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Untold secrets of the slowly charging capacitor

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The slowly charging capacitor is the standard example used to illustrate that the displacement current density is needed in Ampere's law if we want to correctly determine the magnetic field between capacitor plates. However, in any quasi-static situation the magnetic field can also be determined using the Biot-Savart law including only the real current densities. In this work, we will numerically calculate the magnetic field due to the surface currents on the capacitor plates and add it to the magnetic field due to the charging wire and show how they combine to create the correct magnetic field thoughout all space. For regions to the left or right of the capacitor, we find the surprising result that the surface currents replicate the magnetic field that would have been created by the missing section of the charging wire between the plates. For points between the capacitor plates, the magnetic field due to the surface currents mostly cancels the magnetic field from the near-infinite length charging wire, resulting in the well-known reduced field in that interior region. We will also illustrate the impact of finite capacitor plates on these results and briefly comment on how textbook and/or classroom discussions could be improved by carefully discussing these details.

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I. INTRODUCTION

The slowly charging capacitor is the standard example used in introductory[1–3] and advanced[4–7] textbooks to illustrate that the integral form of Ampere's law,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \int_S (\vec{J} + \vec{J}_D) \cdot d\vec{A}, \tag{1}$$

is incomplete without the displacement current density $\vec{J}_D$ (where $\vec{J}_D = (\partial \vec{D}/\partial t)$ and $\vec{D}$ is the electric displacement field). To aid our discussion of the standard textbook analysis, refer to Fig. 1. Imagine that we are trying to determine the magnetic field intensity at point P. In these solutions, we assume that the capacitor plates are infinitely large so the electric field between the plates is uniform and there is no fringing field. From a practical standpoint, the plates are "effectively infinitely large" if the plate radius is large compared to the plate separation ($a \gg d$) and if the field point is both sufficiently close to the plates ($|z| \ll a$) and sufficiently far from their edges. Given the symmetry here, we will use a circular Amperian loop containing point P which is concentric with the current I. That specifies the left side of Eq. (1) but there are an infinite number of choices for the right side since the surface just has to be bounded by the loop. Let's first use the circular area inside the Amperian loop (labeled S_1 in Fig. 1). The current I passes through S_1 and we obtain

$$\vec{B}_{\mathrm{P}} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}, \tag{2}$$

where $\hat{\phi}$ is the standard cylindrical coordinate unit vector. However, we could also use S_2 which is an ellipsoidal-shaped structure with a circular hole on its right side which encloses the right capacitor plate. The current I doesn't pass through S_2, and so we would conclude that there is no magnetic field. Since $\vec{B}_{\mathrm{P}}$ can't depend upon our choice of surface and since we can easily measure the (non-zero) magnetic field, something must be wrong. At this point in the standard analysis, the displacement current density is introduced and then Eq. (1) with S_2 also results in Eq. (2). Unfortunately, this gives students the impression that real currents are the source of $\vec{B}$ for S_1 while displacement currents are the source for S_2[8] and this is certainly misleading.

One important question is never asked during this analysis: "Is the magnetic field at point Q (in Fig. 1) any different from the magnetic field at point P? Why or why not?" Equation (1) makes it clear that the exact same field exists at point Q. However, this result is *very* unexpected since there is no translational invariance here. There must be something deeper involved. In this work (see Sec. IV), we will

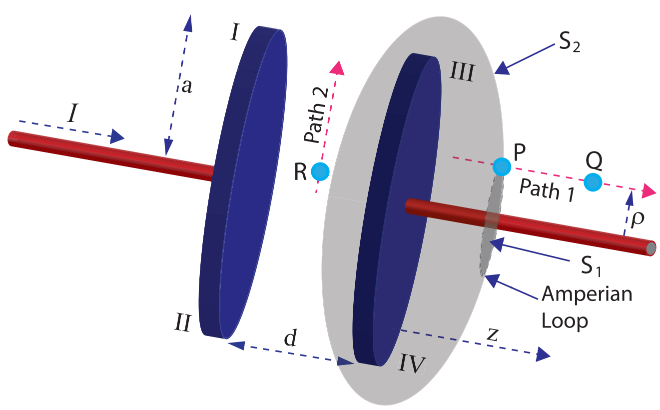

Fig. 1. This illustrates the system geometry. The two capacitor plates (dark blue online) have a radius a and a separation d. The upper and lower halves of each plate are labeled separately (I-IV). Paths 1 and 2 (magenta online) are the locus of points along which we will determine the magnetic field in Secs. IV and V. Three field points (P, Q and R, sky blue online) are shown along these paths. A circular Amperian loop (medium dashed line) passes through point P and it is the bounding contour for two surfaces. S_1 (darker shading) is the perfect circular area inside that contour and S_2 (lighter shading) is an ellipsoidal-shaped structure, with a circular hole on its right side, which encloses the right capacitor plate. We use a standard cylindrical coordinate system (ρ, ϕ, z). The current I (red online) lies on the z axis and $z = 0$ on the right capacitor plate. The charging wire is assumed to be an infinitely long point current and the capacitor plates (while appearing somewhat three-dimensional here) are two-dimensional.

determine how the magnetic fields at points P and Q are identical and how both match the infinite straight current field.

