SENIOR HONORS THESIS:

RANDOM WALKS AND RANDOM SELF-AVOIDING WALKS

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Abstract. It is well-known that the continuum limit of a random walk on a lattice is Brownian motion. Similarly it is believed (but not known) that the continuum limit of a self-avoiding walk is so called Schramm-Loewner evolution, and that the continuum limit of a random self-avoiding polygon is a random loop measure recently constructed by Wendelin Werner. In this paper we give an exposition of these objects. The main questions we eventually focus on are combinatorial issues regarding numbers of self-avoiding walks and polygons. It is a surprise to find that the connective constant for self-avoiding walks and self-avoiding polygons are the same. Some of these issues have been recently settled by Mardras, Slade, Lawler, Duminil-Copin, Smirnov, and Hammond.

0. Introduction

An example of a random walk is the erratic movement of a particle within a medium, for example, a dust particle in the atmosphere. Mathematically, there are two different types of random walks. The first is discrete, in which one jumps from one point of a lattice to a neighboring point of the lattice. This type is also called simple random walk. The other is continuous, which is usually referred to as Brownian motion. The second is a 'continuum limit' of the first (Chapter 10 and 12 of [1]). Both are examples of Markov processes.

A random self-avoiding walk (SAW) is a walk in which the motion is not allowed to return to a previously occupied site (there are again two types, discrete and continuous). SAWs are far more complicated than simple random walks. SAW first arose as a model of polymer chains, modeled as a sequence of moves on a lattice that does not revisit the same point more than once ([2]). Studying random self-avoiding walk can help to understand a range of phenomena in various fields, such as share prices in financial economics and estimating the size of a website in computer science (Section 3 of [3]). A discrete SAW is not a Markov process. A continuous SAW (i.e. an SLE curve) has a Markov type property, but it is far more complicated than in the random walk case.

A Self-avoiding polygon(SAP) is a simple closed curve in the lattice with neither starting point nor orientation. It can be regarded as it's built from a self-avoiding walk, but with one point that have been walked through twice. It is believed, but not known, that random self-avoiding polygons tend to outer boundaries of a Brownian motion (this has been conjectured by Werner [15]).

I shall study the following results and questions:

Theorem 0.1. *Given a discrete time Markov process with countable state space and transition probabilities $p_{i,j}(n)$, the state i is recurrent if and only if*

$$\sum_{n=0}^{\infty} p_{i,i}(n) = \infty$$

Theorem 0.2. *(Polya's theorem) The symmetric random walk on Z^d is recurrent if $d = 1$, 2 and transient if $d \geq 3$. This means if starting from the origin, it will go back to the origin infinitely often in dimensions 1 and 2, and with probability 1, it will never return to the origin in dimension 3.*

Question 1. *How many SAWs are there with n steps? Does this number change in different dimensions?*

Conjecture 1. *Consider the set of walks on the lattice L which start at the origin and have length n. Then the number of such walks is asymptotically $c_N \sim A\mu^N N^{\gamma-1}$, where A, μ, γ are positive constants that depend on dimension. We refer to μ as the connective constant.*

Question 2. *How many SAPs are there with n steps? How do self-avoiding loops, which surround the origin with length n, behave differently from self-avoiding walks that start at the origin with length n? ([4])*

Theorem 0.3. *Given $c_n(x,y)$ is the number of n-step self-avoiding walks from x to y, for every integer $N \geq 2$. There exists a constant C depending only on d, such that*

$$\mu_{polygon}{}^{2M} e^{-cM^{1/2}} \leq c_{2M+1}(0,e) \leq \frac{2(M+1)(d-1)}{d} \mu_{polygon}{}^{2M+2},$$

for all $M \geq 1$.

Theorem 0.4. *Connective constant for self-avoiding walk on square lattice is the same as for self-avoiding polygon.*

0.1. **Plan of the Thesis.** This thesis is organized as follows. In Section 1, random walk is introduced by first introducing Discrete time Markov Chain, based on what I learned from courses Math 466 and Math 468. It contains the proof of theorem 0.1 and 0.2. Section 2 constructs a Brownian motion, which is based on Xia's paper. Section 3 discusses the number of SAWs with n steps and some properties of SLE, based on what I learned from Madras, Slade's book, and Lawler's paper. In Section 4, the relation between SAP and SAW is derived, based on Hammond's paper. Then, I studied Werner's paper and tried to construct a measure satisfying weak conformal restriction property using Brownian loop measures. But it's incomplete since I ran out of time. Finally, The appendix is based on the course, Math 422.

discrete processes	continuous processes
Symmetric discrete random walk	Brownian motion
Self-avoiding walk	Schramm-Loewner Process
Self-avoiding discrete loops	self-avoiding loops

Among these processes, symmetric discrete RW and Brownian motion both have Markov property, Schramm-Loewner process has a domain Markov property, others has no Markov property.

0.2. **Notation.** A summation, $\sum$, is an addition of a sequence of any kind of numbers.

Assume that X is a random variable, then $E(X)$ denotes the expected number of X.

$\mathbb{P}(A|B)$ denotes the probability of the occurrence of event A given event B happens.

The factorial of a number x, denoted by x!, is the product of x and any positive integers that's less than x.

For a real number t, $[t]$ denotes the greatest integer $\leq t$.

For a sequence a_n, $\lim_{n\to\infty} a_n$ is the value that the sequence a_n tend to be when n goes to infinity.

For a sequence a_n, $\inf_{a_n}$ of a subset S of a set T is the greatest element in T that's less than or equal to all the elements in S.

$f : A \to B$ means that the domain of function f is A, the range of function f is B, i.e., f is a function from A to B.

exp(x) is the exponential function of x, i.e., $exp(x) = e^x$.

For a function f, the big O notation of f, denoted by O(f), describes the asymptotic behavior of f.

The tensor product of two vector space V and W, denoted by $V \otimes W$ is a way of creating a new vector space analogous to multiplication of integers. ([17])

1. Discrete Time Markov Chains

1.1. **Basic Definitions.** Throughout we suppose that there is a probability space $(\Omega, \mathcal{E}, \mathbb{P})$ in the background. Unless specified otherwise, it will be understood that random variables are defined on Ω.

Definition 1. *Suppose $\mathcal{S}$ is a space which we call the state space.*

(a) A discrete time stochastic process with values in $\mathcal{S}$ is a sequence of random variables $X_0, X_1, X_2, ...$ taking values in the set $\mathcal{S}$. Recall that, stochastic processes are mathematical models of systems, which vary in time in random manner.

(b) A continuous time stochastic process with values in S is a family of random variables X_t, taking values in the set S, indexed by 'time' t in some interval (e.g. $\{t \geq 0\}$).

Note the state space S could be finite, countable, or a continuum. This divides both discrete time and continuous time stochastic processes into different classes.

Definition 2. *A discrete time stochastic process satisfies the Markov property if*

$$\mathbb{P}(X_{n+1} = i_{n+1}|X_0 = i_0, X_1 = i_1, ..., X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1}|X_n = i_n),$$

where $n > 0$, and $i_0, i_1, ..., i_n, i_{n+1} \in \mathcal{S}$. In this case we refer to X as a Markov chain.

