Topological actions via gauge variations of higher structures

Hisham Sati [a,*], Matthew Wheeler [b]

[a] *Division of Science and Mathematics, New York University, Abu Dhabi, United Arab Emirates*
[b] *Department of Mathematics, The University of Arizona, Tucson, AZ 85721, USA*

A R T I C L E   I N F O

A B S T R A C T

In this note we provide a new perspective on the topological parts of several action functionals in string and M-theory. We show that rationally these can be viewed as large gauge transformations corresponding to variations of higher structures, such as String, Fivebrane, and Ninebrane structures.

© 2018 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP[3].

1. Introduction

Dirac's formulation of spinors establishes that the tangent space with its rotation group (in the Euclidean setting) or Lorentz group (in the Minkowski case) is not enough to account for their behavior and properties. The proper formulation is via lifting to the Spin group, which is the double cover of the orthogonal group. From a topological point of view, one requires that the spacetime manifold M^n admit a Spin structure, whose existence can be established at the level of classifying space $\mathrm{BSpin}(n)$ and requires the vanishing of the second Stiefel–Whitney class w_2 of M^n. Had the Spin group not already been known, one could have further defined the Spin group to be an appropriate loop space of $\mathrm{BSpin}(n)$.

Generalizing from point particles to strings, one realizes that a Spin structure is no longer able to fully account for anomalies, and an appropriate formulation requires lifting to a String structure [1]. This still allows for using Dirac operators and to extending to other backgrounds, such as orbifolds [2]. To account for other signatures, including Lorentzian, semi-Riemannian analogues of String structures are constructed in [3]. From the point of view of (higher) groups, one has the String group as the loop space of the classifying space $\mathrm{BString}(n)$ of such structures. Similar, but more subtle, arguments hold in part for the generalization to the fivebrane [4][5] and the ninebrane [6], where one can also lift to groups (in a more general homotopy sense) that are loop spaces of the corresponding classifying spaces $\mathrm{BFivebrane}(n)$ and $\mathrm{BNinebrane}(n)$, respectively. The structures are related via the Pontrjagin classes

$$\mathrm{SO} \rightsquigarrow \mathrm{Spin} \xrightarrow{p_1^{\mathbb{Q}}} \mathrm{String} \xrightarrow{p_2^{\mathbb{Q}}} \mathrm{Fivebrane} \xrightarrow{p_3^{\mathbb{Q}}} \mathrm{Ninebrane}.$$

The above structures are in the spirit of the higher algebraic approaches to M-branes [7][8][9], but more on the topological side. By uncovering such structures (see [10] for another recent illustration), one hopes that more insight is gained into the fundamental nature of M-theory and its relation to string theory. A priori there are several subtleties involved in considering the above higher structures; most notably the presence of torsion in cohomology and the need to use higher generalization of bundles, i.e., higher gerbes or n-bundles. While such formulations have been applied in useful ways, e.g., to the worldvolume theories of M-branes (see [11][12][13][14][15] for various approaches), both of these complications can be set aside by rationalizing the structures, i.e., by taking the corresponding cohomology to be over the rational (or real) numbers. The outcome is a resolution of both subtleties with the same token: not only does torsion get evaded, but also the higher structures themselves can now be described using a formulation via only the much more familiar Spin structures and corresponding bundles, which after all is sufficient for many purposes in physics.

Indeed, in [16] we defined and characterized these rational Spin-String, Spin-Fivebrane, and Spin-Ninebrane structures. It is our aim here to apply these to describe how topological action functionals in M-theory and string theory can be interpreted as global gauge transformations corresponding to variations of such structures. Such a point of view on the actions via variations of structures has been proposed in [17], where variations of framing (essentially a parallelism, i.e. with a trivialization of the tangent bundle) was highlighted. Our current description is a generalization to the case when the structure is topologically (highly) nontrivial.

---

* Corresponding author.
E-mail address: hsati@nyu.edu (H. Sati).

We consider systems arising from string theory and M-theory via action functionals which, to some extent, allow us to classically describe the dynamics of the system. However, we are interested in considering the "topological terms" which are independent of the metric on the underlying manifold. These terms have the virtue that they are trusted to some extent upon probing the quantum regime. Therefore, focusing on the topological terms allows for at least setting up a starting point for a corresponding quantum field theoretic construction, namely the partition function Z (see [18][19][4] for detailed treatments associated with such a problem). It is natural to ask whether the physical entities, e.g. the action functional and the partition function, depend on the underlying geometric and topological structures imposed on spacetime. For instance, (in)dependence on the underlying smooth structure plays an important role in global anomalies [20]. Similarly, dependence on the Spin structure is a classical question in string theory [21]. One can also ask a related question: How do the physical entities behave under the variation of a corresponding structure? While we do not attempt a full and general treatment here, we do describe the connection in the M-theory and string theory setting: for the case of Spin-Fivebrane structures in relation to the NS5/M5-brane action, and then for Spin-Ninebrane structures for the Green–Schwarz anomaly and its dual as well as for the Chern–Simons and one-loop terms in M-theory.