Let's now imagine that we want to find the magnetic field at point R in Fig. 1. In the standard textbook solution, we apply Eq. (1) to a circular Amperian loop which contains point R and use a circular area analogous to S_1. This surface is only pierced by displacement current and we find that

$$\vec{B}_{\mathrm{R}} = \begin{cases} \dfrac{\mu_0 I \rho}{2\pi a^2}\,\hat{\phi} & 0 \le \rho \le a, \\ \dfrac{\mu_0 I}{2\pi\rho}\,\hat{\phi} & \rho \ge a. \end{cases} \tag{3}$$

Thus, the magnetic field increases linearly with radius inside the capacitor and then matches an infinite straight current field outside the capacitor. In Sec. V, we will determine (in detail) why the field has the structure given in Eq. (3). In 1985, Bartlett and Corle[9] performed a challenging measurement of this linear increase and more recently Scheler and Paulus[10] described an experiment that could demonstrate this in the classroom. Unfortunately, this calculation gives students the impression that the magnetic field between the plates can only be calculated using the displacement current. In fact, other authors have calculated $\vec{B}$ using different Amperian loops which have "different sources."[11,12]

II. WHAT ARE THE SOURCES OF THE MAGNETIC FIELD?

For this slowly charging capacitor, it is the current in the charging wire and the surface currents on the capacitor plates which are the sources of the magnetic field. The displacement current doesn't contribute. For quasi-static situations with no induced electric fields we know that $\vec{\nabla} \times \vec{E} = 0$. In this situation, in addition to using Ampere's law to determine the magnetic field (Eq. (1)), one can also use the Biot-Savart law,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r'}) \times (\vec{r} - \vec{r'})d\tau'}{|\vec{r} - \vec{r'}|^3}, \tag{4}$$

with only *real* current densities ($\vec{J}$). In Eq. (4), $\vec{r}$ and $\vec{r'}$ describe the field and source points and $d\tau'$ is the source volume element. Bierman[13] provided the relevant mathematical proof that only $\vec{J}$ is needed. He showed that if one takes the curl of the Biot-Savart law and assumes there are no induced electric fields then the resulting expression is, in fact, the differential form of Ampere's law. Thus, they are formally equivalent under those conditions. This is a surprising result and one worth emphasizing. It has been stated by many other authors.[6,14–19] While it is formally correct for Eq. (4) also to include the displacement current density,[19] Bartlett[15] and Purcell[6] both clearly explain why its contributions integrate to zero. Thus, while the displacement current density facilitates the calculation of the magnetic field for some choices of Amperian loops/surfaces when using Eq. (1), it is the real currents that are producing the field.

Solutions to Maxwell's equations for situations with slowly varying charge and current densities have also been studied by other authors. For example, Wolsky[20] studied an approximation (valid for light travel times much less than the source variation timescale) and also found that the Biot-Savart law (using only $\vec{J}$) gives the correct magnetic field. Additionally, Griffiths and Heald[21] use the Jefimenko equations[22] (the causally correct expressions for $\vec{E}$ and $\vec{B}$ which express the fields entirely in terms of charge and current densities and their time derivatives) to illustrate that the Biot-Savart Law and Coulomb's law are both correct in some time-varying situations. They state that this "should be regarded as a curiosity" but knowing that the displacement current does not contribute here provides us with a unique opportunity to better understand the magnetic field structure due to a charging capacitor. Other authors have previously used the equivalence of Ampere's law and the Biot-Savart law to examine some aspects of charging and/or discharging capacitors.[14,15]

III. ADDITIONAL CALCULATION DETAILS

We will numerically integrate Eq. (4) to determine the magnetic field due to the surface currents on the capacitor plates and add it to the magnetic field due to the charging wire and show that we can reproduce the magnetic fields in the standard textbook solutions (Eqs. (2) and (3)). We will make the standard assumption that the charge is instantly distributed over the plates (which is perfectly reasonable given the rapid relaxation time in conductors) and we will also assume that the plates are always uniformly charged. Since we will be comparing our results to Eqs. (2) and (3) which assume uniformly charged plates, we will ignore edge effects at/near $\rho = a$. By applying conservation of charge, it can be shown that the surface current density on the two plates is[4]

$$\vec{K} = \frac{I}{2\pi\rho}\left(1 - \frac{\rho^2}{a^2}\right)(\pm\hat{\rho}), \tag{5}$$

where the $\pm$ in $\pm\hat{\rho}$ distinguishes the left (+) and right (-) plates.