In intuitive terms the future value (X_a) only depends on the present$(X_b : b < a)$ and is independent of the past$(X_c : c < b)$. This is analogous to an ordinary differential equation, where the behavior of a function is determined by an initial condition and the law (i.e. the differential equation) that the solution obeys.

1.2. **Discrete Time, Discrete State Space Markov Chain.** In order to compute the probabilities for such a chain, we need to know the transition matrix and initial distribution. These are analogous to the differential equation and initial condition satisfied by a solution to a differential equation

Here's the definition of two quantities of Markov chain from book [1], page 206.

Definition 3. *(a) The transition matrix is often denoted by* $P = (p_{i,j} : i, j \in \mathcal{S})$ *given by* $p_{i,j} = \mathbb{P}(X_1 = j | X_0 = i)$*, where* $p_{i,j}$ *represents the probability of jumping from state* i *to* j*.*

(b) The initial distribution $\lambda = (\lambda_i : i \in \mathcal{S})$ *given by* $\lambda_i = \mathbb{P}(X_0 = i)$ *is a vector, where* $\lambda_i \geq 0$ *for* $i \in \mathcal{S}$*, and* $\sum_{i\in\mathcal{S}} \lambda_i = 1$*.*

Example 1. *Some examples of Markov chain from Hoel, Paul's book, section 1.4 ([13]), which I studied from one of my class, Math 468.*

(1). Birth and death chain.

Consider a chain either on $\varphi = \{0, 1, 2, ...\}$ *or on* $\varphi = \{0, 1, ..., d\}$ *such that starting from x the chain will be at x - 1, x, or x + 1 after one step. In this application, a transition from state x to x + 1 corresponds to a "birth", while a transition from state x to state x - 1 corresponds to a "death".*

The transition function of Birth and Death chain is given by

$$P(x, y) = \begin{cases} q_x, & y = x - 1, \\ r_x, & y = x, \\ p_x, & y = x + 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{1.1}$$

where q_x, r_x, p_x *are nonnegative numbers such that* $q_x + r_x + p_x = 1$*.*

(2). Queuing chain.

Consider a service facility such as a checkout counter at a supermarket. People arrive at the facility at different times and get served eventually. Those customers who have arrived at the facility but have not been served form a queue. Suppose that if there are any customers waiting for service at the beginning of any given period, exactly one customer will be served during that period. If there are no customers waiting for service at the beginning of a period, none will be served during that period.

Let ξ_n *denote the number of new customers arriving during the nth period. Assume that* $\xi_1, \xi_2, ...,$ *are independent nonnegative integer-valued random variables having common density f. Let* X_0 *denote the number of customers present initially, and for* $n \geq 1$*, let* X_n *denote the number of customers present at the end of the nth period. If* $X_n = 0$*, then* $X_{n+1} = \xi_{n+1}$*; and if* $X_n \geq 1$*, then* $X_{n+1} = X_n + \xi_{n+1} - 1$*. Therefore,* X_n *is a Markov chain whose state space is the nonnegative integers and whose transition function P is given by*

$$P(0, y) = f(y)$$

and

$$P(x, y) = f(y - x + 1), x \geq 1.$$

(3). Branching chain.

Consider particles such as neutrons or bacteria that can generate new particles of the same type. The initial set of objects is referred to as belonging to the 0th generation. Particles generated from nth generation are said to belong to the (n + 1)th generation.

Suppose that each particle gives rise to ξ *particles in the next generation, where* ξ *is a nonnegative integer-valued random variables having density f. Suppose that the number of offspring of the various particles in the various generations are chosen independently according to the density f. Then,* $X_n, n \geq 0$*, forms a Markov chain whose state space is the nonnegative integers. State 0 is an absorbing state, which means if a particle gives rise to 0 particles in the next generation, it will stay at 0 in the later generations.*

The transition function for Branching chain is $P(x, y) = P(\xi_1 + ... + \xi_x = y)$*, for* $x \geq 1$*, where* $\xi_1, ..., \xi_x$ *are independent random variables having common density f. In particular,* $P(1, y) = f(y), y \geq 0$.

Theorem 1.1. *(Extended Markov Property)*

Let X *be a Markov chain. A discrete time stochastic process satisfies the extended Markov property if*

$$\mathbb{P}(X_b|X_n = i, X_a) = \mathbb{P}(X_b|X_n = i),$$

for $n \geq 0$*,* $i \in \mathcal{S}$*, for a past event* X_a*, and a future event* X_b*, where* $a < n, n < b$.

Examples 1. *(1). Symmetric Random Walk: A symmetric random walk on* $\mathbb{Z}^d$ *is a random walk with equal probability of jumping to any direction.*

(2) Gambler's Ruin Problem: A gambler starts gambling with a certain amount of capital. The gambler makes successive one dollar bets against the house, and the game ends if the gambler possess zero dollar. Suppose X_n *represent the amount of capital of the gambler after n bets, then* X_{n+1} *can be either* X_n*+1 or* X_n*-1* $(X_n \neq 0)$*. So* X_{n+1} *only depends on* X_n.

Let probability of winning a dollar be p, then the probability of losing a dollar is 1-p. If p = 1/2, then this is the same as a symmetric random walk in one dimension, where the starting point is the initial amount of capital and the ending point is the origin.

(3) Branching chain: Consider bacteria that generate new generation of bacteria. Let X_n *be the number of bacteria in the nth generation. Since each generation of bacteria produce offsprings independently,* X_{n+1} *only depends on* X_n.

1.3. **Chapman-Kolmogorov equations.**

Definition 4. *We call* $p_{i,j}$ *one-step transition probabilities, and* $p_{i,j}(n)$ *n-step transition probabilities, given that* $p_{i,j}(n) = \mathbb{P}(X_n = j|X_0 = i)$*. The n-step transition probabilities form a matrix called the n-step transition matrix* $P(n) = (p_{i,j}(n) : i, j \in \mathcal{S})$.

Here is a theorem from book [1], page 209.

Theorem 1.2. *(Chapman-Kolmogorov equation)*

$$p_{i,j}(m+n) = \sum_{k \in \mathcal{S}} p_{i,k}(m) p_{k,j}(n)$$

for $i, j \in \mathcal{S}$ *and* $m, n \geq 0$*. That is,* $P(m+n) = P(m)P(n)$.

1.4. **Recurrence and transience.**

In this subsection we consider a discrete time Markov chain with countable state space.

Let $T_j = min\{n \geq 1 : X_n = j\}$ represent the first-passage time, and $f_{i,j}(n) = \mathbb{P}_i(T_j = n)$ represent the first-passage probabilities.

Definition 5. *A state i is called recurrent if $\mathbb{P}_i(T_i < \infty) = 1$, which means a walk starting from state i will eventually go back to state i. A state is called transient if it is not recurrent.*

The following theorem gives a useful criterion for recurrence, which is from book [1], page 214.

