The Lived Experience of Linear Algebra: A Counter-story about Women of Color in Mathematics

Aditya P. Adiredja (✉)
Faculty of Mathematics, The University of Arizona, 617 N. Santa Rita Ave. P.O. Box 210089 Tucson, AZ 85721-0089 USA
e-mail: adiredja@math.arizona.edu
tel: +1-520-621-0978
ORCID 0000-0003-3262-5112

Michelle Zandieh
Faculty of Science and Mathematics, Arizona State University, College of Integrative Sciences and Arts, 6073 S. Backus Mall, MC 2780, Mesa, AZ 85212-6420
e-mail: zandieh@asu.edu
tel: +1-480-727-5014
ORCID 0000-0002-2279-8231

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Abstract

This paper focuses on the mathematical sense making of women of color in the United States as part of the global effort of dismantling deficit narratives about historically marginalized groups of students. Following Adiredja's anti-deficit framework for sense making, this cognitive study invited a group of women to share their understanding of basis from linear algebra to construct a sensemaking counter-story. Extending the framework, this study examines a task that explores the boundaries and nuances of a concept to support the effort of going beyond students' deficits. Eight women extended the concept of basis (and vector spaces) to 22 distinct everyday contexts, drawing from their everyday lives as well as topics from their academic experiences. Their explanations revealed analytical codes describing roles and characteristics of a basis. These codes suggest ways that students can mobilize the concept of basis beyond its logical underpinnings. Contrasting interpretations using a deficit and an anti-deficit perspective construct a counter-story that showcases these women's creativity and flexibility in understanding the concept, and potential resources for the teaching and learning of linear algebra.

Key words: student thinking, basis, linear algebra, anti-deficit narratives, sociopolitical perspective

1. Introduction

Deficit narratives about particular groups of students (e.g., ethnic minority students, women, and students with low socioeconomic status) have a sustained presence and influence on mathematics education research and practice globally (Frade, Acioly, & Jun, 2013). According to Adiredja and Louie (2020), deficit narratives, as part of the larger deficit discourse in mathematics education, focus on students' academic and intellectual shortcomings and attribute them to deficiencies located in the students, their values, families, and/or cultures. Deficit narratives rarely acknowledge students' existing understandings and strengths and the sociopolitical and historical contexts of education. These narratives influence classroom practices and broader educational policies. For example, deficit narratives have been associated with lowered teacher expectations and decreased access to quality curriculum (e.g., for Pāsifika students in New Zealand in Hunter & Hunter, 2017; for African American students in the United States in Nasir, Snyder, Shah, and Ross, 2013).

In a review about the development of anti-deficit models of learning in mathematics education, Frade et al. (2013) argued that theoretical turns away from individual cognition in the field (e.g., the social turn, Lerman, 2000) and the development of different research programs (e.g., Ethnomathematics) occurred as an effort to counter deficit narratives about particular groups of students. Recognizing the contributions of studies of individual mathematical cognition and their ongoing presence particularly at the undergraduate level, Adiredja (2019) has offered a framework for cognitive studies to challenge deficit narratives for marginalized groups of students. This paper follows and builds on the framework offered in Adiredja (2019).

The current study aims to challenge deficit narratives about the mathematical participation of women of color in the United States.[1] We join existing studies in exploring the impact of racism and sexism on the experiences of women of color in science, engineering, and mathematics (Carlone & Johnson, 2007; Leyva, 2016; Ong, Wright, Espinosa, & Orfield, 2011; Varma, Prasad, & Kapur, 2006). In this paper, we aim to construct *sensemaking counter-stories* (Adiredja, 2019) by investigating the sense making of women of color about basis in linear algebra and by analyzing their sense making from an anti-deficit perspective. These counter-stories serve to challenge deficit narratives about women of color in mathematics, while simultaneously reflecting on dominant practices in studying mathematical sense making. Scholars have highlighted the connections between dominant narratives about what counts as mathematics and dominant practices to assess students' understanding (Adiredja & Louie, 2020; d'Ambrosio, 1985; Martin, Gholson, & Leonard, 2010).

One component of dominant practices for studying sense making, which was not emphasized in Adiredja (2019), is the kind of mathematical tasks used in research. For this study, we examine the potential of a particular task to investigate students' understanding beyond their deficits. We investigate students' understanding of basis using a task that asks students to extend a mathematical concept to novel contexts, thereby exploring its boundaries and nuances. We explore students' understanding of basis beyond their ability to apply the formal definition and solve standard problems. De Freitas and Sinclair (2017) have conceptualized this distinction between extending and applying knowledge as the difference between focusing on the ontological aspect of the concept versus using the concept as a logical tool:

[1] In the United States, women of color include Black and African American women, Latinx, Asian American, Pacific Islander women, and Indigenous women. We consider them together as a group to recognize their shared underrepresentation in STEM higher education and careers.

> When the concept is used only as a logical tool, or as an object or relation with a finite set of properties, the ontological aspect /…/ is abandoned, and activity is reduced to adhering either to naming exercises (identifying instances of the concept /…/) or to applications of the concept as a rule or logical constraint. In such cases, the ossified concept fails to sustain the mobility and potentiality which is its nature (p. 84).

In this paper, we ask students to extend the concept of basis from Linear Algebra to an everyday context where it has not previously been applied, thereby stretching (or "mobilizing") the concept. De Freitas and Sinclair (2017) have argued that such mobilizing of a concept has supported mathematicians in "generating something ontologically new, like a complex number or a point at infinity or projective geometry—neither of which emerges from the rules of logic alone" (p. 81). We explore basis with students in this way in hopes of uncovering a different kind of mathematical engagement and creativity.

Therefore, in this study we ask the following questions:

1. What insights into students' understanding of basis can we glean by focusing on an activity that asks students to mobilize (or stretch) the concept to other contexts?
2. What resulting counter-stories can we construct from the women's engagement with this task?

We situate these research questions in a brief literature review on students' understanding of basis, and two closely related concepts: span and linear independence.

