Finite alphabet iterative decoders for LDPC codes surpassing floating-point iterative decoders

S.K. Planjery, D. Declercq, L. Danjean and B. Vasic

Introduced is a new type of message-passing (MP) decoders for low-density parity-check (LDPC) codes over the binary symmetric channel. Unlike traditional belief propagation (BP) based MP algorithms which propagate probabilities or log-likelihoods, the new MP decoders propagate messages requiring only a finite number of bits for their representation in such a way that good performance in the error floor region is ensured. Additionally, these messages are not quantised probabilities or log-likelihoods. As examples, MP decoders are provided that require only three bits for message representation, but surpass the floating-point BP (which requires a large number of bits for representation) in the error-floor region.

Introduction: Traditional message-passing (MP) algorithms for decoding low-density parity-check (LDPC) codes are based on an iterative decoding algorithm known as belief propagation (BP). The design of quantised BP decoders and other low-complexity variants has gained importance for practical realisations (see [1] for references). The quantised decoders are typically chosen to have the best possible asymptotic decoding thresholds via density evolution [2]. However, their performance is not guaranteed to be good on codes of practical length in the high signal-to-noise (SNR) region where the problem of error floor typically occurs.

In this Letter, we present a new type of finite precision MP decoders termed as finite alphabet iterative decoders (FAIDs) for LDPC codes. These were first introduced in [1] and can surpass floating-point BP in the error floor with a fraction of its complexity. The messages belong to a finite alphabet and the update functions are simple maps designed to improve the guaranteed error-correction capability of the code. Additionally in this Letter, we provide an alternate representation for FAIDs in the form of symmetric plane partitions in order to enumerate the number of possible decoders. We then provide good 3-bit precision MP decoders different from the ones presented in [1] that were designed specifically for a practical quasicyclic code. We also show that decoders designed solely based on density evolution are not the best FAIDs for this code.

Framework: Let G denote the Tanner graph of an *(N, M)* binary LDPC code which has N variable nodes and M check nodes. Let $V = \{v_1, \ldots, v_N\}$ be the set of variable nodes. A multilevel FAID $\mathcal{F}$ is a 4-tuple given by $\mathcal{F} = (\mathcal{M}, \mathcal{Y}, \Phi_v, \Phi_c)$ [1]. The messages are levels confined to an alphabet $\mathcal{M}$ which is defined as $\mathcal{M} = \{0, \pm L_i : 1 \le i \le s\}$ where $L_i \in \mathbb{R}^+$ and $L_i > L_j$ for any $i > j$. The set $\mathcal{Y}$ denotes the set of possible *channel values*. For the case of BSC, $\mathcal{Y}$ is defined as $\mathcal{Y} = \{\pm C\}$, and for each variable node v_i in G where r_i is the received bit, the channel y_i value is determined as $y_i = (-1)^{r_i} C$. Let $m_1, \ldots, m_{l-1}$ denote the incoming messages to a node with degree l that are used to calculate the extrinsic message.

Φ_c: $\mathcal{M}^{d_c-1} \to \mathcal{M}$ is the update function used at a check node with degree d_c defined as

$$\Phi_c(m_1, \ldots, m_{d_c-1}) = \prod_{j=1}^{d_c-1} \operatorname{sgn}(m_j) \min_{j \in \{1,\ldots,d_c-1\}} (|m_j|)$$

where sgn is the standard signum function.

$\Phi_v : \mathcal{Y} \times \mathcal{M}^{d_v-1} \to \mathcal{M}$ is a symmetric update function used at a variable node with degree d_v and is defined as

$$\Phi_v(y_i, m_1, \ldots, m_{d_v-1} = Q\left(\sum_{j=1}^{d_v-1} m_j + \omega_c \times y_i\right)$$

Q is a function defined based on a threshold set $\mathcal{T} = \{T_i : 1 \le i \le s+1\}$, where $T_i \in \mathbb{R}^+$, $T_i > T_j$ if $i > j$, and $T_{s+1} = \infty$. Q is defined as $Q(x) = \operatorname{sgn}(x) L_i$ if $T_i \le |x| < T_{i+1}$, and $Q(x) = 0$ otherwise. The weight ω_c is computed using a symmetric function $\Omega : \mathcal{M}^{d_v-1} \to \{0, 1\}$. Based on this, Φ_v can be described as a linear-threshold (LT) or nonlinear-threshold (NLT) function. If Ω = constant, then it is an LT function, else it is an NLT function. Note that due to possible nonlinearity in Ω, these decoders are different from existing quantised decoders. Φ_v can also be represented simply as a map.