Figure 1 illustrates the geometry but we have a few additional comments. There is a vacuum between the plates, and so $\vec{D} = \epsilon_0\vec{E}$. We will use a constant current in the charging wire. Otherwise, the standard textbook solutions for the electric and magnetic fields don't satisfy Faraday's law.[23,24] Specifically, we set $I = 0.01$ A. Finally, since a particular current element can always be paired with a current element halfway around the plate which has its current in the exact opposite direction, the magnetic fields due to the upper/lower halves (at least) partially cancel each other. Thus, it will be useful to discuss those magnetic fields separately so we will label these four half-plates I–IV.

IV. DETERMINING THE MAGNETIC FIELD FOR POINTS OUTSIDE THE CAPACITOR

So how are the magnetic fields at points P and Q both equal to the magnetic field due to an infinite straight current (see Eq. (2)), when the current is not infinite since it is missing the current element of length d between the capacitor plates? In this quasi-static situation, the real currents must collectively produce that magnetic field. This means that the net magnetic field created by the surface currents on the plates must be EXACTLY the same as the magnetic field which would have been produced by that missing current element! Note that we can also reach the same conclusion by considering a discharging capacitor.[25]

To illustrate how this is possible, let's consider the magnetic field due to each of the four half-plates (see Fig. 1). Figure 2 shows a face-on view of half-plate I as viewed from point P. Since these surface currents are radial, Eq. (4) tells us that they produce magnetic fields in $\pm\hat{z}$ and/or in $\pm\hat{\phi}$. Using Eq. (4) (or a "right hand rule"), we find that all the surface current elements produce magnetic fields in $-\hat{\phi}$ at point P (or inward for points above the charging wire in Fig. 1). The surface current elements on the left side also produce fields in $-\hat{z}$ while those on the right side produce fields in $+\hat{z}$ and given the symmety those contributions cancel. Thus, the net magnetic field due to those surface currents is in $-\hat{\phi}$. Similarly, the surface currents on half-plate IV also produce a field in $-\hat{\phi}$ while those on half-plates II and III produce fields in $+\hat{\phi}$. Since half-plate III is the closest to P, it is conceivable that the vector sum of these fields is in $+\hat{\phi}$ just like that due to the missing current element.

When we consider finite plates, we must account for the fringing electric field. French and Tessman[14] have previously explained this for point R in Fig. 1 if $\rho > a$. We will continue their analysis as it applies to points P and Q. Imagine applying Eq. (1) to a circular Amperian loop which contains point P and using surface S_1. Then perform a similar calculation with an analogous loop and surface for point Q instead. We find that the magnetic fields at P and Q are different from each other and both are smaller than the infinite straight current field. The electric field is no longer zero so electric field lines pass through S_1 and terminate on the negative charges on the right capacitor plate. Since that electric field is increasing with time as the plates charge, this is equivalent to a displacement current which is opposite the real current, and so the resulting magnetic field calculated using Eq. (1) is smaller. Since we expect the electric field (and the corresponding displacement current) to be smaller at point Q than at point P, the magnetic field will be closer to the infinite straight current value at point Q and will approach it exactly as $z \rightarrow \infty$.

In the following calculations, we explicitly calculate the net magnetic field due to the surface currents (via numerical integration of Eq. (4) and using Eq. (5)) and compare it to the magnetic field due to the missing current element across the plates. That latter field can be determined analytically using Eq. (4) and the result is

$$\vec{B}_{\text{missing current element}} = \frac{\mu_0 I}{4\pi\rho}\left(\frac{z+d}{\sqrt{\rho^2+(z+d)^2}} - \frac{z}{\sqrt{\rho^2+z^2}}\right)(\hat{\phi}). \qquad (6)$$

If point P is very close to the plates and far from their edges, then we expect the net magnetic field due to the surface

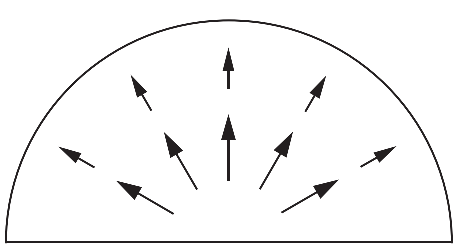

Fig. 2. This is a face-on view of half-plate I as viewed from point P in Fig. 1. It qualitatively illustrates that the surface current density is smaller at larger radii as is described by Eq. (5).