Theorem 1.3. *The state i is recurrent if and only if*

$$\sum_{n=0}^{\infty} p_{i,i}(n) = \infty$$

Remark 1. *The left hand side is the expected value of the number of visits to the site i. More precisely if we set*

$$R = \sum_{n=0}^{\infty} 1_{\{X_n = i\}}$$

then

$$E(R) = \sum_{n=0}^{\infty} p_{i,i}(n)$$

In the following lemma, we introduce the generating functions

$$P_{i,j}(s) = \sum_{n=0}^{\infty} p_{i,j}(n) s^n$$

where $p_{i,j}(0) = \delta_{i,j}$, and

$$F_{i,j}(s) = \sum_{n=0}^{\infty} f_{i,j}(n) s^n$$

where $f_{i,j}(0) = 0$.

Lemma 1. *For $i, j \in \mathcal{S}$, we have that*
$P_{i,j}(s) = \delta_{i,j} + F_{i,j}(s) P_{j},_{j}(s), -1 < s \leq 1$

Proof. Firstly, by partition theorem,

$$p_{i,j}(n) = \sum_{m=1}^{\infty} \mathbb{P}_i(X_n = j | T_j = m) \mathbb{P}_i(T_j = m),$$

where $n \geq 1$. If $m > n$, then the summand is zero, because the probability that $X_n = j$ will be zero if the first passage time if $m > n$.

Secondly, according to homogeneity and extended Markov property,

$$\mathbb{P}(X_n = j, T_j = m) = \mathbb{P}(X_n = j | X_m = j) = \mathbb{P}(X_{n-m} = j),$$

which is $p_{j,j}(n - m)$.

Therefore, we get

$$p_{i,j}(n) = \sum_{m=1}^{n} p_{j,j}(n-m) f_{i,j}(m)$$

where $n \geq 1$, and we recall that $\mathbb{P}_i(T_j = m) = f_{i,j}(m)$. This is a convolution type equation, hence it is advantageous to express this in terms of generating functions (which turn convolution into multiplication). By multiplying through this equation by s^n and sum over $n \geq 1$, we

$$P_{i,j}(s) - p_{i,j}(0) = F_{i,j}(s) P_{j},_{j}(s)$$

(On the right hand side, the convolution has been replaced by a product). By setting $p_{i,j}(0) = \delta_{i,j}$, we can finally derive the equation in the lemma.

□

Proof. (of Theorem 1.3)

Define $f_{i,j} = F_{i,j}(s = 1) = \mathbb{P}_i(T_j < \infty)$.

By definition, we know that i is recurrent if and only if $f_{i,i} = F_{i,i}(s = 1) = \mathbb{P}_i(T_i < \infty) = 1$.

Then, by applying Lemma 1 with $i = j$, we can get $\mathbb{P}_{i,i}(s) = \frac{1}{1-\mathbb{F}_{i,i}(s)}$. When $s = 1$, $\mathbb{F}_{i,i}(1) = f_{i,i}(1)$. Therefore, $f_{i,i} = 1$ implies $P_{i,i}(1) = \infty$. Thus, i is recurrent if and only if $P_{i,i}(1) = \infty$.

Since $P_{i,j}(1) = \sum_{n=0}^{\infty} p_{i,i}(n)$, we know that i is recurrent if and only if $\sum_{n=0}^{\infty} p_{i,i}(n) = \infty$.

□

1.5. Random walks in one, two, and $d \geq$three dimensions.

Theorem 1.4. *(Polya's theorem) The symmetric random walk on $\mathbb{Z}^d$ is recurrent if d = 1, 2 and transient if $d \geq 3$.*

Proof. I learned this proof from Grimmett and Welsh's book ([1]).

(1) Suppose that $d = 1$ and $X_0 = 0$.

The walker can only return to state 0 after an even number of steps, so the probability of returning to state 0 is choosing n steps out of 2n steps to go to the right.

So, $p_{0,0}(2n) = \binom{2n}{n}(\frac{1}{2})^n(\frac{1}{2})^n = (\frac{1}{2})^{2n}\binom{2n}{n} = (\frac{1}{2})^{2n}\frac{(2n)!}{(n!)^2}$.

By Stirling's formula, $p_{0,0}(2n) = (\frac{1}{2})^{2n}\frac{(2n)!}{(n!)^2} \sim \frac{1}{\sqrt{\pi n}}$ as $n \to \infty$. The sum $\sum n^{-1/2}$ diverges. Therefore $\sum_n p_{0,0}(2n) = \infty$, and the state 0 is recurrent by theorem 1.3.

(2) Suppose that d = 2 and the walk starts at the origin (0, 0).

The walk can only return to the origin at time 2n if the number of steps going left is equal to the number of going right, and the number of steps going up is equal to going down. So the number of leftward steps plus upward or downward should be equal to n.

Therefore, we have $p_{0,0}(2n) = (\frac{1}{4})^{2n}\binom{2n}{n}\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = (\frac{1}{2})^{4n}\binom{2n}{n}^2$, which is the square of $p_{0,0}(2n)$ for d=1.

So, $p_{0,0}(2n) \sim \frac{1}{\pi n}$. The sum $\sum n^{-1}$ is divergent. Therefore $\sum_n p_{0,0}(2n) = \infty$ and the state 0 is recurrent.

(3) Finally suppose $d \geq 3$, and the walk start at the origin.

Let l_i denote the number of leftward steps on i-th dimension, r_i denote the number of rightward steps on i-th dimension. Then, only if $l_i = r_i$, for 1≤ $i \leq d$, the walk can return to the origin. Thus

$$p_{0,0}(2n) = (\frac{1}{2d})^{2n} \sum_{l_1+...+l_d=n, r_1+...r_d=n} \frac{(2n)!}{(l_1!...l_d!)(r_1!...r_d!)}$$

$$= (\frac{1}{2d})^{2n} \sum_{l_1+...+l_d=n} \frac{(2n)!}{(l_1!...l_d!)^2}$$

$$= (\frac{1}{2d})^{2n} \frac{(2n)!}{(n!)^2} \sum_{l_1+...+l_d=n} (\frac{n!}{l_1!...l_d!})^2$$

$$= (\frac{1}{2})^{2n} \binom{2n}{n} M \sum_{l_1+...+l_d=n} (\frac{n!}{d^n l_1!...l_d!}) \tag{1.2}$$

where

$$M = max\{\frac{n!}{d^n l_1!...l_d!} : l_1, l_2, ..., l_d \geq 0, l_1 + ... + l_d = n\}$$

We can get M when $l_1, l_2, ..., l_d$ are all closest to $\frac{1}{d}n$, which leads to

$$M \leq \frac{n!}{d^n([\frac{1}{d}n]!)^d}$$

Since the summand in (1.2) is the probability of allocating n balls randomly to d urns, the urns contain $l_1, l_2, ..., l_d$ balls respectively, the summand adds up to 1.