2. Literature Review: Understanding Basis

2.1. Recognition Task and Example Generation Task

In a review of recent articles on linear independence and span, Stewart, Andrews-Larson, and Zandieh (2019) identified 15 studies between 2009 and 2018, two of which focused on basis specifically: Stewart and Thomas (2010) and Bagley and Rabin (2016). Building on their review, we examine the types of tasks used in these studies. We found that the most common types of tasks were recognition tasks (e.g., recognizing linearly independent sets, Ertekin, 2010) and example generation tasks (e.g., constructing linear independent sets with a set of constraints, Rasmussen, Wawro, & Zandieh, 2015). Other studies developed curriculum to engage students in computer simulations (e.g., Dogan, 2017), a modeling situation (e.g., Trigueros & Possani, 2013) or an experientially real starting point (e.g., Cárcamo, Fortuny, & Fuentealba, 2017) to begin to develop their understanding of these concepts. These studies often used recognition or example generation tasks after the initial exploratory situations to further develop or assess student understanding.

The tasks in the literature occur along what we describe as a progression from tasks that are more closely linked to deduction from the formal mathematics (recognition tasks) through tasks that allow for creativity within standard definitions and registers (example generation tasks) to tasks that ask students to create conjectures and justify them within standard registers or in new contexts. We see our work as taking a step beyond this arc in asking students to create everyday examples of basis in new contexts of their own creation in which vector spaces may not work as they are "supposed to" in standard mathematics. In this way we see a connection to the work of Zandieh and Rasmussen (2010) when they asked students to take a definition of triangle on the plane and apply it to the surface of the sphere, describing this activity as the students creating "a new mathematical reality" (Zandieh & Rasmussen, 2010, p. 58). We see this arc as connected with the contrast that De Freitas and Sinclair (2017) make between the logical and the ontological aspect of a concept.

2.2. Reproduction of Deficit Narratives about Student Understanding in Linear Algebra

Bagley and Rabin (2016) have noted that much of the early literature about the learning of linear algebra focuses on identifying students' deficiencies. Many of these studies focused on students' struggles with the particular tasks, then interpreted such struggles as deficiencies in the students. Similar deficit approaches continue to exist in contemporary studies that describe student understanding of span, linear independence, and basis. These studies reproduce and sustain a deficit narrative about students' understanding of these topics.

One dominant deficit narrative about students is that they approach concepts in linear algebra in terms of superficial computations or procedures, and that they cannot meaningfully understand the structure of the concepts or the theory behind them (Aydin, 2014; Çelik, 2015; Harel, 2017). For example, this was one interpretation of students' incorrect responses about linear independence in one study.

> "The 86 students completed all the questions, with 53% incorrect responses to the second question. The *basic fault of these students* was that they *could not think carefully* on the different structures of linear dependence/independence relations between the rows of M" (Aydin, 2014, p. 821, emphases added).

Harel (2017) used a similar interpretation of students' understanding after a teaching experiment episode:

> The participants seemed to have retained *virtually none of the knowledge* pertaining to the theory of linear systems they learned—or were supposed to learn—in their linear algebra courses. They, for example, did not recall, or perhaps *did not initially understand*,

> *basic concepts*, such as the concept of linear independence /…/. The observations made in Unit 2 point to *deficiencies* in applying ways of thinking *critical* to dealing with theoretical questions in linear algebra. (p. 90, emphases added).

Students' errors are taken to be evidence of lack of "basic" understanding, the inability to retain knowledge, and having the critical knowledge to understand the theory. This summary in Harel (2017) was followed by 16 observations, all of which were things students could not do, "weakness," and "lack of ability" (p. 91). The literature review for the study also largely included the early studies that Bagley and Rabin (2016) discussed as deficit oriented, thus illustrating the reproduction of deficit narratives in contemporary literature.

In contrast to the previous studies, Bagley and Rabin (2016) focused on the coordination of different modes of thinking in constructing a basis, and found that students used "Computational thinking in a variety of sophisticated, productive, and reflective ways, including generating Computational justifications for claims and making strategic choices to limit the complexity of their calculations" (p. 100). The study ended up challenging deficit narratives about students' computational skills by highlighting their sophistication and justifications.

2.3. Embodied Knowledge for Students' Explanations of Basis

Some studies suggest that students' embodied knowledge and experiences can support students' conceptions of span, linear independence, and basis. Stewart and Thomas (2010) found that students struggled to formally define and explain basis. However, those who were exposed to instruction that leveraged students' embodied knowledge and experiences (e.g., geometry) were able to make better connections between basis, span, and linear independence, and linear combination.

Plaxco and Wawro (2015) explored student understanding of span and linear (in)dependence while engaging in different mathematical activities. Four categories emerged from their data: travel, geometric, vector algebraic, and matrix algebraic. The travel category captures span and independence in terms of “getting to” or “moving to” locations in the vector space (p. 91), whereas the geometric category captures the use of spatial reasoning or graphical representations without use of travel-oriented language. Vector and matrix algebraic capture students’ use of algebraic operations on vectors and matrices, respectively.

Other aforementioned studies that focused on modeling and using experientially real starting points have also challenged deficit narratives about students’ understanding by leveraging students’ resources to support their learning. Our goal is to challenge deficit narratives about students’ mathematics, as well as deficit narratives about women of color in mathematics by engaging students in a generation task that extends basis to new contexts.

3. Conceptual and Theoretical Frameworks

3.1. An Anti-deficit Framework for Mathematical Sense-making by Students of Color

Informed by Critical Race Theory (Ladson-Billings & Tate, 1995; Solórzano & Yosso, 2002) and sociopolitical perspectives (Gutiérrez, 2013; Valero, 2004), Adiredja (2019) has identified a self-sustaining system of deficit narratives about students of color in mathematics in the United States (see Figure 1). This system illustrates the impact of narratives on how we value knowledge and how we view students of color (Abreu, 1995; Apple, 1992).