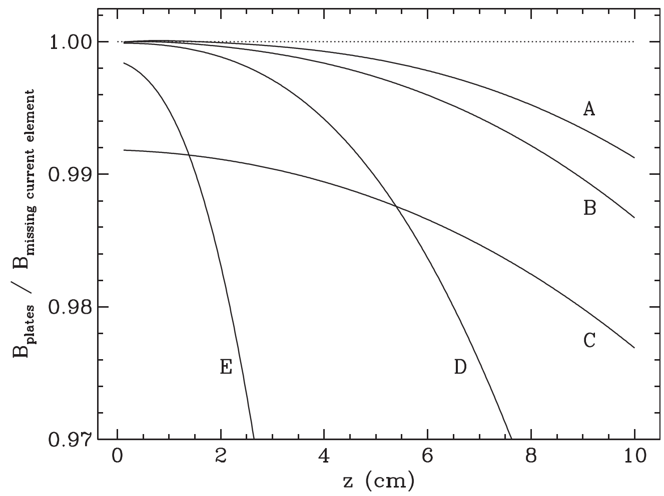

Fig. 3. This plots the magnetic field due to the surface currents on the plates (calculated using Eqs. (4) and (5)) divided by the magnetic field which would have been created by the missing current element across the plates (see Eq. (6)) along Path 1 in Fig. 1. In the limit of infinitely large plates, all five curves here would match the dotted line at 1.0. The model differences can be seen in Table I. All models use $I = 0.01$ A.

currents on the plates to match Eq. (6). However, if point P is at too large a z and/or too close to the plate edge, then the total magnetic field is smaller than that given by Eq. (2), and so the net magnetic field from the surface currents on the plates must be *smaller* than that given by Eq. (6).

Let's now examine Fig. 3. We have plotted the magnetic field due to the surface currents on the plates divided by Eq. (6) along Path 1 in Fig. 1. Let's focus on curve A (see Table I for each model's parameters). For this model, $a = 50$ cm and $\rho = 1$ cm so at small z we expect the field ratio to be very close to 1.0 and it is for $z \lesssim 2$ cm.[26] However, at $z = 10$ cm, $z/a = 0.2$, and so the fringing electric field plays a role and the magnetic field due to the surface currents must be smaller than that given by Eq. (6). As Fig. 3 indicates, it is ~1% smaller.

For the other four curves in Fig. 3, point P is at a location where the fringing electric field plays a larger role. In Model B, the capacitor plates are now 5 cm apart so while the magnetic field ratio still starts very close to 1.0, it drops more rapidly than for Model A. In Model C, $\rho = 10$ cm so $\rho/a = 0.2$ and there is a noticeable difference even at small z. In Models D and E, the plate radii are now 25 and 10 cm, respectively, and so the ratio changes significantly more rapidly with z. In fact, this allows the curves for Models D and

Table I. Model parameters.

Model #	Path	Plate radius a (cm)	Plate separation d (cm)	Path radius ρ (cm)
A	1	50	1	1
B	1	50	5	1
C	1	50	1	10
D	1	25	1	1
E	1	10	1	1
F	2	25	0.25	-
G	2	25	0.5	-
H	2	25	1.0	-
I	2	12.5	1.0	-

E to cross the curve for Model C. At $z = 10$ cm, the field ratio for Model E has dropped to $\sim$0.62. At this location, $z/a = 1.0$, and so the plates begin to look more like an electric dipole and less like infinite parallel plates. Conversely, at $z = 10$ cm for Model C the plates still look quite large (but just not infinitely large).

Note that as long as we are in the quasi-static limit, the results in Fig. 3 do not depend upon the current's magnitude. When the ratio is less than one, it reflects the displacement current passing through S_1. If we were to increase the current by a factor of ten, we would simultaneously increase the displacement current by the same factor since $|\vec{J}_D| \propto d\sigma/dt \propto I$ (where σ is the surface charge density on the plates), and so the ratio wouldn't change.

Let's now examine the magnetic fields from each half-plate and compare them with the magnetic field due to the near-infinite current. In Fig. 4, we plot the magnetic field due to the surface currents on each half-plate for Model A (solid lines), the net field due to those surface currents (short dashed line, B_{plates}), the field due to the near-infinite current (dotted line, $B_{near-infinitecurrent}$) which is the difference between Eqs. (2) and (6) and the net field due to all the real currents (long dashed line, B_{net}) which is the sum of B_{plates} and $B_{near-infinitecurrent}$. The magnetic field is positive if its direction at point P is in $+\hat{\phi}$ (or outward in Fig. 1). In these calculations, $\mu_0/2\pi$ has been removed so 1.0 is equivalent to 2×10^{-7} T.