So,

$$p_{0,0}(2n) \leq \frac{1}{2^{2n}} \frac{(2n)!}{n!n!} \frac{n!}{d^n([\frac{1}{d}n]!)^d}$$

$$= \frac{(2n)!}{(4d)^n n!([\frac{1}{d}n]!)^d}$$

By Stirling's formula, $p_{0,0}(2n) \leq Cn^{-\frac{d}{2}}$ for some constant C. Therefore, $\sum_{n=0}^{\infty} p_{0,0}(2n) < \infty$. Thus, the origin is transient.

□

2. Brownian motion

2.1. Basic Definitions. Brownian motion is a stochastic process that models continuous random walk. Let X_t represent the position of a particle at time t, where t takes values in nonnegative real numbers and X_t takes values in the real line (or plane or space). So, Brownian motion is an example of a stochastic process with continuous time and continuous state space. We can construct Brownian motion as a limit of random walks, according to Xia's paper ([18]).

2.2. Construction of Brownian motion. Suppose S_n is an unbiased one-dimensional simple random walk. We can write $S_n = X_1 + X_2 + ... + X_n$, where the random variables X_i are independent. $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$.

Now we scale random walk by changing the time increment from 1 to $\Delta t = \frac{1}{N}$, where N is an integer.

Define the random function

$${W_t}^{(\mathrm{N})} = a_N S_k \text{ when } t = \frac{k}{N}$$

and we linearly interpolate between these values for other t,

$$W_t^{(N)} = W_{\frac{[tN]}{N}}^{(N)} + (t - \frac{[tN]}{N}) W_{\frac{[tN]+1}{N}}^{(N)}$$

Since $Var(S_N) = N$, we choose a_N to be $\frac{1}{\sqrt{N}}$. So then, ${W_t}^{(\mathrm{N})} = \frac{1}{\sqrt{N}} S_k$ when $t = \frac{k}{N}$.

Recall the statement of the Central Limit Theorem:

Theorem 2.1. *Suppose that $Y_1, Y_2, ...$ is a sequence of i.i.d. random variables with mean 0 and finite common variance σ^2. Then the random variable $N^{-1/2} \sum_{n=1}^{N} Y_n$ has variance σ^2 and converges in distribution (i.e. weakly) to the normal random variable $N(0, \sigma^2)$ as $N \to \infty$.*

Corollary 1. *For fixed* $t = \frac{k}{N}$*, the distribution of* ${W_t}^{(N)} = \sqrt{t}\frac{S_k}{\sqrt{k}}$ *approaches a normal distribution with mean 0 and variance* t *as* $N \to \infty$ *(or equivalently as* $k \to \infty$*).*

As $N \to \infty$, this linearly interpolated and scaled random walk approaches a continuous-time, continuous-space process which is called standard Brownian motion which has the following properties.

Proposition 1. *A one dimensional standard Brownian motion satisfies the following properties:*

(a). $X_0 = 0$.

(b). If $s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n$*, then* $X_{t_1} - X_{s_1}, \ldots, X_{t_n} - X_{s_n}$ *are independent.*

(c). For any $s < t$*, the random variable* $X_t - X_s$ *has a normal distribution with mean 0, and variance* $(t - s)$*.*

(d). The function from t *to* X_t *is a continuous function of* t*.*

A standard Brownian motion in d dimensions is an $\mathbb{R}^d$ valued process $X = (X_1, .., X_d)$ such that the X_i are independent one dimensional standard Brownian motions. Given $x_0 \in \mathbb{R}^d$, $\sigma X + x_0$ is a Brownian motion with root at x_0 and variance σ^2.

2.3. Recurrence and Transience.

Theorem 2.2. *For one-dimensional Brownian motion, if* X_t *is a standard Brownian motion, then* X_t *is recurrent. More precisely, if* $Z = \{t : X_t = 0\}$*, then* Z *is an uncountable closed set which does not have any isolated points.*

Lemma 2. *Let* $Z = \{t : X_t = 0\}$*, then* Z *satisfies two facts:*

(1). Z *is a closed set. This means, if a sequence of points* $t_i \in Z$ *and* $t_i \to t$*, then* $t \in Z$*.*

(2). None of the points of Z *are isolated points.*

Z is similar to a Cantor ternary set from a topological perspective, where the Cantor ternary set is created by deleting the middle third from a set of line segments.

Theorem 2.3. *In dimension* $d = 2$*, Brownian motion is neighborhood recurrent but not point recurrent, which means the Brownian motion returns arbitrarily close to 0 infinitely often, but never actually returns to 0.*

The Brownian motion in dimension $d \geq 3$ *is transient, which implies that the probability that* $X_t = 0$ *for* $t > 0$ *is zero.*

Remark 2. *The Hausdorff dimension of a Brownian motion is 2. In dimension* $d = 4$ *a Brownian motion tends to intersect itself in points. In dimension* $d > 4$ *the Brownian motion does not intersect itself at all. This means that it becomes self-avoiding, which is the next topic.*

3. Self-avoiding walk

3.1. Basic Definition.

A discrete self-avoiding walk is a random walk on a lattice where the walk does not visit a site more than once.

3.2. **Subadditive Sequences.**

Definition 6. *A sequence a_n, $n \geq 1$, is called subadditive if it satisfies the property:*

$$a_{n+m} \leq a_n + a_m,$$

for all m and n.

Lemma 3. *Fekete's subadditive lemma:*

If a sequence $\{a_n\}_{n=1}^{\infty}$ is subadditive, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $inf_{n\geq 1} \frac{a_n}{n}$.

Proof. Let $a_1, a_2, a_3, ...$ be a sequence of non-negative real numbers with the subadditive property: $a_{i+j} \leq a_i + a_j$, for all i, j $\geq$1.

Let L $= inf_{n\geq 1} \frac{a_n}{n}$. It is trivially the case that $L \leq \frac{a_n}{n}$ for all n. Fix $\epsilon > 0$. We need to show that for sufficiently large m, $\frac{a_m}{m} < L + \epsilon$.

By the definition of infimum, there exists an $n \geq 1$ such that $\frac{a_n}{n} < L + \epsilon$, which leads to $n(L+\epsilon) > a_n$. Let b $= max_{1\leq i\leq n}\, a_i$. If $m \geq n$, let m = qn + r, $0 \leq r < n$. By the subadditive property, $a_m = a_{nq+r} = a_{n+n+...+n+r} \leq a_n + a_n + ... + a_n + a_r \leq qa_n + b$. Thus, $\frac{a_m}{m} \leq \frac{qa_n}{m} + \frac{b}{m} < \frac{qn(L+\epsilon)}{m} + \frac{b}{m}$. Since $qn/m \to 1$ and $\frac{b}{m} \to 0$ as $m \to \infty$, $\frac{qn(L+\epsilon)}{m} + \frac{b}{m} \to L + \epsilon$ as $m \to \infty$. So, when $m \to \infty$, $\frac{a_m}{m} - L < \epsilon$. Therefore, $\lim_{m\to\infty} \frac{a_m}{m} = L = inf_{n\geq 1} \frac{a_n}{n}$. $\square$

3.3. **The Connective Constant.** Suppose that we fix a lattice L in $\mathbb{R}^d$, e.g. $\mathbb{Z}^d$. There are other examples such as the hexagonal lattice, pictured below:

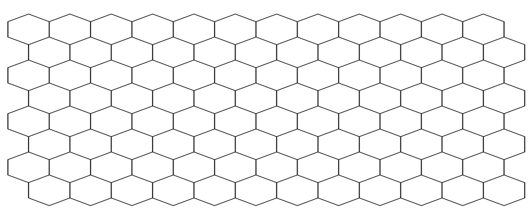

FIGURE 1. Picture of a hexagonal lattice

For a walk on $\mathbb{Z}^d$ which starts at the origin, the number of walks of length n is equal to 2^{nd}. When we require that the walk is self-avoiding, then there are fewer walks of length n. It is likely impossible to write down an exact formula for this number.