Adiredja (2019) has highlighted that deficit master narratives about students (and women) of color in the United States and general deficit narratives about students’ mathematical thinking from research (see literature review) are mutually reinforcing. Together they inform deficit perspectives with which researchers often interpret the mathematical work of women of color. A deficit perspective on student mathematical thinking (1) focuses on the problems in students' knowledge, with little or no focus on their contributions, and (2) interprets those problems as students' shortcomings (Adiredja, 2019). Such interpretations in turn produce deficit stories about sense-making by these students, which sustain and contribute to the existing deficit master narratives (Teo, 2008).

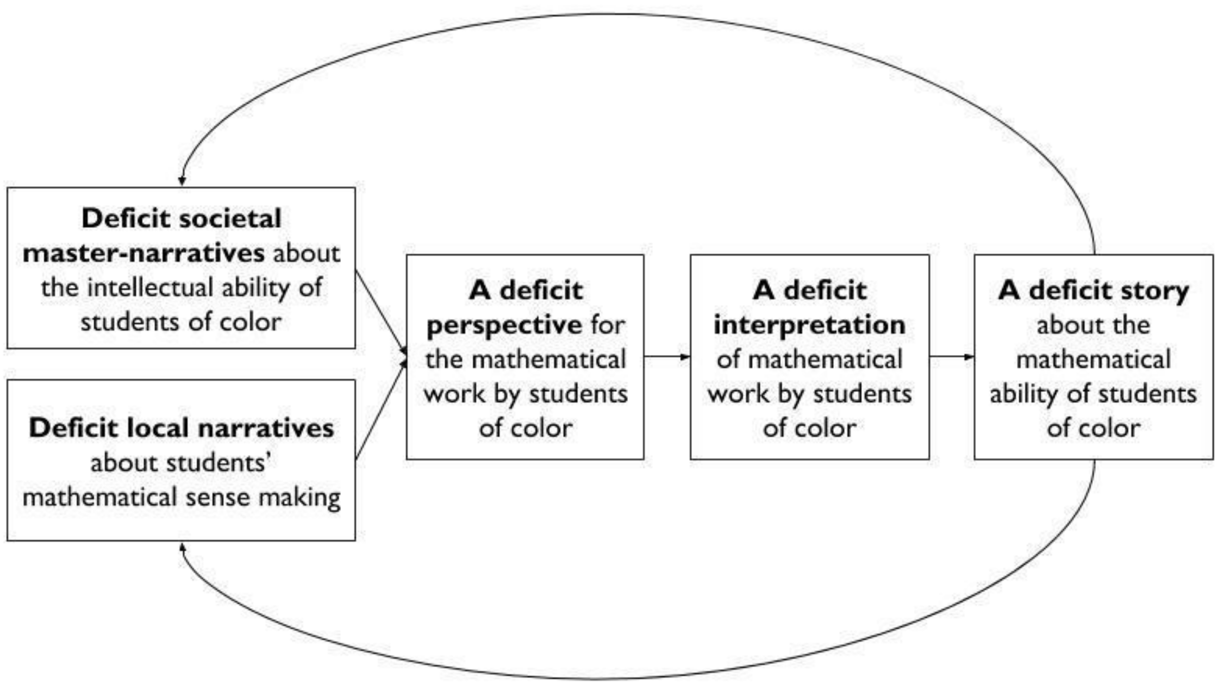

Figure 1. The self-sustaining system of deficit narratives about students of color in mathematics. Reprinted with permission from *Journal for Research in Mathematics Education*, copyright 2019, by the National Council of Teachers of Mathematics.

Adiredja (2019) has argued that research on mathematical sense-making can disrupt this deficit system by adopting an anti-deficit perspective and constructing a sense-making counter-story. Malloy and Jones' (1998) study of problem-solving practices of a group of African

American students, and Lewis' (2014) study of cognitive differences, instead of deficits, of students with mathematical learning disabilities serve as concrete examples.

An anti-deficit perspective about sense making takes as a given that students of color have resources to successfully reason about and with mathematics (see also, axiomatic brilliance of Black students, Leonard & Martin, 2013). The perspective maintains flexibility with respect to the type of resources and the form of sense making. Learning resources can come from students' experiences both in and out of the classroom, and productive sense making can be expressed in imperfect mathematical language and with inconsistencies. Inconsistencies and imperfections are treated as sites for researchers and teachers to explore students' resources and productive sense making (Adiredja, Bélanger-Rioux, & Zandieh, 2020, p. 521).

Methodologically, Adiredja (2019) recommends purposeful selection of research participants and adopting a cognitive framework that focuses on resources for sense making. Analytically, the author uses contrasting interpretations from a deficit and an anti-deficit perspective to construct the counter-story. By using the term "anti-deficit" instead of "asset-based' (e.g., NCTM Research Committee, 2018), Adiredja (2019) emphasizes the analytical need to explicitly challenge deficit interpretations while also focusing on students' resources. This emphasis builds on Critical Race Theory's critique on neutrality of approaches and counter-storytelling as a tool to humanize people of color and challenge racism (e.g., Delgado & Stefancic, 2001). The recommendations from Adiredja (2019) guide the design and the analysis of the current study.

3.2. Theoretical Orientation toward Analyzing Mathematical Sense-making

We each brought to the analysis different but complementary perspectives on analyzing mathematical sense-making. These two perspectives, Knowledge in Pieces (KiP; diSessa, 1993) and conceptual metaphors (e.g., Lakoff & Núñez, 2000) influenced the data analysis in this paper even though neither was used directly as the analytical framework. We provide a brief description of each.

KiP focuses on the productivity of students' knowledge and the foundational role their prior knowledge plays in learning. Studies using KiP prioritize student's own way of reasoning about a topic over its correctness with respect to a normative standard or a conceptual hierarchy (diSessa, Sherin, & Levin, 2016). It is common for studies using this framework to uncover productive sense-making in students' use of everyday language (e.g., diSessa, 2014), and investigate students' knowledge stemming from their experiences in engaging with the physical world (diSessa, Sherin, & Levin, 2016).