Figure 4 has many interesting features. Although $\rho = 1$ cm (so $\rho/a = 0.02$), the upper half-plates I & III create significantly larger fields than the lower ones since the cross product in Eq. (4) biases surface current elements which are both exactly halfway across Fig. 2 and which have the same radius as point P. That cross product also helps explain why the field due to the surface currents on half-plate IV increases in magnitude with z until $z \simeq 1.2$ cm. At small z, the surface current density $\vec{K}$ and the vector from the source point to the field point are nearly parallel so those current elements contribute very little to the field. We also see that for points very close to the plates, $B_{plates} \simeq \frac{1}{3} B_{net}$.[27] Conversely, for $z \gtrsim 2$ cm the surface current magnetic fields become negligible and the near-infinite current dominates.

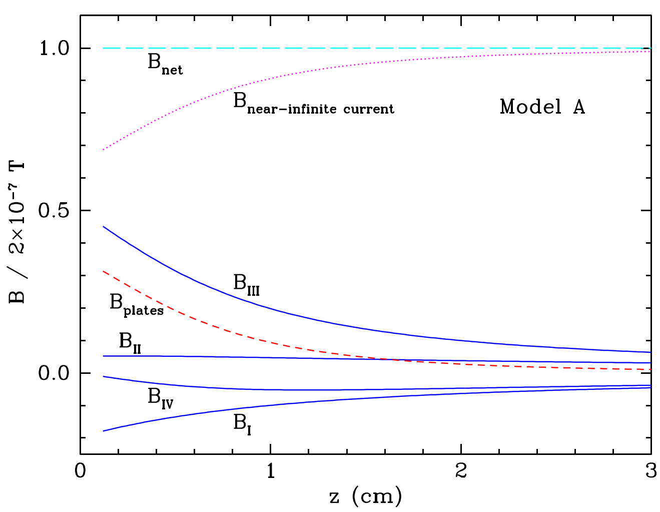

Fig. 4. The solid lines here (blue online) are the magnetic fields due to the surface currents on each half-plate along Path 1 for Model A (see Table I). The short dashed line labeled B_{plates} (red online) is the net magnetic field due to the surface currents on those four half-plates. The dotted line labeled $B_{near-infinitecurrent}$ (magenta online) is the magnetic field due to the near-infinite current and the long dashed line labeled B_{net} (cyan online) is the net magnetic field due to all the real currents (in other words, the sum of B_{plates} and $B_{near-infinitecurrent}$). The magnetic field is positive if its direction at point P is in $+\hat{\phi}$ (or outward in Fig. 1). In these calculations, $\mu_0/2\pi$ has been removed so 1.0 is equivalent to 2×10^{-7} T.

While it isn't plotted here, the standard textbook solution (Eq. (2)) would be equal to 1.0 on this graph so this illustrates that B_{net} matches that solution nearly perfectly. This is to be expected since the plates appear very much like two infinite parallel plates in this geometry. For example, at $z \sim 3$ cm the surface currents on the plates contribute $\sim$1% of the total magnetic field (see Fig. 4) and their field is $\sim$0.033% smaller than would have been produced by the missing current element between the plates (see Fig. 3). Thus, overall the magnetic field is within $0.01 \times 0.00033 \sim 0.00033\%$ of that predicted by Eq. (2).

V. DETERMINING THE MAGNETIC FIELD FOR POINTS "BETWEEN" THE CAPACITOR PLATES

To complete this study, we need to better understand the magnitude of the magnetic field between the capacitor plates so let's consider how the real currents create this field. We will locate Path 2 (see Fig. 1) along a line equidistant from each plate so that $\vec{B}_I = \vec{B}_{III}$. We will group them together as $\vec{B}_{top}$. Similarly, $\vec{B}_{II} = \vec{B}_{IV}$ so we will group them together as $\vec{B}_{bottom}$. The magnetic field due to the charging wire can be determined analytically using Eq. (4) and the result[28] is

$$\vec{B}_{wire} = \frac{\mu_0 I}{2\pi\rho}\left[1 - \frac{d}{\sqrt{4\rho^2 + d^2}}\right](\hat{\phi}). \tag{7}$$

Along Path 2, $\vec{B}_{wire}$ and $\vec{B}_{bottom}$ are in $+\hat{\phi}$ while $\vec{B}_{top}$ is in $-\hat{\phi}$. At $\rho = 0$, $\vec{B}_{top}$ and $\vec{B}_{bottom}$ perfectly cancel each other and $\vec{B}_{wire}$ is identically zero. As we increase ρ, $\vec{B}_{top}$ and $\vec{B}_{bottom}$ no longer perfectly cancel and $\vec{B}_{wire}$ starts to contribute. However, there is still significant cancellation so the resulting field is quite small. For $\rho \gg d$, the second term in Eq. (7) vanishes since the missing current element becomes negligible so $\vec{B}_{wire}$ is just that of an infinite straight current. Additionally, if $\rho > a$ we expect the contributions of the surface currents on the plates to vanish since those currents will be too far away.

