicit calculations, which I describe in the following.

3. Pionless EFT

EFT is the framework to generate the most general S matrix consistent with assumed symmetries. In the case of QCD, we are concerned with Lorentz, parity, time-reversal and baryon-number symmetries —breaking of all of which can be accounted for when/if desired— and color $SU(3)_c$ and electromagnetic $U(1)_{\text{em}}$ gauge invariance. At the energies of interest, the relevant degrees of freedom are nucleons whose annihilation is encoded in a color-singlet, isospin-doublet, two-component spinor field ψ , for which Lorentz invariance can be implemented through a k/m_N expansion. The most general action is

$$S = \int \frac{dt d^3r}{2m} \left\{ \psi^\dagger \left(2im \frac{\partial}{\partial t} + \nabla^2 \right) \psi - 4\pi \sum_{I=0,1} C_{0I} \psi^\dagger \psi^\dagger P_I \psi \psi - \frac{(4\pi)^2}{3} D_0 \psi^\dagger \psi^\dagger \psi^\dagger \psi \psi \psi + \dots \right\}, \quad (3)$$

where P_I stands for a projector onto isospin $I = 0, 1$, and C_{0I} , D_0 , *etc.* are “low-energy constants” (LECs). The “...” include terms with more derivatives, fields, and powers of $e = \sqrt{4\pi\alpha}$ and $m_d - m_u$. The mild constraints of QCD symmetries in this energy regime imply that essentially the same EFT applies also to very different systems, such as atoms at distances larger than the van der Waals length l_{vdW} defined from the coordinate-space potential $V(r) = -l_{vdW}^4/(2mr^6)$. For ${}^4\text{He}$ atoms, for example, ψ is a bosonic field representing the annihilation of an atom and there is only one C_{0I} interaction, with $P_I = 1$.

The action involves singular interactions which need to be regularized as in any other quantum field theory. The momentum regulator parameter Λ arbitrarily splits short-range physics between loops and LECs. Renormalization is the procedure that fixes the cutoff dependence of the LECs in such a way that observables are independent of the arbitrary regularization, up to terms that go as inverse powers of Λ . At any given order, this residual cutoff dependence can be made as small as higher-order terms by taking $\Lambda \gtrsim M_{\text{hi}}$. Failure to eliminate non-negative powers of Λ for one choice of regulator implies model dependence on the form of the regulator function and signals the omission of important interactions at that order.

The EFT’s non-relativistic character simplifies its solution tremendously. Unless they are present in initial or final states, antiparticles do not appear explicitly, and the A -body system can be solved without worrying about systems with more bodies.

4. $A = 2$

The order-by-order solution of the two-body system is straightforward [7]. At LO it consists of the sum of all ladder diagrams with the LO components of C_{0I} interactions. Renormalization at unitarity requires the latter to be $C_{0I}^{(0)}(\Lambda) = -(\theta_0\Lambda)^{-1}$, where $\theta_0 = \mathcal{O}(1)$ is a number that depends on the specific form of the regulator. The LO amplitude is the first term in Eq. (1). The first two terms in the action (3) are invariant under continuous scale transformations,

$$r \rightarrow \alpha r, \quad t/m \rightarrow \alpha^2 t/m, \quad \Lambda \rightarrow \alpha^{-1} \Lambda, \quad \psi \rightarrow \alpha^{-3/2} \psi, \quad (4)$$

with a parameter $\alpha > 0$. The system is also conformally invariant [8] and Wigner $SU(4)_W$ -symmetric [9].