Interpreting the cup products from [16] as topological Lagrangians for brane sigma models, means to regard these branes as propagating not on spacetime M, but on the total space of the Spin bundle Q, i.e., we have sigma models on spacetimes which are "extended". Indeed, in [22], the authors describe the heterotic string not as an algebraically defined CFT, but as a geometric sigma model on the Spin bundle Q of spacetime M. In this "fibered $(0, 2)$ WZW model", the Green–Schwarz-twisted B-field is viewed as a bundle gerbe on Q which restricts on each fiber to the canonical bundle gerbe on Spin, hence to the topological term for the fiberwise WZW model. This means, in particular, that the degree three class of the bundle gerbe on Q restricts on each fiber to the canonical 3-form on Spin. We interpret this here as the String structure $\mathcal{S}$ (see below).

Since the action is fiberwise a WZW model, these additional fiberwise degrees of freedom are exactly what is expected to give the extra current algebra degrees of freedom seen in the heterotic string, but not seen in an ordinary geometric sigma-model on spacetime M [22]. Our approach then can be viewed as extending this case to other situations in M-theory and string theory, namely for the M-theory action and for the NS5/M5-brane action, albeit at the rational level. On the other hand, and in a dual sense, here we are looking at the higher brane version of Distler–Sharpe's heterotic string sigma models on Q (instead of on M) [22], as an analogous "fibered" version of the heterotic NS5-brane sigma-model. Such a heterotic perspective fits well with the general setup as Fivebrane structures [4][5] are motivated from anomalies seen specifically for the heterotic NS5-brane in the "magnetic dual" formulation of heterotic string theory. Other instances of extended spacetime can be found in [10] and references therein.

2. Variations of structures as global gauge transformations

Generally, insisting that the partition function Z of the topological part of the action functional S be defined and well-defined as a function on the moduli space of fields imposes conditions on the underlying manifolds and bundles on them, in the form of constraints on primary (and possibly secondary) characteristic classes. A very classical example, mentioned above, is Dirac's theory of fermions, which requires a Spin bundle lift of the tangent bundle, i.e. the underlying spacetime M has to have a vanishing

second Stiefel–Whitney class, $w_2(M) = 0$. Other instances arise in string theory and M-theory where a lift to one of the higher connected covers discussed earlier is needed (see [4][5][6]).

It is natural to ask whether the physical entities, e.g. S or Z, depend on the underlying geometric and topological structures on spacetime: How do the physical entities behave under the variation of a corresponding structure? For instance, for Chern–Simons theory, the variation of the Spin structure was considered in [23], where two Spin structures essentially differ by some real line bundle. Hence once $w_2(M) = 0$ is satisfied we seek possible dependence on the elements in $H^1(M; \mathbb{Z}_2)$ or, equivalently, real line bundles. In going to higher structures, in [11] dependence of the M2-brane partition function on the underlying String structure imposed was studied via gerbes, as elements in $H^3(M; \mathbb{Z})$. Since the M2-brane worldvolume is 3-dimensional, then it automatically admits a String structure. The main point in [11] is to identify the action functional as a variation of the String structure, using the constructions in [24]. Here what we really mean by a variation of a $\mathcal{O}$-structure is

$$\text{Variation of } \mathcal{O}\text{-structure} = \delta\mathcal{O} = \mathcal{O} - \mathcal{O}', \tag{1}$$

the difference of two elements in the space of $\mathcal{O}$-structures. When a certain field ϕ is identified with a structure $\mathcal{O}$ (pulled back to spacetime or worldvolume) then we view (1) as a global or large gauge transformation for ϕ.