As in Sec. IV, we must account for having finite plates. The use of Eq. (1) to obtain Eq. (3) required a uniform electric field perfectly parallel to the symmetry axis and completely confined to the region between the plates. For finite plates, that calculation *overestimates* the displacement current flux (and hence the magnetic field) at all ρ.[29] Equation (3) will be most accurate for $\rho \ll a$ since the electric field is very uniform there and least accurate at $\rho = a$ since the percentage overestimate will be largest there.

In Fig. 5, we display our results. We plot the magnetic field due to all the real currents divided by Eq. (3). As we just explained, for finite plates this ratio must be < 1.0. For each model (again, see Table I), there is a dip at the plate radius for that model and the ratios depart more from 1.0 when d/a is largest as is expected. For the "worst" case here (Model I with $a/d = 12.5$), the textbook solution overestimates the field by $\sim$11% at $\rho = a$.

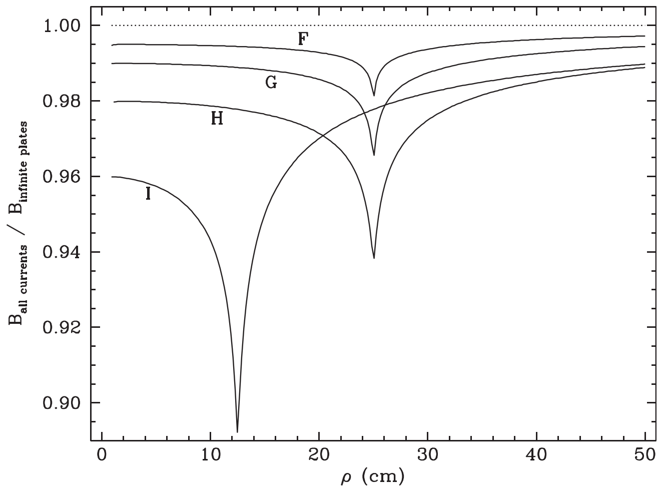

Fig. 5. This plots the net magnetic field due to all currents divided by the standard textbook solution given in Eq. (3) along Path 2. For infinitely large plates, all the ratios would match the dotted line at 1.0. The model differences can be seen in Table I. All models use $I = 0.01$ A.

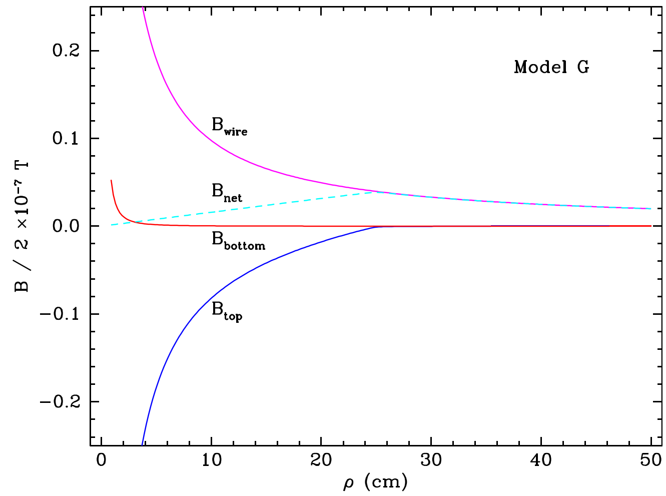

Fig. 6. This plots the (net) magnetic field due to the surface currents on half-plates I and III (labeled B_{top}, blue online), that due to the surface currents on half-plates II and IV (labeled B_{bottom}, red online) and that due to the current in the charging wire (labeled B_{wire}, magenta online) along Path 2 for Model G. It also plots the net magnetic field (dashed line labeled B_{net}, cyan online) due to all these sources. For $\rho \gtrsim 25$ cm, B_{net} and B_{wire} are essentially identical. The magnetic field is positive if its direction at point P is in $+\hat{\phi}$ (or outward in Fig. 1). It is normalized such that 1.0 is equivalent to 2×10^{-7} T.