Higher-order terms in Eq. (1) arise from perturbative insertions of higher-order interactions. At NLO, which consists of one insertion of NLO interactions, scale invariance is broken explicitly by $(ka_2)^{-1}$, kr_2 and αmk^{-1} effects. The scattering lengths enter through the NLO components of C_{0I} , $C_{0I}^{(1)}$, in the combination $(a_2\Lambda)^{-1}$. The effective ranges appear through two-derivative, four-field interactions in the “...” of Eq. (3). Electromagnetic effects arise from (long-range) Coulomb photon exchange stemming from the gauge-covariant derivatives also not written explicitly in Eq. (3), and from an isospin-breaking contact interaction needed for renormalization, which

accounts for the splitting between $I_3 = +1$ and $I_3 = 0$ isotriplet inverse scattering lengths. At this order the predictions, apart from the improved energy dependence of $T_2(k)$, are the equality of $I_3 = -1$ and $I_3 = 0$ isotriplet scattering lengths and of all isotriplet effective ranges, which is reasonably satisfied experimentally. Higher orders include isospin breaking from $m_d - m_u$. For more details, see Refs. [4, 1, 5].

5. $A = 3$

The three-body problem in Pionless EFT also requires an exact solution with the LO two-body interactions. The EFT is still renormalizable, but only with the D_0 interaction in Eq. (3) [10, 11, 12]. With a sharp-cutoff regulator at unitarity,

$$D_0^{(0)}(\Lambda) \propto \frac{1}{\Lambda^4} \frac{\sin\left(s_0 \ln(\Lambda/\Lambda_\star) - \arctan s_0^{-1}\right)}{\sin\left(s_0 \ln(\Lambda/\Lambda_\star) + \arctan s_0^{-1}\right)}, \quad (5)$$

where $s_0 \simeq 1.00624$ and Λ_\star is a physical parameter, which is however only defined up to a factor $\exp(n\pi/s_0) \simeq (22.7)^n$, with n an integer. This is a renormalization-group (RG) limit cycle.

The appearance at LO of a single dimensionful parameter Λ_\star breaks continuous scale invariance down to a discrete subgroup with $\alpha \rightarrow \alpha_n \equiv \exp(n\pi/s_0)$, as can be verified directly from Eqs. (3) and (5). DSI manifests itself in a geometric tower of bound ‘‘Efimov states’’ [13], with the tower position determined by Λ_\star . The tower is truncated from below, with states more bound than the physical ground state falling outside the regime of validity of the EFT. In the nuclear case we can fix Λ_\star from the triton binding energy B_t (equal to the helion energy $B_h^{(0)}$ at this order). Three-body observables are correlated so that variation in Λ_\star leads to a curve on the plane of two observables, such as the empirical ‘‘Phillips line’’ [14] relating B_t and the spin-doublet neutron-deuteron (nd) scattering length. The 3BF is $SU(4)_W$ -symmetric [12, 15].

At NLO, the amplitude is renormalized by a shift $D_0^{(1)}$ in the 3BF LEC, which can be determined by demanding that B_t does not change. Isospin-breaking NN interactions split helion from triton and we can predict [1]

$$B_h^{(1)} - B_t = -(0.92 \pm 0.18) \text{ MeV}, \quad (6)$$

to be compared with the experimental value, -0.764 MeV. Subleading corrections affect the shallow Efimov states, which are sensitive to the finite values of the scattering lengths. Indeed, a virtual state in nd scattering becomes the first triton excited state as Q_2 is decreased [16].

6. $A = 4$

For the various regulators that have been studied, the four-body system seems to be renormalizable in Pionless EFT without a four-body force in LO [17, 18, 19, 20, 21, 22, 23]. As a consequence, DSI should be approximately valid. Indeed, for bosons two towers of bound states have been identified [19], where each Efimov state spawns two four-body states: one $\simeq 1.002$ times deeper than the trimer-particle threshold, the other $\simeq 4.6$ times deeper [24]. The correlation with the three-body tower manifests itself in a line in the space of $A = 3, 4$ observables produced by changing Λ_\star , such as the empirical ‘‘Tjon line’’ [25] on the plane of ground-state energies.

There has been no $A = 4$ calculation where the full NLO interaction is accounted for in perturbation theory. Existing calculations [20, 26] treat NLO interactions nonperturbatively, which precludes a study of renormalizability.