In the current context, the space of such structures is the cohomology group in degree one less than that of the obstruction for the given structures, i.e. $H^3(M; \mathbb{Z})$, $H^7(M; \mathbb{Z})$, and $H^{11}(M; \mathbb{Z})$, for String, Fivebrane, and Ninebrane structures, respectively. We are rationalizing the structures, so that these cohomology groups take rational coefficients. Furthermore, these 'moduli spaces' are vector spaces, so that there is no obstruction to moving in that space, i.e., to forming the variations (1). We will see situations where the left-hand side is the difference between two rational Fivebrane structures $\mathcal{F} - \mathcal{F}'$ or two rational Ninebrane structures $\mathcal{N} - \mathcal{N}'$, respectively. These will be given essentially by corresponding lower structures on the right-hand sides, namely, a String structure and a Fivebrane structure.

Varying underlying topological structures has been advocated in [17] for the case of framed manifolds, where topological parts of the action functionals for the membrane (M2-brane) and the fivebrane (M5-brane) worldvolume theories were interpreted via a change of framing formula, in the spirit of the variational principle. A similar discussion holds for the NS5-brane [25]. Here we adopt that point of view, leading us to interpret the topological action functionals as variations of higher structures. Being in the rational setting makes the identification more direct and transparent. We emphasize that, while we start with de Rham expressions, we will be interested in Lagrangians and action functionals promoted (explicitly or implicitly) to the level of cohomology, as in [26][27][11][17]. The action functionals involve classes on the total spaces of the principal bundles, but we are considering expressions on the base spacetime. This can be done either by assuming a section, as is the case for Chern–Simons theory, or via a more elaborate process such as using a Hodge decomposition as in [24]. This will be implicit below and, as such, we are not aiming for the most general case.

The constructions and results in [16] are established for rational numbers, but they readily extend over the reals, especially for smooth manifolds as we consider here. Note that we are considering only topological terms and our expressions are cohomological, and so we are in a setting akin to that of a topological field theory. By variation we mean large gauge transformations that are not expected to hold locally, hence the parameters do not vary smoothly.

3. Variations on rational Fivebrane classes

Here we show that the topological action associated with the NS5-brane and the M5-brane can be interpreted via variations of Fivebrane classes. Starting with the String group as a structure group, a String-principal bundle $\pi_{\text{String}} : P \to M$ has a rational Fivebrane structure if there is a lift of the rationalized classifying map $f : M \to B\text{String}_{\mathbb{Q}}$ to the homotopy fiber of (a fraction) of the second rational Pontrjagin class $p_2^{\mathbb{Q}}$ (note that since we are working rationally, the fractions are inconsequential). A rational Fivebrane structure class is a cohomology class $\mathcal{F} \in H^7(P; \mathbb{Q})$ such that $\iota_x^* \mathcal{F} = a_7 \in H^7(\text{String}; \mathbb{Q})$ for each fiber inclusion $\iota_x : \text{String} \to P$. Such higher structures (beyond Spin) have been recast using Spin structures. We have shown in [16] that

(i) For every rational Spin-Fivebrane class $\mathcal{F} \in H^7(Q; \mathbb{Q})$, the pullback $\rho^* \mathcal{F}$ is a rational Fivebrane class.
(ii) For any rational Fivebrane structure $\mathcal{F} \in H^7(P; \mathbb{Q})$ there is a Spin-Fivebrane class $\tilde{\mathcal{F}} \in H^7(Q; \mathbb{Q})$ such that $\rho^* \tilde{\mathcal{F}} = \mathcal{F}$.
(iii) Two classes $\mathcal{F}, \mathcal{F}' \in H^7(Q; \mathbb{Q})$ give the same Fivebrane structure if $\mathcal{F} - \mathcal{F}' = \mathcal{S} \cdot \pi_{\text{Spin}}^* \phi_4$ where $\mathcal{S} \in H^3(Q; \mathbb{Q})$ is the String structure class and $\phi_4 \in H^4(M; \mathbb{Q})$ is a rational cohomology class.

Thus two Fivebrane structures are identified rationally if their difference corresponds to a torsion class in $H^7(M; \mathbb{Z})$. This says that, rationally, all the information on Fivebrane structures is essentially encoded in the underlying Spin bundles.