Work on conceptual metaphor focuses on conceptual structures and ways that entities in them interact. Conceptual metaphors may be used without reflection or even awareness that a metaphor is involved. In the undergraduate mathematics education literature, conceptual metaphor has been used to examine student understanding of a number of concepts including limit (e.g., Oehrtman, 2009) and derivative (e.g., Zandieh & Knapp, 2006). In this paper, we consider students' everyday examples as potential conceptual metaphors that allow insight into each student's understanding of basis, and more generally into delineating a framework for what it means to understand this concept.

4. Methods

4.1. Data Collection

Participants were undergraduate women of color at a large public university in the U.S. With the assistance of the local mathematics department, we invited students who had received instruction about basis in an introductory linear algebra course to participate in the interview. We also reached out to some instructors who were teaching introductory linear algebra. In total, eight women responded and agreed to participate in the study.

We include students' self-reported racial and ethnic backgrounds and their past mathematics courses in Table 1. We drew this information from a student background survey that we administered at the end of the interview. With the exception of Morgan,[2] a biomedical engineering major, all the other students were mathematics majors or minors.

Each interview lasted for 90 minutes. Adiredja led the interview. Zandieh videotaped the interview and also participated in asking questions. Students started the interview by solving four tasks related to basis that might be asked in a beginning linear algebra course, followed by more general questions about students' understanding of basis. The questions that are the focus for this study are:

1. Can you think of an example from your everyday life that describes the idea of a basis?
2. How does your example reflect your meaning of basis? What does it capture and what does it not?

At the end of the interview the students were given another opportunity to share an example from their everyday lives. The full interview protocol is in the Appendix.

Table 1. *Students' racial/ethnic background and mathematics course history*

Student	Racial/Ethnic Background	Linear Algebra Completion	Grade	Other Mathematics Courses
Leonie	African American	Spring 2016	A	Calculus I, II, and III
Morgan	Asian/Asian American	Spring 2016	A	Calculus I, II, and III, Differential Equations, Biostatistics

[2] All names are pseudonyms.

Annissa	Hispanic/Latinx	Fall 2014	B	Calculus I and II
Eliana	Hispanic/Latinx	Spring 2014	C	Calculus I and II
Nadia	Hispanic/Latinx	Fall 2015	A	Calculus I, II, and III
Jocelyn	Hispanic/Latinx	Spring 2015	B	Calculus I, II, and III, Statistics
Stacy	Hispanic/Latinx	Spring 2016	C	Calculus I, II, and III
Liliane	Hispanic/Latinx/White	Fall 2015	B/C	Calculus I, II, and III

4.2. Data Analysis

We transcribed the interviews and organized the transcripts by turns, marked by changes in speaker. Our analysis primarily focused on turns during which students responded to the two questions listed above. We also marked turns when students introduced an example from their everyday lives to explain basis, even if they were not in response to the two questions.

The first step of the analysis identified the everyday contexts in students' explanations in each marked turn. An everyday context had a clearly identified non-mathematical object (e.g., recipe). The explanation included objects for both the basis vectors and the vector space (e.g., ingredients and recipes with those ingredients).

The second step of the analysis captured aspects of understanding basis by developing a set of analytical codes. We started with individual open coding of students' responses to the two questions, which was loosely influenced by different theoretical frameworks. Adiredja ended up focusing on the characteristics of the basis vectors. Many studies guided by KiP have focused on students' determination of a concept (Wagner, 2006). In other words, what characteristics do students consider to be important for a vector to be included in a basis? Zandieh focused on verbs that metaphorically describe the behaviors of and relationships between the basis vectors and the vector space. These verbs described what roles students see the basis vectors playing in relationship to the larger vector space. The discussion of these two individual analyses provided a list of preliminary codes organized into *characteristics codes* and *roles codes* for basis (Adiredja & Zandieh, 2017). They were later refined by applying both sets of codes to the data.

In our presentation of the transcript, we use **bolded text** to indicate phrases that capture the codes. We use regular *italics* to refer to codes in the text. We removed most hedges (e.g., “like,” “kinda,” “um”) from the presentation of transcripts and used ellipsis (/…/) to assist the reader in following the students’ explanations. We also included direct objects to help contextualize the verbs students used in their explanations.

5. Results

The eight women explored the concept of basis in 22 distinct everyday contexts, which were influenced by their everyday lived experiences (friendship, family chores assignment, religion) as well as topics from their academic experiences (economics, computer programming, astronomy). While some of the women included specific contexts they heard from instruction, most of the other contexts were constructed during the interview, at times using objects available in the room (e.g., a pen, corners of the room). Table 2 includes all the contexts that each student discussed in the interview.

Students’ examples are not to be *essentialized* by attributing them as inherent to a particular race and gender. Students did discuss basis using a few historically gendered contexts, and religion and collective family chores, which have been associated with some Latinx cultures (Delgado-Gaitan, 1994). However, the diversity of contexts suggests that many everyday examples were likely inspired by individual experiences. Because of that these contexts would not be relevant to all women of color. In the next two sections, we elaborate on the details of some of these contexts to illustrate what students considered to be important characteristics and roles of the basis vectors.

5.2. Roles of Basis Vectors

In this section we focus on the verbs students used to discuss the role that basis vectors have in relationship to the larger vector space. Through our analysis we arrived at the following five codes that categorize the women's explanations of the examples: generating, covering, structuring, traveling and supporting. Examples from different students follow the description of each code.