For nucleons at unitarity [1], we have considered LO and an ‘‘incomplete NLO’’ where only $(ka_2)^{-1}$ corrections were included in first-order perturbation theory. We indeed observe

convergence with the cutoff in both cases. At LO the alpha-particle binding energy $B_\alpha^{(0)}$ is consistent with that for four bosons, while at incomplete NLO

$$B_\alpha^{(1,\text{inc})} = 29.5 \pm 8.7 \text{ MeV}, \quad (7)$$

to be compared with the experimental value, 28.4 MeV. We also observed an excited state close to the triton-nucleon threshold. Thus, an expansion around unitarity captures the basic elements of the four-nucleon system.

7. $A > 4$

We have recently shown that no higher-body force appears at LO, either: for $A = 5, 6$ bosons, ground-state energies converge with the cutoff [22], and the same seems true for nucleons [27], as expected on the basis of the further suppression of many-body contact forces coming from the derivatives required by the exclusion principle. This has far-reaching consequences. First, at unitarity the same parameter Λ_\star , which can be traded for the three-body binding energy B_3 , determines the spectrum of all systems within the range of applicability of Pionless EFT, generating “generalized Tjon lines” [22]. For ground states [28, 29, 22], for example,

$$\frac{B_A}{A} = \kappa_A \frac{B_3}{3}, \quad (8)$$

where κ_A is a set of universal numbers, which however depend on whether the particles are bosons or fermions. Second the spectrum should display DSI in the form of geometric towers, just as for $A = 4$, which indeed has been demonstrated explicitly for bosons [28, 30, 31, 32].

Moreover, using potential-model results for $A = 2, 3$ ^4He atoms in order to determine the LECs, LO EFT works well for $A = 4, 5, 6$ [22]. This suggests that small ^4He -atom clusters are within the range of applicability of the EFT, and are thus universal: they depend essentially on two parameters, which one can take as the energies of the dimer and one trimer state. This is in spite of the fact that binding energies grow rapidly, $\kappa_A \simeq 3(A - 2)^2/A$ [22]. The growth tapers off, and a liquid-drop extrapolation of potential-model results gives saturation with $\lim_{A \rightarrow \infty} \kappa_A \simeq 180$ [33]. It seems that large ^4He -atom clusters are not universal [34]. In contrast, at unitarity bosonic clusters saturate with $\lim_{A \rightarrow \infty} \kappa_A \simeq 90$ [35]. The corresponding distance remains larger than that associated with the next-deeper trimer, which is beyond the validity of the EFT. In this sense, unitary bosonic clusters are universal.

Since for nucleons binding energies grow much more slowly, we have speculated [1] that also larger nuclear systems can be described in a first approximation in terms of Λ_\star alone. If that is true we might expect DSI to be relevant for nuclear matter as well, in which case at LO [36]

$$\lim_{A \rightarrow \infty} \frac{B_A(\rho)}{A} = - \left(\frac{3\pi^2}{2} \right)^{2/3} \frac{\rho^{2/3}}{m} \Upsilon \left(\frac{s_0}{3} \ln \left(\frac{3\pi^2 \rho}{2\Lambda_\star^3} \right) \right), \quad (9)$$

where ρ is the particle density and Υ is a (real) periodic function with period π , which can be expanded in a Fourier series in terms of real coefficients $\gamma_{n \geq 0}$ and phases $\phi_{n \geq 1}$,

$$\Upsilon(x + l\pi) = \Upsilon(x) = \frac{3\gamma_0}{10} - \sum_{n=1}^{\infty} \gamma_n \sin(2nx + \phi_n). \quad (10)$$

The minimum of Eq. (9) describes a quadratic curve under variation of Λ_\star , just as the empirical “Coester line” [37]. A truncation of the Fourier series in Eq. (10) can describe empirical properties of nuclear matter around the minimum. As before, the two-body scattering lengths and effective ranges enter at NLO.

8. Conclusion

The answer to the opening question, if we are right, is: a single-parameter three-body force needed for renormalization-group invariance around the unitarity limit. Evidence is that this works for $A = 3, 4$ systems. There is a long way to go, though, to show that this framework works also for larger nuclei. I am already looking forward to reporting new results at the next Ischia meeting!

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