Example 1 *(The M5/NS5-brane action and variation of Spin-Fivebrane structures).* Consider the M5/NS5-brane on an extended worldvolume, which is a seven-dimensional Spin manifold X^7, as in [28][29]. The action functional of the fivebranes has been considered in [11][13][14] from the point of view of String bundles with String connections. At the level of differential forms, the topological part is given as

$$S_{\text{M5/NS5}} = \int_{X^7} C_3 \wedge G_4 , \qquad (2)$$

where C_3 and G_4 have the usual M-theory meaning for the M5-brane and are different for the NS5-brane (see [30] where the corresponding 6-dimensional term is studied). Passing to cohomology, we consider the pairing of corresponding cohomology classes with the fundamental homology class of the manifold

$$S_{\text{M5/NS5}}^{\text{coh}} = \langle [C_3] \cup [G_4], [X^7] \rangle , \qquad (3)$$

with $G_4 = \text{non-exact} + dC_3$, so that we are able to take both C_3 and G_4 to be closed (see [27]). The heterotic perspective with target Q mentioned in the Introduction and based on [22] fits naturally here as, due to Green–Schwarz, $C_3 = H_3 + CS_3$ contains a closed 3-form as a summand, without itself being closed, due to the Chern–Simons term. Indeed, the heterotic NS5-brane action has been considered from a superspace perspective in [30]. Some aspects of this, but in a restrictive situation, extends to M-theory in the flat case, due to the quantization condition [26], or alternatively upon working in heterotic M-theory [31].

Note that the class of C_3 or H_3 can be interpreted [11] (up to a shift) as a String structure $\mathcal{S}$. The integrand then corresponds to the class $\mathcal{S} \cup \phi$, with $\phi = [G_4]$ being a rational degree four cohomology class. Indeed, $[G_4]$ integrally satisfies the quantization condition (see [26]) $G_4 + \frac{1}{2}\lambda \in H^4(Y^{11}; \mathbb{Z})$, where Y^{11} is the ambient spacetime into which X^7 (and its bounding space) is mapped.

When we rationalize, the requirement that the first Spin characteristic class $\lambda = \frac{1}{2}p_1$ is divisible by two (i.e. Y^{11} admitting a "Membrane structure" [12]) is automatically satisfied, in which case $[G_4] \in H^4(Y^{11}; \mathbb{Q})$. Therefore, we identify the integrand, i.e., the Lagrangian at the level of cohomology, as the difference of two rational Spin-Fivebrane structures on Y^{11}. That is,

$$\mathcal{L}_{M5}^{\text{coh}} = \mathcal{F}_{\mathbb{Q}} - \mathcal{F}_{\mathbb{Q}}' . \qquad (4)$$

This means that for the M5/NS5-brane action, there exist two Spin-Fivebrane structures such that their difference is that part of the Lagrangian. As in the case of the 3-dimensional M2-brane worldvolume admitting a String structure [11], the extended M5/NS5-brane worldvolume automatically admits a Fivebrane structure by virtue of it being 7-dimensional.

Note that if $H^4(M; \mathbb{Q})$ is torsion, then the set of Fivebrane classes and Spin-Fivebrane classes coincide [16]. Compactification manifolds for which this occurs include the following manifolds as in the realistic Kaluza–Klein list [32].

Examples.

(i) The Witten manifolds $M_{k,\ell}$, which are S^1 bundles over the product of complex projective spaces $\mathbb{C}P^2 \times \mathbb{C}P^1$, are classified in [33] according to two integers k and ℓ. They have $H^4(M_{k,\ell}; \mathbb{Z}) = \mathbb{Z}/\ell^2$.
(ii) Generalized Witten manifolds N_{kl} are defined as the total spaces of fiber bundles with fiber the lens space $L_k(\ell_2, \ell_2)$ and structure group S^1. They have $H^4(N_{kl}; \mathbb{Z}) \cong \mathbb{Z}_{|\ell_1 \ell_2|}$ [34].
(iii) Quaternionic line bundles E over closed Spin manifolds of dimension $4k - 1$ with $c_2(E) \in H^4(M)$ been torsion considered in [35] via generalizations of the Kreck–Stolz invariants.

4. Variations on rational Ninebrane classes

We now extend the results from the last section to the next higher connected cover of the orthogonal group O in the sequence in the Introduction. This allows to describe the terms in the Green–Schwarz anomaly cancellation and its dual, as well as the M-theory topological terms. Note that here things become a bit subtle, as there are two structures sitting in between Fivebrane and Ninebrane, denoted 2Orient and 2Spin, respectively [6]. However, these are defined via mod 2 obstructions, so that rationalization will make them equivalent to a Fivebrane structure. So, to follow along the lines of rational Fivebrane structures, we may define rational Ninebrane structures [16].

- A Fivebrane-principal bundle $\pi_{\text{Fivebrane}} : T \to M$ admits a rational Ninebrane structure if there is a lift of the rational classifying map $f : M \to B\text{Fivebrane}_{\mathbb{Q}}$ to the homotopy fiber $F(\frac{1}{240}p_3)^{\mathbb{Q}}$.
- A rational Ninebrane class is a cohomology class $\mathcal{N}_{\mathbb{Q}} \in H^{11}(T; \mathbb{Q})$ such that $\iota_x^* \mathcal{N} = a_{11} \in H^{11}(\text{Fivebrane}; \mathbb{Q})$ for each inclusion $\iota_x : \text{Fivebrane} \to T$.