Table 2. *Everyday contexts used to explain basis and vector spaces*

Student	Context (for basis and vector space)
Leonie	● different individuals representing different desired personalities in a friendship ● three friends creating a cube of desirable characteristics ● different continents contributing to the world market
Morgan	● 2x2 and 2x3 Legos to build a dinosaur ● driving (on a grid) to get anywhere in the city ● sugar and eggs for making a recipe ● a single pen to generate groups of pens ● a corner (basis vectors) for a physical 3D space (e.g., a room) ● 1x1 Lego to generate all Legos
Annissa	● different ideas to solve a problem with multiple solutions
Eliana	● a piece of paper (space) carries two pens (basis vectors) ● using length of arms and body to cover the space of the room ● dimensions of a storage room or a house ● building from a skeleton ● writing from an outline of a paper
Nadia	● floor (as basis for the ground) ● edges of the floor (as a basis for the floor) ● planets and stars (as basis for the universe) ● natural elements (e.g., water or hydrogen, as basis for the earth) ● syntax in a programming language (as a basis for a program code)

Jocelyn	● pieces of clothing (as a basis for different outfits) ● cooking ingredients (as a basis for different recipes you can make) ● art sculpture & collage (capturing the Spanning Set Theorem)
Stacy	● directions to walk in a room as a basis for (all the places you can get in the room) ● edges of the floor (as a basis for the room and the room next door) ● band, music, and the field (basis vectors) as components for a marching band formation (space) ● self and siblings covering all of mom's chores assignments
Liliane	● using vectors to go home/house at a non-origin location ● most basic teaching of a religion (as a basis for all decisions)

5.2.1. Generating

This code focuses on the creation of the space. Elements of the space (basis vectors) are combined to create other elements and ultimately the space. Verbs like "make," "build," "combine" are common indicators for this code.

> *Morgan*: You're given the 3 by 2 Lego [pieces] and you have like a 2 by 2 Lego [piece]. You can **build on to that to create** that, I guess [the] space that you have.
>
> *Nadia*: You need to know the minimum syntax to be able **to make any sort of program** and **make it useful**.

5.2.2. Covering

This code focuses on the basis vectors as filling in or encompassing the space. It is less about how the elements are combining to create the space, and more about how the elements accumulate to cover the entire space. The verbs "fill" and "cover" are common indicators for this code

> *Eliana*: I could **cover** the basis of this room by doing this motion. Like, the length of an arm and an arm, and from here up and that would be the basis of this room. Because I mean it's three vectors and it's **covering the room** and they're all going in opposite

directions. So it's not like I'm going like this and I'm only **covering** two directions. I'm **covering the entire space of the room**.

Stacy: My mom telling us to do **all** the chores. Each of us had a different thing to do in the house. So like mine was taking out all the trashes from the house. My brother was vacuuming the living room. And then Lionel's was cleaning up the kitchen.

These examples anticipate a characteristic of basis vectors (*different*) that is discussed in the characteristic codes section. Each of the students used the word "opposite" or "different" in describing their example. The students emphasized the necessity to use opposite directions to cover the space, and everyone being assigned a different chore to get all the chores done.

5.2.3. Structuring

This code focuses on the idea of basis vectors being foundational for the space, and having all the crucial information to fully describe the space. As part of the structuring, the basis also identifies the kinds of vectors that exist in the space.

Eliana: That's the whole point of the basis so you can **see on a smaller scale** what the rest of the space is gonna look like or what type of space here. And I could go with anything depending. If you're writing a paper. It would be the outline.

Lilianne: "[The basic religious teachings] convey the crucial information **to define the rest of your life, the rest of the space**."

In these examples structuring offers a scale model that shows students what they needed and where everything was. Structuring is a way of describing the space using only a few vectors.

5.2.4. Traveling

This code focuses on using the basis vectors to move through the space, or along different pathways within the space. Travel can discuss a physical space but could also discuss metaphorically moving through a non-physical space.

> *Morgan*: You could use your basis of one block north, one block south, not south, east-west. I think of those as like [a] basis **to get anywhere** in the world.
>
> *Lilianne*: You have the Scriptures, and you have the Prophets, and you have your connection with God and, all of the decisions and all of things that **come from that** and you can **reach all of the other points** with this basis.

Morgan discussed physical motion whereas Lilianne hinted at a metaphorical travel in discussing the utility of basic components of a religion to reach all life decisions.

5.2.5. Supporting

This code focuses on the elements being an important part of the space that make significant contributions to the space.

> *Leonie*: You can like geometrically take the continents and say that they are a basis of the world, and the way that **each of them contributes to the world market** could be like the span.
>
> *Leonie*: I think I could describe it as like my different friend groups that I'm in … /…/ They don't really like associate with each other, but I hang out with them, like the three of those groups, and /.../ **I get something from all of them**. And it makes like a whole. So it's almost like I'm the space and then they're spanning me.

The emphasis in this set of examples is on the important role the basis vectors play in the space. The basis “vectors” contribute to the world market and provide different types of friendship support to a person.

5.3. Characteristics of Basis Vectors

We now focus on important characteristics for basis vectors according to these students. These characteristics helped many of them decide which objects to include/exclude from their set of basis vectors. Through our analysis we arrived at the following five codes: minimal, essential, non-redundant, and different.

5.3.1. Minimal

This characteristic focuses on the set being the smallest size or having the least amount of vectors to generate the space. Phrases like "the least amount of," and "minimum" were common indicators of this characteristic.

> *Eliana*: So it's like what's **the least amount of myself** I need to cover the space of the room.
>
> *Nadia*: Say for C programming specifically /…/ you need to know **the minimum syntax** to be able to make any sort of program.

5.3.2. Non-redundant

This characteristic focuses on not wanting extraneous objects in the set [of basis vectors], and/or the need to reduce or remove the extraneous information. Reasons for being extraneous include belonging to the same essential category as an element already in the set, being able to be created using an element already in the set, and the set already including sufficient information to determine (e.g., *covering* or *generating*) the space.

> *Eliana*: [W]ith a skeleton you could have things. **A lot of extra information in there that you don't really need**. It would need to be a very minimal skeleton. In order for it to match with the definition of basis because basis is just the minimum amount of information.

Morgan: You have a group of, you have one pen on its own, and that's one group [group 1]. You have a group of two pens [group 2]. Those two groups **wouldn't stand on their own** because you could have two groups of group 1. That would be the same as group 2.

To Eliana, while a skeleton satisfied the role of providing a structure on which to build the body (*structuring* and *generating*), she recognized that there would be redundancy in considering the full skeleton, for example having two of the same leg bone. Morgan's example highlighted the fact that one set could *generate* the other resulting in a different kind of redundancy.