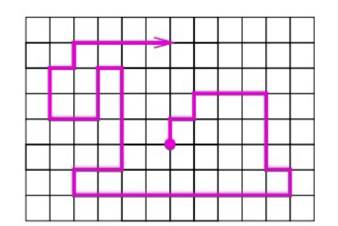

FIGURE 2. Picture of a self-avoiding walk of length 40

Here is a conjecture I learned from Madras and Slade's book ([2]).

Conjecture 2. *Consider the set of walks on the lattice L which start at the origin and have length n. Then the number of such walks is asymptotically*

$$C_n \sim ce^{\beta_c n} n^{\gamma-1}$$

Let c_N denote the number of N-step self-avoiding walks beginning at the origin (for a general lattice). In Lawler's paper, $\mu = e^{\beta_c}$. ([4]) The conjectured behavior of c_N is $c_N \sim A\mu^N N^{\gamma-1}$, where A, μ, γ are positive constants that depend on dimension. ([2])It is believed that μ is dependent on the lattice, whereas γ is supposed to be independent of the choice of lattice. It is known that γ is finite in two, three and four dimensions, and $\gamma \geq 1$ is true in all dimensions.

Since every (n + m)-step self avoiding walk can be decomposed into an n-step self-avoiding walk and an m-step self-avoiding walk, it follows that $c_{n+m} \leq c_n c_m$ and the equality only holds when n or m is zero. Taking logarithms in $c_{n+m} \leq c_n c_m$ gives us $log(c_{n+m}) \leq log(c_n) + log(c_m)$, which shows that the sequence $a_n := log(c_n)$ is subadditive as in the previous subsection. Lemma 3 implies the existence of the limit $log(u) := \lim_{N\to\infty} N^{-1} log(c_N)$, and hence gives $\mu = lim_{N\to\infty} c_N^{1/n}$. The positive real number μ is called the **connective constant**.

According to Madras and Slade's book ([2]) (page 12), the current best rigorous upper and lower bound for μ, and the estimate of the precise value for the hypercubic lattice in dimension 2 and 3 are listed below.

Example 2. *Consider $L = \mathbb{Z}^d$.*

(a) d = 1. The number of self-avoiding walks of length n is equal to 2, because the walk is either strictly increasing to the right or to the left.

(b) d = 2. The lower bound of μ is 2.61987, the upper bound of μ is 2.69576, and the estimates of the precise value is 2.6381585$\pm$0.0000010. The conjectured value of γ is $\frac{43}{32}$. ([2])

(c) d = 3. The lower bound of μ is 4.43733, the upper bound of μ is 4.756, and the estimates of the precise value is 4.6839066$\pm$0.0002. The conjectured value of γ is approximately 1.162. ([2])

(d) For d = 4, the conjectured value of γ is 1 with logarithmic corrections.

(e) For d > 4, $\gamma = 1$.

Theorem 3.1. *In the case of hexagonal lattice in d = 2, $\mu = \sqrt{2+\sqrt{2}}$.*

This is proved by Hugo Duminil-Copil and Stanislav Smirnov ([5]).

The exact value of μ is not known for the lattice in dimension $d > 2$.

3.4. **Conformal Inariance.** Suppose there exists a family of probability measures $\mu(D; z, w)$, indexed by a collection of domains D and distinct boundary points $z, w \in \partial D$, supported on curves connecting z and w in D. Assume that the family satisfies conformal invariance.

Definition 7. *This family of measures is conformally invariant iff the following is always true: If $f : D \to f(D)$ is a conformal transformation, then $f \circ \mu(D; z, w) = \mu(f(D); f(z), f(w))$. More precisely, if γ is a curve in D from z to w, then $f \circ \gamma(t) = f(\gamma(t))$ gives a curve in f(D) from f(z) to f(w) with the same law in proability. ([4])*

3.5. **Domain Markov property.**

Definition 8. *Given $\gamma[0,t]$, the distribution of the remainder of the path is given by $\mu(D_t;\gamma(t),w)$.*

The Riemann mapping theorem shows that all simply connected domains in $\mathbb{C}$ with nontrivial boundary are conformally equivalent. In particular, if D is a simply connected domain with distinct boundary points z, w, then exists a conformal transformation

$$F:\mathbb{H}=x+iy:y>0\to D, F(0)=z, F(\infty)=w.$$

([4])

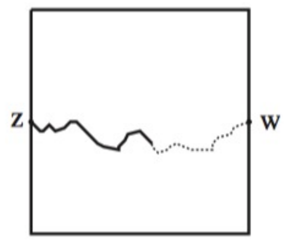

FIGURE 3. Domain Markov property

3.6. **Schramm Loewner Evolution.** Consider a simply connected domain in the plane with smooth boundary, and fix two points $z_0, z_1 \in L$ which are also on the boundary. The probability of a self-avoiding walk ω' from z_0 to z_1 is equal to

$$\frac{1}{\sum exp(-\beta|\omega|)} exp(-\beta|\omega'|)$$

where the sum is over all self-avoiding walks between the two points.

This formula is from the Gibbs measure, which is a probability measure frequently seen in many problems of probability theory and statistical mechanics.

Conjecture 3. *SAW has a continuum limit, so called Schramm Loewner evolution (SLE_κ with $\kappa = 8/3$).*

The following is a famous theorem of Loewner.

Theorem 3.2. *Suppose that γ is a self-avoiding curve in the upper half plane which starts at $z=0$. Let D_t equal the upper half space minus $\gamma([0,t])$. By the Riemann mapping theorem, there is a unique conformal isomorphism $g_t : D_t \to D$, where D denotes the full upper half space, normalized such that*

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2})$$

Then g_t satisfies the Loewner differential equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - U(t)}$$

where $U(t) = g_t(\gamma(t))$.