Now, just as we did in the case of Fivebrane structures, we can relate these classes to ones on the underlying Spin bundle. In order to do this, as we compared degree 7 rational cohomology between Spin and String, we need to compare the degree 11 rational cohomology of Spin and Fivebrane. The map $\rho : \text{Fivebrane} \to \text{Spin}$ induces an isomorphism $\rho^* : H^{11}(\text{Spin}; \mathbb{Q}) \xrightarrow{\cong} H^{11}(\text{Fivebrane}; \mathbb{Q})$ [16], which were used to relate rational Ninebrane classes to classes on the underlying Spin bundle. A rational Spin-Ninebrane class is then a cohomology class $\mathcal{N}_{\mathbb{Q}}$ in $H^{11}(Q; \mathbb{Q})$ such that

$\iota_x^* \mathcal{N}_{\mathbb{Q}} = \tilde{a}_{11} \in H^{11}(\text{Spin}; \mathbb{Q})$ for each $x \in M$. In this case, we have shown in [16]:

(i) For every rational Spin-Ninebrane class $\mathcal{N}_{\mathbb{Q}} \in H^{11}(Q; \mathbb{Q})$, the pullback $\rho^* \mathcal{N}_{\mathbb{Q}}$ is a rational Ninebrane class.

(ii) Any rational Ninebrane structure $\mathcal{N}_{\mathbb{Q}} \in H^{11}(T; \mathbb{Q})$ can be described by a class in $H^{11}(Q; \mathbb{Q})$.

(iii) Two classes $\mathcal{N}_{\mathbb{Q}}, \mathcal{N}'_{\mathbb{Q}} \in H^{11}(Q; \mathbb{Q})$ will give the same rational Ninebrane structure if

$$\mathcal{N}_{\mathbb{Q}} - \mathcal{N}'_{\mathbb{Q}} = \mathcal{S} \cdot \pi_{\text{Spin}}^* \psi_8 + \mathcal{F} \cdot \pi_{\text{Spin}}^* \phi_4 , \tag{5}$$

where $\mathcal{S} \in H^3(Q; \mathbb{Q})$ is the String structure class, $\mathcal{F} \in H^7(Q; \mathbb{Q})$ is the Fivebrane structure class, while $\psi_8 \in H^8(M; \mathbb{Q})$ and $\phi_4 \in H^4(M; \mathbb{Q})$ are rational cohomology classes.

We now consider in the right degree a fundamental example in M-theory, as the topological part of 11-dimensional supergravity [36] together with the one-loop term [37]. Large gauge transformations in M-theory have been considered from a geometric perspective in [38], while the full symmetries of the C-field are explained in [27][39].

Example 2 (*The M-theory action and variation of Spin-Ninebrane structures*). We now consider M-theory on a String manifold Y^{11}, as in [40]. The known topological action functional of M-theory is given as

$$\int_{Y^{11}} \left(\tfrac{1}{6} G_4 \wedge G_4 \wedge C_3 - I_8 \wedge C_3 \right) \tag{6}$$

where I_8 is called the one-loop polynomial, whose corresponding cohomology class is $\frac{1}{48}(p_2 - \lambda^2)$. From a cohomological point of view, properly describing the action and the corresponding partition function (or path integral) is involved due to the presence of subtle torsion (see [19]). However, we will again evade such subtle issues when we rationalize. Having already the interpretation of $[C_3]$ as a String class $\mathcal{S}$, we now interpret $[\tfrac{1}{6} G_4 \wedge G_4 - I_8]$ as a rational cohomology class $y_8 \in H^8(Y^{11}; \mathbb{Q})$, rationalizing the interpretation at the integral level in [27][4][5]. Consequently, we have that the Lagrangian, again at the level of rational cohomology, is a variation of a rational Spin-Ninebrane class

$$\mathcal{L}_M^{\text{coh}} = \mathcal{N}_{\mathbb{Q}} - \mathcal{N}'_{\mathbb{Q}} , \tag{7}$$

where we identify y_8 with $\pi_{\text{Spin}}^* \psi_8$, and where we take $\pi_{\text{Spin}}^* \phi_4$ to be zero. By dimension reasons, we now automatically have a Ninebrane structure on Y^{11}, and so the action functional captures the trivialization of that structure in the form of a Spin-Ninebrane structure. This is similar to having a String structure on the M2-brane by virtue of dimension [11].