For this Letter, we restrict ourselves to codes with $d_v = 3$. Let $|\mathcal{M}| = N_s$. In this case, the map Φ_v of for $y_i = -C$ is given by an array $[l_{i,j}]_{1 \le i,j \le N_s}$, where each $l_{i,j} \in \mathcal{M}$ specifies the output of Φ_v. Tables 1 and 2 show examples of the maps for $N_s = 5$ and $N_s = 7$. A particular choice of $[l_{i,j}]_{1 \le i,j \le N_s}$ gives rise to a particular map for Φ_v. For a given value of N_s, since there could be a very large number of N_s-level FAIDs that can potentially be used for a given code, we restrict the decoder selection to a set of valid N_s-level FAIDs, which is obtained by placing constraints on the choice of $l_{i,j}$. We specify this as a lexicographic ordering on the possible outputs of Φ_v as given below:

$$|\Phi v(|m_1|, |m_2|, -C)| \ge \Phi_v(|m_1'|, |m_2'|, -C)|$$

$\forall |m_1| \ge |m_1'|, |m_2| \ge |m_2'|$. A map Φ_v satisfying this is considered to be a valid map. We found this ordering to be a reasonable constraint that still allows good decoders.

Table 1: Map for 5-level FAID with $y_i = -C$

$m_1 \backslash m_2$	$-L_2$	$-L_1$	0	$+L_1$	$+L_2$
$-L_2$	$-L_2$	$-L_2$	$-L_2$	$-L_2$	0
$-L_1$	$-L_2$	$-L_2$	$-L_2$	$-L_1$	0
0	$-L_2$	$-L_2$	$-L_1$	0	$+L_1$
$+L_1$	$-L_2$	$-L_1$	0	0	$+L_1$
$+L_2$	0	0	$+L_1$	$+L_1$	$+L_2$

Table 2: Map for 7-level FAID with $y_i = -C$

$m_1 \backslash m_2$	$-L_3$	$-L_2$	$-L_1$	0	$+L_1$	$+L_2$	$+L_3$
$-L_3$	$-L_3$	$-L_3$	$-L_3$	$-L_3$	$-L_3$	$-L_3$	$-L_1$
$-L_2$	$-L_3$	$-L_3$	$-L_3$	$-L_3$	$-L_2$	$-L_1$	$+L_1$
$-L_1$	$-L_3$	$-L_3$	$-L_2$	$-L_2$	$-L_1$	$-L_1$	$+L_1$
0	$-L_3$	$-L_3$	$-L_2$	$-L_1$	0	0	$+L_1$
$+L_1$	$-L_3$	$-L_2$	$-L_1$	0	0	$+L_1$	$+L_2$
$+L_2$	$-L_3$	$-L_1$	$-L_1$	0	$+L_1$	$+L_1$	$+L_3$
$+L_3$	$-L_1$	$+L_1$	$+L_1$	$+L_1$	$+L_2$	$+L_3$	$+L_3$

Symmetric plane partitions: A symmetric plane partition π is an array of nonnegative integers $(\pi_{i,j})_{\{i \ge 1, j \ge 1\}}$ such that $\pi_{i,j} \ge \pi_{i+1,j}$, $\pi_{i,j} \ge \pi_{i,j+1} \forall i, j \ge 1$, and $\pi_{i,j} = \pi_{j,i}$. If $\pi_{i,j} = 0 \ \forall i > r, j > s$, and if $\pi_{i,j} \ge t$ $\forall i, j$, then the plane partition is said to be *contained* in a box with side lengths (r, s, t). The value $\pi_{i,j}$ is represented as a box of height $\pi_{i,j}$ positioned at (i, j) co-ordinate on a horizontal plane. Due to the imposition of the lexicographic ordering and symmetry of Φ_v, there exists a bijection between the array $[l_{i,j}]_{1 \le i,j \le N_s}$ and a symmetric plane partition contained in a $(N_s \times N_s \times N_s - 1)$ box, where each $\pi_{i,j}$ is determined based on $l_{i,j}$. Fig. 1 shows an example.

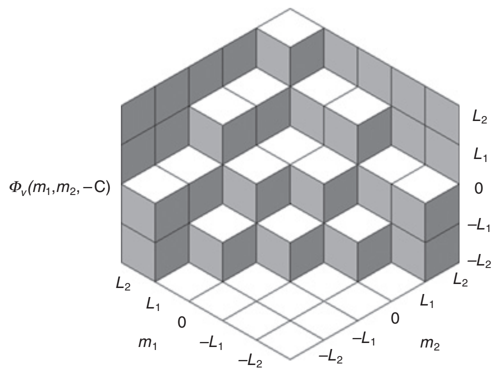

Fig. 1 *Visualisation of plane partition as stacked boxes for 5-level FAID shown in Table 1*

The total number of symmetric plane partitions contained in the $(N_s \times N_s \times N_s - 1)$ box corresponds to the total number of valid maps for Φ_v. In [3], Kuperberg gave an elegant formula for their enumeration. Using this formula, the number of N_s-level FAIDs is given by:

$$\frac{H_2(3N_s)H_1(N_s)H_2(N_s - 1)}{H_2(2N_s + 1)H_1(2N_s - 1)}$$

where $H_k(n) = (n-k)!(n-2k)! \ldots$ is called the staggered hyperfactorial function. The total number of valid maps for $N_s = 5$ and $N_s = 7$ is 28,314 and 530,803,988, respectively.