5.3.3. Different

This characteristic focuses on comparing objects (vectors) based on their difference/similarity. The different code is closely related to the non-redundant code. If objects were sufficiently different, then they would be non-redundant and thus can be kept in the set. Some students explicitly used the word "different" to describe this characteristic. Our examples include other ways that students described *different.*

Leonie: Well you can like geometrically take the continents and say that they are a basis of the world. /.../ Each continent is **physically independent** from the other ones. So that could be the linear independent basis.

Stacy: So have my bedroom and I start from my bed. **I go one way** and that there is a vector for me /.../ Then I see **I don't need to go that way** which I kind of **go another way,** which is a basis in that plane of my room.

Leonie focused on the physical separation between all the continents of the world by looking at them "geometrically." She provided a specific criterion to determine the difference. Other students, like Stacy, emphasized more the idea of selecting a different object (a different direction).

5.3.4. Essential

This characteristic focuses on critical importance of the object or set of objects that are in the basis beyond being different and non-redundant. It highlights both the identity or essence of each object and/or that it is necessary to include an object with this identity as a member of the basis.

> *Lilianne*: [The basic religious teachings] conveys all of the **crucial** information to define the rest of your life.
>
> *Stacy*: Well, I remember in marching band where we have to create a circle. And you need the people to create the circle. You need the field. And you need the music. You **need all of three of them** to help move on to the different parts of the field.

In summary, these roles and characteristic codes represent intuitive ideas reflected in students' everyday examples. They represent general aspects of the concept of basis that students explored across the 22 different contexts. We now turn to students' own analyses of their examples.

5.4. Extending the Mathematics: Students' Analyses

Students recognized limitations of their everyday contexts to explain basis, and at times offered another example to address a particular limitation. Several other ideas about basis emerged in students' analyses of their examples. Students volunteered these ideas without being asked specifically about them. In addition to discussing the notion of linear combinations, some students also discussed important characteristics of the vector space (and subspaces).

5.4.1. Linear combinations and scalar multiples

Jocelyn and Morgan discussed linear combinations of vectors in their examples. Jocelyn resolved the issue of being required to use all the basis vectors in her fashion example by offering a recipe example.

Jocelyn: A basis is literally **any combination you can make**. And with the outfits example, you couldn't take any combination. Cause, like shoes and pants, it's a combination but you can't wear just that. So that's what the downfall is. But with the recipe example, if you really wanted to, you could just mix salt with milk and you'd have something, but you might not want that. You could do that type of thing.

We interpret Jocelyn's explanation to be about the importance of any linear combinations of the basis vectors to be a part of the vector space. She highlighted that the fashion example did not allow wearing just shoes and pants as a possible outfit, whereas the recipe example allowed for a mixing just salt and milk, even if such outcome was not desirable.

Morgan also raised the concern about linear combination like Jocelyn. However, unlike Jocelyn, Morgan did not believe the context of cooking addressed the issue. Eggs and sugars (or their combination) would not "span" anything.

Morgan: Yeah. I don't think that captured a basis. /…/ [T]he point of a basis is **to span** something. I don't think eggs and sugars [capture basis], they're like apples and oranges. Like, it doesn't really span anything really.

Morgan resolved her concern with the example of driving on a grid. Morgan was prioritizing a context that involved combining things that were more alike (driving north-south vs. east-west).

5.4.2. Infinite nature of vector space

Eliana noted the importance for a vector space to be infinite. Her context focuses on the dimensions of a house or a storage room as a space to fill.

Eliana: Well if you were to have a storage room. So, like any house or something, the dimensions of it. /…/ You can **fill it** based on how much you have. How much room you have, sort of. I don't think that really applies because a basis is **the least amount** in order

> **to cover** an entire space. That doesn't really work because the space would be limited. And with a basis **it's just a representation**, like a representation of an unlimited area.

Students in the study would bring up a context that they would quickly dismiss. However, instances like this often reveal an important aspect of basis that the student considered to be important. Like her skeleton and paper outline examples, Eliana emphasized the role of basis vectors to *structure* and *cover* the larger space. She dismissed her example because it was of a "limited" space, whereas a basis would need to *structure* an "unlimited area." The next example from Nadia explores the importance of the vector space and its subspaces.

5.4.3. Relationship between vector space and subspaces

Nadia used planets and stars as a basis for the universe. She then broke down earth into natural elements that create it, such as hydrogen, as part of its basis. The universe was a vector space, of which the earth was a subspace.

> *Nadia*: Yeah. But all of the elements that **describe** earth also **describe** the universe but might not completely **describe it**. Maybe there's another element, like on Jupiter or something, just making this up, that doesn't exist on earth. So, it wouldn't **describe** earth /…/ but you need it to **describe** the whole universe. So, it's like a subspace!

Nadia emphasized a distinction between a basis for a subspace (earth) and a basis for the vector space (universe). To Nadia, a subspace could share some of the same basis vectors with the larger space, but the larger space had to have additional basis vectors. As Nadia focused on the role of the basis vectors in *structuring* the space, she raised the idea of subspaces and there being multiple subspaces in a vector space (Earth and Jupiter). She elaborated:

> *Nadia*: [Y]ou could **do** another planet with **a different span**, I guess, within the universe. cause the earth probably has a lot of stuff that other planets don't have. I honestly, I have

no idea [*laughs*] because I'm an electrical engineering major. So, you could **describe** all of the elements on that planet [they] would be the basis of that planet. Yeah that makes sense to me.

This episode illustrates the way that Nadia constructed this example to make sense of a particular definition of basis. She came up with a context where she could focus on natural elements (set of basis vectors) that made up different planets (subspaces) within the universe (the vector space). Nadia was unique in attending to the subspace idea with basis.