The next two examples deal with the Green–Schwarz anomaly cancellation, which is one of the main highlights of string theory [41]. We will first consider the usual mechanism where we do the matching with expression (5) using $\pi_{\text{Spin}}^* \phi_4 = 0$, and then consider the dual to the Green–Schwarz formulation (in the sense of [18][5]) by taking $\pi_{\text{Spin}}^* \psi_8 = 0$.

Example 3 (*The Green–Schwarz anomaly cancellation mechanism and variation of Spin-Ninebrane structures*). This anomaly cancellation arises in heterotic string theory (or type I supergravity). The system involves a ten-dimensional manifold M^{10} with its natural Spin bundle, a Yang–Mills bundle E, a closed 3-form H_3, with corresponding cohomology class $[H_3]$. The bundle E enters the expressions via its Chern character $\text{ch}(E)$, while the natural bundles are

accounted for via the Pontrjagin classes $p_i(TM^{10})$. The corresponding action functional includes the term

$$\mathcal{L}_{\text{GS}} = H_3 \wedge J_8 , \tag{8}$$

where J_8 is a closed 8-form with cohomology class $-\text{ch}_4(E) + \frac{1}{48} p_1(M) \text{ch}_2(E) - \frac{1}{64} p_1(M)^2 + \frac{1}{48} p_2(TM)$. Identifying the class $[H_3]$ with the String class $\mathcal{S}$, the expression at the level of cohomology is of the form $\mathcal{L}_{\text{GS}}^{\text{coh}} = \mathcal{S} \cdot \psi_8$, where we have also identified the rational cohomology class ψ_8 with $[J_8]$. This gives a special instance of a variation of rational Spin-Ninebrane structures

$$\mathcal{L}_{\text{GS}}^{\text{coh}} = \mathcal{N}_{\mathbb{Q}} - \mathcal{N}'_{\mathbb{Q}} . \tag{9}$$

Note that considerable constraints would be needed in order to ensure that $[J_8]$ is an integral class (see [4] for a discussion on when this is the case). A virtue of working rationally is also highlighted here in the evasion of these complications.

Example 4 (*The dual Green–Schwarz anomaly and variation of Spin-Fivebrane structures*). The dual Green–Schwarz anomaly cancellation arises in heterotic string theory (or type I supergravity). The system still involves a ten-dimensional manifold M^{10} with its natural Spin bundle, a Yang–Mills bundle E, except that now we have the (Hodge-dual) closed form H_7, with corresponding cohomology class $[H_7]$, of degree seven. The action functional involves a term of the form

$$\mathcal{L}_{\text{dual GS}} = H_7 \wedge J_4 , \tag{10}$$

where J_4 is a 4-form with cohomology class $p_1(TM^{10}) - \text{ch}_2(E)$. Then the cohomology class corresponding to this action functional is given as the difference

$$\mathcal{L}_{\text{dual GS}}^{\text{coh}} = \mathcal{N}_{\mathbb{Q}} - \mathcal{N}'_{\mathbb{Q}} , \tag{11}$$

where we identify $[H_7]$ with the Spin-Fivebrane class $\mathcal{F}$, $[J_4]$ with the form $\pi_{\text{Spin}}^* \phi_4$, and where we take $\pi_{\text{Spin}}^* \psi_8$ to be zero.

Rationally, there is an isomorphism $H^{12}(B\text{Spin}; \mathbb{Q})/(p_1^{\mathbb{Q}}, p_2^{\mathbb{Q}}) \cong H^{12}(B\text{Fivebrane}; \mathbb{Q})$. If $H^4(M; \mathbb{Z})$ and $H^8(M; \mathbb{Z})$ are pure torsion, then the set of Ninebrane classes and Spin-Ninebrane classes coincide.

Example. We can give a nontrivial example of a spacetime manifold X which has torsion $H^4(X; \mathbb{Z})$, vanishing $H^8(X; \mathbb{Z})$ and non-torsion $H^{12}(X; \mathbb{Z})$. As in [42], let SU(2) be the subgroup of SU(4) consisting of all block diagonal matrices $\text{diag}(A, A)$ where $A \in \text{SU}(2)$. Then the 12-dimensional quotient $X = \text{SU}(4)/\text{SU}(2)$, viewed as the base of an S^3 bundle, is stably parallelizable with $H^4(X; \mathbb{Z}) = \mathbb{Z}_2$, $H^8(X; \mathbb{Z}) = 0$ and $H^{12}(X; \mathbb{Z}) = \mathbb{Z}$.