Example 3. *If* $\gamma(t) = 2\sqrt{t}i$*, then* $g_t(z) = \sqrt{z^2+4t}$*. In the Loewner equation,* $U(t) = 0$*. This is especially simple, because the Loewner equation is separable in this case.*

Definition 9. *The SLE curve is obtained by solving the Loewner equation when* $U(t) = \sqrt{\kappa}W(t)$ *where* $\kappa \geq 0$ *and* $W(t)$ *a standard Brownian motion starting at the origin of* $\mathbb{R}$*.*

The following is a difficult result:

Theorem 3.3. SLE_κ *is a self-avoiding curve for* $\kappa \leq 4$*. It is at worst self-touching for* $4 < \kappa \leq 8$*, and it is space filling for* $\kappa > 8$*.*

4. Self-Avoiding Polygons

Definition 10. *Let N be an integer and* $N \geq 2$*. An N-step self-avoiding polygon is a set* **P** *of N nearest-neighbour bonds with the property: there exists a corresponding self-avoiding walk* ω *having* $\mid \omega(N-1) - \omega(0) \mid = 1$ *such that* **P** *consists of precisely the bond joining ω(N - 1) to ω(0) and the N - 1 bonds joining ω(i - 1) to ω(i), i = 1, ..., N-1. ([2])*

Lemma 4. *Each N-step self-avoiding polygon has precisely 2N corresponding self-avoiding walks, since there are N choices of starting point and two choices of orientation.*

Example 4. *For Dimension 2, if we only consider different shapes and ignore starting point and orientation, we can get*
a. only one 4-step self-avoiding polygon, which is a unit square.
b. two 6-step self-avoiding polygons, which are rectangles with one being a 90° rotation of the other.

Two self-avoiding polygons can be concatenated to form a larger self-avoiding polygon. It's clear to see the procedure in two dimensions.

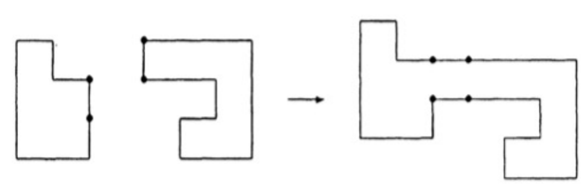

Figure 4. Example of a concatenation of two self-avoiding polygons

Let P_N denote the number of self-avoiding polygons surrounding the origin, where N is the number of steps. Then it's not difficult to see that, in two dimensions, $P_N = 0$, for $N < 8$. In particular, $P_8 = 1$.

Since a polygon is a closed curve, it needs to have even steps, which is structurally similar to a symmetric random walk. Therefore, we have the following theorem.

Theorem 4.1. *The number of self-avoiding polygons with odd number of steps is 0, i.e.,* $P_{2N+1} = 0$*, for* $N \in \mathbb{N}$*.*

By observations, we can get to know that, in two dimensions, $P_{10} = 12$, $P_{12} = 31$.

Question: Can we find an upper bound and a lower bound for P_N in two dimensions?

Definition 11. *([13])*

Two N-step self-avoiding polygons are said to be equivalent up to translation if there is a vector v in R^d such that translation by v defines a one-to-one correspondence from the set of bonds of one polygon to the set of bonds of the other polygon.

We denote q_N as the number of distinct equivalence classes up to translation of N-step self-avoiding polygons.

Lemma 4 implies the following

Proposition 2. *If e is one of the 2d nearest neighbours of the origin in 2d, then*

$$2Nq_N = 2dc_{N-1}(0, e),$$

where $c_n(x, y)$ is the number of n-step self-avoiding walks from x to y, for every $N \geq 2$.

Definition 12. *(Lexicographic ordering on $\mathbb{Z}^d$)*

$(a_1, ..., a_d) < (b_1, ..., b_d)$ if for some j(with $1 \leq j \leq d$), we have $a_i = b_i$ whenever $1 \leq i < j$, and $a_j < b_j$.

Theorem 4.2. *([2]) For $M, N \in \mathbb{N}$, and $M, N \geq 4$,*

$$\frac{q_M q_N}{d-1} \leq q_{N+M}$$

and

$$q_N \leq q_{N+2}.$$

Proof. ([13]) For even integer $N \geq 4$, let $Q[N]$ be the set of N-step self-avoiding polygons whose lexicographically smallest point is the origin. Then $Q[N]$ has exactly q_N members. For each $i = 1, ..., d$, let $e^{(i)}$ be the neighbour of the origin with $e_i^{(i)} = 1$ and $e_j^{(i)} = 0$ for $j \neq i$.? For $i = 2, ..., d$ and for even $M \geq 4$, let $Q_i[M]$ be the set of M-step self-avoiding polygons that lie in the half-space $x_1 \geq 0$ and that contain the bond joining the origin to $e^{(i)}$. Then, $Q[M]$ is contained in the union of $Q_2[M], ..., Q_d[M]$, and so, by symmetry,

$$|Q_2[M]| = ... = |Q_d[M]| \geq \frac{q_M}{d-1}.$$

Choose an arbitrary N-step polygon P in $Q[N]$, and let p be its lexicographically largest point. There are two values of i ($1 \leq i \leq d$) such that P contains the bond joining p to p - $e^{(i)}$; let I be the larger one between these two values. So, we have $I \geq 2$. Then, let Q be an arbitrary self-avoiding polygon in $Q_I[M]$.

?Now, we can concatenate P and Q. First, we translate Q by the vector p - $e^{(I)} + e^{(1)}$ (so the resulting polygon lies in the half-space $x_1 \geq p_1 + 1$ and contains the bond joining $p - e^{(I)} + e^{(I)}$ to $p + e^{(I)}$). Then, we take all of the bonds in the translated Q except the bond joining $p - e^{(I)} + e^{(I)}$ to $p + e^{(I)}$, and all of the bonds of P except the bond joining p to $p - e^{(I)}$, and also take the two bonds that join $p - e^{(I)}$ to $p - e^{(I)} + e^{(I)}$ and p to $p + e^{(I)}$. Since P is contained in the half-space $x_1 \leq p_1$, the result is a self-avoiding polygon in $Q[N + M]$.

Conversely, given an (N + M)-step polygon constructed in this way, we can reconstruct P and Q, because the N sites with smallest first coordinate are precisely the points of P.

Since there were q_N ways to choose P, and at least $\frac{q_M}{d-1}$ ways to choose Q given P by inequality $|Q_2[M]| = ... = |Q_d[M]| \geq \frac{q_M}{d-1}$, $\frac{q_M q_N}{d-1} \leq q_{N+M}$ holds.

Remove the bond joining p to $p - e^{(I)}$ from P, and add the three bonds of the walk $(p, p+e^{(I)}, p+e^{(I)}-e^{(I)}, p-e^{(I)})$ to P. The result is a self-avoiding polygon in $Q[N+2]$ from which Q can be unambiguously determined as above. This proves $q_N \leq q_{N+2}$.

□

Corollary 2. *([14])* $q_n{}^2 \leq q_{2n} \leq (d-1)\mu_{polygon}{}^{2n}$.