5.5. Anti-deficit Interpretations and a Counter-story

Following Adiredja (2019), our analysis section closes with contrasting interpretations of the data (i.e., sensemaking stories) using a deficit and an anti-deficit perspective. A deficit perspective focuses on deficiencies in the work and interprets them as shortcomings of the students. We present in this section deficit claims that can be drawn from our data. An anti-deficit interpretation then challenges each claim by focusing on resources and productive sense making from the group of women of color.

The first two claims focus on deficiencies in the women's examples. The first one interprets the deficiencies as evidence for a shortcoming of the women. The second one prioritizes the value of efficiency and formalism to dismiss the contribution from the women's sense making. The last claim devalues the utility of these examples in learning based on their potential conflicts with formal knowledge. These three claims highlight some of the presumed values and practices in mathematics that might inadvertently support deficit narratives about women of color in mathematics (Adiredja, 2019).

Claim 1. *There are many inconsistencies and ambiguities in their examples and explanations. These issues suggest that these women do not have command over the concepts and the formal language.*

This claim focuses on inconsistencies within the work and interprets them as a deficiency of the women. For example, a deficit perspective would point to Nadia's statement, "do another planet with a different span" and interpret it as problematic. Was Nadia using "basis" and "span" interchangeably? With such ambiguity, a deficit perspective would conclude that Nadia was not careful in her use of the mathematical language and likely did not have a strong understanding of basis. Such an interpretation constructs a sensemaking story that supports existing deficit narratives about women of color and their participation in mathematics.

An anti-deficit perspective would instead recognize ambiguities and their potential inconsistencies in students' responses as a natural part of learning mathematics and its formal language. The group of women were aware of ambiguities and other conceptual limitations of their examples, and at times addressed the issues themselves. An anti-deficit interpretation would highlight the mathematical ideas Nadia did raise (e.g., bases constructing different subspaces), not just the ones she might have omitted. Some studies have highlighted that allowing for ambiguity opens up opportunities for creativity (e.g., Leikin & Pitta-Pantazi, 2013). Loosening some of the constraints with the concept of basis may have been what supported this group of women in constructing these creative everyday examples.

Kleiner (2007) noted that historically the notion of basis appeared without a precise formal definition as early as 1844 in different mathematical contexts (e.g., algebraic number theory and hypercomplex number systems). The formal definition was introduced much later in Peano's formalization of vectors spaces in 1888. In other words, precise formalization is not necessarily part of the origin of concept development, both historically and for students. Instead, working through ambiguities related to the concept is a process by which one may arrive at a formal definition.

Claim 2. *The intuitive examples and explanations from these women are idiosyncratic and are challenging to align with one another. The formal definition of basis, linear independence, and span are much more efficient and most accurate in explaining the concept.*

Mathematical definitions are constructed in a precise manner to efficiently describe concepts. In this way, the women's examples did not meet such a standard of precision and efficiency. Deficit interpretations arise when this otherwise productive value of efficiency is used to dismiss the value of both the product and process of the women's mathematical sense making. An anti-deficit perspective would recognize the creativity in these examples. It would also acknowledge the flexibility in each student's understanding that allows her to extend formal mathematics to contexts that might not be seen as immediately mathematical. The fact that some of the contexts students used included theoretical ideas beyond students' real life experiences (e.g., the world market) further showcases their flexibility and creativity.

Moreover, the implicit learning goal within Claim 2 is for students to understand basis as it has been formally defined. Focusing instead on exploring the richness and ambiguities of the concept, we can interpret these women's examples as exploring and extending the notion of basis and vector spaces into novel contexts. De Freitas and Sinclair (2017) have attributed new constructions in the history of mathematics to this type of mathematical activity where mathematicians stretch and extend concepts beyond their current use. For example, Lebesgue's construction of function (vector) spaces was only formalized in the 1920s, nearly forty years after vector spaces were formalized. The women in our study were engaged in a disciplinary practice akin to ones that mathematicians use to explore new ideas in mathematics (Rasmussen et al., 2015; Zandieh & Rasmussen, 2010).

Claim 3. *These are fine examples, but students still need to learn the formal definition. Potential conflicts between the examples and the definition might mislead these students and their classmates in learning the formal definition.*

This claim reflects a dominant narrative about teaching mathematics: intuitions would likely lead students to erroneous conceptions and thus we need to present students with the most succinct and accurate definition. Unlike Claim 1 that argues against the validity of students' conceptions, this claim recognizes some productivity of the examples but still dismisses their utility in learning the formal definition and basis more broadly. In this way, the claim interprets the women's contributions as deficient, and in turn devalues their ability to contribute to learning.

These intuitive examples provide useful and accessible ideas for students to develop the more formal mathematical knowledge. Plaxco and Wawro (2015) have documented students' use of the intuitive notions of *different, covering,* and *traveling* (as captured in our codes) in discussing span and linear independence while engaging with a formal mathematical task. Research using the Realistic Mathematics Education design framework (Gravemeijer, 1999) has also shown the utility of students' intuitions or experientially real starting points as a resource to develop the formal mathematics. An anti-deficit perspective would focus on exploring ways to use these examples to explicitly build towards or provide insights into the formal definitions.

6. Discussions

Our anti-deficit analysis constructs a counter-story wherein a group of women of color made sense of basis. Analysis of their sense making reveals creative engagement with generalizing an abstract concept to diverse contexts from their lived experiences. The group of women attended to and responded to ambiguities and other conceptual issues in their examples. Constructed examples, like the ones constructed by these women of color, are resources that can be drawn out in a classroom. The novelty and the creativity reflected within these examples are further highlighted by the fact that most of the contexts have not been documented in existing literature about the learning of span, linear independence, and basis. The anti-deficit analysis includes as its focus strengths in the students' work.

The anti-deficit analysis also explicitly challenges deficit interpretations of the women's mathematical engagement. Leveraging historical accounts of the concept, it treats imprecisions and ambiguities in students' examples as a natural part of conceptual development. The anti-deficit analysis draws connections between the women's engagement with the task to disciplinary practices of mathematics. It conjectures the role of the ambiguities in opening up the space for the women's creativity. By also leveraging research literature with *expansive* framing of mathematical practice and ability (Louie, 2017), the anti-deficit analysis further highlights the limitations of deficit interpretations.