Remarks. The above formulation extends to other theories as well:

(i) Another string theory, namely type I, admits a duality-symmetric formulation, with the dual formulation using H_7 given in [43]. A similar connection to the above higher structures holds in this case.

(ii) It is also possible to consider duality-symmetric M-theory action [44] by supplementing the dual (see [45]) to the action to (6) where we would have both $\pi_{\text{Spin}}^* \psi_4$ and $\pi_{\text{Spin}}^* \phi_8$ nonzero in general. Hence this would involve variations of both Spin-Fivebrane and Spin-Ninebrane structures.

(iii) Similarly, the duality-symmetric heterotic action [46][18] (by combining Examples 3 and 4) would involve both $\pi_{\text{Spin}}^* \psi_4$ and $\pi_{\text{Spin}}^* \phi_8$, hence also leading to variation of both structures.

(iv) With appropriate interpretation and conditions on classes, other theories such as topological (super)gravity [47], Chern–Simons (super)gravity [48], and higher Chern–Simons theories [49][50] can also be made to fit the above description.

We find it noteworthy that the topological parts of the three main action functionals in M-theory, i.e., that of the M2-brane, the M5/NS5-brane, and of classical M-theory, as well as the Green–Schwarz anomaly and its dual, can be interpreted as trivializations of higher obstructions with the rational Spin-Fivebrane and Spin-Ninebrane structures able to account for the form as well as the compositeness of the actions. The above discussion can be summarized in the following schematic table, where the last column displays the new interpretation as a variation of a higher structure.

System	Existing structure	Variation of
Chern–Simons	Riemannian	Spin
M2-brane	Spin	String
M5/NS5-brane	String	Spin-Fivebrane
M-theory, (dual) GS	String/Fivebrane	Spin-Ninebrane

The first row follows [23][11][24], while the other three start with existing descriptions, as presented in [4][11][5][6], and then provides an alternative description, as given in the above examples. More explicitly, the new description for the NS5/M5-brane theory is given in Example 1 that for M-theory in Example 2, as well as the Green–Schwarz anomaly and its dual are given in Examples 3 and 4, respectively.

What we provided above are only glimpses of connections, and we believe that this is the starting point of interesting constructions, which deserve to be elaborated on in a lot of detail. In particular, a deeper discussion on variations of the structures would have to be in the context of partition functions, rather than just action functionals. This would be considerable, as it requires the two notions that we suppressed, namely torsion and higher bundles, and would go way beyond this note, but we plan to take it up elsewhere.