Results: Given a set of valid N_s-level FAIDs, a FAID for a given code is now selected based on its guaranteed error-correction capability rather than its asymptotic decoding threshold. By doing so, a superior performance in the error-floor region is ensured.

To validate this, we make performance comparisons on a rate 0.75 (2388,597) quasicyclic LDPC code over the binary symmetric channel (BSC). This code has been designed for good BP performance using the methodology in [4] and serves as a good test code since structured codes with similar lengths are typically used in practice. Density evolution was performed on the set of valid 5-level and 7-level FAIDs in order to determine the decoders with the best (highest) decoding threshold α^* [2].

Fig. 2 shows the frame error rate (FER) against α results for different decoders (maximum 100 iterations) with α being the cross-over probability of the BSC. The 5-level and 7-level FAIDs designed for this code are specified by Tables 1 and 2, which have thresholds of $\alpha^* = 0.022546$ and $\alpha^* = 0.023242$, respectively. They clearly outperform BP in the error-floor region. Note that Φ_v for these FAIDs can only be described by NLT functions, and hence they are different from quantised BP decoders. For instance, Φ_v for the 5-level FAID can be described by setting $L_2 = L_1 + C, C = 1.5L_1, T_1 = L_1, T_2 = L_2, \Omega(m_1, m_2) = 1 - (\text{sign}(m_1) \oplus (\text{sign}(m_2)) \times \delta(|m_1| + |m_2| - 2L_2)$ where $\text{sign}(x) = 1$ if $x < 0$, and $\text{sign}(x) = 0$ otherwise .

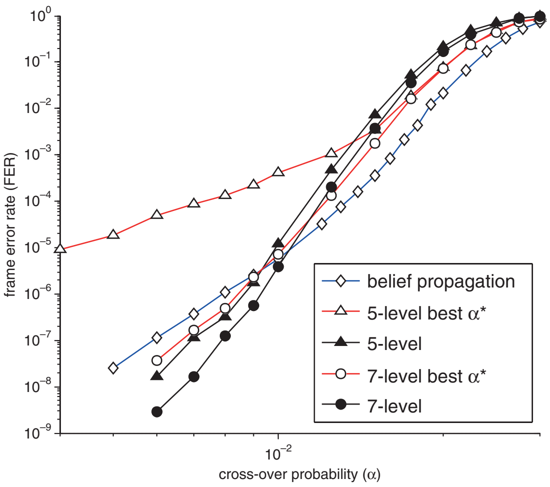

Fig. 2 *Frame error rate performance comparison on rate 0.75 (2388,597) quasicyclic code*

Also included are the results for 5-level and 7-level FAIDs with best decoding thresholds of $\alpha^* = 0.025134$ and $\alpha^* = 0.025244$, respectively. It is clearly evident that these are not the best decoders for this code. In fact the 5-level FAID with best α^* has a high error floor.

Conclusion: We have introduced a new framework of finite precision decoders termed as FAID which, when chosen appropriately, can surpass floating-point BP in the error-floor region. Since the 5-level and 7-level FAIDs are 3-bit precision MP decoders, they have only a fraction of the complexity of floating-point BP (32-bit precision was used for simulation).

Acknowledgments: This work was funded by NSF grants CCF-0830245 and CCF-0963726, and by the Nano2012 project.

6 May 2011
doi: 10.1049/el.2011.1184
One or more of the Figures in this Letter are available in colour online.

S.K. Planjery and B. Vasic (*Department of Electric Engineering, University of Arizona, 1230 E Speedway Blvd, Tucson AZ 85721, USA*)

E-mail: shivap@ece.arizona.edu

D. Declercq and L. Danjean (*ETIS, ENSEA/UCP/CNRS UMR 8051, Cergy-Pontoise 95000, France*)

References

1 Planjery, S.K., Declercq, D., Chilappagari, S.K., and Vasic, B.: 'Multilevel decoders surpassing belief propagation on the binary symmetric channel'. Proc. Int. Symp. Information Theory, (ISIT'10), Austin, TX, USA, July 2010, pp. 769–773

2 Richardson, T., and Urbanke, R.: 'Capacity of low-density parity-check codes under message-passing decoding', *IEEE Trans. Inf. Theory*, 2001, **47**, pp. 599–618

3 Kuperg, G.: 'Symmetries of plane partitions and the permanent-determinant method', *J. Comb. Theory A*, 1994, **68**, (1), pp. 115–151

4 Nguyen, D.V., Vasic, B., and Marcellin, M.: 'Structured LDPC codes from permutation matrices free of small trapping sets'. Proc. Information Theory Workshop, (ITW'10), Dublin, Ireland, September 2010, pp. 1–5