Proof. By theorem 4.2, the sequence $-log(\frac{q_{2n}}{d-1})$, $n \in \mathbb{N}$, is subadditive, so that Fekete's lemma implies the existence of $\mu_{polygon} = lim_{n\in\mathbb{N}} q_{2n}^{1/2n}$ and bound $q_{2n} \leq (d-1)\mu^{2n}$. □

Question: How do self-avoiding loops, which surround the origin with length n, behave differently from self-avoiding walks that start at the origin with length n? Here is a corollary from Madras and Slade's book.

Corollary 3. *There exists a constant C depending only on d, such that*

$$\mu^{2M} e^{-cM^{1/2}} \leq c_{2M+1}(0, e) \leq \frac{2(M+1)(d-1)}{d}\mu^{2M+2},$$

for all $M \geq 1$.

Proof. The first inequality can be proved by equation, $\mu^{N-1}e^{-BN^{1/2}} \leq b_N \leq \mu^N$, and a theorem from Madras and Slade's book. The theorem is stated as the following. Theorem: let e be a nearest neighbour of the origin in Z^d, there exist a constant K, depending only on d, such that for all integer $M \geq 1$, $c_{2M+1}(0, e) \geq KM^{-d-2}(b_M)^2$.

The second inequality can be derived by Corollary 2, Corollary 3 and the fact that $\mu_{polygon} \leq \mu$. □

Combining corollary 2 and corollary 4, we can finally get the result below.

Corollary 4.

$$\mu_{polygon} = \lim_{n\to\infty} (q_2 n)^{1/2n} = \mu.$$

5. Conformally invariant self-avoiding loops

5.1. **Background.** In Werner's paper([15]), he claimed that there exists a unique and natural measure on simple loops in the plane and on each Riemann surface such that the measure is conformally invariant and also invariant under restriction, which means the measure on a Riemann surface S' contained in another Riemann surface S, is just the measure on S restricted to those loops that stay in S. He constructed and described a natural measure on the set of self-avoiding loops in the plane and on any Riemann surface.

Remark 3. *A Riemann surface is a surface that represents the domain of a multiple-valued complex function.*

Here the Strong conformal invariance property in terms of a measure on the set of self-avoiding loops in the plane stated in Werner's paper.

Definition 13. *A measure μ satisfies conformal restriction if for any two conformally invariant domains D and D', the image of the measure μ restricted to the set of loops that stay in D, via any conformal map Φ from D onto D', is exactly the measure μ restricted to the set of loops that stay in D'.*

This property implies that μ is translation-invariant, scale-invariant in particular, and if μ satisfies the conformal restriction, then so does $c\mu$ for any positive constant c.

5.2. **Uniqueness of the measure.** Here are some settings for proving the uniqueness of the measure. We only consider self-avoiding loops without time-parametrizations, which means two self-avoiding loops are the same if their traces are the same. We consider a bounded open set A as an annular region in the plane if it is conformally equivalent to some annulus $\{z : 1 < |z| < R\}$, i.e., A is a bounded open set and $\mathbb{C} \setminus A$ has two connected components.

Definition 14. *Given that μ is a measure supported on the set of self-avoiding loops in the complex plane, μ_D is the measure μ restricted to the set of loops that stay in D, for each simply connected domain D.*

Definition 15. *(Weak Conformal Restriction) A measure μ on the set of self-avoiding loops in the plane satisfies weak conformal restriction if for any two simply connected domains D, D' and any conformal map Φ from D onto D',*

$$\Phi \circ \mu_D = \mu_{D'}.$$

Remark 4. *The difference between the strong conformal restriction and the weaker conformal restriction is that, the domains D and D' for the weaker conformal restriction is simply connected.*

Proposition 3. *Up to multiplication by a positive constant, there exists at most one non-trivial measure μ on the set of self-avoiding loops satisfying weak conformal restriction.*

Furthermore, for any simply connected sets $\widetilde{D} \subset D$, and any $z \in \widetilde{D}$,

$$\mu(\gamma : \gamma \subset D, \gamma \not\subset \widetilde{D}, \gamma disconnects z from \partial D) = c log \Phi'(z),$$

where c is a positive constant that depends on μ only, and Φ denotes the conformal map from $\widetilde{D}$ onto D such that $\Phi(z) = z$ and $\Phi'(z)$ is a positive real.

Now, suppose that ν is a non-trivial measure supported on the set of self-avoiding loops in the complex plane that disconnect 0 from infinity.

Lemma 5. *Up to multiplication by a positive constant, there exists at most one such measure ν such that for any simply connected domain D containing the origin and any conformal map Φ defined on D such that $\Phi(0) = 0$, one has $\Phi \circ \nu_D = \nu_{\Phi(D)}$. Furthermore, there exists a positive constant c such that for any simply connected subset U of the unit disc $\mathbb{U}$ such that $0 \in U$,*

$$\nu(\gamma : \gamma \subset \mathbb{U}, \gamma \not\subset U) = c log(\Phi'(0)),$$

where Φ is the conformal map from U onto $\mathbb{U}$ such that $\Phi(0) = 0$ and $\Phi'(0) > 0$.

Proofs of proposition 3 and lemma 5 are provided in Werner's paper [15], page 9 to 11.

5.3. **Brownian Loop Measure.**

Definition 16. *Planar Brownian motion is conformally invariant.*

Suppose that Z is a Brownian motion started from $z \in D$, and stopped at its first exit time T of the domain open $D \subset \mathbb{C}$. Suppose that Φ is a conformal map from D onto $\Phi(D)$. Then, up to time-reparametrization, $\Phi(Z[0,T])$ is a Brownian motion started from $\Phi(z)$ and stopped at its first exit time of $\Phi(D)$.

Theorem 5.1. *$log|Z_t|$ is a local martingale when Z is a planar Brownian motion started away from the origin.*

Remark 5. *A sequence of random variables having the property that, the expected value of X_{n+1} given the past and present values of $X_0, ..., X_n$ equals the present value of X_n (i.e., $E[X_{n+1}|X_0 = x_0, ..., X_{n-1} = x_{n-1}, X_n = x] = x$), is called a martingale.([13])*

This theorem can be easily derived from two facts:

For each z, one can define a measure N^z supported on Brownian loops that start and end at z as follows. Consider, for each $\epsilon > 0$, the law $P_{z,\epsilon}$ of a Brownian motion started uniformly on the circle of radius ϵ around z and stopped at its first hitting of the circle of radius ϵ^2. Then, consider the limit N^z when $\epsilon \to 0$ of $4log(1/\epsilon)xP_{z,\epsilon}$. This limit is an infinite measure supported on the set of Brownian paths Z that start and end at z.

Conformal invariance of planar Brownian motion implies readily that this measure N^z is also conformally invariant.

Since we won't be considering the time-parametrization of the Brownian loop, we identify two Brownian loops if one can be obtained from the other by an "increasing" reparametrization, which is not only identifying the traces since the Brownian motions have double points.

A Brownian loop Z has a natural time-length $T(Z)$ which is not a conformally invariant quantity.

Consider the product measure $\widehat{M} = d^2z \otimes N^0$, where d^2z denotes the Lebesgue measure in the complex plane.