Deficit interpretations of the data coincide with a pattern of skepticism towards women of color in mathematics (McGee and & Martin, 2011; Varma, Prasad, & Kapur, 2006). Deficit claims from the previous section, if left unexamined, would sustain and contribute to deficit master-narratives about women of color in mathematics. These deficit claims would also contribute to deficit research narratives about students' inability to explain basis and to understand linear algebra more broadly (Figure 1). The anti-deficit analysis disrupts this system.

Engaging students in the type of task in this paper is also uncommon in the research literature. The task of constructing everyday examples about basis allows students to explore both the logical and the ontological aspect of the concept (De Freitas & Sinclair, 2017). Students stretched basis to novel contexts and explored the boundaries of the concept in these contexts. Beyond engaging students in this high-level mathematical activity, the examples also provide insights into students' sense making in a way that may not be captured by the use of more formal mathematical language and symbols.

Our analysis in the previous section shows that the product of students' engagement with this creative task could still be interpreted in a deficit way. Researchers (and teachers, see next section) could examine the students' work with a focus on deviations from the formal definitions and interpret them as deficiencies. In fact, our analytical codes also have the potential of being used as a new standard for "normative" knowledge. Doing so would contradict the anti-deficit perspective of this paper.

The analytical codes about the roles and characteristics of basis capture intuitive ideas associated with the concept of basis and aspects of the concept that have not been documented by other authors. To our knowledge only Plaxco and Wawro (2015) have discussed some of the ideas reflected in our codes as related to students' understanding of span and linear independence. In another paper, we have used the codes that we developed in the current paper to reveal patterns in how a group of students in Germany made sense of basis (Zandieh, Adiredja, & Knapp, 2019).

Returning to the De Freitas and Sinclair's (2017) notion of concepts as generative devices, we conjecture that these analytical codes point to aspects of the concepts that could be mobilized to novel contexts to explore new ontologies. The general nature of the codes and their applicability to novel contexts can illuminate the process of extending a concept beyond its logical underpinnings. We posit this conjecture as a productive direction for future research.

7. Implications

7.1. Classroom Implications

The task from this current study has been implemented in one classroom as a homework problem. The instructor of the course gained important insights into her students' understanding, and an anti-deficit approach to teaching (Adiredja, Bélanger-Rioux, & Zandieh, 2020). The instructor, Bélanger-Rioux observed how the task gave her students agency and ownership of ideas. This was especially important for the instructor as she was teaching a course for non-engineering and mathematics majors, who were predominantly women and students of color. In this class, the student who wrote the best response according to the instructor was a woman in the course who had been "struggling" in traditional assessments. Her response to this task offered a different story about her mathematical understanding. This is to say that counter-stories can also be developed in the classroom.

In Adiredja et al. (2019), the instructor also shared some of her initial deficit interpretations of the students' work, which included Claim 1 from the earlier discussion. Prioritizing standards related to formal concepts and language distracted her from recognizing important nuances in students' work, which she later uncovered. Our collective work in Adiredja et al. (2019) and in the current paper complements existing research in K-12 teacher education that has explored challenges teachers face to notice students' strengths instead of deficiencies (Louie, 2017; 2018), and others that focus on professional development around this issue (Jilk, 2016). Adiredja and Louie (2020) have explored the intersections of their respective work in teacher education and students' cognition to understand the reproduction of deficit discourses in mathematics education more broadly.

7.2. Situated and Intersectional Nature of Deficit Narratives

Deficit master narratives are situated in the history and politics of a region or a nation. While this paper focuses on deficit master narratives about race and gender in the United States., studies situated in other countries have examined the influence of similar types of narratives on the educational experiences of students from different marginalized groups. Straehler-Pohl, Gellert, Fernandez, and Figueiras (2014) documented differentiated learning opportunities for students from low socio-economic backgrounds who were new immigrants to Barcelona. Darragh and Valoyes-Chávez (2019) specifically discuss the impact of narratives with students with special education needs in Chile. These studies also suggest that deficit narratives often intersect with other systems of oppression (e.g., ableism, classism, racism, and xenophobia) and are intertwined with discourses about mathematics and mathematical ability (Adiredja & Louie, 2020).

The studies about student learning of linear algebra we reviewed in this paper included studies from the United States as well as Turkey and New Zealand. This is to say that deficit research narratives about students' mathematical ability also exist globally. One of the main aims of the current study is to raise awareness of the existence and persistence of both deficit research narratives and societal master narratives about different groups of students. Directly challenging such narratives using a study of mathematical sense-making by a group of women of color in the United States also supports the effort of examining dominant values and methods in studying mathematical sense making.

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Appendix
Basis Interview Protocol

Q1. (a) What does a basis mean to you?
Follow up:
(i) [*If students felt like their course did not cover basis formally*]: Some students have said that a basis of a vector space is a linear independent set that spans the vector space. What do you think that student means by that? Then go to (iii).

(ii) [*If they only mention one but not the other*]: Some students have said that a basis of a vector space is a linear independent set that spans the vector space. What do you think that student means by that? Then go to (iii).

(iii) [*If they mention both span or linear independence*]: What does each of those things mean to you?

Q1. (b) How would you explain it to a student who is about to take a Linear Algebra course?

Q2. (a) Can you think of an example from your everyday life that describes the idea of a basis?
(b) How does your example reflect your meaning of basis? What does it capture and what does it not?

Q3. Could basis be relevant for any of the tasks you did? If so, how?

Q4. Can you see a basis as a way to describe something? If so, what is the something? How?

Q5. Can you see basis as a way to generate something? If so, what is the something? How?

Q6. Go through each task, and ask if they CAN possibly see basis in them.
Follow up: Can you express #3 in parametric form?

[If time permits] Q7. (a) Some students say that a basis is a minimal spanning set, what do you think the student means by that?
(b) Some students say that a basis is a maximal linear independent set, what do you think the student means by that?