We can actually use the $\rho \ll a$ behavior in Fig. 5 to verify the accuracy of our numerical integrations. Our results must match those from Eq. (1) so let's apply Eq. (1) to a circular Amperian loop which contains point R and use a circular area analogous to S_1. For $\rho \ll a$, the field point is approximately on the axis of two uniformly charged disks, and so the electric field must be *very* close to that on-axis electric field. A straightforward integration of Coulomb's law[30] on the axis tells us that

$$|\vec{E}(t)| = \frac{\sigma(t)}{\epsilon_0}\left[1 - \left(1 + \frac{4a^2}{d^2}\right)^{-1/2}\right], \tag{8}$$

where the pre-factor, $\sigma(t)/\epsilon_0$, is the resulting electric field for infinite plates. For the four models shown in Fig. 5, a/d = 100, 50, 25, and 12.5 and the corresponding factors in brackets in Eq. (8) are 0.995, 0.99, 0.98 and 0.96 respectively (to many digit accuracy). These are also easily obtained using $|\vec{E}(t)| = (\sigma(t)/\epsilon_0)\,[1 - d/(2a) + \mathcal{O}(d/a)^3 \cdots]$ which is the Taylor expansion of Eq. (8) valid for $a \gg d$. Since the displacement current densities must be reduced by the same factors, Eq. (1) tells us that the magnetic fields are similarly reduced. In Fig. 5, the four models have exactly the correct ratios at small ρ. Summarizing, our calculations and an analytic electric field approximation both tell us that departures from the standard textbook solution for $\rho \ll a$ are linearly proportional to d/a when $a/d \gg 1$. In fact, Bartlett[15] obtained the same result using an analytic approximation to Eq. (4).

As $\rho \to 50$ cm in Fig. 5, the magnetic field is again approaching the infinite straight current field. Assuming that the charging wire is the only source of magnetic field, Eq. (7) predicts that the ratios for the four models (F-I) would be 0.9975, 0.995, 0.99 and 0.99 respectively (to many digits) at $\rho = 50$ cm. This definitely matches Fig. 5 and verifies what we said earlier that for these locations above/below the capacitor the effects of the surface currents are negligible and the net magnetic field is just that of the near-infinite straight current.

Let's now see how the fields created by the upper and lower halves of the plates help create the net magnetic field. In Fig. 6, we plot $\vec{B}_{top}$, $\vec{B}_{bottom}$ and $\vec{B}_{wire}$ for Model G along Path 2. The figure also displays $\vec{B}_{net}$ so the approximately linear increase within the capacitor and the approximately ρ^{-1} decrease above/below the capacitor are visible.[31] On this scale, $\vec{B}_{net}$ is essentially identical to $\vec{B}_{wire}$ for $\rho \gtrsim a$. $\vec{B}_{bottom}$ is clearly negligible except at very small radii so for much of the interior region of the capacitor ($\rho \lesssim a$), the magnetic field is determined by the vector sum of $\vec{B}_{wire}$ and $\vec{B}_{top}$. At small radii, these fields are much larger than $\vec{B}_{net}$ so accurate numerical integration of Eq. (4) is required. For example, at $\rho = 0.5$ cm, $|\vec{B}_{wire}|$ and $|\vec{B}_{top}|$ are $\sim$1400 and $\sim$1600 times larger respectively than $|\vec{B}_{net}|$.

The curve representing $\vec{B}_{top}$ in Fig. 6 appears to have a "kink" but this is an artifact of the linear plot. $|\vec{B}_{top}|$ drops very rapidly toward zero as $\rho \to a$ since the surface current density (see Eq. (5)) drops rapidly toward zero in that region.

VI. CONCLUSION AND COMMENTS ABOUT EDUCATION

In this work, we have utilized the equivalence of the Biot-Savart Law and Ampere's Law in quasi-static situations to better understand the magnetic field structure due to a slowly charging capacitor. For points to the left or right of the capacitor (e.g., point P in Fig. 1) the surface currents replicate the magnetic field that would have been created by the missing current element across the plates as long as we have infinitely large plates. For points between the capacitor plates (e.g., point R in Fig. 1 for $\rho < a$), the field is primarily determined by the magnetic field due to the charging wire and that due to the surface currents on the half-plates on the same side of the charging wire as R. When the plates are finite in size, our results tell us about the effects of the fringing electric field.

The fact that the magnetic field can be determined using only the real currents is mostly absent from the most commonly used textbooks. In 1998, Roche[32] wrote that "It is rather more surprising that the conclusions of French, Tessman, Purcell and Bartlett seem to have had very little impact on the textbook tradition." Unfortunately, this is still true today. Many students obtain the impression (even if this misrepresents the text) that the real current generates the magnetic field to the left or right of the capacitor while the displacement current generates it between the plates. This is certainly undesirable. This point does appear in a slightly modified form in a homework problem in Griffiths' text[5] and it is nicely explained in Purcell's text.[6] In 2013, a new edition of Purcell's text appeared (in conjunction with David Morin)[33] so hopefully this issue will become more widely acknowledged.