Definition 17. $\frac{dM}{d\widehat{M}} = \frac{1}{T(Z^0)}$ *is the Brownian loop measure.*

5.4. **Construction of a measure satisfying weak conformal restriction property.**

Lemma 6. *For any open domain $D \subset \mathbb{C}$, and for any conformal map Φ from D onto $\Phi(D)$, $\Phi \circ M_D = M_{\Phi(D)}$.*

This lemma shows that M does satisfy the weak conformal restriction property.

Since the Brownian loop has no cut-points, M induces a measure on self-avoiding loops in the plane. So, this measure exists and it satisfies conformal restriction in simply connected domains, because the outer boundary of the image of Z under a conformal map Φ defined on a simply connected domain D, is the image under Φ of the outer boundary of Z.

Proposition 4. *There exists a non-trivial measure μ on the set of self-avoiding loops in the plane that does satisfy weak conformal restriction. It's equal to the measure on outer boundaries of Brownian loops defined via M.*

Proposition 5. *There exists a non-trivial measure ν that satisfies the conditions of Lemma 5, It is equal to the measure on outer boundaries of Brownian loops defined under N^0. It is also qual to the measure on outer boundaries of Brownian loops surrounding 0 defined under M.*

Appendix A. Conformal Mapping

The set of equations u=u(x, y), v=v(x, y) determines a transformation or mapping that builds a relation between points in uv plane and xy plane, where u=u(x, y), v=v(x, y) are called transformation equations. If each point in uv plane corresponds to one and only one point in xy plane, and conversely, then we call it a one to one transformation or mapping. In this case, we say that a set of points in the xy plane is mapped into a set of points in uv plane.

Examples 2. *Some General Transformations*

(1). Rotation. Consider w=$e^{i\theta_0}z$. In this transformation, figures in the z plane are rotated through an angel θ_0. If $\theta_0 > 0$, the rotation is counterclockwise; if $\theta_0 < 0$, the rotation is clockwise.

(2). Stretching. Consider w=az, where a is a real constant. In this transformation, figures in the z plane are stretched in the direction of z if $a > 1$, and they are contracted if $0 < a < 1$.

(3). Succesive Transformations. If $w = f_1(x)$ maps region R_x of the x plane into region R_w of the w plane, and $x = f_2(z)$ maps region R_z of the z plane into region R_x of the x plane, then $w = f_1[f_2(z)]$ maps R_z into R_w. The functions f_1 and f_2 are called successive transformations.

A.1. **Definition.** Suppose that a point (x_0, y_0) in xy plane is mapped into point (u_0, v_0) in uv plane while curves C_0 and C_1 are mapped into curves C_0' and C_1'. If the orientation and angel at (x_0, y_0) between C_0 and C_1 is the same with the orientation and angel at (u_0, v_0) between C_0' and C_1', then the transformation or mapping is called **conformal** at (x_0, y_0).

Examples 3. *Examples of Conformal Mapping ([8])*

(1)Mapping of a half plane onto a circle

Let z_0 be any point P in the upper half of the z plane denoted by R. Then, $w = e^{i\theta_0}(\frac{z-z_0}{z-z_0^-})$, is a transformation mapping the upper half plane onto R' of the unit circle —w— = 1 in a one-to-one manner. Each point of the x-axis is mapped to the boundary of the circle.

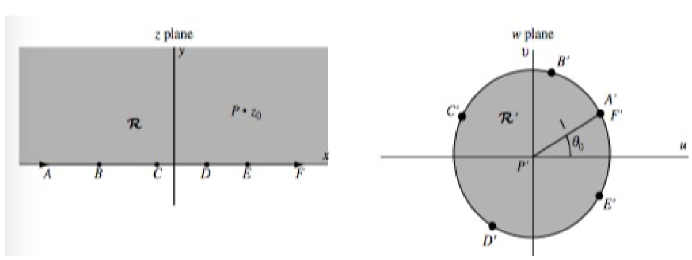

Figure 5. Mapping of a half plane onto a circle

(2) Half plane with semicircle removed

$w = \frac{a}{2}(z + \frac{1}{z})$ is a transformation which maps the upper half plane with a semicircle, centered at the origin, removed onto a a upper half plane.

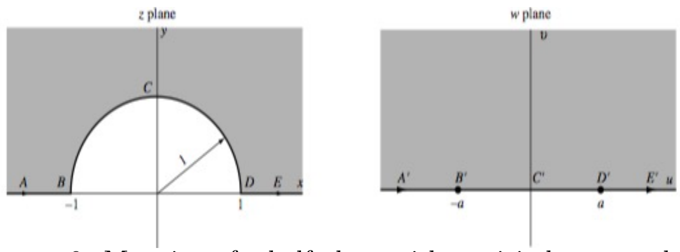

FIGURE 6. Mapping of a half plane with semicircle removed onto a half plane

(3) Semicircle

$w = (\frac{1+z}{1-z})^2$ is a transformation which maps a semicircle, centered at the origin, on the upper half plane onto a upper half plane.

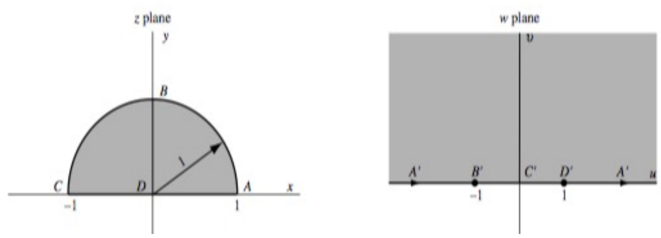

FIGURE 7. Mapping of a Semicircle onto a half plane

A.2. **Definition.** Suppose that $f : U \subset \mathbb{C} \to \mathbb{C}$ is a complex valued function, $w = f(z)$. We will typically write $z = x + iy$ and $w = f(z) = u(x, y) + iv(x, y)$.

Definition 18. *(a) f is holomorphic (or complex analytic) at a point $z_0 \in U$ if*

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We refer to $f'(z_0)$ as the complex derivative of f at z_0.

(b) If $f : U \to V$ is 1-1, onto and holomorphic, then we say that f is a conformal isomorphism from U to V.

Suppose f(x, y) is a complex function with real part, u(x, y), and imaginary part, v(x, y). To say that f is holomorphic is essentially equivalent to the Cauchy-Riemann equations:

$$\frac{du}{dx} = \frac{dv}{dy}, \frac{du}{dy} = -\frac{dv}{dx}.$$

Theorem A.1. *If f(z) is analytic and f'(z)$\neq$ 0 in a region R, then the mapping w=f(z) is conformal at all points of R.*

Definition 19. *A domain $U \subset \mathbb{C}$ is simply connected iff the complement in the Riemann sphere is connected.*

Example 5. *Let C be a simple closed curve in the z plane forming the boundary of a region R. Then R is simply connected.*

Theorem A.2. *(Riemman's Mapping Theorem)*
Suppose that R is a simply connected proper domain in $\mathbb{C}$, then there exists a conformal isomorphism:

$$f : \Delta := |z| < 1 \to R$$

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