Acknowledgements

References

[1] T.P. Killingback, Class. Quantum Gravity 5 (9) (1988) 1169–1185.
[2] K. Pilch, N.P. Warner, Commun. Math. Phys. 115 (1988) 191–212.
[3] H. Sati, H.-b. Shim, arXiv:1504.02088.
[4] H. Sati, U. Schreiber, J. Stasheff, Rev. Math. Phys. 21 (10) (2009) 1197–1240, arXiv:0805.0564.
[5] H. Sati, U. Schreiber, J. Stasheff, Commun. Math. Phys. 315 (1) (2012) 169–213, arXiv:0910.4001.
[6] H. Sati, Int. J. Geom. Methods Mod. Phys. 12 (2015) 1550041, arXiv:1405.7686.
[7] J. Bagger, N.D. Lambert, Phys. Rev. D 77 (2008) 065008, arXiv:0711.0955.
[8] A. Gustavsson, Nucl. Phys. B 811 (2009) 66, arXiv:0709.1260.
[9] O. Aharony, O. Bergman, D.L. Jafferis, J.M. Maldacena, J. High Energy Phys. 0810 (2008) 091, arXiv:0806.1218.
[10] H. Sati, U. Schreiber, Phys. Lett. B 781 (2018) 694–698, arXiv:1805.00233.
[11] H. Sati, Superstrings, geometry, topology, and C^*-algebras, in: Proc. Symp. Pure Math., vol. 81, Amer. Math. Soc., Providence, RI, 2010, pp. 181–236, arXiv:1001.5020.
[12] H. Sati, Int. J. Geom. Methods Mod. Phys. 8 (5) (2011) 1097–1116, arXiv:1008.1755.
[13] H. Sati, J. Aust. Math. Soc. 90 (2011) 93–108, arXiv:1007.5419.
[14] D. Fiorenza, H. Sati, U. Schreiber, Adv. Theor. Math. Phys. 18 (2) (2014) 229–321, arXiv:1201.5277.
[15] C. Saemann, L. Schmidt, J. Math. Phys. 59 (2018) 043502, arXiv:1712.06623.
[16] H. Sati, M. Wheeler, J. Geom. Phys. 130 (2018) 229–248, arXiv:1612.06983.
[17] H. Sati, J. Math. Phys. 59 (2018) 062304, arXiv:1310.1060.
[18] D.S. Freed, Surveys in differential geometry, in: Surv. Differ. Geom., vol. VII, Int. Press, Somerville, MA, 2000, pp. 129–194, arXiv:hep-th/0011220.
[19] D. Freed, G.W. Moore, Commun. Math. Phys. 263 (1) (2006) 89–132, arXiv:hep-th/0409135.
[20] E. Witten, Global anomalies in string theory, in: Symposium on Anomalies, Geometry, Topology, Chicago, Ill., 1985, World Sci. Publishing, Singapore, 1985, pp. 61–99.
[21] N. Seiberg, E. Witten, Nucl. Phys. B 276 (2) (1986) 272–290.
[22] J. Distler, E. Sharpe, Adv. Theor. Math. Phys. 14 (2010) 335–398, arXiv:hep-th/0701244.
[23] J.A. Jenquin, arXiv:math/0504524 [math.DG].
[24] C. Redden, Pac. J. Math. 249 (2) (2011) 447–484, arXiv:0912.2086.
[25] H. Sati, arXiv:1109.4834.
[26] E. Witten, J. Geom. Phys. 22 (1997) 1–13, arXiv:hep-th/9609122.
[27] E. Diaconescu, G. Moore, D.S. Freed, The M-theory 3-form and E_8 gauge theory, in: Lond. Math. Soc. Lect. Note Ser., vol. 342, Cambridge Univ. Press, Cambridge, 2007, pp. 44–88, arXiv:hep-th/0312069, 2007.
[28] E. Witten, J. Geom. Phys. 22 (1997) 103–133, arXiv:hep-th/9610234.
[29] E. Witten, J. High Energy Phys. 0005 (2000) 031, arXiv:hep-th/9912086.
[30] K. Lechner, Phys. Lett. B 693 (2010) 323–329, arXiv:1005.5719.
[31] P. Horava, E. Witten, Nucl. Phys. B 460 (1996) 506–524, arXiv:hep-th/9510209.
[32] E. Witten, Nucl. Phys. B 186 (1981) 412–428.
[33] M. Kreck, S. Stolz, Ann. Math. (2) 127 (2) (1988) 373–388.
[34] C.M. Escher, Geom. Dedic. 115 (2005) 79–120.
[35] D. Crowley, S. Goette, Trans. Am. Math. Soc. 365 (6) (2013) 3193–3225.
[36] E. Cremmer, B. Julia, J. Scherk, Phys. Lett. B 76 (1978) 409–412.
[37] M.J. Duff, J.T. Liu, R. Minasian, Nucl. Phys. B 452 (1995) 261–282, arXiv:hep-th/9506126.
[38] J. Kalkkinen, K.S. Stelle, J. Geom. Phys. 48 (2003) 100–132, arXiv:hep-th/0212081.
[39] D. Fiorenza, H. Sati, U. Schreiber, Commun. Math. Phys. 333 (1) (2015) 117–151, arXiv:1202.2455.
[40] H. Sati, Int. J. Mod. Phys. A 26 (13) (2011) 2177–2197, arXiv:0807.4940.
[41] M.B. Green, J.H. Schwarz, Phys. Lett. B 149 (1984) 117–122.
[42] W. Singhof, Math. Ann. 260 (1) (1982) 101–116.
[43] A.H. Chamseddine, Phys. Lett. B 367 (1996) 134–139.
[44] I. Bandos, N. Berkovits, D. Sorokin, Nucl. Phys. B 522 (1998) 214–233.
[45] P. Fré, Class. Quantum Gravity 1 (6) (1984) L81–L90.
[46] S.J. Gates Jr., H. Nishino, Phys. Lett. B 173 (1986) 52–58.
[47] A.H. Chamseddine, Nucl. Phys. B 346 (1990) 213–234.
[48] J. Zanelli, arXiv:hep-th/0502193.
[49] X. Bekaert, A. Gomberoff, J. High Energy Phys. 0301 (2003) 054, arXiv:hep-th/0212099.
[50] D. Fiorenza, H. Sati, U. Schreiber, J. Geom. Phys. 74 (2013) 130–163, arXiv:1207.5449.