It would also be useful and educational to bring some of the rich physics here into the classroom (particularly in upper division classes). Imagine asking your students "Is the magnetic field at point Q (in Fig. 1) any different from the magnetic field at point P? Why or why not?" Or asking them "Why does Ampere's Law suggest that the magnetic fields at points P and Q are both identical to the magnetic field due to an infinite straight current when our current actually has a gap in it where the capacitor is?" These questions would naturally lead to discussions of the lack of translational invariance, the impact of the surface currents on the capacitor plates, the impact of fringing electric fields when comparing finite and infinite plates, etc. Overall, a discussion of these details would result in a better understanding of the interesting physics in this system.

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[8]Imagine a third surface with the same shape as S_2 but with a smaller radius so it cuts across capacitor plates III and IV. That surface would be pierced by the surface currents on the plates and by the displacement currents.

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[22]Oleg D. Jefimenko, *Electricity and Magnetism: An Introduction to the Theory of Electric and Magnetic Fields* (Appleton-Century-Crofts, New York, 1966), p. 516.

[23]C. J. Papachristou, "Maxwell equations for a charging capacitor," META Publishing, **57** (2019), available at <http://metapublishing.org/index.php/MP/catalog/book/57>.

[24]Since introductory textbooks always discuss charging or discharging *RC* circuits, it is likely that students think that the standard solutions in Eqs. (2) and (3) apply to those circuits. However, since those currents are exponentially decreasing with time, those solutions wouldn't satisfy Faraday's Law.

[25]Consider an isolated capacitor whose infinitely large plates have equal and opposite charge densities. Now connect a wire between the plates along their symmetry axis. If we apply Eq. (1) to a point analogous to point P in Fig. 1 and use a circular Amperian loop with surface S_1, we would conclude that there is no magnetic field to the left or right of the plates. This means that the magnetic field due to the surface currents on the plates perfectly cancels the magnetic field due to the discharging wire across them. Since these surface currents are in exactly the opposite direction from those in the charging capacitor problem we are focusing on here, this also means that the surface currents for the charging plates must create exactly the *same* magnetic field as the missing current element across the plates would have created.

[26]In fact, in the region $z \lesssim 2$ cm, the ratio is within $0.002\% - 0.008\%$ of 1.0. This difference is so small that we are reaching the limits of the numerical algorithm. If one looks *very* closely at Fig. 3, one can see that curve A lies slightly above 1.0 at $\sim z = 0.8$ cm (by $\sim 0.008\%$). This certainly isn't physically possible. We can definitely say that in that small z region the magnetic field due to the surface currents is essentially identical to the magnetic field that would have been produced by the missing current element across the capacitor but the exact difference between them isn't accurately calculated here. At larger z, the numerical limitations are not relevant and we have an accurate picture of the ratio.

[27]Note that the maximum contribution of the surface currents on the plates to the net magnetic field is 50%. Imagine a geometry where the capacitor plates are infinitely large so that Eq. (2) applies and also assume that $d \gg \rho \gg z$. In this geometry, the charging wire to the right of point P is a semi-infinite current so it must contribute exactly half the infinite straight current field. That means that the surface currents on the plates and the charging wire far to the left must contribute the other half. In this limit, the influence of the charging wire on the left will be negligible so the surface currents must create the other 50% of the field. Since we know that the magnetic field due to the surface currents replicates the field that would have been created by the missing current element, we can see this mathematically using Eq. (6). Notice that the prefactor in front is exactly half the infinite straight current field and that both terms in parenthesis are guaranteed to be <1. In this limit, the first term approaches 1.0 and the second approaches zero so the resulting expression is at most 50% of Eq. (2).

[28]Equation (7) is clearly the magnetic field due to the near-infinite current analogous to what is plotted in Fig. 4. However, since we are determining the magnetic field along Path 2, $z = -d/2$ and ρ is varied. We have chosen to refer to it as $\vec{B}_{\rm wire}$ to avoid any confusion.

[29]Note that the curving of the fringing electric field lines does not affect our use of Eq. (1) since our Amperian loop is equidistant from each capacitor plate. Thus, the electric field lines (and hence the displacement current) are completely perpendicular to the loop.

[30]$\vec{E}(\vec{r}) = (1/4\pi\epsilon_0)\int_V((\rho(\vec{r'})(\vec{r} - \vec{r'})d\tau')/|\vec{r} - \vec{r'}|^3)$ where ρ in this context is the volume charge density.

[31]Of course, Eq. (3) has an exactly linear increase for $\rho \leq a$ and a ρ^{-1} decrease for $\rho \geq a$ but that is the infinite parallel plate solution and not the correct one for the finite plates which we calculate here.

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